Linear Optimization: Model Building and Fitting

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Linear Optimization

Outline

- In general, optimization is a tool for decision making.
  - Tells us the optimal input to our objective function $f(b)$
- So far, only considered $f(b)$ that is not linear in b.
- What about $f(b)$ linear in b?
  - EX: $f(b) = b_1 x_1 + b_2 x_2 - b_3 x_3$
  - What if there are restrictions on x_1, x_2, x_3?
  - Can we just apply the same tools?

Linear Optimization Methods

- A linear optimization problem depends on
  - decision variables, $x_1, x_2, \ldots, x_n$
  - objective function $f = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n$
  - known $b_1, \ldots, b_n$
  - linear constraints: restrictions on $x_1, \ldots, x_n$.
- Specific case of optimization where f is linear.

Linear Optimization Methods

- Allow us to deal with cases where
  - $f(b)$ linear in b?
  - there are restrictions on x_1, x_2, x_3.
  - Solve real world problems.
- AIM: minimize or maximize some objective function f **(same as before)**
  - Subject to restrictions, called **constraints**, on x_1, x_2, x_3.
  - Linearity of $f(b)$ in b.

Linear optimization Structure

- $\max_{x_1,\ldots,x_n} b_1x_1 + b_2x_2 + \ldots + b_nx_n$ subject to
- $a_{11}x_1 + \ldots + a_{1n}x_n \leq RHS_1$
- $\ldots$
- $a_{m1}x_1 + \ldots + a_{mn}x_n \leq RHS_n$
- $x_1, \ldots, x_n \geq 0$

Example

-A farmer has 240 acres of land to plant. She needs to decides how many acres of corn to plant and how many acres of oats. She can make $40 per acre profit for corn and $30 per acre for oats. However, the corn takes 2 hours of labour per acre to harvest and the oats take only 1 hour per acre. She only has 320 hours of labour she can invest. To maximize her profit, how many acres of each should she plant?

```
- What is the objective function?
- What are the constraints?
```

Example

-A farmer has 240 acres of land to plant. She needs to decides how many acres of corn to plant and how many acres of oats. She can make **$40 per acre profit for corn** and **$30 per acre for oats**. However, the *corn takes 2 hours of labour per acre to harvest* and the *oats take only 1 hour per acre*. She only has *320 hours of labour* she can invest. **To maximize her profit**, how many acres of each should she plant?

```
- What is the objective function?
- What are the constraints?
```

From words to Math

- Decision variables: corn x, oats y (in acers)
- Obj. Fun.
  - Profit: $\Pi = 40x + 30y$
- Constraints?
  - $2x + y \leq 320$
  - $x + y \leq 240$
- Mathematically:

$$\max_{x,y} \Pi \; s.t. \;\; 2x + y \leq 320; \;\; x + y \leq 240.$$

Modelling Assumptions in Linear Programming

- **Proportionality**: the contribution to the objective function and the left-hand-side of each constraint by each decision variable is proportional to the value of the decision variable.
- **Additivity**: the contribution to the objective function and the left-hand-side of each constraint by any decision variable is completely independent of any other decision variable and is additive (involving?addition).
- **Divisibility**: the quantities represented by each decision variable are innately divisible (i.e., fractional answers make sense).
- **Certainty**: the coefficients in the objective function and constraints are known with certainty.

Example: Par, INC.

-The PAR, Inc company produces golf bags. -The managers are interested in expand production and also want to move to **medium** & **high** price bags. -Bag production involves 4 steps:

```
- Cutting and dyeing the material
- Sewing
- Finishing
- Inspection & packaging
```

Example: Par, INC.

-The director of manufacturing estimated that each of the two options will result in the following number of hours at each department:

```
- Cutting and dyeing the material
    -Standard Bag: 7/10, Delux Bag: 1
- Sewing
    -Standard Bag: 1/2, Delux Bag: 5/6
- Finishing
    -Standard Bag: 1, Delux Bag: 2/3
- Inspection & packaging
    -Standard Bag: 1/10, Delux Bag: 1/4
```

Example: Par, INC.

-After studying each department, the director estimates the following production capacities:

```
- 630 hours of cutting and dyeing,
- 600 hours of sewing,
- 708 hours of finishing, and
- 135 hours for inspection and packaging
```

-The accounting manager estimates that the company's profit from a standard bag and a deluxe bag will be **$10** and **$9**, respectively.

Example: Par, INC.

Decision Variables?

Objective function?

Constraints?

Example: Par, INC.

