Inverse Free Kalman Filter Using Approximate Inverse of Diagonally Dominant Matrices

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***Abstract*—Conventional Kalman filter (KF) requires matrix inversion. But the pervasive applications of KF cannot at times afford inversion. Especially, embedded implementations do not have the capabilities to compute inverse using methods such as Cholesky decomposition. For large matrices, inversion could be computationally prohibitive even for non-embedded implementations. To address this problem, an inverse free Kalman filter (IFKF) is proposed in this letter. The inverse of innovation covariance matrix required in the update step of the KF is approximated using Taylor series expansion. The approximate inverse has a closed form expression in the elements of the original matrix. Bounds on the error covariance of proposed IFKF are also established. The proposed IFKF does not require any iterations to converge.**

***Index Terms*—Kalman filtering, estimation, filtering, sensor fusion.**

I. Introduction

KALMAN filter has become ubiquitous because of its analytical solution and robustness against deviations from assumptions. Not requiring online optimizations, Kalman filter (KF) remains the computationally most feasible estimation technique for a wide range of applications. However, KF implementations for resource constrained systems, face a computational bottleneck because of the inverse in the update step, which requires $O(n^3)$ computations [1], [2]. To this end we use a Taylor series based approximate inversion method which scales on $O(n^2)$ computations. It is sufficient to assume that the innovation covariance matrix is strictly diagonally dominant to apply this approximate inversion method to KF. However, diagonal dominance is only a sufficient condition and not necessary as will be discussed.

A matrix is said to be row diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \forall i = 1, 2, \ldots, N \tag{1}$$

where, $a'_{ij}s$ are the matrix elements. A matrix is strictly row diagonally dominant if the inequality is strict. The special structure of diagonally dominant (DD) matrices gives way to exploit the properties of its inverse. Recently, in [3] and [4] a Neumann series (NS) based approximate inverse of diagonally dominant matrices was proposed. This NS based approximate inverse was used in multi-input and multi-output (MIMO) techniques for wireless communications, where diagonally dominant matrices arise naturally. In [5], bounds on the error in NS based approximations are presented. In this letter, we additionally show how the NS based approximate inverse retains the positive definiteness of the original matrix, which is desirable in many applications such as KF. The approximate inverse also allows us to rewrite Kalman filter equations to make them inverse free.

Though there have been inverse free Kalman filter formulations such as rank one update algorithms and sequential Kalman filter [6], they rely on sequential processing of measurements, increasing computational time. Sequential methods also assume uncorrelated measurement noises (diagonal $\mathbf{R}$), restricting their applicability. Other methods which try to address computational issues of KF broadly take either of the approaches involving low rank or banded approximation of covariance matrix. Methods which use banded approximations or covariance tapering do not always guarantee a positive definite covariance matrix (leading to divergent estimates [7]) and require special measures or assumptions to maintain positive definiteness [1]. Low rank approximation methods either approximate the covariance matrix as a low rank matrix or as a low rank perturbation of system noise covariance [8], [9]. The drawback of this assumption is that it is reasonable only for large dimensional systems with fewer unstable modes.

Despite the processors becoming cheap and powerful over the years, researchers look at alternate KF formulations as the standard KF implementation is still not viable in many applications. This is because, on one hand applications like cyber physical systems, wireless sensor networks, weather prediction etc., involve large matrices. On the other hand, applications like satellite control systems, underwater vehicles, aircrafts etc., have tight on-board resource constraints. These limitations have pushed many researchers to resort to approximate KF formulations [9]–[11] which are often less accurate. An inverse free KF which does not sacrifice accuracy is thus much required. An inverse free KF not only merits reduced computational complexity but also saves power, leaves more time for execution of system critical functions, allows faster sampling rates, faster fault diagnosis etc.

Manuscript received March 6, 2018; revised May 31, 2018; accepted June 26, 2018. Date of publication July 9, 2018; date of current version July 23, 2018. Recommended by Senior Editor G. Cherubini. *(Corresponding author: Ketan Detroja.)*

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Digital Object Identifier 10.1109/LCSYS.2018.2854238

The inverse free Kalman filter (IFKF) equations proposed in this letter not only reduce computational complexity but are also highly parallelizable, reducing the computation time. The proposed method places a less restrictive diagonal dominance property on the *innovation covariance matrix* as a sufficiency condition, than assuming a banded structure or low rank approximation of the *state covariance matrix*. The diagonal dominance property ensures convergence of Taylor series expansion and consequently guarantees bounds on the error covariance matrix. The bounds ensure non-divergent estimates. Accuracy of proposed IFKF is established with the help of numerical examples.

