Term Project Report on-

Timoshenko Beam Theory

Prepared by-

Group No- 1

Neeruganti Ranjith Kumar (*19ME63R01*)

Sayan Batabyal (*19ME63R05*)

Sagar Das (*19ME63R13*)

Mechanical Systems Design

IIT Kharagpur

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Abstract

Euler-Bernoulli beam theory as defined in text, neglects shear stresses across the cross-section. Hence, if the beams are thick enough, then a considerable amount of result will be erroneous. In this report, a new beam theory called *Timoshenko beam theory* is studied. In this beam theory, considerable amount of shear stresses developed in the beam is accounted for computation. As a result, thicker beams will give more accurate results unlike in case of the former theory. Moreover this theory can be extended to study dynamics of beams. The effect of shear forces and rotary moments on vibration of beams have been studied. Finally, buckling of beams using this theory has also been presented. Buckling, being a nonlinear phenomenon, which is very difficult to predict beforehand will give errors because shear stresses are neglected. On the other hand, *Timoshenko beam theory* will give more accurate results. A comparison between the two theories have been provided in each section for easy reference.

Contents

Chapter 1

Introduction to Beam Theories

1.1 Literature Review

Study of beam bending and its associated problems form an integral part of mechanical engineering. The beam theories which mostly start with the legendary *Euler-Bernoulli* beam theory takes most of the credit as being the torchbearer for the problem. However, little do most engineers and educators know that the study was initiated some five centuries back by *Leonardo da Vinci*. It was again studied by *Galileo* some four centuries back and finally several of their successors gave some results which eventually lead to the *Euler-Bernoulli* beam theory.

In the year 1967, a book titled *"The Codex Madrid"* was found in the National Library of Spain. It was written by *Leonardo da Vinci* and was published in 1493. The book was summarized, translated, and discussed by Vincian scholars in *The Unknown Leonardo*[1], a book that should be of significant historical interest to the mechanical engineering community. Leonardo was not armed with the Hooke's Law and the Calculus to derive an exact expression for strength of the beam, however he developed the stress and strain distribution along a cross-section of the beam very accurately.

After another century came *Galileo Galilei*. He, in his book *"Dialogues concerning two new sciences"*[2], illustrated the problem with an alarmingly unstable looking cantilever beam supported by a wall. Galileo assumed that the beam rotated about the base at its point of support, and that there was a uniform tensile stress across the beam section equal to the tensile strength of the material. The assumption was incorrect and thus, his theory was rejected in the very first place.

According to the book *"History of Strength of Materials"*[3] by Stephen Timoshenko, the alternative approach to solve Galileo's error was published by Edme Mariotte in 1686 where he assumed the stress distribution to be of triangular shape where the highest value came near the neutral axis. He also proposed the neutral axis to be at the centre(without proof). However the result proved to be quite inaccurate for any general cross-section.

It was in the year 1713, when Antoine Parent derived the correct stress distribution across a beam cross-section. Unfortunately Parent's work had little impact in the field. Around the same time, the Bernoulli family was rising to prominence. Jacob Bernoulli in his final year(1705) gave out a form for stress distribution, which later took 2 generations to come out in its final form and some 2 centuries to be used in real world. It was Daniel Bernoulli, Johann Bernoulli's son and Leonhard Euler, Johann Bernoulli's pupil who together gave the final *'Euler-Bernoulli'* beam theory. It was Daniel Bernoulli who proposed the idea to Leonhard Euler in a famous letter, where he said-

> "Since no one is so completely the master of the isoparametric method as you are, you will very easily solve the following problem in which it is required that $\int \frac{ds}{r^2}$ shall be a minimum." [3]

Euler-Bernoulli beam theory came in 1750. However it was till late 19th century, people could not trust the work. It was in 1889, the Eiffel Tower was built based on the theory. People believed no structure taller than 300m would survive and objected its building. However it stands erect till today. From then, *Euler-Bernoulli* beam theory became a very common name to engineers and it is so till today. Many successors to the beam theory have come and gives better results, but none saw the popularity as was seen by Euler-Bernoulli beam theory.

1.2 Euler-Bernoulli beam theory

A brief introduction to the Euler-Bernoulli beam theory is very necessary for advancing the topic to the main goal of *Timoshenko beam theory.* Detailed study can be found in the textbook *"Solid Mechanics- A Variational Approach"*[4] by Dym and Shames.
Some assumptions in the *Euler-Bernoulli* beam theory are-

1. Plane sections remain plane after bending, thereby $\sigma_{xx} = \frac{My}{I_z}$ holds true.
2. Plane section of a beam remain perpendicular to the neutral axis after bending, thus there is no shear strain along beam cross-section.

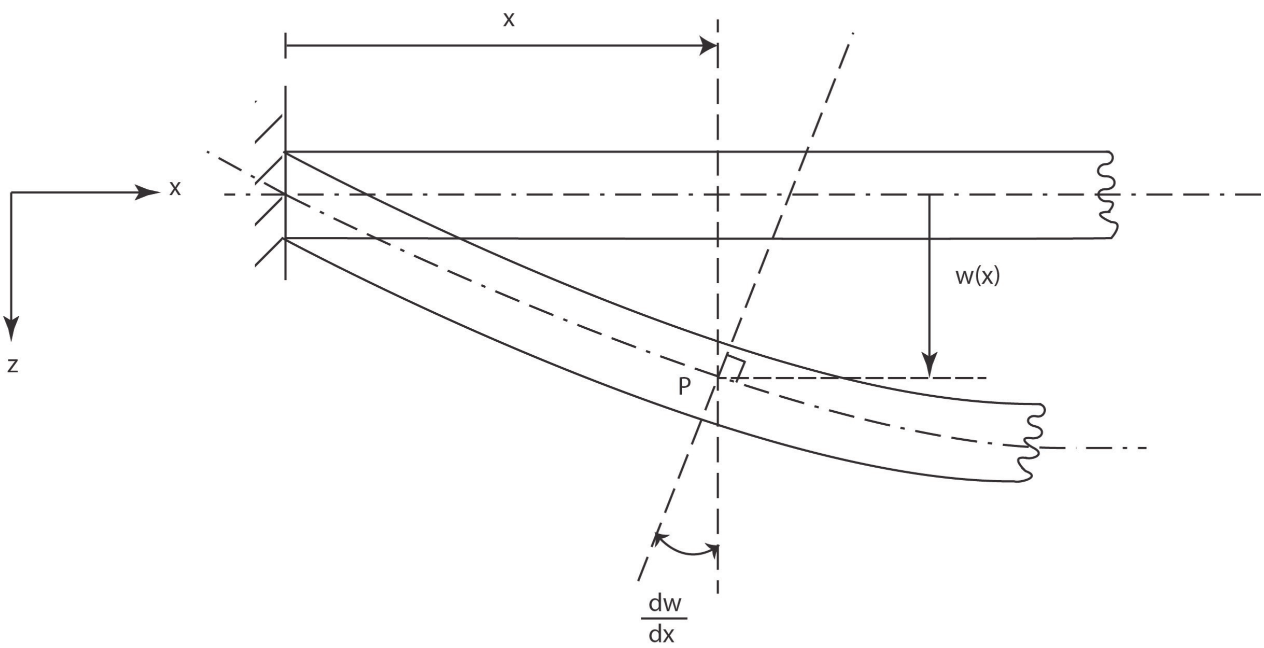

Figure 1.1: Deformation of Euler-Bernoulli beam

Thus, the proposed displacement field are-

$$u = u_s - z\frac{dw}{dx} \tag{1.1a}$$

$$v = 0 \tag{1.1b}$$

$$w = w(x) \tag{1.1c}$$

Using *Strain-Displacement relations*, the strains can be determined easily.
[*Note*- The nonlinear components of strains are ignored]

$$\epsilon_{xx} = \frac{du}{dx} = \frac{du_s}{dx} - z\frac{d^2w}{dx^2} \tag{1.2}$$

Except this, all other strains turn out be 0.
Accordingly, the stress and strain values are implemented in the Principle of Virtual Work-

$$\int_V \sigma_{xx}\delta\epsilon_{xx}dV = \int_S q(x)\delta w dS \tag{1.3}$$

Substituting the strain as obtained in equation (1.2), the governing equation can be determined as-

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x) \tag{1.4}$$

where $q(x)$ is the loading applied on the beam in the transverse direction.