- Decision Variables?
  - Number of bags of each type:
    - S = standard bags
    - D = delux bags

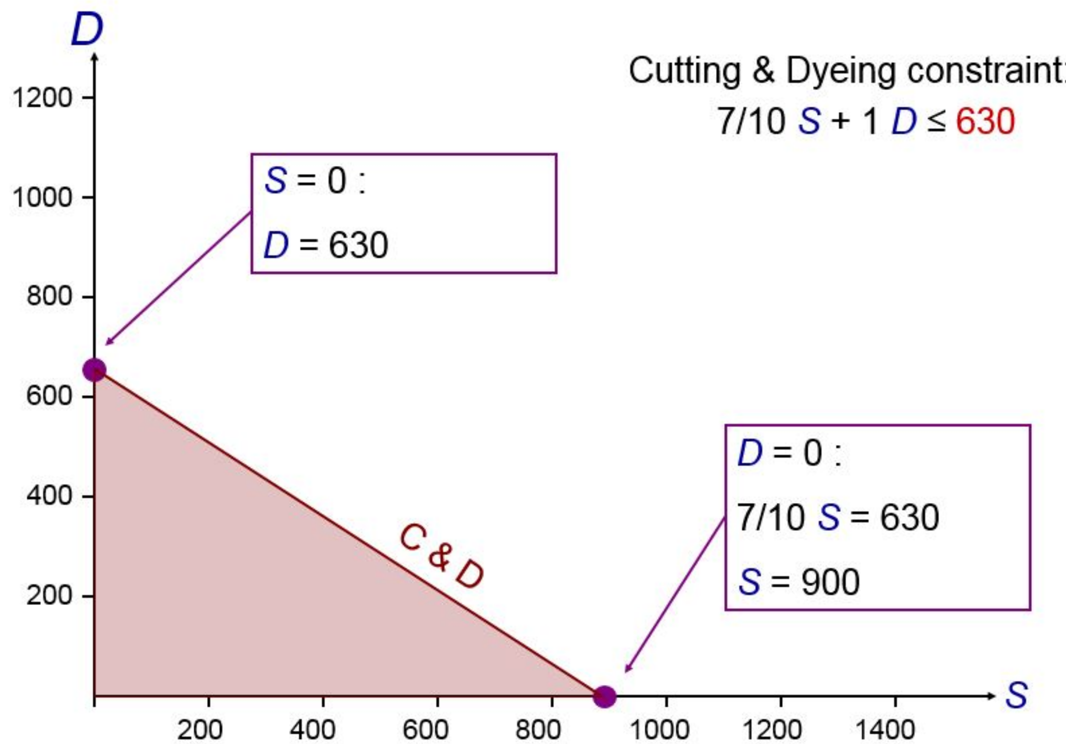
Example: Par, INC.

- Objective function?
  - Maximize profit: $max_{S,D} 10S + 9D$

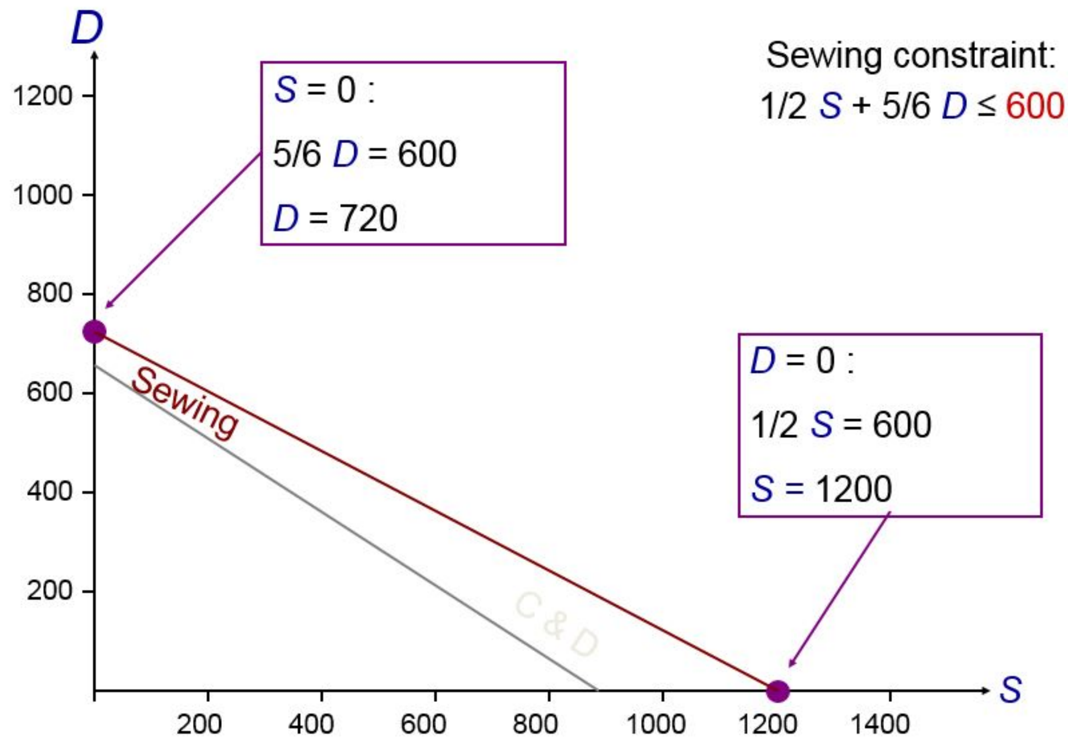
Example: Par, INC.