Section II provides the theory of approximate inverse and its characteristics along with numerical simulations. In Section III the theory developed is applied to Kalman filter to derive inverse free Kalman filter equations. Section III also provides bounds on the error covariance and Section IV gives numerical results. The letter concludes with some observations and possible future work in Section V.

II. Theory of Approximate Inverse

In this section, the NS based approximate inversion proposed in [3] and [4] is restated using Taylor series. We also establish some of the additional properties possessed by the approximate inverse.

Consider a square matrix with non-zero diagonal entries. Rewriting the matrix as a sum of two matrices.

$$\mathbf{M} = \mathbf{A} + \mathbf{B} \tag{2}$$

Now, the Taylor series expansion of function $f(x)$ about a point h is given as

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots\cdots + f^{n-1}(x)\frac{h^{n-1}}{(n-1)!} + f^n(x+\lambda h)\frac{h^n}{n!} \tag{3}$$

Applying Taylor series expansion for inverse function on (2) using the fact that $(\mathbf{A}^{-1})' = -\mathbf{A}^{-1}\mathbf{A}'\mathbf{A}^{-1}$, the inverse of a matrix can be written as

$$\mathbf{M}^{-1} = (\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} - \cdots \tag{4}$$

For the above Taylor series expansion to converge, it is necessary as well as sufficient that $||\mathbf{A}^{-1}\mathbf{B}|| < 1$ or $||\mathbf{B}\mathbf{A}^{-1}|| < 1$ for some matrix norm. As given in [3] and [4], let $\mathbf{A}$ be the diagonal component of $\mathbf{M}$ and $\mathbf{B}$ be the off-diagonal component of $\mathbf{M}$ with diagonal entries as zero. The striking advantage of this decomposition is that, the inverse only appears on the diagonal matrix ($\mathbf{A}$) which can be easily found by inverting individual elements in the diagonal. It can be readily shown that for a strictly diagonally dominant matrix, $(\mathbf{A}+\mathbf{B})$, $||\mathbf{A}^{-1}\mathbf{B}||_\infty < 1$, where $||\cdot||_\infty$ is the infinity norm of the matrix. Hence, the Taylor series expansion in (4) converges for strictly diagonally dominant matrices.

Remark 1: As such, the expansion in (4) can be used on any matrix as long as the condition for convergence is satisfied for some matrix norm. Thus diagonal dominance is only a sufficient condition and not necessary.

If the matrix $\mathbf{M}$ is a strictly diagonally dominant, symmetric positive definite matrix, we show that the Taylor series expansion considered up to n terms retains the positive definiteness of the matrix in its approximate inverse.

Theorem 1: If $\mathbf{M}$ is a strictly diagonally dominant, symmetric positive definite matrix, then the approximate inverse obtained by considering up to n terms ($\tilde{\mathbf{M}}_n^{-1}$) in the expansion (4), is positive definite $\forall n$.

Proof: Considering the expansion (4) up to the first term yields the approximate inverse of $\mathbf{M}$ as

$$\tilde{\mathbf{M}}_1^{-1} = \mathbf{A}^{-1}$$

Since, $\mathbf{M} > 0 \implies \mathbf{A} > 0,\ \tilde{\mathbf{M}}_1^{-1} > 0$.

Considering the expansion up to the second term yields the approximate inverse of $\mathbf{M}$ as

$$\tilde{\mathbf{M}}_2^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$$

Since, $\mathbf{M} > 0 \implies \mathbf{A}+\mathbf{B} > 0$ we also have

$$\mathbf{A} - \mathbf{B} > 0 \tag{5}$$

because $\mathbf{A}-\mathbf{B}$ is symmetric, diagonally dominant matrix with positive diagonal entries, it is positive definite. Pre and post multiplying (5) with $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} > 0 \implies \tilde{\mathbf{M}}_2^{-1} > 0 \tag{6}$$

Pre and post multiplying (6) with $\mathbf{A}^{-1}\mathbf{B}$ and $(\mathbf{A}^{-1}\mathbf{B})^T$

$$\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}(\mathbf{A}^{-1}\mathbf{B})^T - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}(\mathbf{A}^{-1}\mathbf{B})^T > 0$$
$$\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1} > 0$$

Similarly, it can be shown that $\mathbf{A}^{-1}(\mathbf{B}\mathbf{A}^{-1})^{2n} - \mathbf{A}^{-1}(\mathbf{B}\mathbf{A}^{-1})^{2n+1} > 0$. Thus $\tilde{\mathbf{M}}_n^{-1} > 0$ when n is even.

