Homework 0

Subject: Reinforcement Learning

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Ans: 3

Maximum-likelihood estimation of

(a) Binomial Distribution

We have,

$$f_X(x|\theta) = \binom{n}{x} p^x (1-p)^{1-x}, \quad \theta = p$$

Now, the likelihood function is,

$$\mathcal{L}(X;\theta) = \prod_{i=1}^{N} \binom{n}{x_i} p^{x_i} (1-p)^{1-x_i} \quad \text{(}\because\text{ the samples are i.i.d)}$$

$\longrightarrow$ (1)

Now,

the log-likelihood function is,

$$\ell(X;\theta) = \log(\mathcal{L}(x;\theta))$$

$$= \sum_{i=1}^{N} \log\left[\binom{n}{x_i} p^{x_i} (1-p)^{1-x_i}\right]$$

$$= \sum_{i=1}^{N} \left(\log\binom{n}{x_i} + x_i \log(p) + (1-x_i)\log(1-p) \right)$$

$\longrightarrow$ (2)

Now,

$$\theta_{MLE} = p_{MLE} = \arg\max_{\theta} \ell(x;\theta)$$

$$\therefore \left.\frac{\partial \ell}{\partial p}\right|_{p_{MLE}} = 0$$

$$\Rightarrow \frac{\partial}{\partial p}\left[\sum_{i=1}^{N}\left(\log\binom{n}{x_i} + x_i \log(p) + (1-x_i)\log(1-p)\right)\right] = 0$$

(For p_{MLE}, MLE subscript is dropped for convenience)

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{x_i}{p} - \frac{(1-x_i)}{1-p}\right] = 0$$

$$\Rightarrow \sum_{i=1}^{N}\frac{x_i}{p} = \sum_{i=1}^{N}\frac{(1-x_i)}{1-p}$$

$$\Rightarrow \frac{1-p}{p} = \frac{N - \sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i}$$

$$\Rightarrow \frac{1}{p} = \frac{\sum_{i=1}^{N} x_i + N - \sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i}$$

$$\Rightarrow p = \frac{\sum_{i=1}^{N} x_i}{N}$$

$$\therefore \boxed{p_{MLE} = p = \frac{\sum_{i=1}^{N} x_i}{N}}$$

(b) Poisson Distribution

We have,

$$f_X(x;\theta) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad \theta = \lambda$$

Now, the likelihood function is,

$$L(x;\theta) = \prod_{i=1}^{N} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \quad \left(\text{i.i.d assumption of the samples}\right)$$

Now, the log-likelihood function is,

$$\ell(x;\theta) = \log\left(\prod_{i=1}^{N} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$

$$= \sum_{i=1}^{N} \log\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$

$$= \sum_{i=1}^{N} \left(-\log(x_i!) + \log\left(e^{-\lambda}\lambda^{x_i}\right)\right)$$

$$= \sum_{i=1}^{N} \left(-\log(x_i!) + (-\lambda) + x_i \log(\lambda)\right)$$

Now,

$$\theta_{MLE} = \lambda_{MLE} = \arg\max_{\theta} \ell(x;\theta)$$

$$\therefore \quad \frac{\partial \ell}{\partial \lambda_{MLE}} = 0$$

$$\Rightarrow \frac{\partial}{\partial \lambda}\left(\sum_{i=1}^{N}\left(-\log(x_i!) + (-\lambda) + x_i \log(\lambda)\right)\right) = 0$$

$$\Rightarrow \quad -N + \frac{1}{\lambda}\sum_{i=1}^{N} x_i = 0$$

(For λ_{MLE}, MLE subscript is dropped for convenience)

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{N} x_i}{N}$$

$$\therefore \boxed{\lambda_{MLE} = \lambda = \frac{\sum_{i=1}^{N} x_i}{N}}$$

(c) Exponential Distribution

We have,

$$f_X(x;\theta) = \lambda e^{-\lambda x}, \quad \theta = \lambda$$

Now, the likelihood function is,

$$\mathcal{L}(x;\theta) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i}, \quad \left(\text{i.i.d assumption of samples}\right)$$

Now, the log-likelihood function is,

$$l(x;\theta) = \sum_{i=1}^{N} \log\left(\lambda e^{-\lambda x_i}\right)$$

Now,

$$\theta_{MLE} = \lambda_{MLE} = \arg\max_{\theta} \ell(x;\theta)$$

$$\therefore \frac{\partial \ell}{\partial \lambda_{MLE}} = \frac{\partial}{\partial \lambda_{MLE}} \sum_{i=1}^{N} \log\left(\lambda e^{-\lambda x_i}\right) = 0$$