- Constraints?
  - Cutting and dyeing the material
    - Standard Bag: 7/10, Delux Bag: 1; 630 hours
    - $(7/10)S + D \leq 630$
  - Sewing
    - Standard Bag: 1/2, Delux Bag: 5/6; 600 hours
    - $(1/2)S + (5/6)D \leq 600$
  - Finishing
    - Standard Bag: 1, Delux Bag: 2/3; 708 hours
    - $S + (2/3)D \leq 708$
  - Inspection & packaging
    - Standard Bag: 1/10, Delux Bag: 1/4; 135 hours
    - $(1/10)S + (1/4)D \leq 135$
  - Number of bags can't be negative!!
    - $S, D \geq 0.$

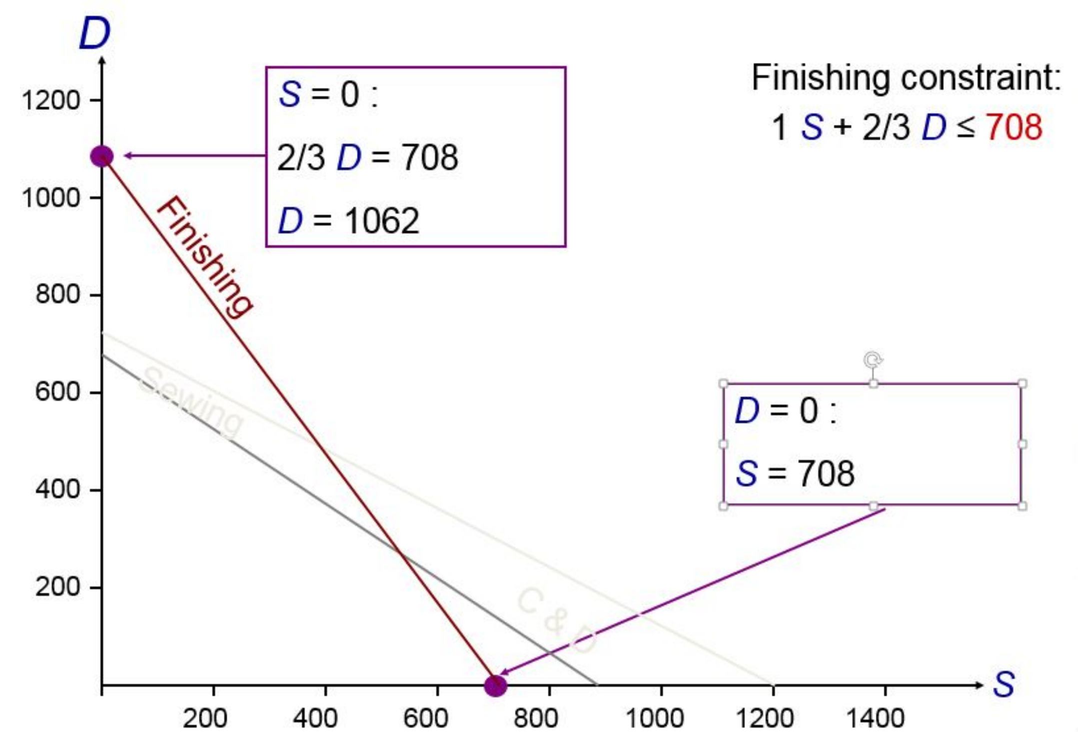
Example: Par, INC.



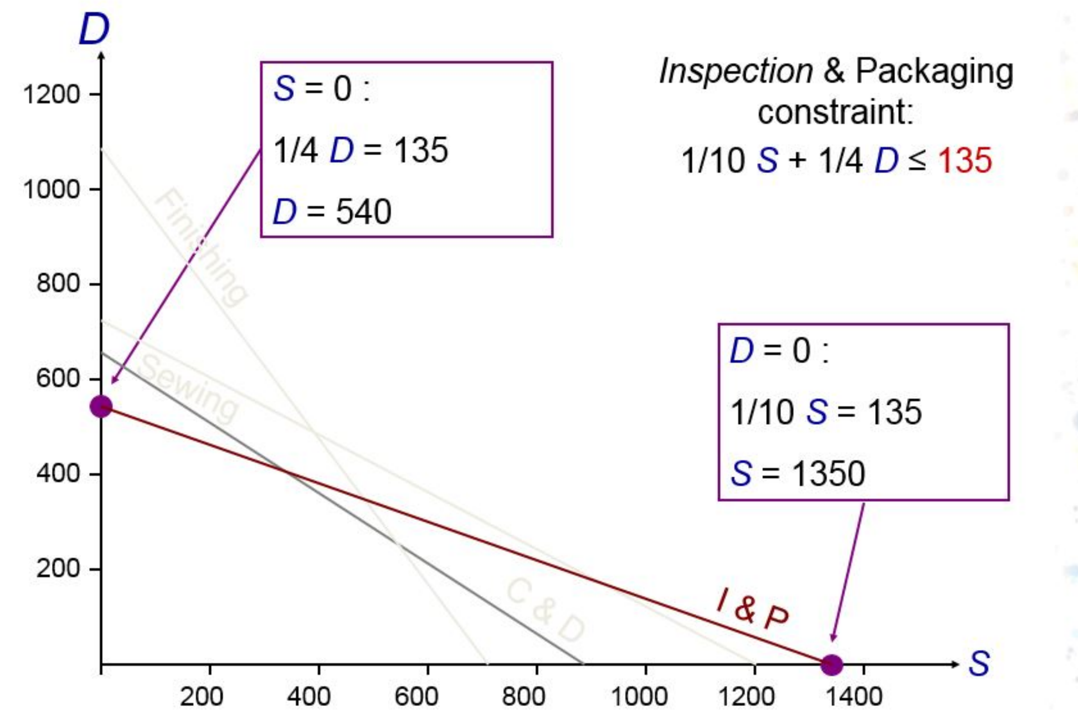
Example: Par, INC.