The odd numbered terms of the Taylor series are all of the form $\mathbf{A}^{-1}(\mathbf{B}\mathbf{A}^{-1})^{2n}$. Since these odd terms are all positive semi-definite in the least, $\tilde{\mathbf{M}}_n^{-1} > 0\ \forall n$ ■

A. Computational Savings and Parallelizability

Since the inverse in (4) only appears on the diagonal component matrix, which only requires n calculations for an $n \times n$ matrix, the approximate inversion method results in substantial computational savings. When terms up to first order in $\mathbf{B}$ are considered, the total number of computations for finding approximate inverse scale at $O(n^2)$. That is an order of n lesser than the Cholesky decomposition based inversion which requires $\frac{1}{3}n^3$ computations in its most efficient implementations. For larger matrix dimension (n) this can result in significant reduction of computation time. The reduction in computational burden using approximate inversion has been reported in [3], [4], and [12]. Particularly Wang *et al.* [12] propose a hardware implementation of the approximate inverse and compare the number of multipliers and adders required with standard methods like Cholesky decomposition. The authors report a 56.7% increase in maximum frequency while hardware consumption is reduced by 58.9% in comparison with Cholesky decomposition.

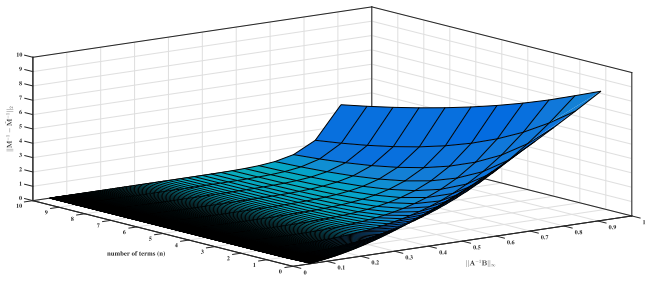

Fig. 1. Plot of norm of error between actual and approximate matrix inverse.

Inverting diagonal matrices and multiplication of a matrix with diagonal matrices fall into the class of 'embarrassingly parallel problems'. The approximate inverse calculation when considered up to first order is hence highly amenable for parallel processing. The matrix multiplications present in the higher order terms can also be parallelized with a linear speedup and an efficiency p. Also, the absence of any sequential steps in the Taylor series approach, translates to a zero wait time among the parallel processors. Thus the approximate inversion, with its low computational complexity and parallelizability, can significantly reduce the computing time depending on the size of the matrix and the number of processors (p) used.

B. Numerical Simulations

In this section, the trend of error between actual inverse and the approximated inverse of numerically constructed diagonally dominant matrices is examined. The Euclidean norm of the error is as shown in Fig. 1. On the x-axis $||\mathbf{A}^{-1}\mathbf{B}||_\infty$, representative of strength of diagonal dominance is varied. On the y-axis, the number of Taylor series terms considered in orders of $\mathbf{B}$ is varied. It can be inferred from Fig. 1 that the error decreases non-linearly with increasing strength of diagonal dominance as well as with increasing number of Taylor series terms. This non-linear relationship shows that for most diagonally dominant matrices, the investment made in the computation of higher order terms does not necessarily pay off. The plot is obtained by averaging the values across multiple runs by varying magnitude of the matrix elements for each run. The Fig. 1 is only representative of the trend of the error, for it is generated using a small subset of the infinitely many diagonally dominant matrices. Unlike the previous studies [3]–[5], the matrices considered for generating Fig. 1 are random and not application specific.

Remark 2: The computational savings (in FLOPS) of the approximate inverse only occur when up to two terms in the Taylor series are considered. When three terms are considered the computational burden of the approximate inverse is of the order of exact inverse but with lesser number of real valued multiplications [4]. Beyond this, the computations required are higher than that of exact inverse.

Remark 3: The Taylor series based approximative method does not require successive inversions and hence is numerically more stable.

Remark 4: For a hardware implementation, finding the approximate inverse using Taylor series is preferable than finding the actual inverse, as it only involves matrix multiplications.

III. Proposed Inverse Free Kalman Filter (IFKF)

In this section, the procedure for obtaining approximate inverse presented in the previous section is applied to the Kalman filter.

Consider a system whose dynamics are of the form

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{u}(k) + \mathbf{w}(k)\\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k)\end{aligned} \tag{7}$$

where, $\mathbf{w}(k)$, $\mathbf{v}(k)$ represent the state and measurement noises with covariances $\mathbf{Q}$ and $\mathbf{R}$, respectively such that $\mathbf{Q} \geq 0$, $\mathbf{R} > 0$. $\mathbf{\Phi}$, $\mathbf{\Gamma}$ and $\mathbf{C}$ are the system, input and observation matrices, respectively. $\mathbf{x}$, $\mathbf{y}$ are the corresponding state and measurement vectors, such that $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{u} \in \mathbb{R}^p$, $\mathbf{\Phi} \in \mathbb{R}^{n\times n}$ and $\mathbf{\Gamma}$, $\mathbf{C}$, $\mathbf{w}$ and $\mathbf{v}$, are of appropriate dimension.

