Linear Optimization: Model Building and Fitting

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Linear Optimization

Outline

- In general, optimization is a tool for decision making.
  - Tells us the optimal input to our objective function $f(b)$
- So far, only considered $f(b)$ that is not linear in b.
- What about $f(b)$ linear in b?
  - EX: $f(b) = b_1x_1 + b_2x_2 - b_3x_3$
  - What if there are restrictions on x_1, x_2, x_3?
  - Can we just apply the same tools?

Linear Optimization Methods

- A linear optimization problem depends on
  - decision variables, $x_1, x_2, \ldots, x_n$
  - objective function $f = b_1x_1 + b_2x_2 + \ldots + b_nx_n$
  - known $b_1, \ldots, b_n$
  - linear constraints: restrictions on $x_1, \ldots, x_n$.
- Specific case of optimization where f is linear.

Linear Optimization Methods

- Allow us to deal with cases where
  - $f(b)$ linear in b?
  - there are restrictions on x_1, x_2, x_3.
  - Solve real world problems.
- AIM: minimize or maximize some objective function f **(same as before)**
  - Subject to restrictions, called **constraints**, on x_1, x_2, x_3.
  - Linearity of $f(b)$ in b.

Linear optimization Structure

- $\max_{x_1,\ldots,x_n} b_1x_1 + b_2x_2 + \ldots + b_nx_n$ subject to
- $a_{11}x_1 + \ldots + a_{1n}x_n \leq RHS_1$
- $\ldots$
- $a_{m1}x_1 + \ldots + a_{mn}x_n \leq RHS_n$
- $x_1, \ldots, x_n \geq 0$

Example

-A farmer has 240 acres of land to plant. She needs to decides how many acres of corn to plant and how many acres of oats. She can make $40 per acre profit for corn and $30 per acre for oats. However, the corn takes 2 hours of labour per acre to harvest and the oats take only 1 hour per acre. She only has 320 hours of labour she can invest. To maximize her profit, how many acres of each should she plant?

```
- What is the objective function?
- What are the constraints?
```

Example

-A farmer has 240 acres of land to plant. She needs to decides how many acres of corn to plant and how many acres of oats. She can make **$40 per acre profit for corn** and **$30 per acre for oats**. However, the *corn takes 2 hours of labour per acre to harvest* and the *oats take only 1 hour per acre*. She only has *320 hours of labour* she can invest. **To maximize her profit**, how many acres of each should she plant?

```
- What is the objective function?
- What are the constraints?
```

From words to Math

- Decision variables: corn x, oats y (in acers)
- Obj. Fun.
  - Profit: $\Pi = 40x + 30y$
- Constraints?
  - $2x + y \leq 320$
  - $x + y \leq 240$
- Mathematically:

$$\max_{x,y} \Pi \ s.t. \ 2x + y \leq 320; \ x + y \leq 240.$$

Modelling Assumptions in Linear Programming

- **Proportionality**: the contribution to the objective function and the left-hand-side of each constraint by each decision variable is proportional to the value of the decision variable.
- **Additivity**: the contribution to the objective function and the left-hand-side of each constraint by any decision variable is completely independent of any other decision variable and is additive (involving?addition).
- **Divisibility**: the quantities represented by each decision variable are innately divisible (i.e., fractional answers make sense).
- **Certainty**: the coefficients in the objective function and constraints are known with certainty.

Example: Par, INC.

-The PAR, Inc company produces golf bags. -The managers are interested in expand production and also want to move to **medium** & **high** price bags. -Bag production involves 4 steps:

```
- Cutting and dyeing the material
- Sewing
- Finishing
- Inspection & packaging
```

Example: Par, INC.

-The director of manufacturing estimated that each of the two options will result in the following number of hours at each department:

```
- Cutting and dyeing the material
    -Standard Bag: 7/10, Delux Bag: 1
- Sewing
    -Standard Bag: 1/2, Delux Bag: 5/6
- Finishing
    -Standard Bag: 1, Delux Bag: 2/3
- Inspection & packaging
    -Standard Bag: 1/10, Delux Bag: 1/4
```

Example: Par, INC.

-After studying each department, the director estimates the following production capacities:

```
- 630 hours of cutting and dyeing,
- 600 hours of sewing,
- 708 hours of finishing, and
- 135 hours for inspection and packaging
```

-The accounting manager estimates that the company's profit from a standard bag and a deluxe bag will be **$10** and **$9**, respectively.

Example: Par, INC.

Decision Variables?

Objective function?

Constraints?

Example: Par, INC.

- Decision Variables?
  - Number of bags of each type:
    - S = standard bags
    - D = delux bags

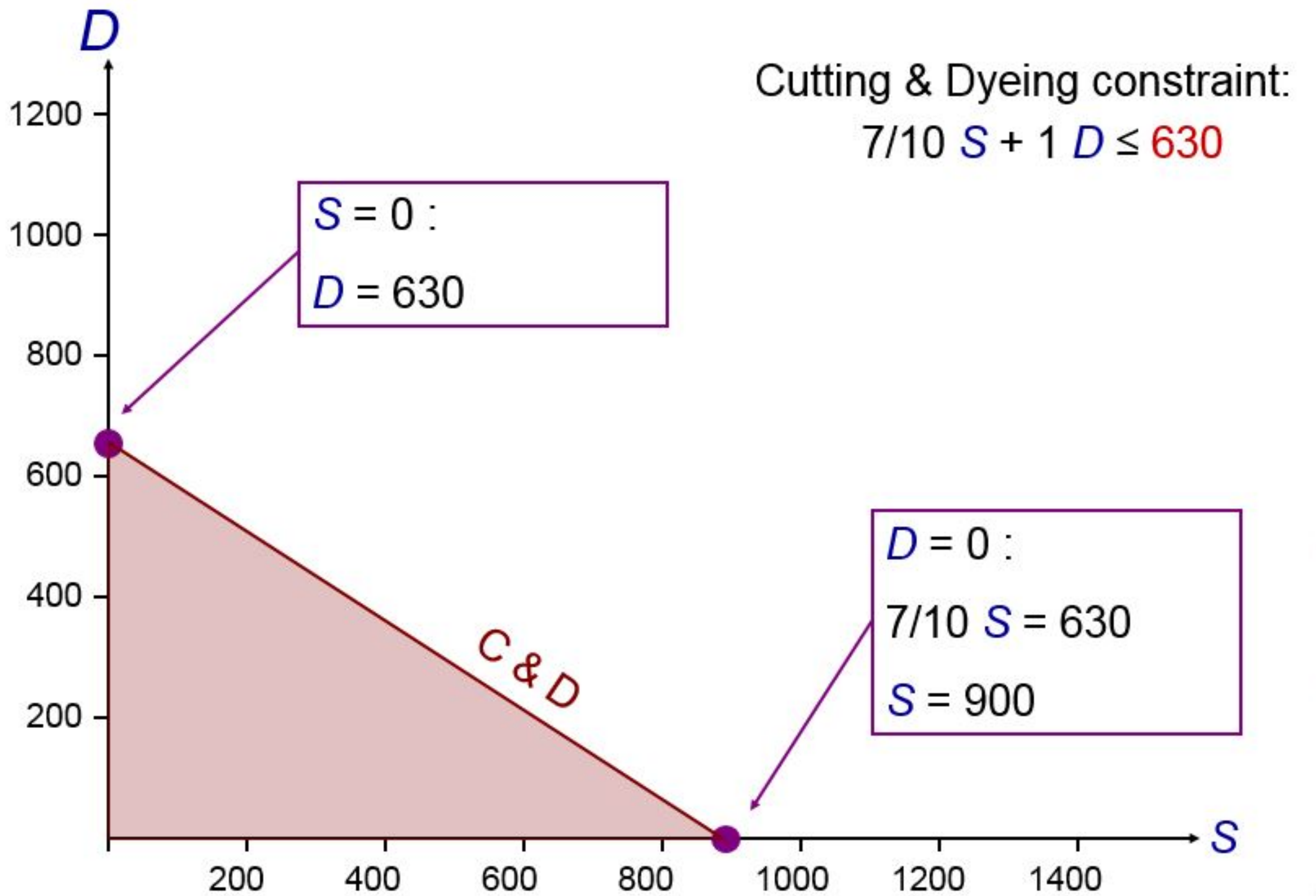
Example: Par, INC.