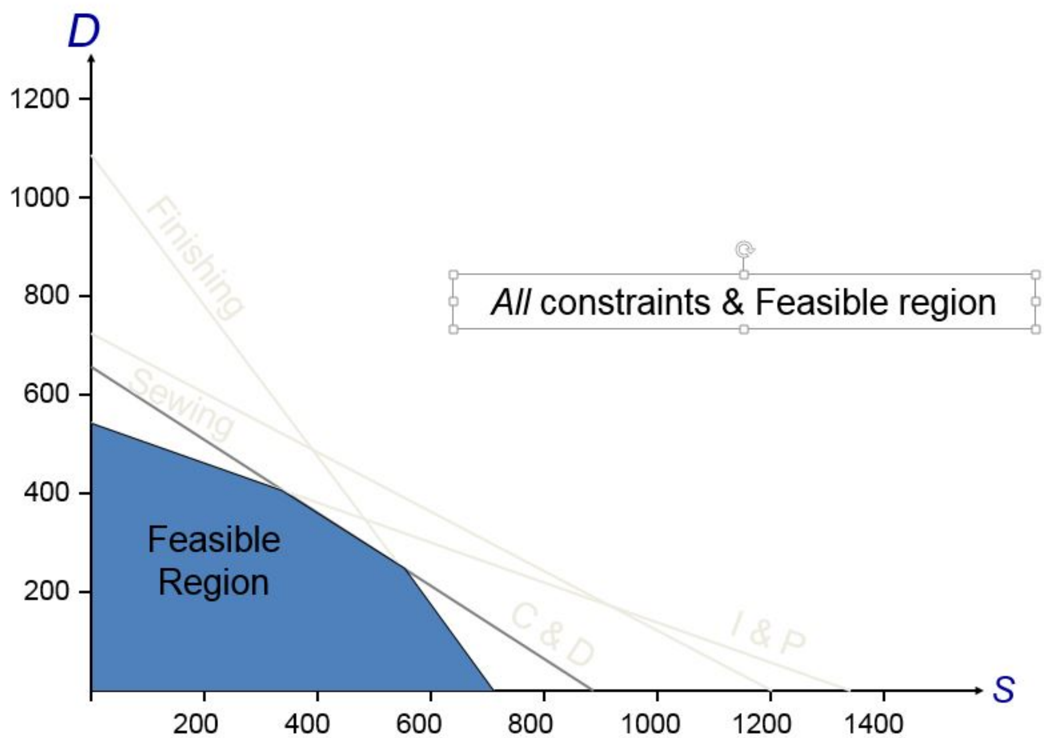
Example: Par, INC.



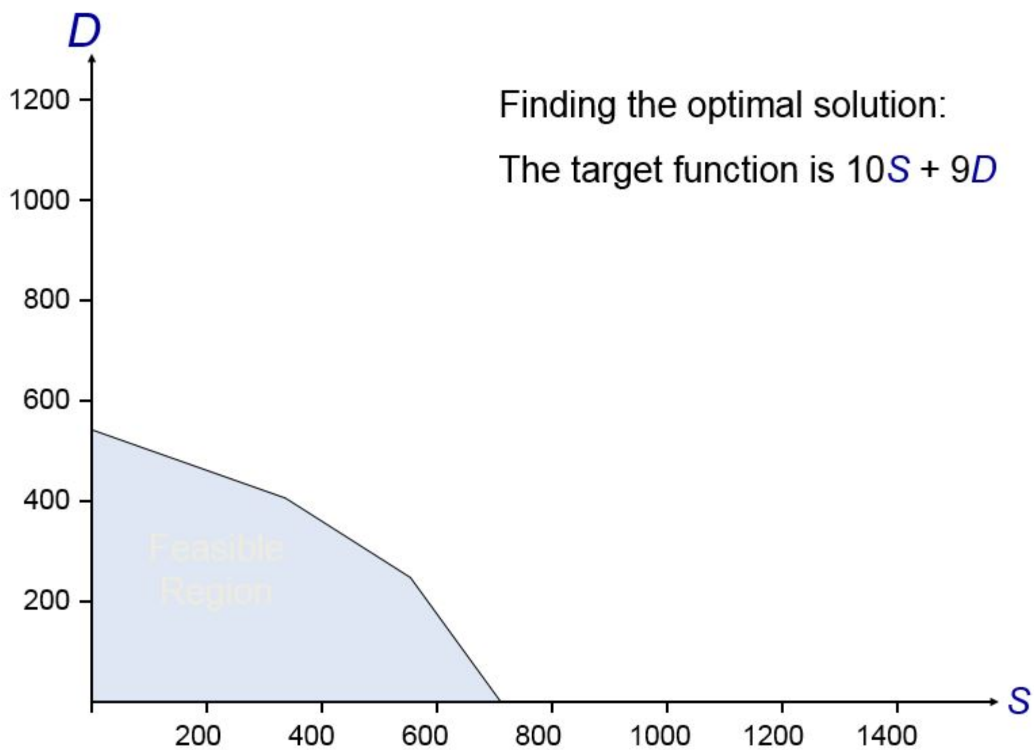
Example: Par, INC.