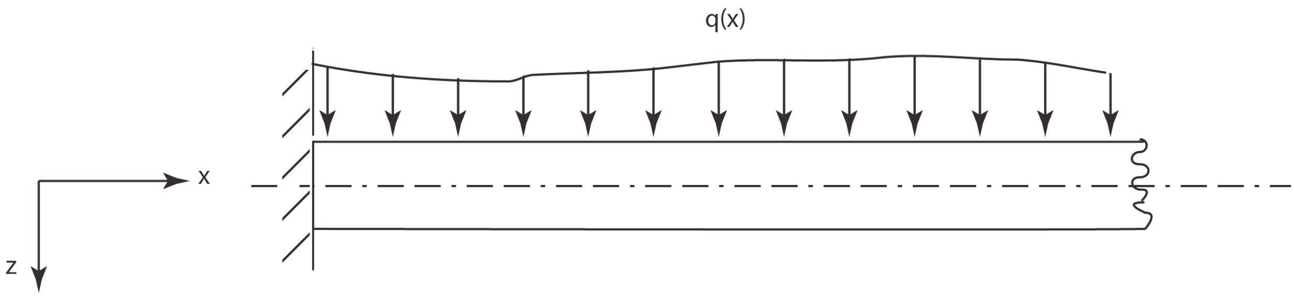

Figure 1.2: Euler-Bernoulli beam under applied load

1.3 Modification to Euler-Bernoulli beam theory

Moving ahead of the Euler-Bernoulli beam theory, the author would like to draw the attention of the readers towards the approximations used in Euler-Bernoulli beam theory. Although these approximations work quite well for thin beams, they falter as beams become thicker. The reason can be well credited to the negligence towards the shear strain developed across the beam cross-section. Thus, in the preceding chapters there will be an extensive discussion of beam theories where shear strains will be taken included. This beam theory, which came up in the early 20th century, developed by S.P. Timoshenko [5], is very commonly called *Timoshenko* beam theory. Several problems involving beam bending, vibration and buckling will be discussed in the upcoming chapters.

Chapter 2

Bending of Timoshenko Beams

2.1 Introduction

There is already a brief section presented in section (1.3) stating about the motivation of moving from Euler-Bernoulli beam theory to Timoshenko beam theory. In the Timoshenko beam theory, the assumption of plane sections remaining plane still is considered, but the plane section does not remain perpendicular to the neutral axis as opposed to the case of Euler-Bernoulli beams. So, there is a provision of the beams to be thick to some extent. We assume the plane section to rotate by an angle ϕ with respect to a perpendicular from the neutral axis, as shown in figure (2.1).

2.2 Mathematical Formulation

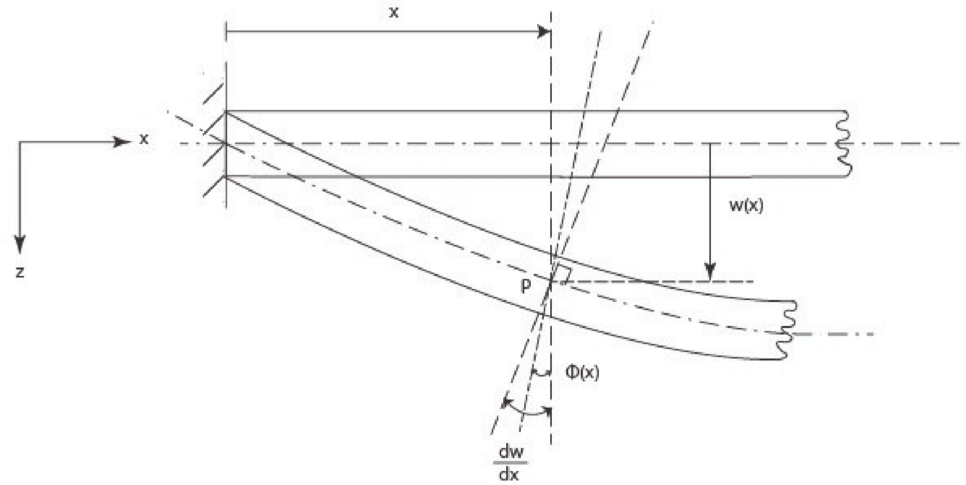

Figure 2.1: Bending of Timoshenko Beam

From the figure, the displacement fields can be taken as-

$$u = u_s - z\phi(x) \tag{2.1a}$$
$$v = 0 \tag{2.1b}$$
$$w = w(x) \tag{2.1c}$$

The strains can be obtained from Strain-Displacement relations-

$$\epsilon_{xx} = \frac{du}{dx} = \frac{du_s}{dx} - z\frac{d\phi}{dx} \tag{2.2a}$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right) = \frac{1}{2}\left(-\phi + \frac{dw}{dx}\right) \tag{2.2b}$$

Except ϵ_{xx} and ϵ_{xz}, all other strains become zero. They can be easily shown by substituting in the corresponding strain-displacement relations.
There is a need to look into the values of stresses from the strain relations, as there are some small approximations involved here too. For a linear, elastic, isotropic solid, the constitutive relation is given by-

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} \tag{2.3}$$

Using equation (2.3), we can write-

$$\sigma_{xx} = (\lambda + 2G)\,\epsilon_{xx} + \lambda\,(\epsilon_{yy} + \epsilon_{zz}) \tag{2.4}$$

It can be easily determined that-

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{2.5}$$

$$\lambda + 2G = \frac{E}{2(1+\nu)(1-2\nu)} \tag{2.6}$$

From equation (2.5), it can be said that if ν is very small, or $\nu \approx 0$ then $\lambda \approx 0$ and $\lambda + 2G \approx E$. Thus the Stress-Strain relation can be simplified as-

$$\sigma_{xx} = E\epsilon_{xx} = E\left(\frac{du_s}{dx} - z\frac{d\phi}{dx}\right) \tag{2.7a}$$

$$\sigma_{xz} = 2G\epsilon_{xz} = G\left(-\phi + \frac{dw}{dx}\right) \tag{2.7b}$$

Note, equation (2.7b) gives an expression for the shear stress, σ_{xz}. However to make the results match with the practical results, a new term called *Timoshenko shear coefficient*, denoted by κ is introduced. So-

$$\sigma_{xz} = G\kappa\left(-\phi + \frac{dw}{dx}\right) \tag{2.8}$$

In order to derive the deflection equations, the *Principle of Virtual Work* is recalled.

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV = \int_S T_i\delta u_i dS \tag{2.9}$$

where T_i is the traction force on the surface and u_i is the corresponding displacement. While deriving, first the two sides of the equation will be simplified and then equated.

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV = \int_V (\sigma_{xx}\delta\epsilon_{xx} + 2\sigma_{xz}\delta\epsilon_{xz})\,dV \tag{2.10}$$

Substituting equation (2.7a) and (2.8) in equation (2.10)-

$$\begin{aligned}&\int_V \sigma_{ij}\delta\epsilon_{ij}dV \\ &= \int_0^L\int_{-h/2}^{h/2}\left[E\left(\frac{du_s}{dx} - z\frac{d\phi}{dx}\right)\left(\frac{d\delta u_s}{dx} - z\frac{d\delta\phi}{dx}\right) + G\left(-\phi + \frac{dw}{dx}\right)\left(-\delta\phi + \frac{d\delta w}{dx}\right)\right]bdzdx\end{aligned} \tag{2.11}$$

Simplifying equation (2.11) will give some important results. The detailed algebra is performed in Jupyter Notebook and presented in Appendix B. The final result is presented here.

$$\begin{aligned}\int_V \sigma_{ij}\delta\epsilon_{ij}dV = &\int_0^L EA\frac{du_s}{dx}\frac{d\delta u_s}{dx}dx + \int_0^L EI\frac{d\phi}{dx}\frac{d\delta\phi}{dx}dx \\ &- \int_0^L GA\kappa\left(-\phi + \frac{dw}{dx}\right)\delta\phi dx + \int_0^L GA\kappa\left(-\phi+\frac{dw}{dx}\right)\frac{d\delta w}{dx}dx\end{aligned} \tag{2.12}$$

Note that, area of the cross-section, $A = \int_{-h/2}^{h/2} bdz$ and moment of inertia, $I = \int_{-h/2}^{h/2} bz^2dz$, where b is the depth of the cross-section.