- Objective function?
  - Maximize profit: $max_{S,D} 10S + 9D$

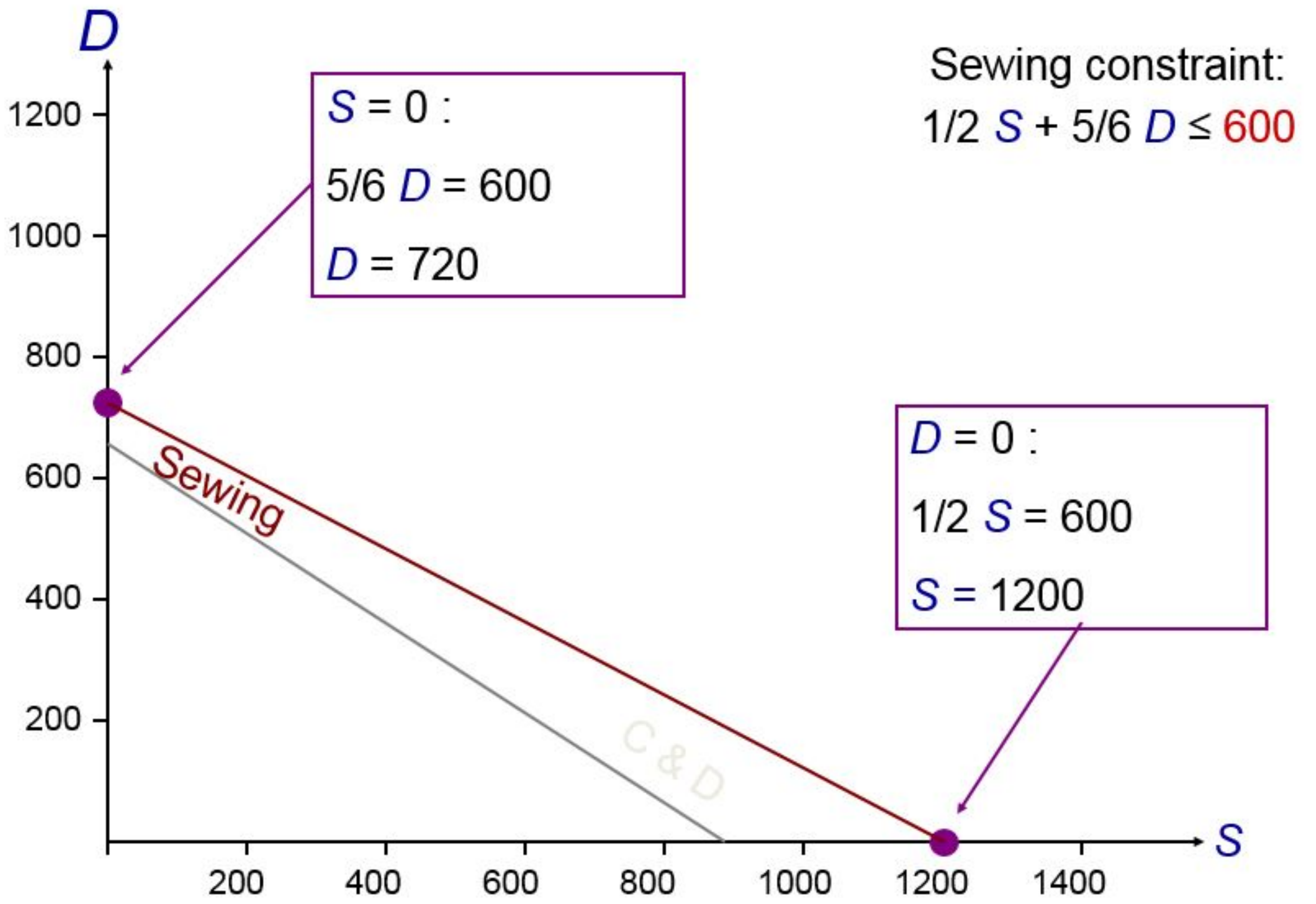
Example: Par, INC.

- Constraints?
  - Cutting and dyeing the material
    - Standard Bag: 7/10, Delux Bag: 1; 630 hours
    - $(7/10)S + D \leq 630$
  - Sewing
    - Standard Bag: 1/2, Delux Bag: 5/6; 600 hours
    - $(1/2)S + (5/6)D \leq 600$
  - Finishing
    - Standard Bag: 1, Delux Bag: 2/3; 708 hours
    - $S + (2/3)D \leq 708$
  - Inspection & packaging
    - Standard Bag: 1/10, Delux Bag: 1/4; 135 hours
    - $(1/10)S + (1/4)D \leq 135$
  - Number of bags can't be negative!!
    - $S, D \geq 0.$

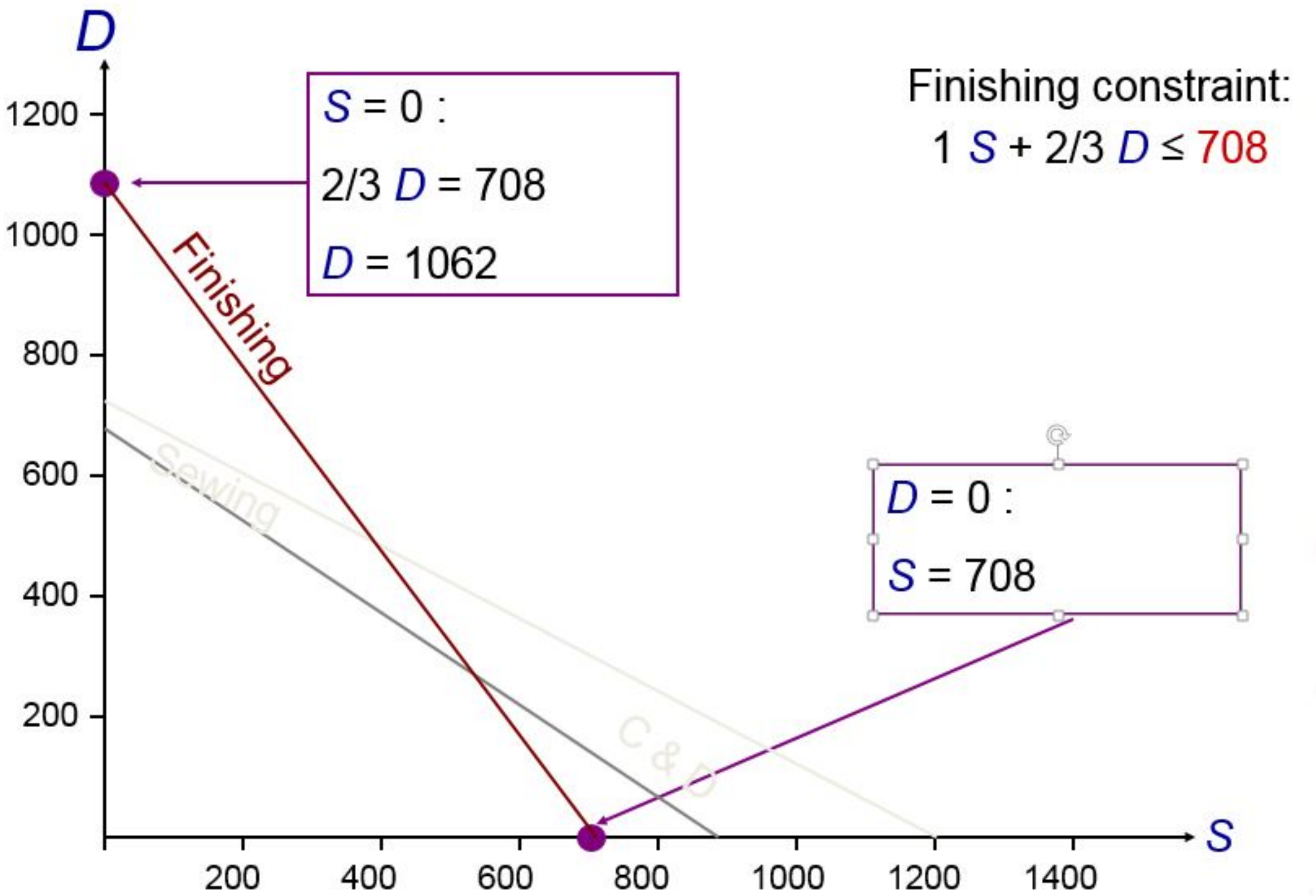
Example: Par, INC.



