Homework 0

Submitted by:
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Ans: 1

Random variable:

A random variable is a measurable function $X: \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where,

Ω = sample space
$\mathcal{F}$ = σ-algebra
$\mathbb{P}$ = probability measure

Examples of random variables

The five examples of random variables from the world around me are:

(i) Temperature.

(ii) Humidity.

(iii) Amount of time I sleep.

(iv) Chances of having a surprise test.

(v) Waiting time at the mess.

Ans: 2

(a) X = sum of the numbers showing up on a pair of fair dice when they are rolled.

support of X:

$$x_i \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Sample space Ω:

$$\Omega = \begin{bmatrix} 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \end{bmatrix}$$ 1

pmf of X:

x	$p_X(X=x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

(b)

X = max of two numbers when two fair dice are rolled

support of X:

$x_i \in \{1, 2, 3, 4, 5, 6\}$

Sample space is given in (1).

pmf of X:

x	$p_X(x)$
1	1/36
2	3/36
3	5/36
4	7/36
5	9/36
6	11/36

(c)

X = number of distinct numbers when three fair dice are rolled.

support of X:

$x_i \in \{1, 2, 3\}$

and $|\Omega| = 6 \times 6 \times 6 = 216$

pmf of X:

x	$p_X(x)$
1	$6/216$
2	$\frac{{}^5C_1 \times 6 \times {}^3C_2}{216} = 90/216$
3	$\frac{6 \times 5 \times 4}{216} = 120/216$

Mean of X:

$$E[X] = \frac{6 + 2\times 90 + 3\times 120}{216} = 2.5277$$

(d)

Given,

X is a random variable taking values from the set $x_i \in \{-2, -1, 0, 1, 2, 3, 4, 5\}$ with equal probability

Let,

$$Y = X^2$$

Then,

$$E[Y] = \sum_{i=1}^{\infty} x_i^2 \, P(x_i)$$

$$= \frac{4 + 1 + 1 + 4 + 9 + 20 + 25}{8} = 7.5$$

(e) Given,

X is a random variable with pmf $f_X(x)$.

Let,

$$Y = X^2 + 3X$$

Then,

$$E[Y] = E[X^2] + 3E[X] \quad \left(\text{linearity of expectation}\right)$$

Ans: 3

(a) Yes, every random variable has a cumulative distribution function (CDF).

Justification:

By definition, a random variable $X: \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable function, i.e., the inverse image $X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\}$ is $\mathcal{F}$-measurable, $B \in \mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ is a probability space and consider the Borel σ-algebra

$$\sigma\left((-\infty, x] \mid x \in \mathbb{R}\right) = \mathcal{B}(\mathbb{R}).$$

Then, $(-\infty, x] \in \mathcal{B}(\mathbb{R})$ and

$\mathbb{P}_X((-\infty, x])$ is well defined.

$$\mathbb{P}_X((-\infty, x]) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \leq x\})$$
$$= F_X(x),$$ is well defined.

(6) Given, the following PDF,

$$f_X(x) = \begin{cases} A(1-x^2), & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Value of A

$$\int_{-1}^{1} f_X(x)\, dx = 1$$

$$\Rightarrow A\int_{-1}^{1} (1-x^2)\, dx = 1$$

$$\Rightarrow A\left[2 - \frac{[x^3]_{-1}^{1}}{3}\right] = 1$$

$$\Rightarrow A\left[\frac{6-2}{3}\right] = 1$$

$$\Rightarrow A = 3/4$$

(ii) $\mathbb{P}(0.5 < x < 1.5)$

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$\mathbb{P}(0.5 < x < 1.5)$$

$$= \int_{0.5}^{1} 3/4 \, (1-x^2)\, dx$$

$$= 3/4 \left[0.5 - \frac{[x^3]_{0.5}^{1}}{3} \right]$$

$$= 3/4 \, [0.5 - 0.29167]$$

$$= 0.156$$

(iii) $F_X(x)$

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \frac{3}{4} \int_{-1}^{x} (1-x^2)\, dx$$

$$= \frac{3}{4} \left[(x+1) - \frac{(x^3+1)}{3} \right]$$

$$= \frac{3}{12} \left[3x + 3 - x^3 - 1 \right] = \frac{1}{4} \left[3x - x^3 + 2 \right]$$

$$\therefore \quad F_X(x) = \begin{cases} 0 \ , & x < -1 \\ \frac{1}{4}\left[3x - x^3 + 2\right], & -1 \leq x \leq 1 \\ 1 \ , & x > 1 \end{cases}$$