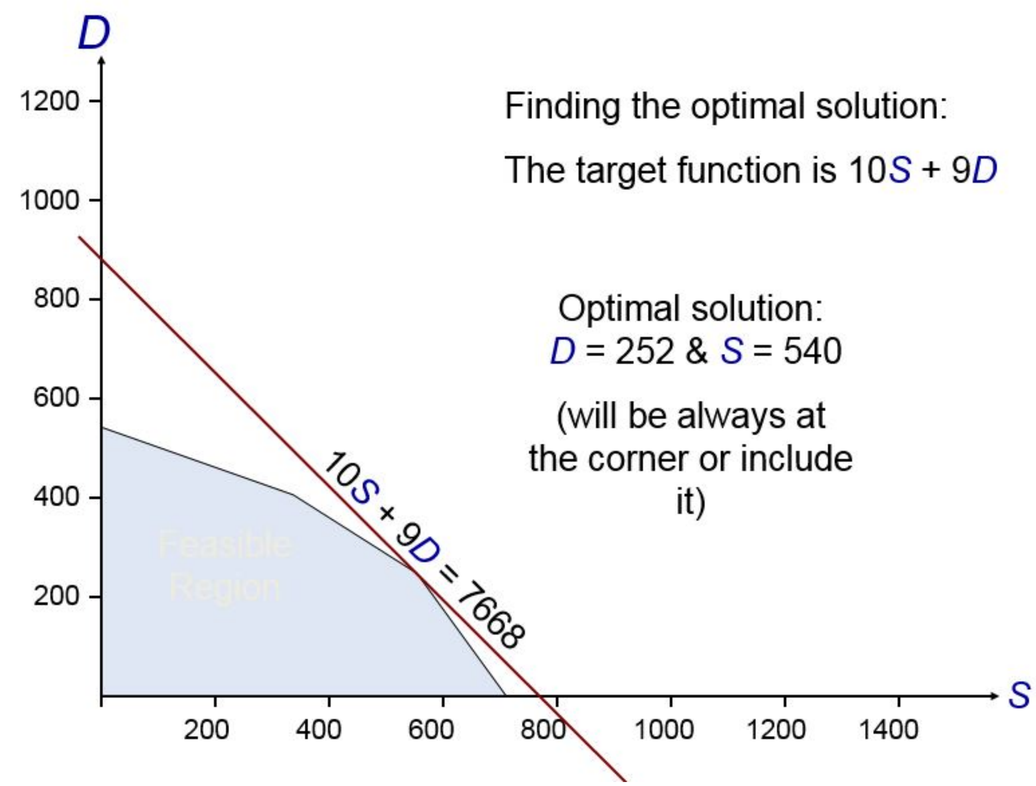
Example: Par, INC.



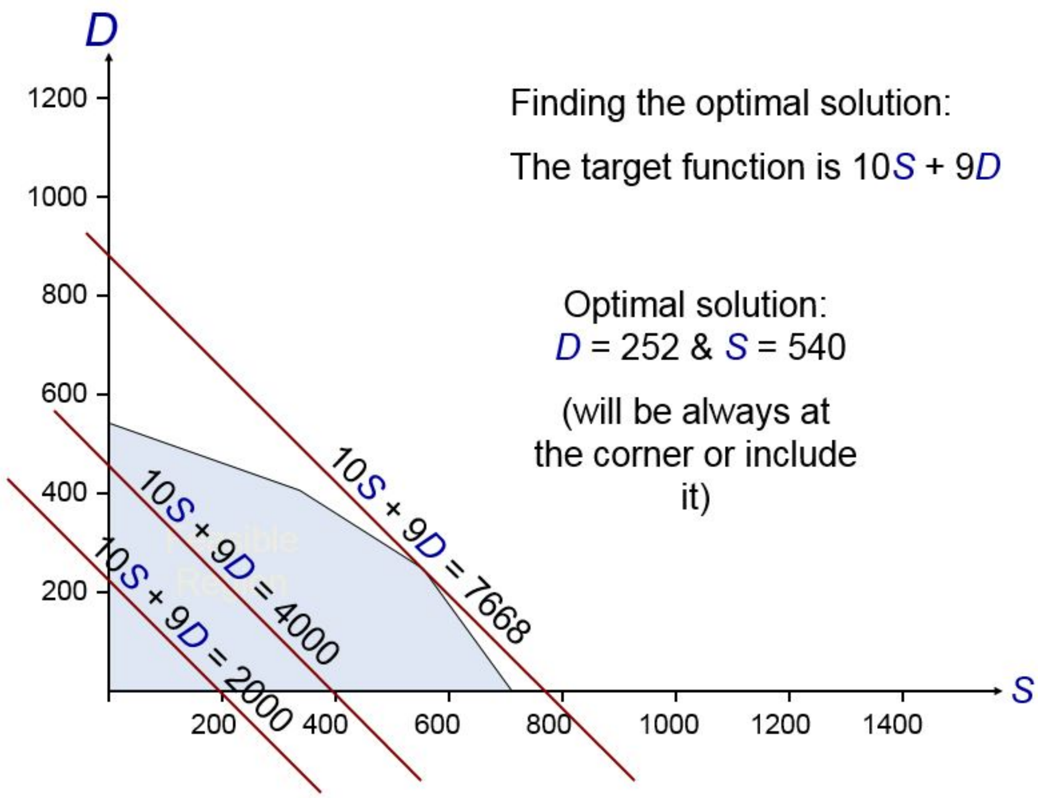
Example: Par, INC.