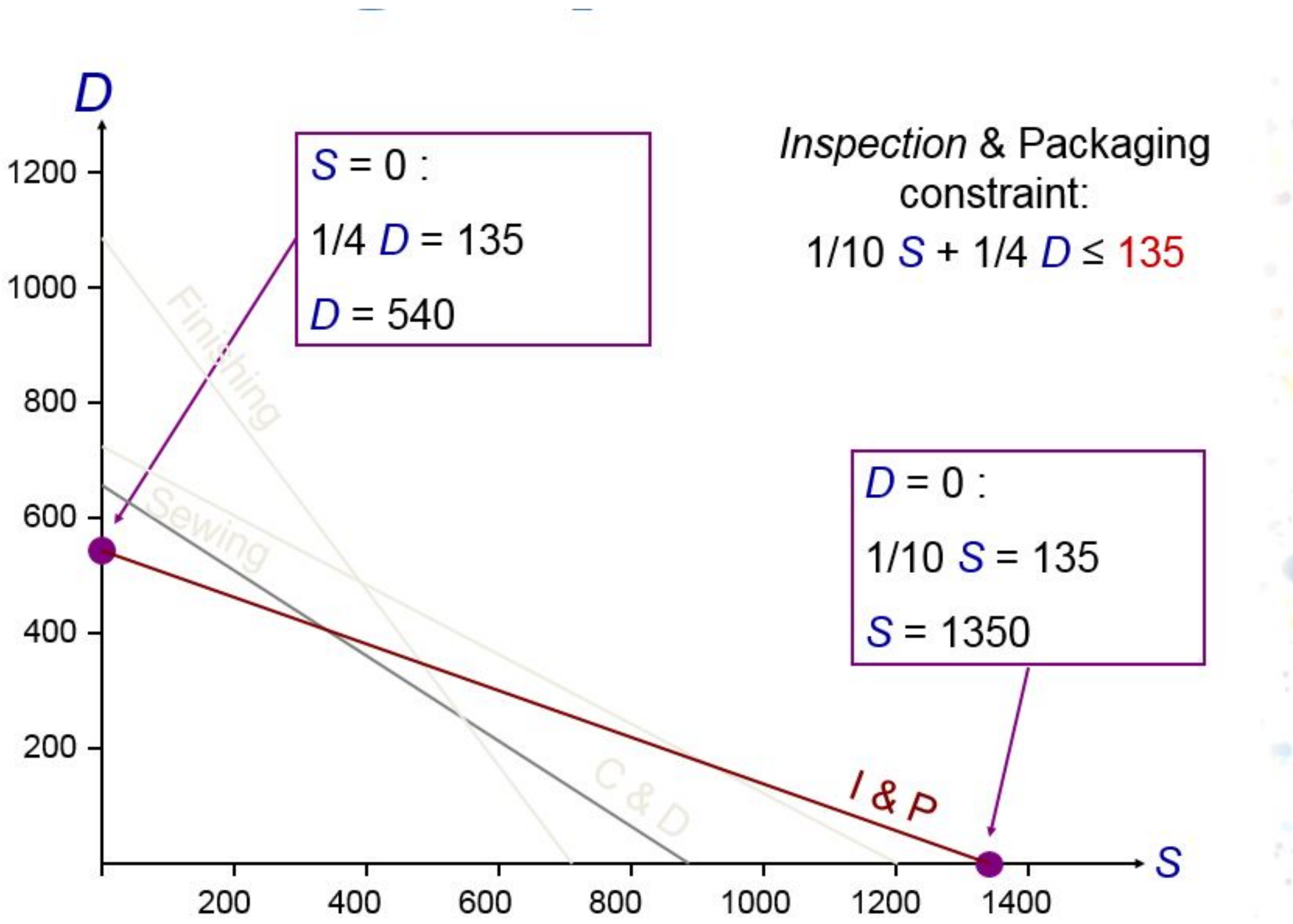
Example: Par, INC.



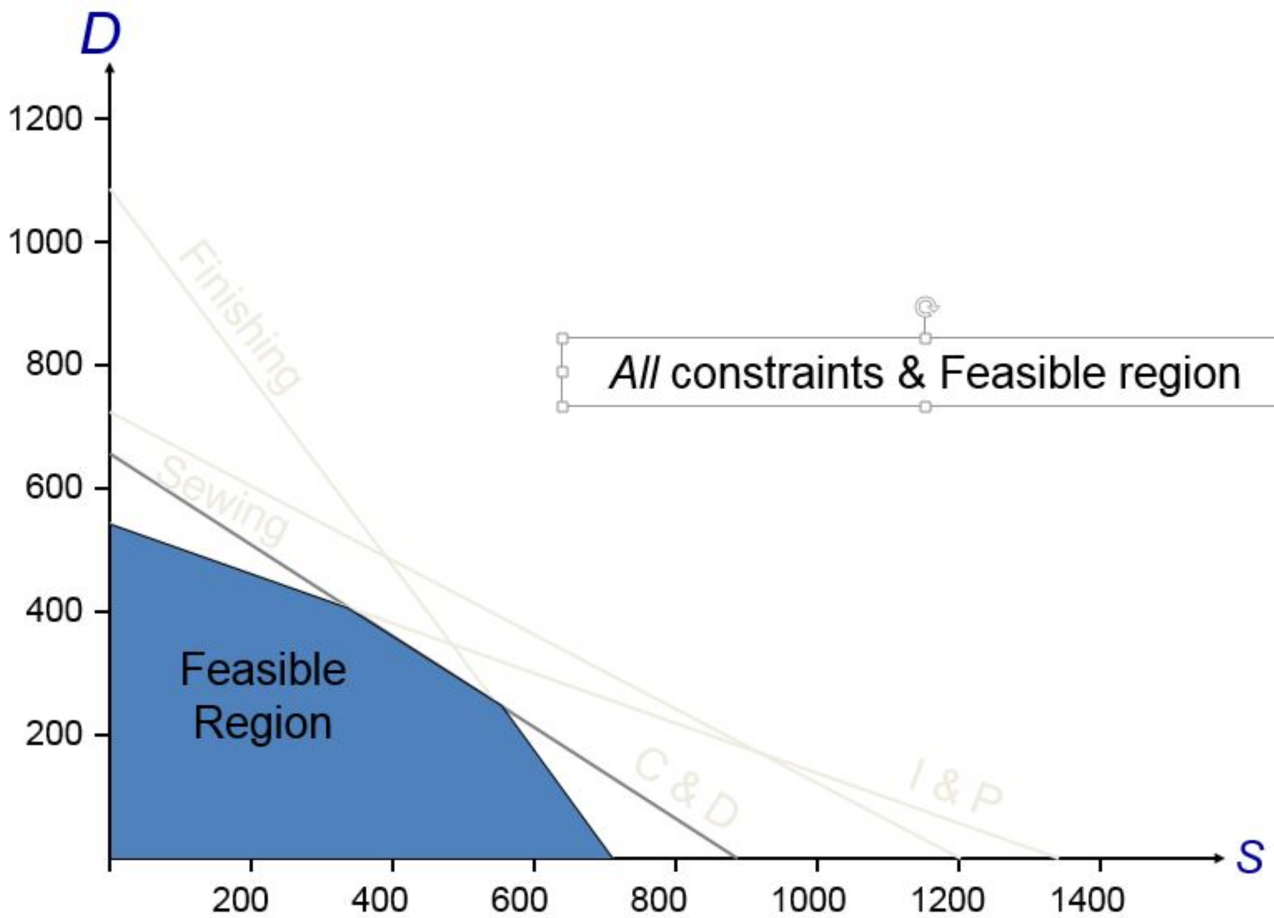
Example: Par, INC.