For a system with dynamics of the form (7), the standard Kalman filter equations are

$$\hat{\mathbf{x}}(k|k-1) = \mathbf{\Phi}\,\hat{\mathbf{x}}(k-1|k-1) + \mathbf{\Gamma}\boldsymbol{u}(k) \tag{8}$$

$$\mathbf{P}(k|k-1) = \mathbf{\Phi}\,\mathbf{P}(k-1|k-1)\mathbf{\Phi}^T + \mathbf{Q} \tag{9}$$

$$\begin{aligned}\hat{\mathbf{x}}(k|k) &= \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\big(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)\big)\\ \mathbf{P}(k|k) &= (\mathbf{I} - \mathbf{K}(k)\mathbf{C})\mathbf{P}(k|k-1)\\ \mathbf{K}(k) &= \mathbf{P}(k|k-1)\mathbf{C}^T\mathbf{S}(k)^{-1}\end{aligned} \tag{10}$$

where, (8) and (10) are the prediction and update steps of Kalman filter, respectively. $\mathbf{K}(k)$ is the Kalman gain and $\mathbf{S}(k) = \mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^T + \mathbf{R}$ is the innovation covariance. $\mathbf{P}$ is the covariance of estimation error such that $\mathbf{P}(k|k-1) = \text{cov}(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k-1))$ and $\mathbf{P}(k|k) = \text{cov}(\mathbf{x}(k) - \hat{\mathbf{x}}(k|k))$.

It can be seen from (8) and (9) that the prediction step of KF only involves matrix multiplications and hence can be easily parallelized. However, the calculation of Kalman gain in (10) requires the inverse of innovation covariance matrix. We employ the Taylor series based approximate inverse method to obtain inverse free Kalman filter equations as given below. For demonstrative purposes we derive the expressions for the case of a diagonal observation matrix. However, the same procedure can be extended to systems with non-diagonal observation matrices, as will be shown later.

Consider a covariance matrix of the form

$$\mathbf{P}(k|k-1) = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N}\\ P_{21} & P_{22} & \cdots & P_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ P_{N1} & P_{N2} & \cdots & P_{NN}\end{bmatrix} \tag{11}$$

and an observation matrix of the form

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 & \cdots & 0\\ 0 & C_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & C_N\end{bmatrix} \tag{12}$$

The inverse of innovation covariance required for calculation of Kalman gain is given by

$$\mathbf{S}(k)^{-1} = \left(\mathbf{CPC}^T + \mathbf{R}\right)^{-1} = \begin{bmatrix} S_{11} & C_1P_{12}C_2 & \cdots & C_1P_{1N}C_N \\ C_2P_{21}C_1 & S_{22} & \cdots & C_2P_{2N}C_N \\ \vdots & \vdots & \ddots & \vdots \\ C_NP_{N1}C_1 & C_NP_{N2}C_2 & \cdots & S_{NN} \end{bmatrix}^{-1} \quad (13)$$

where, $S_{ii} = C_iP_{ii}C_i + R_{ii}$. Applying (4) up to first order in $\mathbf{B}$, by assuming $\mathbf{S}(k)$ to be diagonally dominant, we get the approximate inverse of innovation covariance as

$$\tilde{\mathbf{S}}(k)^{-1} = \begin{bmatrix} \frac{1}{S_{11}} & \frac{-C_1P_{12}C_2}{S_{11}S_{22}} & \cdots & \frac{C_1P_{1N}C_N}{S_{11}S_{NN}} \\ \frac{C_2P_{21}C_1}{S_{22}S_{11}} & \frac{1}{S_{22}} & \cdots & \frac{C_2P_{2N}C_N}{S_{22}S_{NN}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_NP_{N1}C_1}{S_{NN}S_{11}} & \frac{C_NP_{N2}C_2}{S_{NN}S_{22}} & \cdots & \frac{1}{S_{NN}} \end{bmatrix} \quad (14)$$

The innovation covariance matrix ($\mathbf{S}(k) = \mathbf{CP_{k|k-1}C}^T + \mathbf{R}$) has a natural tendency to be diagonally dominant. This is because $\mathbf{Q}$ and $\mathbf{R}$ are generally considered to be diagonal or to have strong diagonal entries and the lower bound of $\mathbf{P}_{k|k-1}$ approaches $\mathbf{Q}$ when $\mathbf{R}$ is very low (in fact when $\mathbf{R} \to 0$ the lower bound is $\mathbf{Q}$). On the other hand, higher $\mathbf{R}$ adds more strength to the diagonal of $\mathbf{S}(k)$ as $\mathbf{R}$ is usually diagonal or centred along the diagonal. Also in many examples, methods which assume nearly banded or tapered structure of covariance matrix proved to be effective [9]. Therefore, it is reasonable and less restrictive to assume that the innovation covariance matrix ($\mathbf{S}(k)$), will be strictly diagonally dominant.