Example: Par, INC.



Example: Par, INC.



How to solve this in R?

- Just like `optim`, we need some numerical method to solve this optimization problem
- We'll make use of the `lpSolve` package in R.
  - "Modeling and Solving Linear Programming with R."
- Similar to `optim` function
  - Takes function as input
  - initial values for parameters
- Main difference: Must now add in constraint information.

Constraints in `lpSolve`

- Constraint information must be entered in a particular way
- Three pieces of information
  - Constraint coefficients
  - Constraint direction
  - RHS value of the constraint
- Must be entered as a matrix
- Each row of the matrix corresponds to a constraint
- Include inequality information in a separate line

Example: Par, INC

- Easiest to see from an example.

```
library(lpSolve)
obj_fun <- c(10,9)
constr <- matrix(c(7/10,1,1/2,5/6,1,2/3,1/10,1/4),ncol = 2
const_dir <- c("<=","<=","<=","<=")
RHS <- c(630,600,708,135)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 540 252
```

Example: Solve Farm Prob.

Problem:

$$\max_{x,y} 40x + 30y,\ s.t.\ x + y \leq 320;\ x + y \leq 240.$$

```
library(lpSolve)
obj_fun <- c(40,30)
constr <- matrix(c(2,1,1,1),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=")
RHS <- c(320,240)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1]  80 160
```

Example

A small business sells two products, named Product 1 and Product 2. Each tonne of Product 1 consumes 30 working hours, and each tonne of Product 2 consumes 20 working hours. The business has a maximum of 2,700 working hours for the period considered. As for machine hours, each tonne of Products 1 and 2 consumes 5 and 10 machine hours, respectively. There are 850 machine hours available.

Each tonne of Product 1 yields 20 Me of profit, while Product 2 yields 60 Me for each tonne sold. For technical reasons, the firm must produce a minimum of 95 tonnes in total between both products. We need to know how many tonnes of Product 1 and 2 must be produced to maximize total profit.

Example: Objective function?

A small business sells two products, named Product 1 and Product 2. Each tonne of Product 1 consumes 30 working hours, and each tonne of Product 2 consumes 20 working hours. The business has a maximum of 2,700 working hours for the period considered. As for machine hours, each tonne of Products 1 and 2 consumes 5 and 10 machine hours, respectively. There are 850 machine hours available.

Each tonne of Product 1 yields **20 Millon of profit**, while Product 2 yields **60 Millon** for each tonne sold. For technical reasons, the firm must produce a minimum of 95 tonnes in total between both products. We need to know how many tonnes of Product 1 and 2 must be produced to **maximize total profit**.

Example: Objective function?

- Mathematically:

$$\max_{p_1,p_2} 20p_1 + 60p_2$$

Example: Constraints?

A small business sells two products, named Product 1 and Product 2. Each tonne of **Product 1 consumes 30 working hours**, and each tonne of **Product 2 consumes 20 working hours**. The business has a maximum of **2,700 working hours** for the period considered. As for machine hours, each tonne of **Products 1 and 2 consumes 5 and 10 machine hours**, respectively. There are **850** machine hours available.

Each tonne of Product 1 yields 20 Millon of profit, while Product 2 yields 60 Millon for each tonne sold. For technical reasons, the firm must produce **a minimum of 95 tonnes in total** between both products. We need to know how many tonnes of Product 1 and 2 must be produced to maximize total profit.

Example: Constraints?

- Mathematically:
  - $30p_1 + 20p_2 \leq 2700$
  - $5p_1 + 10p_2 \leq 850$
  - $p_1 + p_2 \geq 95$
  - $p_1, p_2 \geq 0.$

Example: Solution?

```
library(lpSolve)
obj_fun <- c(20,60)
constr <- matrix(c(30,20,5,10,1,1),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=",">=")
RHS <- c(2700,850,95)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 20 75
```

Additional Examples

- Portfolio selection
- Transportation planning
- Advertising campaign planning

Example: Portfolio Selection

- Select a specific investment stragey
- Variety of investment alternatives
- Goal: maximize profit or minimize risk.

Example: Portfolio Selection

Investment opportunity for a fund of $100,000.

The company has been recommended to invest in the oil industry, steel industry or government bonds.

Five investment opportunities: Atlantic Oil (7.3% return), Pacific Oil (10.3 %), Midwest Steel (6.4%), Huber Steel (7.5%) and Government bonds (4.5%).