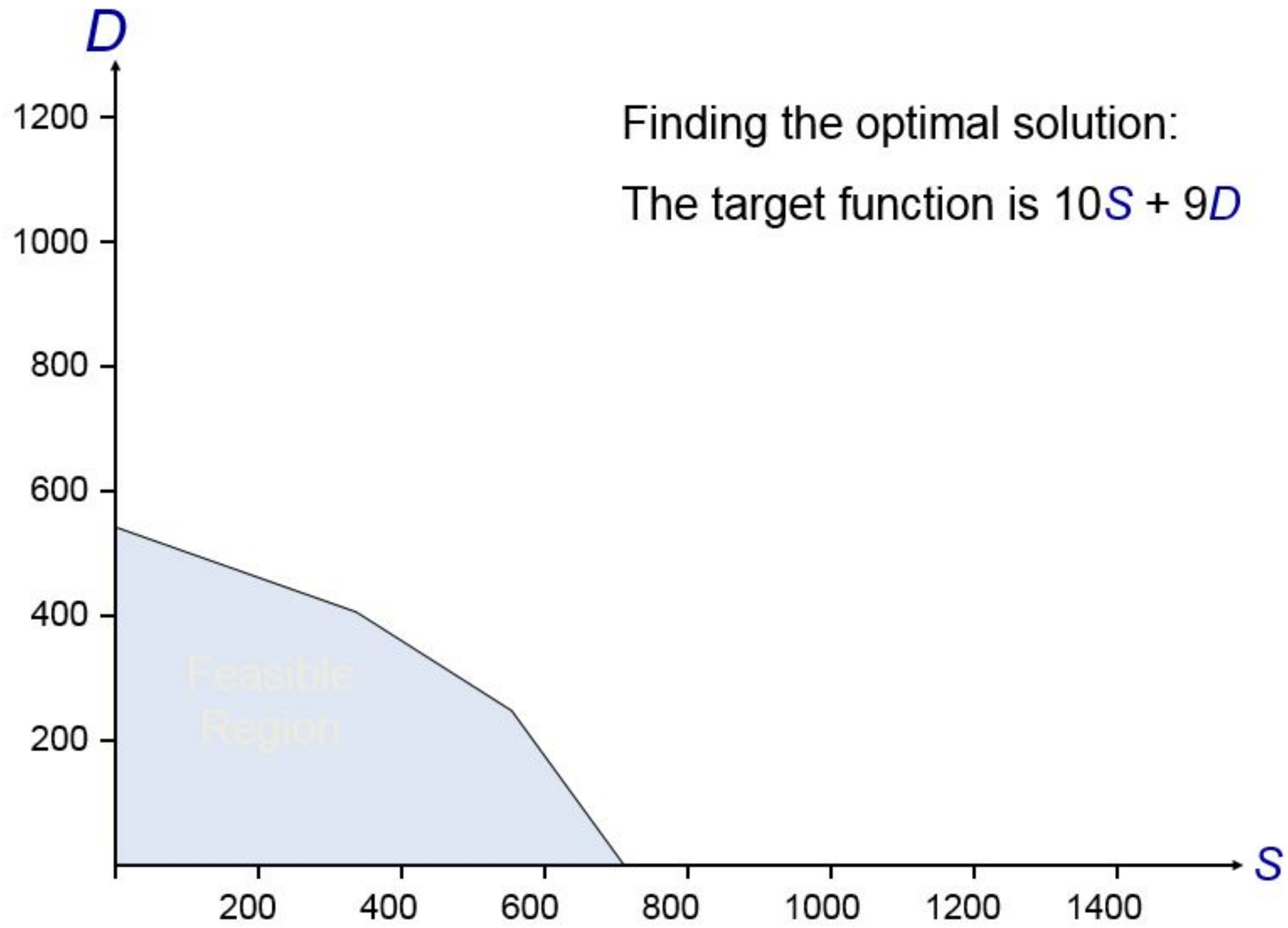
Example: Par, INC.



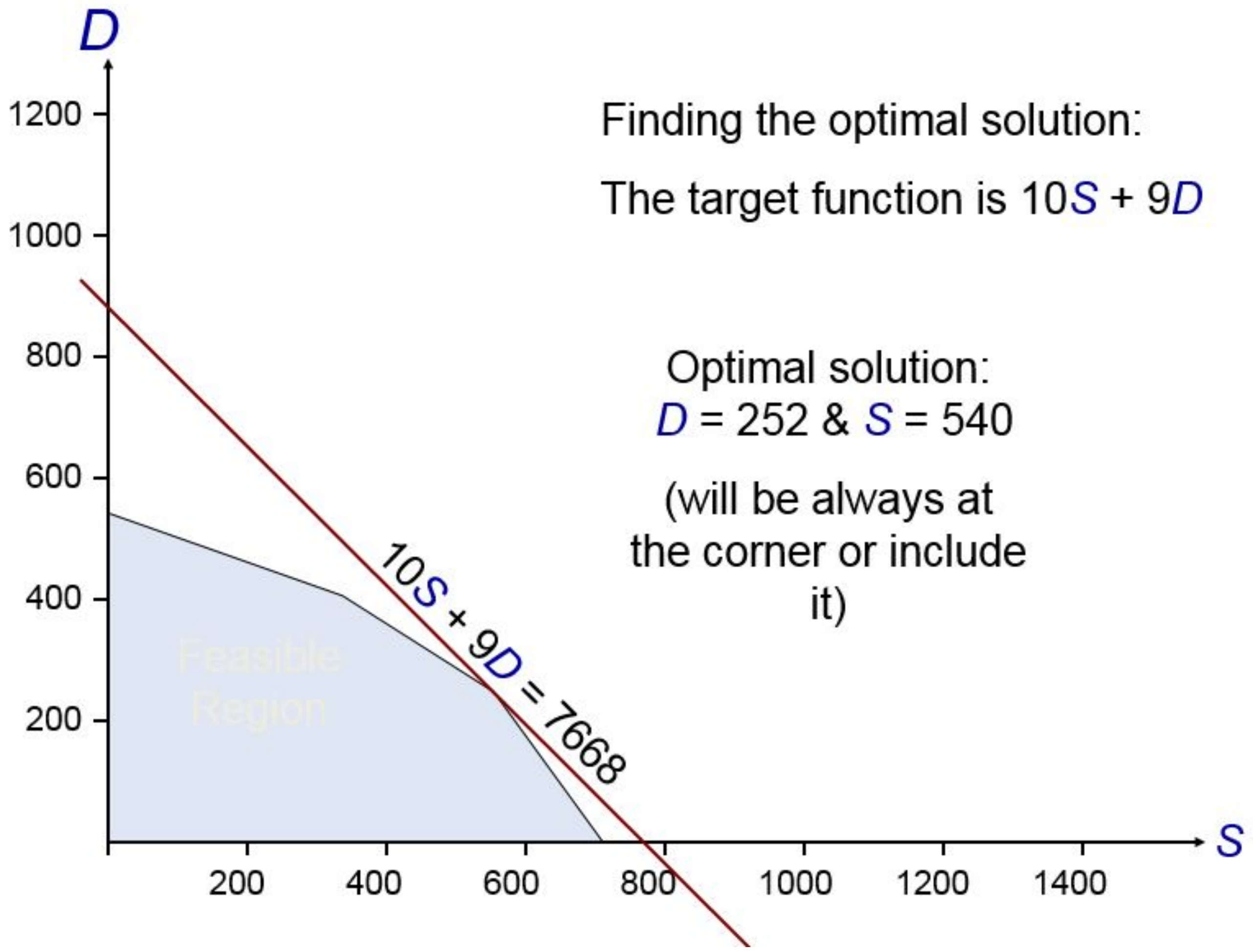
Example: Par, INC.