Since it is sufficient that $\mathbf{S}(k)$ is DD to use the Taylor series expansion (4), this expression of $\tilde{\mathbf{S}}(k)^{-1}$ can be substituted in the update equation (10) of Kalman filter, to get

$$P_{ij}(k|k) = \left[P_{ij} - \sum_{q=1}^{N} \frac{P_{iq}C_q^2P_{qj}}{S_{qq}} + \sum_{m=1}^{N} \frac{P_{im}C_m^2}{S_{mm}} \sum_{l=1,l\neq m}^{N} \frac{P_{ml}C_l^2P_{lj}}{S_{ll}} \right]_{k|k-1} \quad (15)$$

where, $i = 1, \ldots N;\ j = 1, \ldots N$. Extending the above procedure for the case of non-diagonal observation matrices gives the corresponding inverse free update expressions as follows.

$$P_{ij}(k|k) = \left[P_{ij} - \sum_{l=1}^{N} P_{il} \left(\sum_{m=1}^{N} C_{ml} \times \left(\sum_{n=1}^{N} \frac{S_{nm}}{S_{nn}S_{mm}} \left(\sum_{p=1}^{N} C_{np}P_{pj} \right) \right) \right) \right]_{k|k-1} \quad (16)$$

where, $i = 1, \ldots, N; j = 1, \ldots, N$ and

$$S_{nm} = \begin{cases} \sum_{q=1}^{N} C_{nq} \left(\sum_{r=1}^{N} P_{qr}C_{mr} \right) + R_{nn}, & \text{if } n = m \\ -\sum_{q=1}^{N} C_{nq} \left(\sum_{r=1}^{N} P_{qr}C_{mr} \right), & n \neq m \end{cases}$$

The inverse free update expressions defined by (15) and (16) can be used for parallel processing of measurements, as each covariance term can be updated independent of other terms at a given instant. Since (15) and (16) are parallelizable and only require $O(n^2)$ calculations, execution time is reduced.

A. Bounds on the Covariance Matrix

Theorem 2 gives bounds on the innovation covariance matrix of the proposed IFKF. In [13], Jazwinski gave the bounds on error covariance matrix for the standard Kalman filter. In the following theorem, we use these standard results (and notations) from [13] to establish bounds on error covariance of the proposed IFKF.

Before proceeding to establish bounds, we establish the positive definiteness of the innovation covariance matrix at each iteration. Rewriting the covariance update step (10) of Kalman filter using approximate inverse,

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{C}^T\tilde{\mathbf{S}}_k^{-1}\mathbf{CP}_{k|k-1} \quad (17)$$

The above equation can be rewritten as follows

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_{k|k-1}(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{R}\mathbf{K}_k^T \quad (18)$$

where $\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{C}^T\tilde{\mathbf{S}}_k^{-1}$. If $\mathbf{P}_0 \geq 0$, then the propagated covariance $\mathbf{P}_{1|0} \geq 0$ at least, since $\mathbf{Q} \geq 0$. Thus the first term in (18) is at least positive semi-definite. Since $\mathbf{R} > 0$ and as the positive definiteness of $\mathbf{S}_1$ is preserved in its approximate inverse $\tilde{\mathbf{S}}_1^{-1}$ as per Theorem 1, the second term in (18) is positive definite, hence $\mathbf{P}_{1|1} > 0$. In view of (9) and (18), $\mathbf{P}_{k|k}$ remains positive definite throughout the IFKF iterations and consequently the innovation covariance matrix $\hat{\mathbf{S}}_k = \mathbf{CP_{k|k-1}C}^T + \mathbf{R}$ will be positive definite at each time step.

Theorem 2: If the dynamical system (7) is uniformly completely observable and the pair $(\mathbf{\Phi}, \mathbf{Q}^{1/2})$ is uniformly completely controllable and if $\mathbf{P}_0 \geq 0$ then $\tilde{\mathbf{S}}_{k+1}$ is upper bounded by

$$\tilde{\mathbf{S}}_{k+1} \leq \left(\mathbf{C}\left(\mathbf{\Phi}\left(\varsigma^{-1}(k, k-N) + \zeta(k, k-N) \right)\mathbf{\Phi}^T + \mathbf{Q} \right)\mathbf{C}^T + \mathbf{R} \right)\left(\mathbf{I} + \left(\mathbf{S}_d^{-1}\mathbf{S}_o \right)\left(\mathbf{S}_o\mathbf{S}_d^{-1} \right)^T \right)^{-1}$$