$$\begin{aligned}\int_V \sigma_{ij}\delta\epsilon_{ij}dV = &\left[EA\frac{du_s}{dx}\delta u_s\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du_s}{dx}\right)\delta u_s dx + \left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L \\ &- \int_0^L \frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)\delta\phi dx - \int_0^L GA\kappa\left(-\phi+\frac{dw}{dx}\right)\delta\phi dx \\ &+ \left[GA\kappa\left(-\phi+\frac{dw}{dx}\right)\delta w\right]_0^L - \int_0^L \frac{d}{dx}\left(GA\kappa\left(-\phi+\frac{dw}{dx}\right)\right)\delta w dx\end{aligned} \tag{2.13}$$

Moving onto the right hand side of equation (2.9)-

$$\int_S T_i\delta u_i dS = [P\delta u_s]_0^L + \int_0^L q(x)\delta w dx \tag{2.14}$$

Since both the sides of equation (2.9) are simplified, they are substituted in the equation and further proceeded.

$$\begin{aligned}&\left[\left(EA\frac{du_s}{dx} - P\right)\delta u_s\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du_s}{dx}\right)\delta u_s dx + \left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L \\ &- \int_0^L \frac{d}{dx}\left(EI\frac{d\phi}{dx}\right)\delta\phi dx - \int_0^L GA\kappa\left(-\phi+\frac{dw}{dx}\right)\delta\phi dx \\ &+ \left[GA\kappa\left(-\phi+\frac{dw}{dx}\right)\delta w\right]_0^L - \int_0^L \frac{d}{dx}\left(GA\kappa\left(-\phi+\frac{dw}{dx}\right)\right)\delta w dx = \int_0^L q(x)\delta w dx\end{aligned}$$

Bringing the like terms together-

$$\begin{aligned}&\left[\left(EA\frac{du_s}{dx} - P\right)\delta u_s\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\frac{du_s}{dx}\right)\delta u_s dx + \left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L \\ &- \int_0^L \left[\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GA\kappa\left(-\phi+\frac{dw}{dx}\right)\right]\delta\phi dx + \left[GA\kappa\left(-\phi+\frac{dw}{dx}\right)\delta w\right]_0^L \\ &- \int_0^L \left[\frac{d}{dx}\left(GA\kappa\left(-\phi+\frac{dw}{dx}\right)\right) + q\right]\delta w dx = 0\end{aligned} \tag{2.15}$$

The equation (2.15) gives us the Governing Differential Equation and the Boundary Conditions for the bending of a Timoshenko beam.
The governing differential equations are-

$$\frac{d}{dx}\left(EA\frac{du_s}{dx}\right) = 0 \tag{2.16a}$$

$$\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GA\kappa\left(-\phi+\frac{dw}{dx}\right) = 0 \tag{2.16b}$$

$$\frac{d}{dx}\left[GA\kappa\left(-\phi+\frac{dw}{dx}\right)\right] + q = 0 \tag{2.16c}$$

The boundary conditions for the problem are-

1. At x=0 and x=L, either $EA\frac{du_s}{dx} = P$ or u_s is specified.
2. At x=0 and x=L, either $EI\frac{d\phi}{dx} = 0$ or ϕ is specified.
3. At x=0 and x=L, either $GA\kappa\left(-\phi + \frac{dw}{dx}\right) = 0$ or w is specified.

From equation (2.16b), it can be said that-

$$GA\kappa\left(-\phi + \frac{dw}{dx}\right) = -\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) \tag{2.17}$$

Substituting equation (2.17) in equation (2.16c)-

$$EI\frac{d^3\phi}{dx^3} = q \tag{2.18}$$

Also, equation (2.16b) can be broken down as-

$$\frac{dw}{dx} = \phi - \frac{EI}{GA\kappa}\frac{d^2\phi}{dx^2} \tag{2.19}$$

Thus, the final set of governing differential equations are-

$$EA\frac{d^2u_s}{dx^2} = 0 \tag{2.20a}$$

$$EI\frac{d^3\phi}{dx^3} = q \tag{2.20b}$$

$$\frac{dw}{dx} = \phi - \frac{EI}{GA\kappa}\frac{d^2\phi}{dx^2} \tag{2.20c}$$

2.3 Physical Interpretations

For the time being, the axial stretching part is ignored. Considering only the bending part, the bending moment, M_{xx} and shear force, Q_x can be determined.

$$\begin{aligned} M_{xx} &= \int_{-h/2}^{h/2} z\sigma_{xx} b dz \\ &= \int_{-h/2}^{h/2} zE\epsilon_{xx} b dz \\ &= -\int_{-h/2}^{h/2} Ez^2\frac{d\phi}{dx} b dz \\ &= -EI\frac{d\phi}{dx} \end{aligned} \tag{2.21}$$

$$\begin{aligned} Q_x &= \int_{-h/2}^{h/2} \sigma_{xz} b dz \\ &= \int_{-h/2}^{h/2} 2G\kappa\epsilon_{xz} b dz \\ &= \int_{-h/2}^{h/2} G\kappa\left(-\phi + \frac{dw}{dx}\right) b dz \\ &= GA\kappa\left(-\phi + \frac{dw}{dx}\right) \end{aligned} \tag{2.22}$$

So using the above terms, the governing differential equation can be viewed in a different light in terms of moments and shear forces. Equation (2.16b) can be written as-

$$\frac{dM_{xx}}{dx} = Q_x \tag{2.23}$$

Similarly equation (2.16c) can be written as-

$$\frac{dQ_x}{dx} + q = 0 \tag{2.24}$$

Both equation (2.23) and (2.24) are very well perceived from bending moment and shear force diagrams of a beam. Similarly, the boundary conditions can also be thought of using the bending moment and shear force concept.

2.4 Example of a simply supported beam

Consider a simply supported beam of length L with a uniformly distributed load q along its length acting in the transverse direction. The situation is shown in figure (2.2). Since there

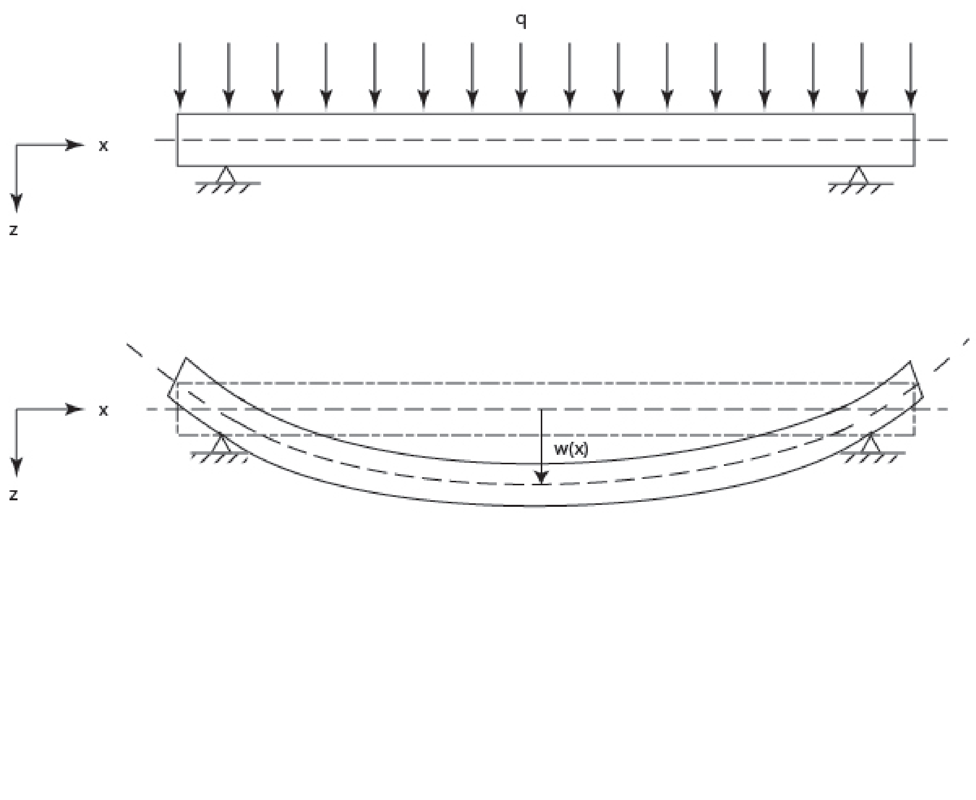

Figure 2.2: Simply supported beam with uniformly distributed load

is no axial load, axial stretching u_s can be neglected, i.e. equation (2.20a) is not considered.