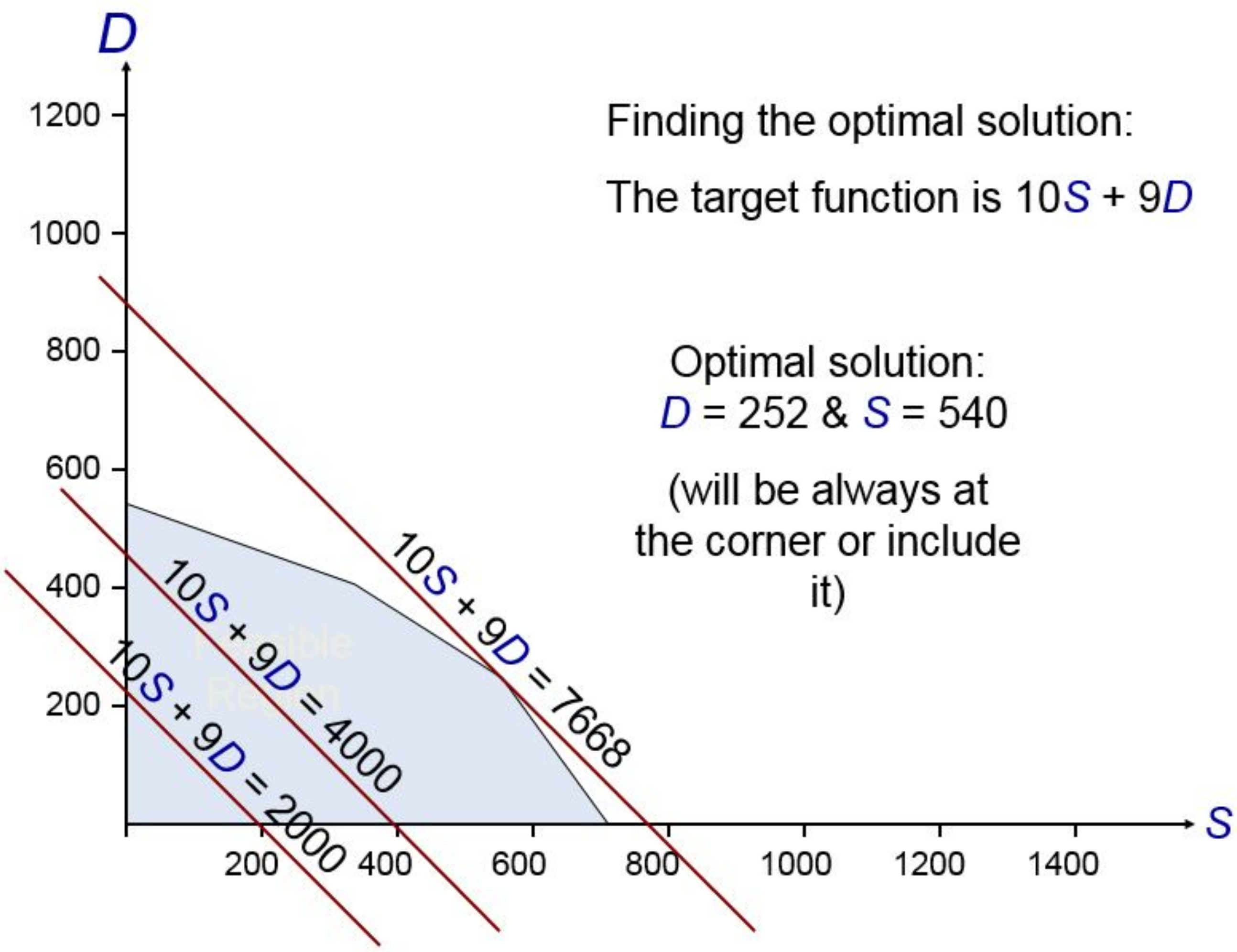
Example: Par, INC.



Example: Par, INC.



Example: Par, INC.



How to solve this in R?

- Just like `optim`, we need some numerical method to solve this optimization problem
- We'll make use of the `lpSolve` package in R.
  - "Modeling and Solving Linear Programming with R."
- Similar to `optim` function
  - Takes function as input
  - initial values for parameters
- Main difference: Must now add in constraint information.

Constraints in `lpSolve`

- Constraint information must be entered in a particular way
- Three pieces of information
  - Constraint coefficients
  - Constraint direction
  - RHS value of the constraint
- Must be entered as a matrix
- Each row of the matrix corresponds to a constraint
- Include inequality information in a separate line

Example: Par, INC

- Easiest to see from an example.

```
library(lpSolve)
obj_fun <- c(10,9)
constr <- matrix(c(7/10,1,1/2,5/6,1,2/3,1/10,1/4),ncol = 2
const_dir <- c("<=","<=","<=","<=")
RHS <- c(630,600,708,135)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 540 252
```

Example: Solve Farm Prob.

Problem:

$$\max_{x,y} 40x + 30y,\ \ s.t.\ \ x + y \leq 320;\ \ x + y \leq 240.$$

```r
library(lpSolve)
obj_fun <- c(40,30)
constr <- matrix(c(2,1,1,1),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=")
RHS <- c(320,240)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1]  80 160
```

Example

A small business sells two products, named Product 1 and Product 2. Each tonne of Product 1 consumes 30 working hours, and each tonne of Product 2 consumes 20 working hours. The business has a maximum of 2,700 working hours for the period considered. As for machine hours, each tonne of Products 1 and 2 consumes 5 and 10 machine hours, respectively. There are 850 machine hours available.

Each tonne of Product 1 yields 20 Me of profit, while Product 2 yields 60 Me for each tonne sold. For technical reasons, the firm must produce a minimum of 95 tonnes in total between both products. We need to know how many tonnes of Product 1 and 2 must be produced to maximize total profit.

Example: Objective function?

A small business sells two products, named Product 1 and Product 2. Each tonne of Product 1 consumes 30 working hours, and each tonne of Product 2 consumes 20 working hours. The business has a maximum of 2,700 working hours for the period considered. As for machine hours, each tonne of Products 1 and 2 consumes 5 and 10 machine hours, respectively. There are 850 machine hours available.

Each tonne of Product 1 yields **20 Millon of profit**, while Product 2 yields **60 Millon** for each tonne sold. For technical reasons, the firm must produce a minimum of 95 tonnes in total between both products. We need to know how many tonnes of Product 1 and 2 must be produced to **maximize total profit**.

Example: Objective function?

- Mathematically:

$$\max_{p_1,p_2} 20p_1 + 60p_2$$

Example: Constraints?