and lower bounded by

$$\tilde{\mathbf{S}}_{k+1} \geq \left(\mathbf{C}\left(\mathbf{\Phi}\left(\varsigma(k, k-N) + \zeta^{-1}(k, k-N) \right)^{-1}\mathbf{\Phi}^T + \mathbf{Q} \right)\mathbf{C}^T + \mathbf{R} \right)\left(\mathbf{I} + \left(\mathbf{S}_d^{-1}\mathbf{S}_o \right)\left(\mathbf{S}_o\mathbf{S}_d^{-1} \right)^T \right)^{-1}$$

where, $\mathbf{S}_d$, $\mathbf{S}_o$ are diagonal and off-diagonal components of $\mathbf{S}_{k+1}$ respectively and

$$\varsigma(k, k-N) = \sum_{i=k-N}^{k-1} \mathbf{\Phi}^T(i, k)\mathbf{C}^T\mathbf{R}_i^{-1}\mathbf{C}\mathbf{\Phi}(i, k)$$

$$\zeta(k, k-N) = \sum_{i=k-N-1}^{k-1} \mathbf{\Phi}(k, i+1)\mathbf{Q}_{i+1}\mathbf{\Phi}^T(k, i+1)$$

Proof: The upper bound of covariance matrix for Kalman filter as given in [13] is

$$\mathbf{P}(k|k) \leq \varsigma^{-1}(k, k-N) + \zeta(k, k-N) \quad (19)$$

The upper bound of covariance matrix at the propagation stage can be obtained as follows

$$\mathbf{P}(k+1|k) \leq \mathbf{\Phi}\bar{\mathbf{P}}\mathbf{\Phi}^T + \mathbf{Q} \tag{20}$$

where, $\bar{\mathbf{P}}$ is the upper bound given in (19). Pre and post multiplying (20) with $\mathbf{C}$ and $\mathbf{C}^T$ respectively and adding $\mathbf{R}$ on both sides gives

$$\begin{aligned}\mathbf{C}\mathbf{P}(k+1|k)\mathbf{C}^T + \mathbf{R} &\leq \mathbf{C}\big(\mathbf{\Phi}\bar{\mathbf{P}}\mathbf{\Phi}^T + \mathbf{Q}\big)\mathbf{C}^T + \mathbf{R} \\ \implies \mathbf{S}_{k+1} &\leq \mathbf{C}\big(\mathbf{\Phi}\bar{\mathbf{P}}\mathbf{\Phi}^T + \mathbf{Q}\big)\mathbf{C}^T + \mathbf{R}\end{aligned} \tag{21}$$

From the Taylor series expansion in (4), the relation between the approximate and exact inverse of the innovation covariance matrix is obtained as

$$\begin{aligned}\mathbf{S}_{k+1}^{-1} &= \tilde{\mathbf{S}}_{k+1}^{-1} + \sum_{n=2}^{\infty}(\mathbf{S}_d^{-1}\mathbf{S}_o)^n\mathbf{S}_d^{-1}\mathbf{S}_o\mathbf{S}_d^{-1}\mathbf{S}_o\mathbf{S}_d^{-1} \\ \implies \mathbf{S}_{k+1}^{-1} - \tilde{\mathbf{S}}_{k+1}^{-1} &= \Big(\mathbf{S}_d^{-1}\mathbf{S}_o\Big)\Big(\mathbf{S}_o\mathbf{S}_d^{-1}\Big)^T\mathbf{S}_{k+1}^{-1}\end{aligned} \tag{22}$$

The relation between the exact and approximate innovation covariance matrix is thus obtained as

$$\mathbf{S}_{k+1} = \tilde{\mathbf{S}}_{k+1}\Big(\mathbf{I} + \Big(\mathbf{S}_d^{-1}\mathbf{S}_o\Big)\Big(\mathbf{S}_o\mathbf{S}_d^{-1}\Big)^T\Big) \tag{23}$$

Substituting (23) in (21) we get

$$\tilde{\mathbf{S}}_{k+1}\Big(\mathbf{I} + (\mathbf{S}_d^{-1}\mathbf{S}_o)(\mathbf{S}_o\mathbf{S}_d^{-1})^T\Big) \leq \mathbf{C}\big(\mathbf{\Phi}\bar{\mathbf{P}}\mathbf{\Phi}^T + \mathbf{Q}\big)\mathbf{C}^T + \mathbf{R}$$