Equation (2.20b) and (2.20c) can be used to obtain values of w and ϕ.

$$EI\frac{d^3\phi}{dx^3} = q$$
$$\Rightarrow \phi = c_1 + c_2x + c_3x^2 + \frac{qx^3}{6EI} \tag{2.25}$$

$$\frac{dw}{dx} = \phi - \frac{EI}{GA\kappa}\frac{d^2\phi}{dx^2}$$
$$\Rightarrow \frac{dw}{dx} = \left(c_1 + c_2x + c_3x^2 + \frac{qx^3}{6EI}\right) - \frac{EI}{GA\kappa}\left(2c_3 + \frac{qx}{EI}\right) \tag{2.26}$$
$$\Rightarrow w = c_1x + \frac{c_2}{2}x^2 + \frac{c_3}{3}x^3 + c_4 + \frac{qx^4}{24EI} - \frac{2EI}{GA\kappa}c_3x - \frac{qx^2}{2GA\kappa}$$

where c_1, c_2, c_3 and c_4 are the arbitrary constants which have to be determined using the boundary conditions.
Invoking the 1st boundary condition i.e. $w = 0$ at $x = 0$-

$$w(0) = c_4 = 0 \tag{2.27}$$

The 2nd boundary condition i.e. at $x = 0$, $M_{xx} = 0$ or $\frac{d\phi}{dx} = 0$-

$$\left.\frac{d\phi}{dx}\right|_{x=0} = c_2 = 0 \tag{2.28}$$

The 3rd boundary condition i.e. at $x = L$, $w = 0$-

$$w(L) = c_1L + \frac{c_3}{3}L^3 + \frac{qL^4}{24EI} - \frac{2EI}{GA\kappa}c_3L - \frac{qL^2}{2GA\kappa} \tag{2.29}$$

The 4th boundary condition i.e. at $x = L$, $M_{xx} = 0$ or $\frac{d\phi}{dx} = 0$-

$$2c_3L + \frac{qL^2}{2EI} = 0 \Rightarrow c_3 = -\frac{qL}{4EI} \tag{2.30}$$

Substituting equation (2.30) in (2.29), gives the value of the final constant c_1.

$$c_1L - \frac{qL^4}{24EI} = 0 \Rightarrow c_1 = \frac{qL^3}{24EI} \tag{2.31}$$

Since all the constants are determined, the values can be substituted in equation (2.26) to find the deflection of the beam, $w(x)$.

$$w(x) = \frac{qxL^3}{24EI}\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) + \frac{qxL}{2GAK}\left(1 - \frac{x}{L}\right) \tag{2.32}$$

In case of Euler-Bernoulli beams, the deflection is given by the 1st part of equation(2.32) which is quite intuitive since shear stress is ignored here. The next section deals with a detailed comparison between the 2 theories.

2.5 Comparison between the two theories

The comparison can be easily initiated with the thought that there is no shear stress term in Euler-Bernoulli beam theory. As a result, for the same problem solved in section (2.4) using Euler-Bernoulli beam theory would yield the deflection as-

$$w(x) = \frac{qxL^3}{24EI}\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) \tag{2.33}$$

Let the deflections arising out of the two cases be denoted by $w_1(x)$ for the Euler-Bernoulli beam and $w_2(x)$ for the Timoshenko beam. Also, $G = \frac{E}{2(1+\nu)}$ and $\nu \approx 0$. So-

$$w_1(x) = \frac{qx}{2E}\left(\frac{L}{h}\right)^3\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) \tag{2.34}$$

$$w_2(x) = \frac{qx}{2E}\left(\frac{L}{h}\right)^3\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) + \frac{qx}{EK}\left(\frac{L}{h}\right)\left(1 - \frac{x}{L}\right) \tag{2.35}$$

At the mid-point where maximum deflection occurs-

$$w_1(x) = \frac{qx}{2E}\left(\frac{L}{h}\right)^3\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) \tag{2.36}$$

$$w_2(x) = \frac{qx}{2E}\left(\frac{L}{h}\right)^3\left(1 - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right) + \frac{qx}{EK}\left(\frac{L}{h}\right)\left(1 - \frac{x}{L}\right) \tag{2.37}$$

At mid-point i.e. $x = \frac{L}{2}$, the deflection is found out as-

$$w_1(x) = \frac{5q}{128E}\left(\frac{L}{h}\right)^3 \tag{2.38}$$

$$w_2(x) = \frac{5q}{128E}\left(\frac{L}{h}\right)^3 + \frac{q}{4E\kappa}\left(\frac{L}{h}\right) \tag{2.39}$$

Taking the ratio of the 2 deflections-

$$\frac{w_2}{w_1} = 1 + \frac{3}{5}\left(\frac{h}{L}\right)^2 \tag{2.40}$$

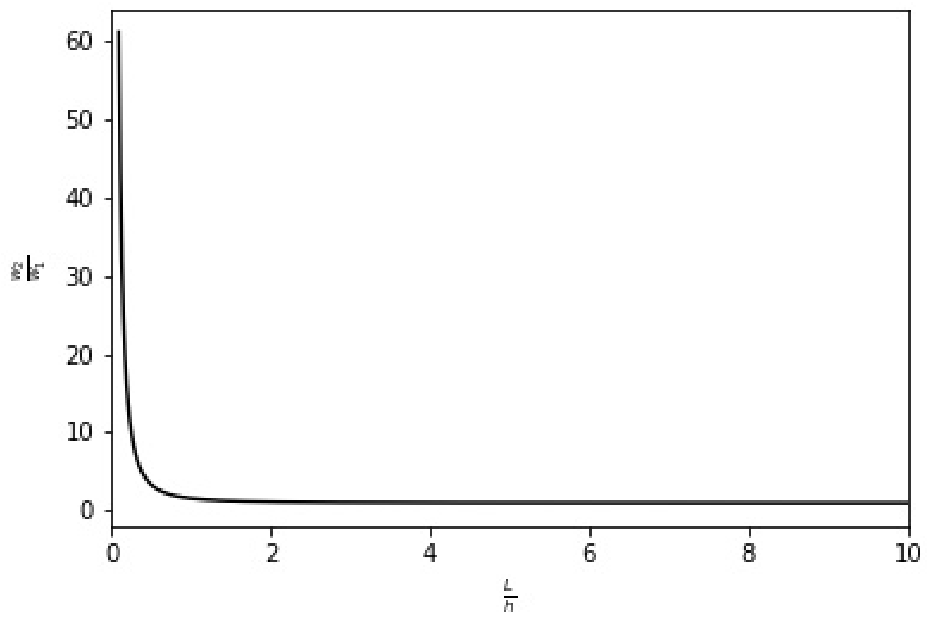

Figure 2.3: Variation of the ratio of deflection with respect to the slenderness ratio

From figure (2.4), we can easily conclude that as $\frac{L}{h}$ increases, deviation between the 2 results decreases. Generally, for stocky beams i.e. $\frac{L}{h} < 10$, Timoshenko theory gives good results. On the other hand, for slender beams, i.e. $\frac{L}{h} > 20$, Euler-Bernoulli beam theory is more appropriate.

Chapter 3

Vibration of Timoshenko Beams

3.1 Introduction

In the previous chapter, the bending of beams was presented at length. However the entire discussion was on static beams. Dynamic problems say, vibration of a beam is also necessary to be studied in details. This chapter deals with study of vibration of beams in detail. To study Timoshenko beams, an introductory study of Euler-Bernoulli beams is very necessary, hence the first section is dedicated to Euler-Bernoulli beams only.

3.2 Vibration of Euler-Bernoulli beams

As given in equations (1.1), the displacement field of Euler-Bernoulli beam remains the same. However being a dynamic problem, there will be an additional kinetic energy, T in this case. The potential energy, U remains the same as before.

$$T = \int_0^l \frac{1}{2}\rho A \left(\frac{dw}{dt}\right)^2 dx \tag{3.1}$$

$$U = \int_0^l \frac{1}{2}EI \left(\frac{d^2w}{dx^2}\right)^2 dx \tag{3.2}$$

where ρ is the density of the beam, A is the area of the cross-section and I is the moment of inertia of the beam.
In the presence of an external load, $q(x)$, the Lagrangian turns out as-

$$\begin{aligned} \mathcal{L} &= T - U \\ \Rightarrow \mathcal{L} &= \int_0^l \frac{1}{2}\rho A \left(\frac{dw}{dt}\right)^2 dx - \int_0^l \frac{1}{2}EI \left(\frac{d^2w}{dx^2}\right)^2 dx + \int_0^l q(x)w dx \end{aligned} \tag{3.3}$$

To obtain the equation of motion, we extremize the Lagrangian, L. A more detailed derivation on how to extremize the Lagrangian is presented in the Appendix A. Also, a similar derivation in a very detailed manner will be presented in the next section. Finally, we get the equation of motion-

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = -\rho A\frac{\partial^2 w}{\partial t^2} + q(x) \tag{3.4}$$

Considering free vibration, the solution procedure is very easy. Any standard textbook[6] will guide through the steps. Substituting appropriate boundary conditions will give the mode shapes for the beam.