A small business sells two products, named Product 1 and Product 2. Each tonne of **Product 1 consumes 30 working hours**, and each tonne of **Product 2 consumes 20 working hours**. The business has a maximum of **2,700 working hours** for the period considered. As for machine hours, each tonne of **Products 1 and 2 consumes 5 and 10 machine hours**, respectively. There are **850** machine hours available.

Each tonne of Product 1 yields 20 Millon of profit, while Product 2 yields 60 Millon for each tonne sold. For technical reasons, the firm must produce **a minimum of 95 tonnes in total** between both products. We need to know how many tonnes of Product 1 and 2 must be produced to maximize total profit.

Example: Constraints?

- Mathematically:
  - $30p_1 + 20p_2 \leq 2700$
  - $5p_1 + 10p_2 \leq 850$
  - $p_1 + p_2 \geq 95$
  - $p_1, p_2 \geq 0.$

Example: Solution?

```r
library(lpSolve)
obj_fun <- c(20,60)
constr <- matrix(c(30,20,5,10,1,1),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=",">=")
RHS <- c(2700,850,95)
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 20 75
```

Additional Examples

- Portfolio selection
- Transportation planning
- Advertising campaign planning

Example: Portfolio Selection

- Select a specific investment strategy
- Variety of investment alternatives
- Goal: maximize profit or minimize risk.

Example: Portfolio Selection

- Investment opportunity for a fund of $100,000.

- The company has been recommended to invest in the oil industry, steel industry or government bonds.

- Five investment opportunities: Atlantic Oil (7.3% return), Pacific Oil (10.3 %), Midwest Steel (6.4%), Huber Steel (7.5%) and Government bonds (4.5%).

Example: Portfolio Selection

- Investment guidelines:
  - Neither oil or steel industry should receive more than $50,000.
  - Amount invested in government bonds should be at least 25 percent of the steel industry investments.
  - The investment in Pacific Oil, the high-return but high-risk investment, cannot be more than 60 percent of the total oil industry investment.

Example: Portfolio Selection

- Define the following decision variables:
  - X_1 - Atlantic Oil
  - X_2 - Pacific Oil
  - X_3 - Midwest Steel
  - X_4 - Huber Steel
  - X_5 - Government bonds

Example: Portfolio Selection

- Objective function?

Example: Portfolio Selection

- Objective function?
  - $\max_{x_1,\ldots,x_5} .073x_1 + .103x_2 + .064x_3 + .075x_4 + .045x_5$

Example: Portfolio Selection

- Constraints?

Example: Portfolio Selection

- Constraints?
- Max fund 10000
  - $x_1 + x_2 + \ldots + x_5 \leq 100,00$
- Oil or steel no more than \$50,000.
  - $x_1 + x_2 \leq 50,000$
  - $x_3 + x_4 \leq 50,000$
- Government bonds at least 25 percent of the steel industry investments.
  - $x_5 \geq .25 * (x_3 + x_4)$
- Pacific Oil investment, no more than 60 percent of the total oil industry investment.
  - $x_2 \leq .6 * (x_1 + x_2)$
- $x_1, \ldots, x_5 \geq 0$

Putting it together

- $\max_{x_1,\ldots,x_5} .073x_1 + .103x_2 + .064x_3 + .075x_4 + .045x_5$
  - $x_1 + x_2 + \ldots + x_5 \leq 100,00$
  - $x_1 + x_2 \leq 50,000$
  - $x_3 + x_4 \leq 50,000$
  - $x_5 \geq .25 * (x_3 + x_4)$
  - $x_2 \leq .6 * (x_1 + x_2)$
  - $x_1, \ldots, x_5 \geq 0$

Integer Programs and Optimization

Integer Optimization

- So far, assumed $x_1, \ldots, x_n$ infinitely divisible
  - i.e., continuous.
- Not true in many cases
  - Can we sell partial apartments?
  -
- **Integer linear programs**: Problems that are modeled as linear programs with the additional requirement that at least one variable must be integer.

Integer Optimization

-Types of Integer Linear Optimization Models:

```
- All-integer linear program
- Mixed-integer linear program
- Binary integer linear program
```

Linear optimization Structure

- $\max_{x_1,\ldots,x_n} b_1x_1 + b_2x_2 + \ldots + b_nx_n$ subject to
- $a_{11}x_1 + \ldots + a_{1n}x_n \leq RHS_1$
- $\ldots$
- $a_{m1}x_1 + \ldots + a_{mn}x_n \leq RHS_n$
  - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1, 2, \ldots$ all $j = 1, \ldots, n$
  - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1, 2, \ldots$; x_i, $i \neq j$, continuous
  - $x_1, \ldots, x_n \geq 0$; $x_j = 0, 1$ for all $j = 1, \ldots, n$

Linear optimization Structure

- Using R, can solve nearly exactly the same way
- All-integer
  - `lp("min", f, const.mat, const.dir, const.rhs, all.int=TRUE)`
- All-binary
  - `lp("min", f, const.mat, const.dir, const.rhs, all.bin=FALSE)`
- Mixed
  - `lp("min", f, const.mat, const.dir, const.rhs, int.vec)`
  - Numeric vector giving the indices of variables that are required to be integer. - The length of this vector will therefore be the number of integer variables

LP Relaxation

- In many cases, we can simply treat the decision variables as continuous
  - Then solve the optimization problem,
  - Round (up or down) to nearest integer.
- Called LP relaxation

LP Relaxation

- Gives a near-optimal solution
- Rounding may not always be a good strategy.
  - If decision variables take on small values
  - Can have major impact on the value of the objective function or feasibility,
  - In such cases, optimal integer solution is needed.

All Integer Example: Eastborne Realty

- Eastborne Realty has $2 million available for the purchase of new rental property.
- After an initial screening, Eastborne reduced the investment alternatives to townhouses and apartment buildings.
- Each townhouse can be purchased for $282,000.
  - Five are available.
- Each apartment building can be purchased for $400,000.
  - The developer will construct as many buildings as Eastborne wants to purchase.

All Integer Example: Eastborne Realty

- Eastborne Realty has \$2 million available for the purchase of new rental property.
- After an initial screening, Eastborne reduced the investment alternatives to townhouses and apartment buildings.
  - T- townhouse, A- apartment.
- Each townhouse can be purchased for 282,000
  - Five available: $T \leq 5$
- Each apartment can be purchased for 400,000. Develop construct as many as desired.
  - $282T + 400A \leq 2000$

All Integer Example: Eastborne Realty

- Property manager can devote 140 hours per month to properties.
- Each townhouse requires 4 hours per month.
- Each apartment requires 40 hours per month.

All Integer Example: Eastborne Realty

- Property manager can devote 140 hours per month to properties.
- Each townhouse requires 4 hours per month.
- Each apartment requires 40 hours per month.
  - $4T + 40A \leq 140$