(for λ_{MLE}, MLE subscript is dropped for convinience)

$$\Rightarrow \frac{\partial}{\partial \lambda} \sum_{i=1}^{N} \left(\log \lambda - \lambda x_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{N} \left[\frac{1}{\lambda} - x_i\right] = 0$$

$$\Rightarrow \frac{N}{\lambda} = \sum_{i=1}^{N} x_i$$

$$\Rightarrow \lambda = \frac{N}{\sum_{i=1}^{N} x_i} = \frac{1}{\sum_{i=1}^{N} x_i / N}$$

$$\therefore \boxed{\lambda_{MLE} = \lambda = \frac{1}{\sum_{i=1}^{N} x_i / N}}$$

(d) Gaussian Distribution

We have,

$$f_x(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad \theta = [\mu \ \sigma]$$

Now,

the likelihood function is,

$$\mathcal{L}(x;\theta) = \prod_{i=1}^{N} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right)$$

$$\Rightarrow \mathcal{L}(x;\theta) = \prod_{i=1}^{N} \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2} \right) \right)$$

$$\Rightarrow \mathcal{L}(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\sum_{i=1}^{N} (x_i-\mu)^2}{2\sigma^2} \right)$$

Now,

the log-likelihood function is

$$l(x;\theta) = \log(\mathcal{L}(x;\theta))$$

$$= -\frac{N}{2} \log(2\pi\sigma^2) - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\sigma^2}$$

Now,

$$\theta_{MLE} = \arg\max_{\theta} l(x;\theta)$$

$$\therefore \frac{\partial l}{\partial \mu} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{N} 2(x_i-\mu)}{2\sigma^2} = 0$$

$$\Rightarrow \sum_{i=1}^{N} x_i = N\mu$$

$$\Rightarrow \mu = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\therefore \boxed{\mu_{MLE} = \mu = \frac{1}{N} \sum_{i=1}^{N} x_i}$$

and,

$$\frac{\partial l}{\partial \sigma^2_{MLE}} = 0$$

$$\Rightarrow \frac{\partial}{\partial \sigma^2}\left[-\frac{N}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial \sigma^2}\left[-\frac{N}{2}\log(2\pi) - \frac{N}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right] = 0$$

$$\Rightarrow -\frac{N}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{N}(x_i-\mu)^2 = 0$$

$$\Rightarrow N = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\Rightarrow \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\therefore \boxed{\sigma^2_{MLE} = \sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2}$$

(e) Laplacian Distribution

We have,

$$f_X(x;\theta) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right), \quad \theta = [\mu, b]$$

The likelihood function is,

$$L(X;\theta) = \prod_{i=1}^{N} \frac{1}{2b} \exp\left(-\frac{|x_i-\mu|}{b}\right)$$

$$= \frac{1}{(2b)^N} \exp\left(-\frac{\sum_{i=1}^{N} |x_i-\mu|}{b}\right)$$

The log-likelihood is given as,

$$l(X;\theta) = \log\left(L(X;\theta)\right)$$

$$= \log\left(\frac{1}{(2b)^N} \exp\left(-\frac{1}{b}\sum_{i=1}^{N} |x_i-\mu|\right)\right)$$

$$= -N\log(2b) - \frac{1}{b}\sum_{i=1}^{N} |x_i-\mu|$$

Now,

$$\theta_{MLE} = \arg\max_{\theta} \, l(X;\theta)$$

$\therefore \quad \frac{\partial l}{\partial \mu} = 0$

$$\Rightarrow \frac{\partial}{\partial \mu}\left[-N\log(2b) - \frac{1}{b}\sum_{i=1}^{N}|x_i - \mu|\right] = 0$$

$$\Rightarrow \frac{1}{b}\sum_{i=1}^{N}\frac{\partial}{\partial \mu}|x_i - \mu| = 0$$

$$\Rightarrow \frac{1}{b}\sum_{i=1}^{N}\frac{x_i - \mu}{|x_i - \mu|} = 0$$

$$\Rightarrow \frac{1}{b}\sum_{i=1}^{N}\operatorname{sgn}(x_i - \mu) = 0 \quad \text{—— (3)}$$

To maximize (3), we choose,

$$\mu_{MLE} = \text{median}\{x_1, \ldots, x_N\}$$

Now,

$$\frac{\partial l}{\partial b} = 0$$

$$\Rightarrow \frac{\partial}{\partial b}\left[-N\log(2b) - \frac{1}{b}\sum_{i=1}^{N}|x_i - \mu|\right] = 0$$

$$\Rightarrow -Nb + \sum_{i=1}^{N}|x_i - \mu| = 0$$

$$\Rightarrow \boxed{b = \frac{1}{N}\sum_{i=1}^{N}|x_i - \mu|}$$

The programmes are implemented in:

EE18MTECH11030_HW0_Question3_final.ipynb.

Ans: 1

The programmes are implemented in.

EE18MTECH11030_HW0_Question1_blob_final.ipynb

EE18MTECH11030_HW0_Question1_image_final.ipynb

Ans: 2

The programmes are implemented in.

EE18MTECH11030_HW0_Question2_image_final.ipynb