3.3 Vibration of Timoshenko beams

The presence of shear strain in Timoshenko beams will make the problem a bit complicated. The displacement fields as defined in equation (2.1) are-

$$u = -z\phi(x) \tag{3.5a}$$
$$v = 0 \tag{3.5b}$$
$$w = w(x) \tag{3.5c}$$

Equations (3.1) and (3.2) i.e. the kinetic energy and the potential energy can be written as-

$$T = \int_0^l \frac{1}{2}\rho A \left(\frac{dw}{dt}\right)^2 dx + \int_0^l \frac{1}{2}\rho I \left(\frac{d\phi}{dt}\right)^2 dx \tag{3.6}$$
$$U = \int_0^l \frac{1}{2}EI \left(\frac{d\phi}{dx}\right)^2 dx + \int_0^l \frac{1}{2}GA\kappa \left(-\phi + \frac{dw}{dx}\right)^2 dx \tag{3.7}$$

Substituting equation (3.6) and (3.7) in the Lagrangian yields-

$$\mathcal{L} = \int_0^l \frac{1}{2}\rho A \left(\frac{dw}{dt}\right)^2 dx + \int_0^l \frac{1}{2}\rho I \left(\frac{d\phi}{dt}\right)^2 dx - \int_0^l \frac{1}{2}EI \left(\frac{d\phi}{dx}\right)^2 dx - \int_0^l \frac{1}{2}GA\kappa \left(-\phi + \frac{dw}{dx}\right)^2 dx \tag{3.8}$$

To obtain the equation of motion, the Lagrangian has to be extremized. According to the *Principle of Least Action*-

$$\delta \int_{t_1}^{t_2} \mathcal{L}dt = 0 \tag{3.9}$$

Since the Lagrangian in equation (3.8) is of the form-

$$\mathcal{L} = \int_0^l \hat{\mathcal{L}}dx \tag{3.10}$$

$\hat{\mathcal{L}}$ is commonly called *Lagrangian Density.* Substituting equation (3.10) in equation (3.9) yields-

$$\delta \int_{t_1}^{t_2} \int_0^l \hat{\mathcal{L}}dxdt = 0 \tag{3.11}$$

A detailed method of the extremization is presented in Appendix A. The final results obtained are-

$$\frac{\partial \hat{\mathcal{L}}}{\partial w} - \frac{d}{dt}\frac{\partial \hat{\mathcal{L}}}{\partial \dot{w}} - \frac{d}{dx}\frac{\partial \hat{\mathcal{L}}}{\partial w'} + q(x) = 0 \tag{3.12a}$$
$$\frac{\partial \hat{\mathcal{L}}}{\partial \phi} - \frac{d}{dt}\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\phi}} - \frac{d}{dx}\frac{\partial \hat{\mathcal{L}}}{\partial \phi'} = 0 \tag{3.12b}$$

where $\dot{()} = \frac{d()}{dt}$ and $()' = \frac{d()}{dx}$.
From equation (3.8), the Lagrangian Density, $\hat{\mathcal{L}}$ can be written as-

$$\hat{\mathcal{L}} = \frac{1}{2}\rho A\dot{w}^2 + \frac{1}{2}\rho I\dot{\phi}^2 - \frac{1}{2}EI\phi'^2 - \frac{1}{2}GA\kappa\left(-\phi + w'\right)^2 \tag{3.13}$$

Substituting equation (3.13) in equation (3.12a) yields-

$$-\rho A\ddot{w} + GA\kappa\left(-\phi' + w''\right) + q(x) = 0 \tag{3.14}$$

Similarly substituting in equation (3.12b) yields-

$$-\rho I\ddot{\phi} + EI\phi'' + GA\kappa\left(-\phi + w'\right) = 0 \tag{3.15}$$

Equations (3.14) and (3.15) thus turn out as-

$$\rho A\frac{\partial^2 w}{\partial t^2} = GA\kappa\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + q(x) \tag{3.16a}$$

$$\rho I\frac{\partial^2 \phi}{\partial t^2} = EI\frac{\partial^2 \phi}{\partial x^2} + GA\kappa\left(-\phi + \frac{\partial w}{\partial x}\right) \tag{3.16b}$$

We can easily use the equations (3.16a) and (3.16b) to eliminate ϕ and thus obtain the equation of motion in terms of w.

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \left(\rho I + \frac{\rho EI}{G\kappa}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{G\kappa}\frac{\partial^4 w}{\partial t^4} = q - \frac{EI}{GA\kappa}\frac{\partial^2 q}{\partial x^2} \tag{3.17}$$

The above equation is the final equation of motion for a dynamic Timoshenko beam. Note that the externally applied force, $q(x)$ is assumed to be independent of time. Many books do consider to add up a term involving a time derivative of the force.

3.4 Free Vibration Analysis

Since this section starts of with an intention to solve the beam vibration equation with no externally applied load, equation (3.17) gets modified as-

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \left(\rho I + \frac{\rho EI}{G\kappa}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{G\kappa}\frac{\partial^4 w}{\partial t^4} = 0 \tag{3.18}$$

Equation (3.18) gives out a fourth order, homogeneous, partial differential equation of the form-

$$A\frac{\partial^4 w}{\partial x^4} + B\frac{\partial^4 w}{\partial x^2 \partial t^2} + C\frac{\partial^4 w}{\partial t^4} + D\frac{\partial^2 w}{\partial t^2} = 0 \tag{3.19}$$

On inspection-

$$\begin{aligned} B^2 - 4AC &= \left(\rho I + \frac{\rho EI}{G\kappa}\right)^2 - 4EI\frac{\rho^2 I}{G\kappa} \\ &= \left(\rho I - \frac{\rho EI}{G\kappa}\right)^2 > 0 \end{aligned} \tag{3.20}$$

Thus, equation (3.18) is a hyperbolic partial differential equation. So the solution would be oscillating with time, hence the solution is taken in the form of-

$$w(x,t) = W(x)e^{i\omega t} \tag{3.21}$$

Substitute this value of $w(x,t)$ in equation (3.18)-

$$\begin{aligned} &EI\frac{d^4 W}{dx^4} - \rho A\omega^2 W + \left(\rho I + \frac{\rho EI}{G\kappa}\right)\omega^2\frac{d^2 W}{dx^2} + \frac{\rho^2 I}{G\kappa}\omega^4 W = 0 \\ \Rightarrow\ &EI\frac{d^4 W}{dx^4} + \rho I\omega^2\left(1 + \frac{E}{G\kappa}\right)\frac{d^2 W}{dx^2} + \rho A\omega^2\left(\frac{\rho I}{GA\kappa}\omega^2 - 1\right)W = 0 \\ \Rightarrow\ &\frac{d^4 W}{dx^4} + \frac{\rho\omega^2}{E}\left(1 + \frac{E}{G\kappa}\right)\frac{d^2 W}{dx^2} + \frac{\rho A\omega^2}{EI}\left(\frac{\rho I}{GA\kappa}\omega^2 - 1\right)W = 0 \end{aligned} \tag{3.22}$$

Let equation (3.22) be assumed in the form-

$$\frac{d^4W}{dx^4} + \alpha\frac{d^2W}{dx^2} + \beta W = 0 \tag{3.23}$$

To solve for the differential equation, substitute $W(x) = e^{mx}$. So-

$$m^4 + \alpha m^2 + \beta = 0 \tag{3.24}$$

Solving equation (3.24) gives out the result in the form-

$$m = -\left[\frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}\right]^{1/2} \tag{3.25}$$