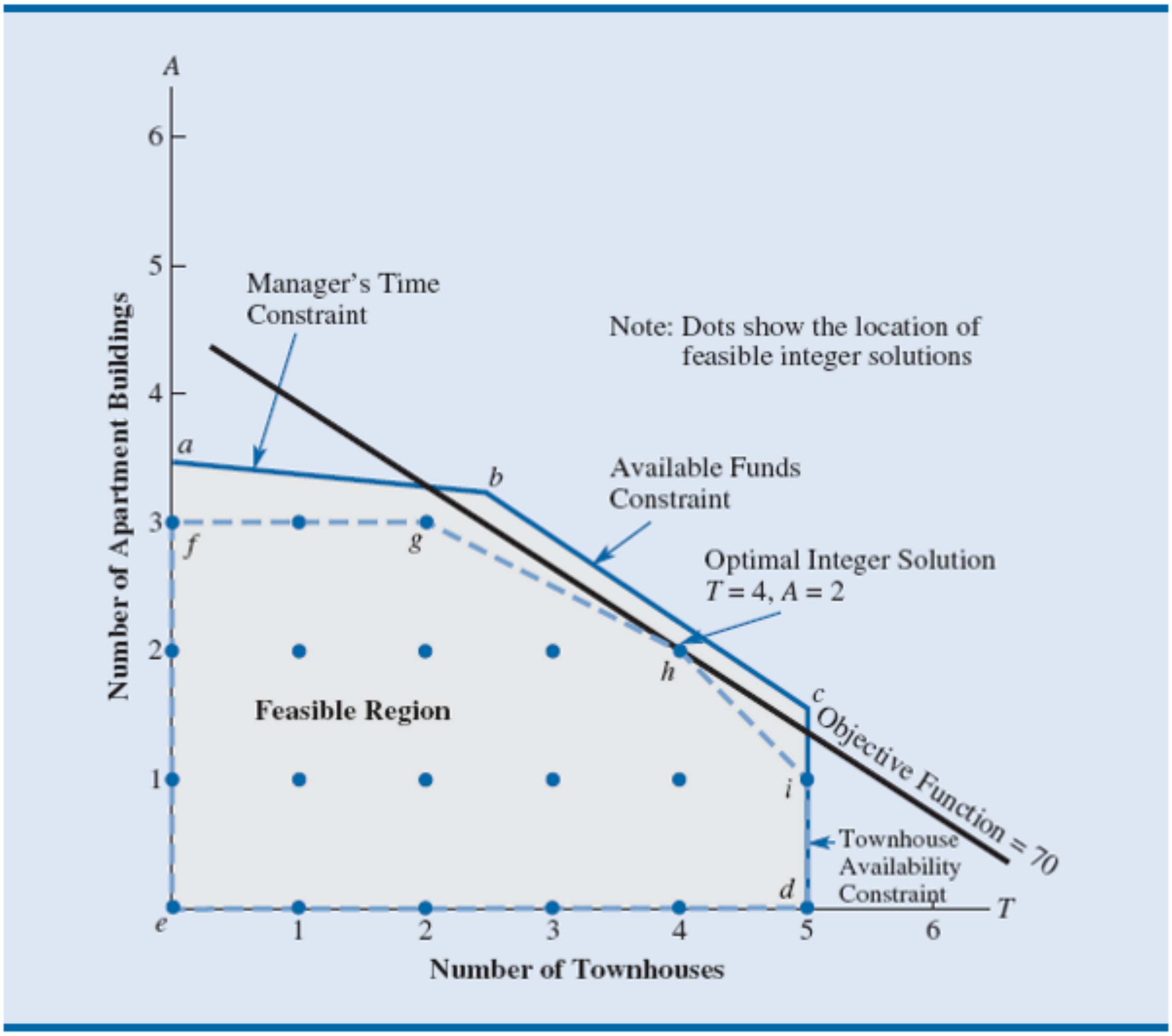
All Integer Example: Eastborne Realty

- Annual cash flow (After deducting mortgage payments and operating expenses) estimated to be:
  - \$10,000 per townhouse.
  - \$15,000 per apartment building.
- The objective: Determine the number of townhouses and the number of apartment buildings to purchase to maximize annual cash flow.

All Integer Example: Eastborne Realty

- Defining the decision variables:
  - T = number of townhouses
  - A = number of apartment buildings
- Objective function for cash flow (in thousands):
  - $max_{T,A} 10T + 15A$
- Constraints?
  - Available funds: $282T + 400A \leq 2000$
  - Management: $4T + 40A \leq 140$
  - Available funds: $282T + 400A \leq 2000$
  - Available Townhouses: $T \leq 5$
  - Non-negative: $T, A \geq 0$
  - Integer: T, A integers

Example: Eastborne Realty



Example: Eastborne Realty

```r
library(lpSolve)
obj_fun <- c(10,15)
constr <- matrix(c(282,400,4,40,1,0),ncol = 2, byrow=TRUE)
const_dir <- c("<=","<=","<=")
RHS <- c(2000,140,5)
# LP relax
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS)
prod_sol$solution
```

```
## [1] 2.479339 3.252066
```

```r
# Exact relax
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS,all.int=
prod_sol$solution
```

```
## [1] 4 2
```

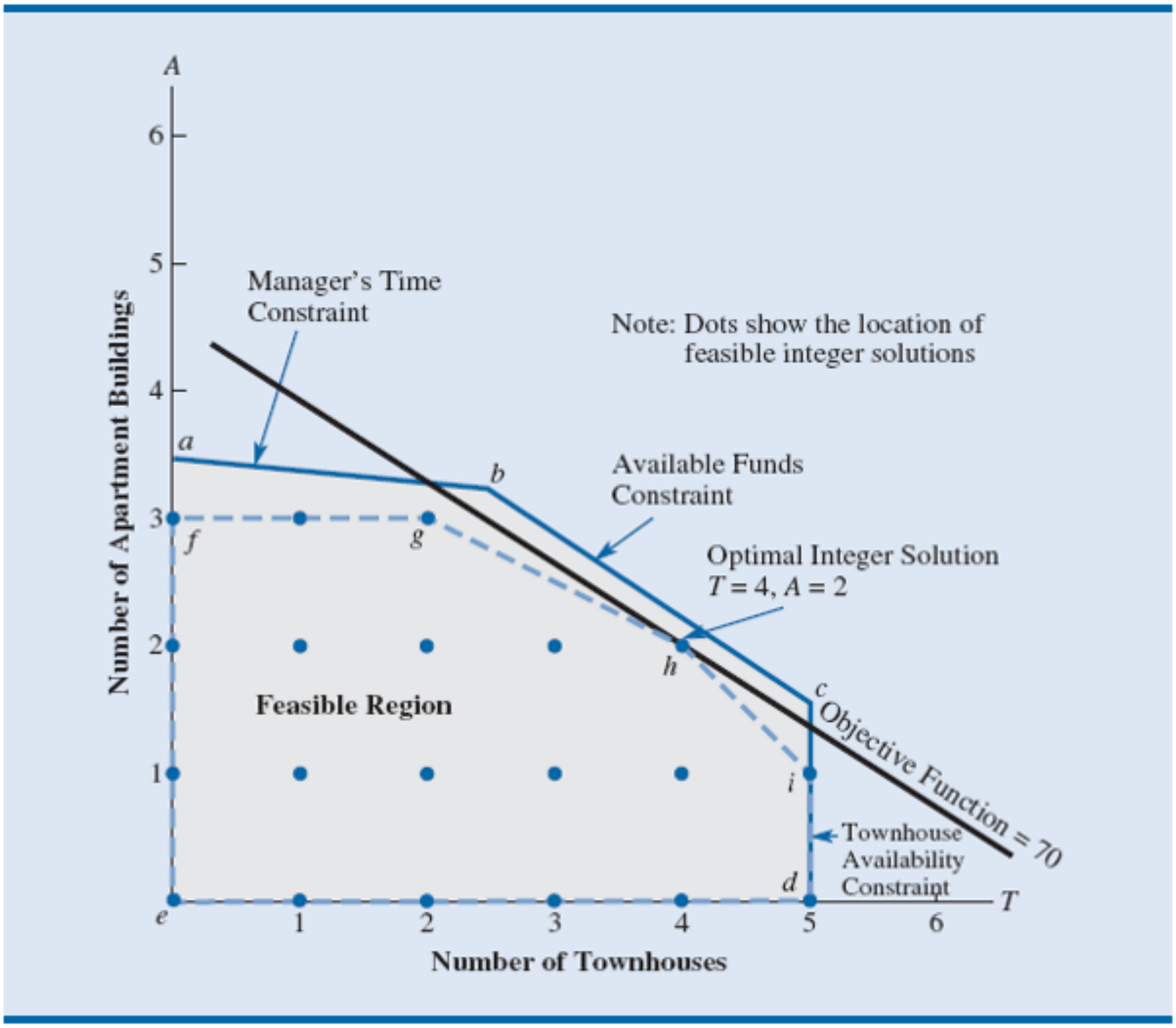
A note on LP

- Rounding is a trial-and-error approach.
- Each rounded solution must be evaluated for feasibility as well as for its impact on the value of the objective function.
- Even when a rounded solution is feasible, there is no guarantee that the optimal integer solution has been found.