(iv) μ_X

$$\mu_X = \int_{-1}^{1} x f_X(x)\,dx$$

$$= \frac{3}{4}\int_{-1}^{1} x(1-x^2)\,dx$$

$$= \frac{3}{4}\int_{-1}^{1} (x - x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{\left[x^2\right]_{-1}^{1}}{2} - \frac{\left[x^4\right]_{-1}^{1}}{4}\right]$$

$$= \frac{3}{4} \cdot 0 = 0$$

(v) $Var(X)$

$$Var(X) = E\left[X^2\right] - \left(E[X]\right)^2$$

$$= E\left[X^2\right]$$

$$= \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)\,dx$$

$$= \frac{3}{4}\left[\frac{[x^3]_{-1}^{1}}{3} - \frac{[x^5]_{-1}^{1}}{5}\right]$$

$$= \frac{3}{4} \times \frac{4}{15} = 1/5 = 0.2$$

(vi) $\sqrt{x}$

$$\sqrt{x} = \sqrt{1/5} = 0.4472$$

(c) Given,

$$\mu(x) = 10, \quad Var(x) = 3$$

(i) X is Gaussian

Now,

$$\mathbb{P}(x < 8.27) = \frac{1}{2}\left[1 + erf\left(\frac{8.27 - 10}{\sqrt{3}\sqrt{2}}\right)\right]$$

$$= \frac{1}{2}\left[1 - 0.6821\right]$$

$$= 0.15895$$

(ii) X is uniform

Now,

$$\mu(x) = \frac{(a+b)}{2} = 10 \Rightarrow a + b = 20 \quad (2)$$

$$Var(x) = \frac{(b-a)^2}{12} = 3 \Rightarrow b - a = \pm 6 \quad (3)$$

Solving, (2), (3),

$b = 13, 7, \; a = 7, 13$

$\therefore a = 7, \; b = 13 \quad \because b > a$

Now,

$$F_X(x) = \begin{cases} 0 & , \; x < 7 \\ \frac{x-7}{6} & , \; 7 \le x \le 13 \\ 1 & , \; x > 13 \end{cases}$$

Now,

$$F_X(8.26) = \frac{8.26-7}{6} = 0.21$$

Ans: 4

Given, (X, Y) are uniformly distributed over the unit circle.

(i) $f_{X,Y}(x, y)$

We have,

$$X^2 + Y^2 \le 1^2$$

$$\Rightarrow -\sqrt{1-x^2} \le Y \le \sqrt{1-x^2}$$

Now,

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X,Y}(x,y) \, dx \, dy = 1$$

$$\Rightarrow \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} c \; dx \, dy = 1 \quad (\because X, Y \text{ are uniformly distributed})$$

$$\Rightarrow C\int_{-1}^{1} 2\sqrt{1-x^2}\, dx = 1$$

Let, $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\theta \in [-\pi/2, \pi/2]$

$$\Rightarrow C\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} |\cos\theta| \cos\theta\, d\theta = 1/2$$

$$\Rightarrow C\int_{-\pi/2}^{\pi/2} \cos^2 d\theta = 1/2 \quad (\because \theta \in [-\pi/2, \pi/2])$$

$$\Rightarrow \frac{C}{2}\int_{-\pi/2}^{\pi/2} (\cos 2\theta + 1)\, d\theta = 1/2$$

$$\Rightarrow C \cdot \pi = 1$$

$$\Rightarrow C = 1/\pi$$

(Eg) $\mathbb{P}(\{X, Y\} \in A)$, $A = \{(x, y) : x \geq 0, y \geq 0\}$

Now,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1/\pi \; dy\, dx$$

$$= \frac{1}{\pi}\int_0^1 \sqrt{1-x^2}\; dy\, dx$$

Let, $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\theta \in [-\pi/2, \pi/2]$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{2\pi}\int_0^{\pi/2} (\cos 2\theta + 1)\, d\theta = \frac{1}{4}$$

(iii) $\mathbb{P}(\{X^2+Y^2 \le r^2\})$, $r \geq 0$

The region $\{(x,y) \mid x^2+y^2 \le r^2\}$ is a disk.

$\therefore$ we have

$$\mathbb{P}(\{X^2+Y^2 \le r^2\}) = \begin{cases} r^2 & , \ 0 \le r \le 1 \\ 1 & , \ r > 1 \end{cases}$$

(iv) $f_X(x)$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X,Y}(x,y)\, dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1/\pi \ dy$$

$$= \frac{2\sqrt{1-x^2}}{\pi}$$

$$\therefore \ f_X(x) = \begin{cases} \dfrac{2\sqrt{1-x^2}}{\pi} & , \quad |x| \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(v) $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-x^2}}$$

$$= \frac{1}{2\sqrt{1-x^2}}$$

$$\therefore f_{Y|X}(y|x) = \begin{cases} \frac{1}{2\sqrt{1-x^2}} & , -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \\ 0 & , \text{otherwise} \end{cases}$$

Ans: 5

The code can be found in the folder EE18MTECH11030_hw0 along with the plots.

The file name (code) is:

q5_N_armed_Bandit_Problem_Epsilon_Greedy_approach_final.ipynb