In the above equations-

$$\alpha = \frac{\rho\omega^2}{M_r} \tag{3.26}$$

where $\frac{1}{M_r} = \frac{1}{E} + \frac{1}{G\kappa}$ and M_r is called the *reduced modulus.*
Also,

$$\beta = \frac{\rho^2\omega^2}{\kappa GE}\left(\omega^2 - \omega_c^2\right) \tag{3.27}$$

where $\omega_c = \sqrt{\frac{GA\kappa}{\rho I}}$ which is called the *critical frequency.*
As is visible from equation (3.27), the sign of β depends on ω_c. If $\omega > \omega_c$, then β is positive, or else negative. Now, from equation (3.24), it can be said that the sign of the solution is also dependent on β. For $\omega < \omega_c$, it can be shown that both roots of m will be real. On the other hand, if $\omega > \omega_c$, both roots will be imaginary. Thus, the final solution to equation (3.23) can be written as-

$$W(x) = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + C_4e^{m_4x} \tag{3.28}$$

where C_1, C_2, C_3 and C_4 are constants which are to be determined from boundary conditions and m_1, m_2, m_3 and m_4 are the roots of equation (3.24) also given in equation (3.25). So-

$$w(x,t) = \left(C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + C_4e^{m_4x}\right)e^{i\omega t} \tag{3.29}$$

Using the above equation, there is a need to obtain the value of ϕ. Substitute equation (3.29) in equation (3.16a) to obtain value of $\frac{d\phi}{dx}$. From equation (3.16a)-

$$\frac{\partial\phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\rho}{G\kappa}\frac{\partial^2 w}{\partial t^2} \tag{3.30}$$

Substituting equation (3.21)-

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{d^2W}{dx^2}e^{i\omega t} + \frac{\rho\omega^2}{G\kappa}We^{i\omega t}\\ \Rightarrow \phi &= \frac{dW}{dx}e^{i\omega t} + \frac{\rho\omega^2}{G\kappa}e^{i\omega t}\int W dx + C_5(t)\end{aligned} \tag{3.31}$$

From equation (3.31), the value of ϕ is wanted in the form of an eigenvalue problem i.e.-

$$\phi(x,t) = \Phi(x)e^{i\omega t} \tag{3.32}$$

Thus, we can conclude that $C_5(t) = 0$ to satisfy equation (3.32). Hence-

$$\Phi(x) = \frac{dW}{dx} + \frac{\rho\omega^2}{G\kappa}\int W dx \tag{3.33}$$

Substituting equation (3.28)-

$$\begin{aligned}\Phi(x) = C_1\left(m_1+\frac{\omega^2\rho}{G\kappa m_1}\right)e^{m_1x}+C_2\left(m_2+\frac{\omega^2\rho}{G\kappa m_2}\right)e^{m_2x}\\+C_3\left(m_3+\frac{\omega^2\rho}{G\kappa m_3}\right)e^{m_3x}+C_4\left(m_4+\frac{\omega^2\rho}{G\kappa m_4}\right)e^{m_4x}\end{aligned}\tag{3.34}$$

At this juncture, it will be useful to remember that m_1, m_2, m_3 and m_4 are all functions of ω. So, the coefficients of $\Phi(x)$ will also contain ω.
Using the obtained values of $W(x)$ an $\phi(x)$, several problems can be solved using appropriate boundary conditions.

3.5 Example of a simply supported beam

For a simply supported beam, the boundary conditions can be specified as-

1. At any time t, $W = 0$ at $x = 0$ and at $x = L$.
2. At any time t, $\frac{d\Phi}{dx} = 0$ at $x = 0$ and at $x = L$.

The two boundary conditions are already discussed in details in section (2.4). Substituting the values, we obtain the matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ e^{Lm_1} & e^{Lm_2} & e^{Lm_3} & e^{Lm_4}\\ m_1\left(m_1+\frac{\omega^2\rho}{G\kappa m_1}\right) & m_2\left(m_2+\frac{\omega^2\rho}{G\kappa m_2}\right) & m_3\left(m_3+\frac{\omega^2\rho}{G\kappa m_3}\right) & m_4\left(m_4+\frac{\omega^2\rho}{G\kappa m_4}\right)\\ m_1\left(m_1+\frac{\omega^2\rho}{G\kappa m_1}\right)e^{Lm_1} & m_2\left(m_2+\frac{\omega^2\rho}{G\kappa m_2}\right)e^{Lm_2} & m_3\left(m_3+\frac{\omega^2\rho}{G\kappa m_3}\right)e^{Lm_3} & m_4\left(m_4+\frac{\omega^2\rho}{G\kappa m_4}\right)e^{Lm_4}\end{bmatrix}\begin{Bmatrix}C_1\\C_2\\C_3\\C_4\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

For non-trivial solutions-

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ e^{Lm_1} & e^{Lm_2} & e^{Lm_3} & e^{Lm_4}\\ m_1\left(m_1+\frac{\omega^2\rho}{G\kappa m_1}\right) & m_2\left(m_2+\frac{\omega^2\rho}{G\kappa m_2}\right) & m_3\left(m_3+\frac{\omega^2\rho}{G\kappa m_3}\right) & m_4\left(m_4+\frac{\omega^2\rho}{G\kappa m_4}\right)\\ m_1\left(m_1+\frac{\omega^2\rho}{G\kappa m_1}\right)e^{Lm_1} & m_2\left(m_2+\frac{\omega^2\rho}{G\kappa m_2}\right)e^{Lm_2} & m_3\left(m_3+\frac{\omega^2\rho}{G\kappa m_3}\right)e^{Lm_3} & m_4\left(m_4+\frac{\omega^2\rho}{G\kappa m_4}\right)e^{Lm_4}\end{vmatrix} = 0 \tag{3.35}$$

Using a computer software[7], the natural frequencies ω_n can be computed. A tabular comparison between the 2 theories is presented below. In the table below, TBT stands for Timoshenko Beam Theory and EBT stands for Euler-Bernoulli beam theory. Also, to keep out the geometrical parameters, λ is defined as-

$$\lambda = \omega_n\sqrt{\frac{\rho l^4}{EI}} \tag{3.36}$$

In the computations, shear modulus, G is taken as 1 and shear correction factor, κ is taken as 0.86. ν value is assumed to be zero to keep in consistency with the theory taken up in bending.

Mode	λ from TBT	λ from EBT
1	8.68	9.87
2	29.77	39.48
3	51.62	88.83

Chapter 4

Buckling of Timoshenko Beams

4.1 Introduction

Beam buckling is generally perceived as a nonlinear phenomenon under compressive loading, wherein the beam begins to deflect randomly once the compressive load crosses a minimum critical value which well below the failure point. Euler predicted a critical load, called *Euler's critical load* which has gained widespread popularity. It uses Euler-Bernoulli beam theory to predict the deflection. Now, beam buckling, being itself an unstable situation, the deflection cannot be predicted with so much accuracy. As a result, to give some relaxation on the assumed deflection, the critical loading can be derived using Timoshenko beam theory.

4.2 Mathematical Formulation

The displacement field remains the same as equations (2.1). However there should be some combined terms of u_s and w to derive the critical load, as the axial compression, u_s will bring about transverse displacement, w. So, the strains are modified by bringing in a nonlinear term in the normal strain.

$$\epsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right] = \frac{du_s}{dx} - z\frac{d\phi}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{4.1a}$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right) = \frac{1}{2}\left(-\phi + \frac{dw}{dx}\right) \tag{4.1b}$$

In equation (4.1a), it is assumed that $\frac{du}{dx} << \frac{dw}{dx}$.
The principle of virtual work is invoked again and a similar procedure is followed as taken up in bending given by equation (2.9) and (2.10). So-

$$\begin{aligned}\int_V \sigma_{ij}\delta\epsilon_{ij}dV = &\int_0^L EA\left\{\frac{du_s}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}\frac{d\delta u_s}{dx}dx + \\ &\int_0^L EA\left\{\frac{du_s}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}\frac{dw}{dx}\frac{d\delta w}{dx}dx \\ &+ \int_0^L EI\frac{d\phi}{dx}\frac{d\delta\phi}{dx}dx - \int_0^L GA\kappa\left(-\phi + \frac{dw}{dx}\right)\delta\phi dx \\ &+ \int_0^L GA\kappa\left(-\phi + \frac{dw}{dx}\right)\frac{d\delta w}{dx}dx\end{aligned} \tag{4.2}$$