Post multiplying with $(\mathbf{I} + (\mathbf{S}_d^{-1}\mathbf{S}_o)(\mathbf{S}_o\mathbf{S}_d^{-1})^T)^{-1}$ on both sides

$$\begin{aligned}\tilde{\mathbf{S}}_{k+1} &\leq \big(\mathbf{C}\big(\mathbf{\Phi}\bar{\mathbf{P}}\mathbf{\Phi}^T + \mathbf{Q}\big)\mathbf{C}^T + \mathbf{R}\big) \\ &\quad\times\Big(\mathbf{I} + \Big(\mathbf{S}_d^{-1}\mathbf{S}_o\Big)\Big(\mathbf{S}_o\mathbf{S}_d^{-1}\Big)^T\Big)^{-1} \\ \implies \tilde{\mathbf{S}}_{k+1} &\leq \Big(\mathbf{C}\Big(\mathbf{\Phi}\Big(\varsigma^{-1}(k,k-N) + \zeta(k,k-N)\Big)\mathbf{\Phi}^T \\ &\quad + \mathbf{Q}\Big)\mathbf{C}^T + \mathbf{R}\Big)\Big(\mathbf{I} + \Big(\mathbf{S}_d^{-1}\mathbf{S}_o\Big)\Big(\mathbf{S}_o\mathbf{S}_d^{-1}\Big)^T\Big)^{-1}\end{aligned}$$

Similarly the lower bound is obtained as

$$\begin{aligned}\tilde{\mathbf{S}}_{k+1} \geq \Big(\mathbf{C}\Big(\mathbf{\Phi}\Big(\varsigma(k,k-N) + \zeta^{-1}(k,k-N)\Big)^{-1}\mathbf{\Phi}^T + \mathbf{Q}\Big)\mathbf{C}^T \\ + \mathbf{R}\Big)\Big(\mathbf{I} + \Big(\mathbf{S}_d^{-1}\mathbf{S}_o\Big)\Big(\mathbf{S}_o\mathbf{S}_d^{-1}\Big)^T\Big)^{-1}\end{aligned}$$

■

The approximate innovation covariance matrix is given by $\tilde{\mathbf{S}}_{k+1} = \mathbf{C}\tilde{\mathbf{P}}(k+1|k)\mathbf{C}^T + \mathbf{R}$. Since $\mathbf{S}_d$, $\mathbf{S}_o$ are diagonal and off-diagonal components of $\mathbf{S}$ (which is bounded) and $\mathbf{C}, \mathbf{R}$ are finite matrices, the propagated covariance matrix $\tilde{\mathbf{P}}(k+1|k)$ of the IFKF is also bounded. Since the approximate inverse preserves the positive definiteness of innovation covariance matrix as given by Theorem 1, the covariance update step only involves subtraction of a positive definite matrix from the propagated covariance matrix. Hence, it follows that the updated covariance matrix $\mathbf{P}(k+1|k+1)$ is also bounded.

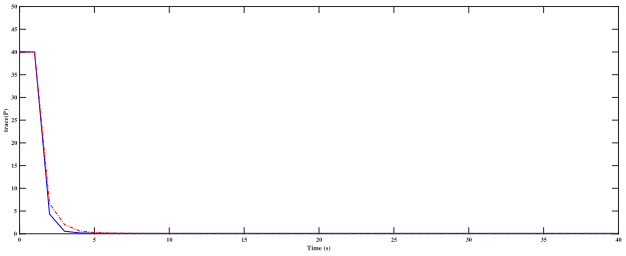

Fig. 2. Evolution of trace of error covariance matrices using exact and approximate inverse of $\mathbf{S}(k)$ for arbitrary system. Solid lines gives the trace for KF and dash-dot for IFKF.

TABLE I
ERROR STATISTICS FOR ARBITRARY SYSTEM

	e_1	e_2	e_3	e_4
ISE_{exact}	1.2736	1.1411	0.7234	1.4940
σ^2_{exact}	1.9685	2.2444	0.4773	1.2277
ISE_{apprx}	1.2832	1.1515	0.7252	1.5003
σ^2_{apprx}	1.9881	2.2580	0.4805	1.2389
ISE_{static}	2.1583	2.0739	0.9073	1.8476
σ^2_{static}	3.6722	3.8968	0.7407	1.9167

IV. SIMULATION RESULTS

In this section, the applicability of the matrix inversion procedure given in Section II for Kalman filter is investigated through numerical examples.

A. Kalman Filter: Numerical Example I

An arbitrary system of the form (7) is considered here with the following system parameters.

$$\mathbf{\Phi} = \begin{bmatrix} 0.811 & -0.348 & 0.049 & 0.331 \\ 0.013 & 0.941 & 0.018 & 0.039 \\ 0.209 & 0.009 & 0.251 & 0.108 \\ -0.318 & -0.025 & -0.144 & 0.411 \end{bmatrix} \quad \mathbf{\Gamma} = \mathbf{0}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{24}$$

The state and measurement noise covariance matrices considered for simulation are $\mathbf{Q} = 0.1 \times \mathbf{I}_4$ and $\mathbf{R} = 0.1 \times \mathbf{I}_2$. The sampling time is 1s. The error covariance matrix is initialized at $\mathbf{P}_0 = 10 \times \mathbf{I}_4$. The initial guess for state $\hat{\mathbf{x}}_0 \in \mathcal{N}(0,1)$. A step variation of $0.2 \times \mathbf{I}_2$ is introduced in $\mathbf{R}$ at 40s.