Example: Portfolio Selection

- Investment guidelines:
  - Neither oil or steel industry should receive more than \$50,000.
  - Amount invested in government bonds should be at least 25 percent of the steel industry investments.
  - The investment in Pacific Oil, the high-return but high-risk investment, cannot be more than 60 percent of the total oil industry investment.

Example: Portfolio Selection

- Define the following decision variables:
  - X_1 - Atlantic Oil
  - X_2 - Pacific Oil
  - X_3 - Midwest Steel
  - X_4 - Huber Steel
  - X_5 - Government bonds

Example: Portfolio Selection

- Objective function?

Example: Portfolio Selection

- Objective function?
  - $\max_{x_1,\ldots,x_5} .073x_1 + .103x_2 + .064x_3 + .075x_4 + .045x_5$

Example: Portfolio Selection

- Constraints?

Example: Portfolio Selection

- Constraints?
- Max fund 10000
    - $x_1 + x_2 + \ldots + x_5 \leq 100,00$
- Oil or steel no more than $50,000.
    - $x_1 + x_2 \leq 50,000$
    - $x_3 + x_4 \leq 50,000$
- Government bonds at least 25 percent of the steel industry investments.
    - $x_5 \geq .25 * (x_3 + x_4)$
- Pacific Oil investment, no more than 60 percent of the total oil industry investment.
    - $x_2 \leq .6 * (x_1 + x_2)$
- $x_1, \ldots, x_5 \geq 0$

Putting it together

- $\max_{x_1,\ldots,x_5} .073x_1 + .103x_2 + .064x_3 + .075x_4 + .045x_5$
  - $x_1 + x_2 + \ldots + x_5 \leq 100,00$
  - $x_1 + x_2 \leq 50,000$
  - $x_3 + x_4 \leq 50,000$
  - $x_5 \geq .25 * (x_3 + x_4)$
  - $x_2 \leq .6 * (x_1 + x_2)$
  - $x_1, \ldots, x_5 \geq 0$

Integer Programs and Optimization

Integer Optimization

- So far, assumed $x_1, \ldots, x_n$ infinitely divisible
  - i.e., continuous.
- Not true in many cases
  - Can we sell partial apartments?
  -
- **Integer linear programs**: Problems that are modeled as linear programs with the additional requirement that at least one variable must be integer.

Integer Optimization

-Types of Integer Linear Optimization Models:

```
- All-integer linear program
- Mixed-integer linear program
- Binary integer linear program
```

Linear optimization Structure

- $\max_{x_1,\ldots,x_n} b_1x_1 + b_2x_2 + \ldots + b_nx_n$ subject to
- $a_{11}x_1 + \ldots + a_{1n}x_n \leq RHS_1$
- $\ldots$
- $a_{m1}x_1 + \ldots + a_{mn}x_n \leq RHS_n$
    - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1, 2, \ldots$ all $j = 1, \ldots, n$
    - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1, 2, \ldots$; x_i, $i \neq j$, continuous
    - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1$ for all $j = 1, \ldots, n$

Linear optimization Structure

- Using `R`, can solve nearly exactly the same way
- All-integer
  - `lp("min", f, const.mat, const.dir, const.rhs, all.int=TRUE)`
- All-binary
  - `lp("min", f, const.mat, const.dir, const.rhs, all.bin=FALSE)`
- Mixed
  - `lp("min", f, const.mat, const.dir, const.rhs, int.vec)`
  - Numeric vector giving the indices of variables that are required to be integer. - The length of this vector will therefore be the number of integer variables

LP Relaxation

- In many cases, we can simply treat the decision variables as continuous
    - Then solve the optimization problem,
    - Round (up or down) to nearest integer.
- Called LP relaxation

LP Relaxation

- Gives a near-optimial solution
- Rounding may not always be a good strategy.
  - If decision variables take on small values
  - Can have major impact on the value of the objective function or feasibility,
  - In such cases, optimal integer solution is needed.

All Integer Example: Eastborne Realty

- Eastborne Realty has $2 million available for the purchase of new rental property.
- After an initial screening, Eastborne reduced the investment alternatives to townhouses and apartment buildings.
- Each townhouse can be purchased for $282,000.
  - Five are available.
- Each apartment building can be purchased for $400,000.
  - The developer will construct as many buildings as Eastborne wants to purchase.

All Integer Example: Eastborne Realty

- Eastborne Realty has $2 million available for the purchase of new rental property.
- After an initial screening, Eastborne reduced the investment alternatives to townhouses and apartment buildings.
  - T- townhouse, A- apartment.
- Each townhouse can be purchased for 282,000
  - Five available: $T \leq 5$
- Each apartment can be purchased for 400,000. Develope construct as many as desired.
  - $282T + 400A \leq 2000$

All Integer Example: Eastborne Realty

- Property manager can devote 140 hours per month to properties.
- Each townhous requires 4 hours per month.
- Each apartment requires 40 hours per month.