Example: Eastborne Realty

- What
- What is the true feasible region for the Eastborne Realty problem?
  - As shown in the following plot, the feasible region is the set of integer points that lie within the feasible region of the LP Relaxation.
  - There are 20 such feasible solutions (designated by blue dots in the figure).
- The region bounded by the dashed lines is the feasible integer solutions.

Example: Eastborne Realty



Binary Variables

Binary Optimization Example

- Binary variables
  - $x_j = 1$ if activity *IS* undertaken
  - $x_j = 0$ if activity *NOT* undertaken
- Many Examples:
  - Capital budgeting
  - fixed cost
  - area covering problems

Capital budgeting

- The Ice-Cold Refrigerator Company is considering investing in several projects that have varying capital requirements over the next four years.
- Faced with limited capital each year, management would like to select the most profitable projects that it can afford.
- Estimated net present value for each project, the capital requirement and the available capital over 4-year period are:

Capital budgeting

- Estimated net present value for each project, the capital requirement and the available capital over 4-year period are:
-

	Plant Expansion ($)	Warehouse Expansion ($)	Project New Machinery ($)	New Product Research ($)	Total Capital ($) Available ($)
Present Value	90,000	40,000	10,000	37,000	
Year 1 Cap Rqmt	15,000	10,000	10,000	15,000	40,000
Year 2 Cap Rqmt	20,000	15,000		10,000	50,000
Year 3 Cap Rqmt	20,000	20,000		10,000	40,000
Year 4 Cap Rqmt	15,000	5,000	4,000	10,000	35,000

Capital budgeting

- Capital budgeting problem: the objective function is to maximize the net present value of the capital budgeting projects.
- Decision variables
  - $P = 1$ plant expansion
  - $W = 1$ warehouse expansion
  - $M = 1$ new machinery
  - $R = 1$ new research
- Constraints? Objective function?

Captial budgeting: Objective Function

- Maximize NPV
- $\max 90P + 40W + 10M + 37R$

Captial budgeting: Constraints

	Project				
	Plant Expansion ($)	Warehouse Expansion ($)	New Machinery ($)	New Product Research ($)	Total Capital Available ($)
Present Value	90,000	40,000	10,000	37,000	
Year 1 Cap Rqmt	15,000	10,000	10,000	15,000	40,000
Year 2 Cap Rqmt	20,000	15,000		10,000	50,000
Year 3 Cap Rqmt	20,000	20,000		10,000	40,000
Year 4 Cap Rqmt	15,000	5,000	4,000	10,000	35,000

Captial budgeting: Constraints

- $15P + 10W + 10M + 15R \leq 40$ (Year 1)
- $20P + 15W + 00M + 10R \leq 50$ (Year 2)
- $20P + 20W + 00M + 10R \leq 20$ (Year 3)
- $15P + 5W + 4M + 10R \leq 35$ (Year 4)
- P, W, M, R are either $0, 1$

Capital budgeting: Putting it all together

- $\max 90P + 40W + 10M + 37R$ subject to
- $15P + 10W + 10M + 15R \leq 40$
- $20P + 15W + 00M + 10R \leq 50$
- $20P + 20W + 00M + 10R \leq 40$
- $15P + 5W + 4M + 10R \leq 35$
- P, W, M, R are either $0, 1$

Captial budgeting: Solution

```
library(lpSolve)
obj_fun <- c(90,40,10,37)
constr <- matrix(c(15,10,10,15,20,15,00,10,20,20,00,10,15,
const_dir <- c("<=","<=","<=","<=")
RHS <- c(40,50,40,35)
# Solution
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS,all.bin=
prod_sol$solution
```

```
## [1] 1 1 1 0
```

Binary mixed-integer Problems

Mixed problems

- Many problems require binary variables, integers, and continuous part
- Same process, but must specify which variables are which...

Fixed-Cost problem

-In many applications, the cost of production has two components

- A setup cost, which is fixed (hence, fixed cost)
- A variable cost, directly related to production
- Fixed cost problem: binary mixed-integer programming problem where binary variables represent whether an activity, such as a production run, is undertaken (variable = 1) or not (variable = 0).

Fixed-Cost problem

		Usage per ton				
	Profit contribution per ton ($)	Material 1	Material 2	Material 3	Set up cost ($)	Maximum production (tons)
Fuel additive	40$	0.4		0.6	200$	50
Solvent base	30$	0.5	0.2	0.3	50$	25
Carpet cleaning fluid	50$	0.6	0.1	0.3	400$	40
Max material available		20	5	21		

Fixed-Cost problem: Decision Variables

- F tons of fuel additive produced
- S tons of solvent produced
- C tons of carpet cleaning fluid produced
- How to incorporate fixed cost?
  - $SF = 1$ if fuel additive is produced
  - $SS = 1$ if fuel additive is produced
  - $SC = 1$ if fuel additive is produced

Fixed-Cost problem: Objective function?

- Maximize profits from production
  - Need to deal with the fixed costs
- Revenue
  - $40F + 30S + 50C$
- Fixed Costs
  - $200SF + 50SS + 400SC$
- Var costs dealt with through material constraints

Fixed-Cost problem: Constraints?

- $.4F + .5S + .6C \leq 20$
  - $F + 0S + 0C \leq 50SF$
- $0F + .2S + .1C \leq 5$
  - $0F + S + 0C \leq 25SS$
- $.6F + .3S + .3C \leq 21$
  - $0F + 0S + C \leq 40SC$
- $F, S, C \geq 0$
- SF, SS, SC are $0, 1$

Fixed-Cost problem: Putting it together

- $\max 40F + 30S + 50C - (200SF + 50SS + 400SC)$ subject to
- $.4F + .5S + .6C \leq 20$
- $F + 0S + 0C \leq 50SF$
- $0F + .2S + .1C \leq 5$
- $0F + S + 0C \leq 25SS$
- $.6F + .3S + .3C \leq 21$
- $0F + 0S + C \leq 40SC$
- $F, S, C \geq 0$
- SF, SS, SC are $0, 1$

Fixed-Cost: Solution

- Need to rewrite some of the constraints
- $.4F + .5S + .6C + 0SF + 0SS + 0SC \leq 20$
- $F + 0S + 0C + 0SS + 0SC - 50SF \leq 0F$
- $0F + .2S + .1C + 0SF + 0SS + 0SC \leq 5$
- $0F + S + 0C + 0SF + 0SC - 25SS \leq 0$
- $.6F + .3S + .3C + 0SF + 0SS + 0SC \leq 21$
- $0F + 0S + C + 0SF + 0SS - 40SC \leq 0$

Fixed-Cost: Solution

```
library(lpSolve)
obj_fun <- c(40,30,50,-200,-50,-400)
constr <- matrix(c(.4,.5,.6,0,0,0,
                   1,0,0,-50,0,0,
                   0,.2,.1,0,0,0,
                   0,1,0,0,-25,0,
                   .6,.3,.3,0,0,0,
                   0,0,1,0,0,-40),ncol = 6, byrow=TRUE)
const_dir <- c("<=","<=","<=","<=")
RHS <- c(20,0,5,0,21,0)
# Solution
prod_sol <- lp("max",obj_fun,constr,const_dir,RHS,binary.v
```

```
## Warning in rbind(const.mat, const.dir.num, const.rhs): number of
## result is not a multiple of vector length (arg 2)
```

```
prod_sol$solution
```

```
## [1] 25 20  0  1  1  0
```