The above equation can be simplified further to decompose into boundary terms. However let $\left\{\frac{du_s}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}$ be expressed as $*_1$ and $\left(-\phi+\frac{dw}{dx}\right)$ be expressed as $*_2$.

$$\begin{aligned}\int_V \sigma_{ij}\delta\epsilon_{ij}dV &= \int_0^L EA\left\{*_1\right\}\frac{d\delta u_s}{dx}dx+\int_0^L EA\left\{*_1\right\}\frac{dw}{dx}\frac{d\delta w}{dx}dx\\ &+\int_0^L EI\frac{d\phi}{dx}\frac{d\delta\phi}{dx}dx-\int_0^L GA\kappa\left(*_2\right)\delta\phi dx\\ &+\int_0^L GA\kappa\left(*_2\right)\frac{d\delta w}{dx}dx\end{aligned}$$

$$\begin{aligned}&\Rightarrow\int_V \sigma_{ij}\delta\epsilon_{ij}dV\\ &=\left[EA\left(*_1\right)\delta u_s\right]_0^L-\int_0^L\frac{d}{dx}\left\{EA\left(*_1\right)\right\}\delta u_s dx+\left[EA\left(*_1\right)\frac{dw}{dx}\delta w\right]_0^L-\int_0^L\frac{d}{dx}\left\{EA\left(*_1\right)\frac{dw}{dx}\right\}\delta w dx\\ &+\left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L-\int_0^L\frac{d}{dx}\left\{EI\frac{d\phi}{dx}\right\}\delta\phi dx-\int_0^L GA\kappa\left(*_2\right)\delta\phi dx+\left[GA\kappa\left(*_2\right)\delta w\right]_0^L\\ &-\int_0^L\frac{d}{dx}\left\{GA\kappa\left(*_2\right)\right\}\delta w dx\end{aligned} \tag{4.3}$$

Combining the like terms-

$$\begin{aligned}&\Rightarrow\int_V \sigma_{ij}\delta\epsilon_{ij}dV\\ &=\left[EA\left(*_1\right)\delta u_s\right]_0^L+\left[EA\left(*_1\right)\frac{dw}{dx}\delta w\right]_0^L+\left[EI\frac{d\phi}{dx}\delta\phi\right]_0^L+\left[GA\kappa\left(*_2\right)\delta w\right]_0^L-\int_0^L\frac{d}{dx}\left\{EA\left(*_1\right)\right\}\delta u_s dx\\ &-\int_0^L\frac{d}{dx}\left\{EA\left(*_1\right)\frac{dw}{dx}+GA\kappa\left(*_2\right)\right\}\delta w dx-\int_0^L\left[\frac{d}{dx}\left\{EI\frac{d\phi}{dx}\right\}+GA\kappa\left(*_2\right)\right]\delta\phi dx\end{aligned} \tag{4.4}$$

On the other hand-

$$\int_0^L T_i\delta u_i ds=\int_0^L q\delta w dx-\left[P\delta u_s\right]_0^L \tag{4.5}$$

Equating equation (4.4) and (4.5)-

$$\frac{d}{dx}\left\{EA(*_1)\right\}=0 \tag{4.6a}$$

$$\frac{d}{dx}\left\{EA\left(*_1\right)\frac{dw}{dx}+GA\kappa\left(*_2\right)\right\}=q \tag{4.6b}$$

$$\frac{d}{dx}\left\{EI\frac{d\phi}{dx}\right\}+GA\kappa\left(*_2\right)=0 \tag{4.6c}$$

The above equations constitute the governing differential equations. The boundary conditions are-

1. At $x=0$ and $x=L$, either $EA\left(*_1\right)=-P$ or u_s is specified.
2. At $x=0$ and $x=L$, either $\left\{EA\left(*_1\right)\frac{dw}{dx}+GA\kappa\left(*_2\right)\right\}=0$ or w is specified.

3. At $x = 0$ and $x = L$, either $EI\frac{d\phi}{dx} = 0$ or ϕ is specified.

From equation (4.6a), it can be confirmed that $EA(*_1) = -P$ in the entire domain. So, equation (4.6b) gets modified as-

$$\begin{aligned}
-P\frac{d^2w}{dx^2} + GA\kappa\frac{d(*_2)}{dx} &= q \\
\Rightarrow -P\frac{d^2w}{dx^2} + GA\kappa\left(-\frac{d\phi}{dx} + \frac{d^2w}{dx^2}\right) &= q \\
\Rightarrow \frac{d\phi}{dx} &= -\frac{q}{GA\kappa} + \left(1 - \frac{P}{GA\kappa}\right)\frac{d^2w}{dx^2} \\
\Rightarrow \frac{d^2\phi}{dx^2} &= -\frac{1}{GA\kappa}\frac{dq}{dx} + \left(1 - \frac{P}{GA\kappa}\right)\frac{d^3w}{dx^3}
\end{aligned} \tag{4.7}$$

Substituting in equation (4.6c)-

$$\begin{aligned}
-\frac{EI}{GA\kappa}\frac{dq}{dx} + EI\left(1 - \frac{P}{GA\kappa}\right)\frac{d^3w}{dx^3} + GA\kappa\left(-\phi + \frac{dw}{dx}\right) &= 0 \\
\Rightarrow \phi = \frac{dw}{dx} - \frac{EI}{(GA\kappa)^2}\frac{dq}{dx} + \frac{EI}{GA\kappa}\left(1 - \frac{P}{GA\kappa}\right)\frac{d^3w}{dx^3} \\
\Rightarrow \frac{d\phi}{dx} = \frac{d^2w}{dx^2} - \frac{EI}{(GA\kappa)^2}\frac{d^2q}{dx^2} + \frac{EI}{GA\kappa}\left(1 - \frac{P}{GA\kappa}\right)\frac{d^4w}{dx^4}
\end{aligned} \tag{4.8}$$

Substitute the above equation in the modified form of equation (4.6b).

$$\begin{aligned}
-P\frac{d^2w}{dx^2} + GA\kappa\left(-\frac{d\phi}{dx} + \frac{d^2w}{dx^2}\right) &= q \\
\Rightarrow -P\frac{d^2w}{dx^2} + GA\kappa\left(-\left\{\frac{d^2w}{dx^2} - \frac{EI}{(GA\kappa)^2}\frac{d^2q}{dx^2} + \frac{EI}{GA\kappa}\left(1 - \frac{P}{GA\kappa}\right)\frac{d^4w}{dx^4}\right\} + \frac{d^2w}{dx^2}\right) &= q \\
\Rightarrow \left(1 - \frac{P}{GA\kappa}\right)\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} + \frac{q}{EI} &= \frac{1}{GA\kappa}\frac{d^2q}{dx^2} \\
\Rightarrow \frac{d^4w}{dx^4} + \frac{P/EI}{(1 - P/GA\kappa)}\frac{d^2w}{dx^2} + \frac{q}{EI(1 - P/GA\kappa)} &= \frac{1}{GA\kappa}\frac{1}{(1 - P/GA\kappa)}\frac{d^2q}{dx^2}
\end{aligned} \tag{4.9}$$

If there is no external loading, i.e. $q = 0$, then equation (4.9) takes up a very common form-

$$\frac{d^4w}{dx^4} + k^2\frac{d^2w}{dx^2} = 0 \tag{4.10}$$

where $k^2 = \frac{P/EI}{1-P/GA\kappa}$.
In case of Euler-Bernoulli beam theory, the value of k^2 was $\frac{P}{EI}$ only.
Solving equation (4.10), gives a solution of the form-

$$w(x) = c_1 cos(kx) + c_2 sin(kx) + c_3 x + c_4 \tag{4.11}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants which can be determined from boundary conditions.

4.3 Example of a beam hinged at both ends

Consider a beam of length L with both ends hinged. On the action of a compressive load P, the beam gets buckled. Using the previously derived theory, the critical load can be determined. The boundary conditions are-

1. At $x = 0$ and $x = L$, $w = 0$.
2. At $x = 0$ and $x = L$, $\frac{d\phi}{dx} = 0$.

From equation (4.11), it can be found as-

$$w(x) = c_1 cos(kx) + c_2 sin(kx) + c_3 x + c_4 \tag{4.12}$$

$$\frac{d\phi}{dx} = c_1 \left(-k^2 cos(kx) + \frac{Pk^2 cos(kx)}{GA\kappa}\right) + c_2 \left(-k^2 sin(kx) + \frac{Pk^2 sin(kx)}{GA\kappa}\right) \tag{4.13}$$

Substituting the boundary conditions results in 4 equations.