All Integer Example: Eastborne Realty

- Property manager can devote 140 hours per month to properties.
- Each townhous requires 4 hours per month.
- Each apartment requires 40 hours per month.
  - $4T + 40A \leq 140$

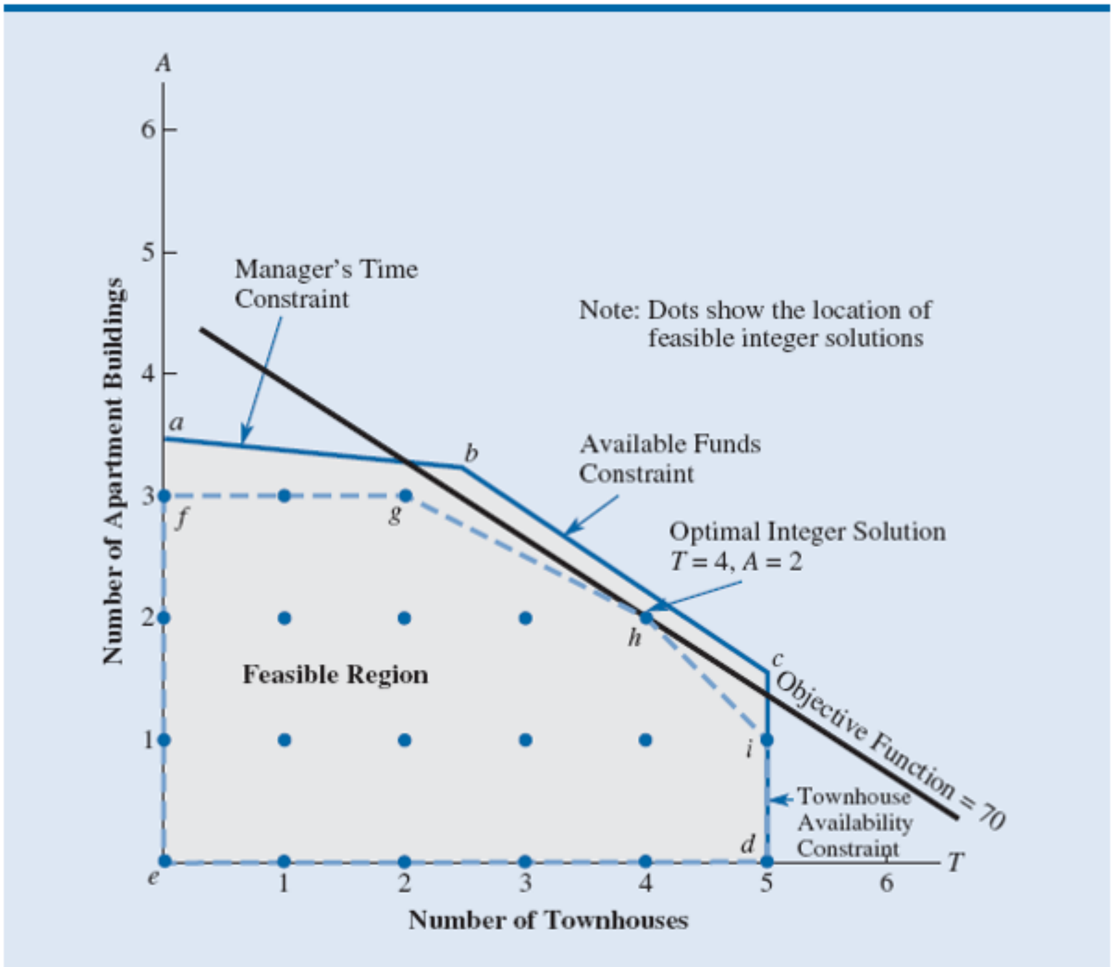
All Integer Example: Eastborne Realty

- Annual cash flow (After deducting mortgage payments and operating expenses) estimated to be:
  - \$10,000 per townhouse.
  - \$15,000 per apartment building.
- The objective: Determine the number of townhouses and the number of apartment buildings to purchase to maximize annual cash flow.

All Integer Example: Eastborne Realty

- Defining the decision variables:
  - T = number of townhouses
  - A = number of apartment buildings
- Objective function for cash flow (in thousands):
  - $max_{T,A} 10T + 15A$
- Constraints?
  - Available funds: $282T + 400A \leq 2000$
  - Management: $4T + 40A \leq 140$
  - Available funds: $282T + 400A \leq 2000$
  - Available Townhouses: $T \leq 5$
  - Non-negative: $T, A \geq 0$
  - Integer: T, A integers

Example: Eastborne Realty



Example: Eastborne Realty

```
library(lpSolve)
obj_fun <- c(10,15)
constr <- matrix(c(282,400,4,40,1,0),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=","<=")
RHS <- c(2000,140,5)
# LP relax
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 2.479339 3.252066
```

```
# Exact relax
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS,all.int=
prod_sol$solution
```

```
## [1] 4 2
```

A note on LP

- Rounding is a trial-and-error approach.
- Each rounded solution must be evaluated for feasibility as well as for its impact on the value of the objective function.
- Even when a rounded solution is feasible, there is no guarantee that the optimal integer solution has been found.

Example: Eastborne Realty

- What
- What is the true feasible region for the Eastborne Realty problem?
    - As shown in the following plot, the feasible region is the set of integer points that lie within the feasible region of the LP Relaxation.
    - There are 20 such feasible solutions (designated by blue dots in the figure).
- The region bounded by the dashed lines is the feasible integer solutions.

Example: Eastborne Realty