For the above defined system, the plot of trace of error covariance matrices using actual and approximate inverse of the innovation covariance matrix at each instant is shown in Fig. 2. The average values of integral square error (ISE) and variance of error in estimation (σ^2), obtained from rigorous simulations are given in Table I. The subscripts *exact*, *apprx* and *static* in Table I correspond to error statistics obtained using exact inverse, approximate inverse and a static gain KF respectively. All ISE values are normalized by a factor of 10^3. It can be observed from Table I that the proposed method gives estimates close to that obtained by KF while reducing computational complexity. Though a static gain KF is easy to implement and reduces computational load, its performance is

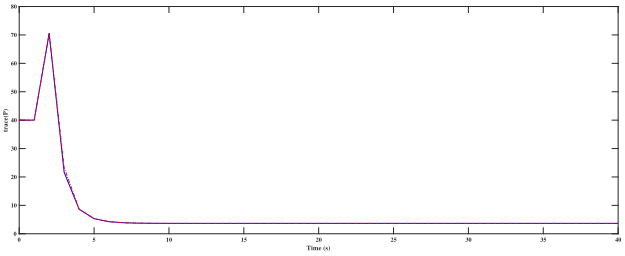

Fig. 3. Evolution of trace of error covariance matrices using exact and approximate inverse of $\mathbf{S}(k)$ for IP system. Solid lines gives the trace for KF and dash-dot for IFKF.

TABLE II
ERROR STATISTICS FOR IP SYSTEM

	e_1	e_2	e_3	e_4
ISE_{exact}	0.0874	0.1524	0.0484	9.1540
σ^2_{exact}	0.1665	0.5553	0.1060	16.7618
ISE_{apprx}	0.1175	0.1532	0.0486	9.1521
σ^2_{apprx}	0.2161	0.5562	0.1064	16.7170
ISE_{static}	0.1337	0.1570	0.0504	9.4530
σ^2_{static}	0.2424	0.5775	0.1087	18.1767

inferior to the proposed method due to the fact that a static KF does not account online changes in noise characteristics.

B. Example II-Inverted Pendulum

To establish the applicability of the proposed method for larger class of systems, an unstable system is considered. The proposed IFKF and the conventional Kalman filter are applied on a Inverted Pendulum (IP) system and the results are compared. As the IP is an unstable system, it allows us to verify the robustness of convergence of the proposed method through simulations. The simulation parameters considered are identical to example I. The discrete system matrix and observation matrix of the IP system are as follows.

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 0.003 & 0.058 & 0 \\ 0 & 1.231 & -0.184 & 0.105 \\ 0 & 0.054 & 0.30 & 0.003 \\ 0 & 4.702 & -3.2269 & 1.179 \end{bmatrix} \mathbf{\Gamma} = \mathbf{0}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{25}$$

The plot of trace of error covariance matrices in Fig. 3 and the error statistics given in Table II, show that the proposed method retains accuracy in comparison with KF even for unstable systems. As expected the trace of covariance matrix obtained using approximate inverse is slightly higher than the one obtained using the exact inverse in both the examples.

Note that though the system matrix ($\mathbf{\Phi}$) is not diagonally dominant in both the above examples, the innovation covariance matrix still has strong diagonal dominance. This resulted in the low error between actual and approximate inverse. The approximative inverse procedure doesn't place any restriction on the nature of system matrix and thus can be applied to a wide range of Kalman filtering applications.

V. CONCLUSION

The theory developed, along with numerical results show that one can use the approximate inverse in place of exact inverse for a large range of DD matrices. This Taylor series based approximate inverse is highly parallelizable, giving it a distinct advantage over sequential inverse finding approaches. The approximate inverse also preserves the positive definiteness of the matrix as shown in Theorem 1. Altogether, the approximate inverse of DD matrices could be used to make Kalman filter inverse free and speed up calculations. Despite the approximation, error covariance of IFKF is bounded as given by Theorem 2. Simulation results for Kalman filter further show that the inverse in update step can be replaced with approximate inverse without losing accuracy. The proposed IFKF equations, to a great effect, could make embedded implementations of Kalman filter realizable for a wide range of applications. Depending on the allowable error tolerance, one can use the inverse free Kalman filter equations proposed, for reducing the associated computational time. Possible extensions of approximate inverse finding techniques could employ different ways of splitting the matrix which results in better convergence properties and are applicable to a wider class of matrices.

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