1. At x=0: $c_1 + c_4 = 0$.
2. At x=0: $c_1 \left(-k^2 + \frac{Pk^2}{GA\kappa}\right) = 0$
3. At x=L: $c_1 cos(kL) + c_2 sin(kL) + c_3 L + c_4 = 0$
4. At x=L: $c_1 \left(-k^2 cos(kL) + \frac{Pk^2 cos(kL)}{GA\kappa}\right) + c_2 \left(-k^2 sin(kL) + \frac{Pk^2 sin(kL)}{GA\kappa}\right) = 0$

In matrix form-

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \cos(Lk) & \sin(Lk) & L & 1 \\ -k^2 + \frac{Pk^2}{AG\kappa} & 0 & 0 & 0 \\ -k^2\cos(Lk) + \frac{Pk^2\cos(Lk)}{AG\kappa} & -k^2\sin(Lk) + \frac{Pk^2\sin(Lk)}{AG\kappa} & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.14}$$

For non-trivial solution, the determinant of the matrix must be zero.

$$\begin{aligned} Lk^4 \left(1 - \frac{P}{GA\kappa}\right)^2 \sin(kL) = 0 \\ \Rightarrow L\left(\frac{P}{EI}\right)^2 sin(kL) = 0 \end{aligned} \tag{4.15}$$

Since $P \neq 0$, so-

$$\begin{aligned} \sin kL = 0 \ \Rightarrow kL = n\pi \quad \text{where n=0,1,2,3,...} \\ \Rightarrow k^2 = \frac{n^2\pi^2}{L^2} \Rightarrow \frac{\frac{P_c}{EI}}{\left(1 - \frac{P_c}{AG\kappa}\right)} = \frac{n^2\pi^2}{L^2} \end{aligned} \tag{4.16}$$

Solving the above equation, gives the critical load for buckling-

$$P_c = \frac{\frac{n^2\pi^2}{L^2}}{\frac{1}{EI} + \frac{n^2\pi^2}{L^2}\frac{1}{GA\kappa}} \tag{4.17}$$

Had the problem been solved using Euler-Bernoulli beam theory, the critical load would have been-

$$P_{c,euler} = \frac{n^2\pi^2 EI}{L^2} \tag{4.18}$$

Taking the ratio between the 2 critical loads-

$$\frac{P_{c,timo}}{P_{c,euler}} = \frac{1}{1 + \left(\frac{n\pi}{L}\right)^2 \frac{EI}{GA\kappa}} \tag{4.19}$$

Quite clearly, for any mode shape, the critical load is lower in Timoshenko beams. This trend increases with increase in mode shape.

Appendix A

Principle of Least Action

A.1 General Statement

In any dynamic system, *action*, $\mathcal{S}$ is defined as the integral of Lagrangian, $\mathcal{L}$ between 2 time constants t_1 and t_2- technically a functional of the N generalized coordinates $q = (q_1, q_2, ..., q_N)$ which define the configuration of the system.

$$\mathcal{S}[\mathbf{q}, t_1, t_2] = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt \tag{A.1}$$

The principle of least action says the path taken by the system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary (no change) to first order.[8]

$$\begin{aligned} \delta\mathcal{S} &= 0 \\ \Rightarrow \delta \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt &= 0 \\ \Rightarrow \int_{t_1}^{t_2} \delta\mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dt &= 0 \end{aligned} \tag{A.2}$$

In the text, the Lagrangian appears in a special form-

$$\mathcal{L} = \int_0^L \hat{\mathcal{L}} dx$$

where $\hat{\mathcal{L}}$ is called the *Lagrangian Density*. Minimizing it-

$$\int_{t_1}^{t_2} \int_0^L \delta\hat{\mathcal{L}}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)dxdt = 0 \tag{A.3}$$

A.2 Lagrange's equation of motion

The Lagrangian Density can be simplified as-

$$\begin{aligned} \delta\hat{\mathcal{L}}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) &= \frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}}\delta\mathbf{q}(t) + \frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\frac{\partial\delta\mathbf{q}}{\partial t} \\ \Rightarrow \delta\hat{\mathcal{L}}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) &= \frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}}\delta\mathbf{q}(t) + \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\delta\mathbf{q}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\right)\delta\mathbf{q} \end{aligned} \tag{A.4}$$

Substituting in equation (A.3)-

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^L \delta\hat{\mathcal{L}}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)dxdt &= \int_{t_1}^{t_2}\int_0^L \left(\frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}}\delta\mathbf{q} + \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\delta\mathbf{q}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\right)\delta\mathbf{q}\right)dxdt \\ &= \int_{t_1}^{t_2}\int_0^L \left(\frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\right)\right)\delta\mathbf{q}dxdt + \left[\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\delta\mathbf{q}\right]_{t_1}^{t_2}\end{aligned} \tag{A.5}$$

Hence, finally-

$$\int_{t_1}^{t_2}\int_0^L \left(\frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\right)\right)\delta\mathbf{q}dxdt + \left[\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\delta\mathbf{q}\right]_{t_1}^{t_2} = 0 \tag{A.6}$$

From equation (A.6), the governing differential equation turns out to be-

$$\frac{\partial\hat{\mathcal{L}}}{\partial\mathbf{q}} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\right) = 0 \tag{A.7}$$

The above equation is called *Lagrange's equation of motion.*
In addition to this, the boundary condition involved is-

$$\left[\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\mathbf{q}}}\delta\mathbf{q}\right]_{t_1}^{t_2} = 0 \tag{A.8}$$

A.3 Application to context

In the theory, $q(x,t) = [w(x,t), \phi(x,t)]^T$. So, equation (A.8) will be modified as-

$$\frac{\partial\hat{\mathcal{L}}}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{w}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\hat{\mathcal{L}}}{\partial w'}\right) = 0 \tag{A.9a}$$

$$\frac{\partial\hat{\mathcal{L}}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\phi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\hat{\mathcal{L}}}{\partial\phi'}\right) = 0 \tag{A.9b}$$

The 2 equations presented above will be used in the text to derive equations of vibration.

Appendix B

Implementation in Computer Program

All the codes and routines are built using Python language. In order to take advantage of the symbolic computations and mathematical display, *Jupyter Notebook* is used. All the codes are uploaded in a GitHub repository titled AMOS_TermProject.

In order to minimize the work of implementing some of the very common theories like strain-displacement relation etc, a python library file named Utilities.py is built and will be used in every subsequent routines.

The first chapter i.e. Bending of Timoshenko beams is coded in a file named Bending.ipynb. The example problem solved in section (2.4) is solved using Jupyter Notebook in a file, BendingExamples.ipynb.

The next chapter is on Vibration of Timoshenko beams. It is coded in a file named Timoshenko_Vibration.ipynb. Just for the sake of acquaintance with the vibration of Euler-Bernoulli beam theory, a file named EB_Vibration is also prepared.

The next chapter deals on Buckling of Timoshenko beams. Buckling, being a nonlinear phenomenon required complicated calculations. Hence coding the theory was difficult as it involved large computation time. However once the theory was developed, an example problem, which is solved in section (4.3) is coded. The code is stored in a file named BucklingExample.ipynb.

Bibliography

[1] Ladislao Reti. *The Unknown Leonardo.* McGraw Hill Co., New York, 1974.

[2] Galileo Galilei. *Dialogues Concerning Two New Sciences.* Macmillan, New York, 1638.

[3] S.P. Timoshenko. *History of Strength of Materials.* Dover Publications, Inc., New York, 1983.

[4] Irving H. Shames Clive L. Dym. *Solid Mechanics- A Variational Approach.* Springer, Claremont, CA, 2013.

[5] S.P. Timoshenko. On the correction factor for shear of the differential equation for transverse vibrations of bars of uniform cross-section. *Philosophical Magazine*, 1921.

[6] Rolland T. Hinkle Francis S. Tse, Ivan E. Morse. *Mechanical Vibrations.* CBS Publishers, New Delhi, India, 2004.

[7] B. Downs. Transverse Vibration of a Uniform Simply Supported Timoshenko Beam Without Transverse Deflection. *Journal of Applied Mechanics*, Dec 1976.

[8] J.L. Safko H. Goldstein, C.P. Poole. *Classical Mechanics.* Addison-Wesley, Boston, 2002.