Formal definition of cyclic granularities

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Quasi-circular

A **quasi-circular** granularity can not be defined using modular arithmetic due to its irregular mapping with the bottom granularity. However, they are still formed with linear granularities, one of which "groups periodically into" the other. We will take the example of days and months to illustrate the notion of "groups periodically into" and "quasi-circular" granularities.

The grouping (day, month) has a period of 1 year. Ignoring leap years, this would mean that the behavior of days within months repeat every 365 days. That is, each month would consist of the same number of days in every year. We assume there are D linear granularity "day" and M linear granularity "month" in total.

$$
\begin{aligned}
month(0) &= day(0) \bigcup day(1) \cdots \bigcup day(30) \\
month(1) &= day(31) \bigcup day(32) \cdots \bigcup day(58) \\
month(2) &= day(59) \bigcup day(60) \cdots \bigcup day(89) \\
&\cdots \\
month(11) &= day(334) \ldots day(363) \bigcup day(364)
\end{aligned}
\tag{1}
$$

If we know the composition of each of the months in terms of days for one year, we can find the composition of any month beyond 1 year since the "pattern" repeats itself along the time domain due to the periodic property.

In other words, if

$$
month(j) = day(a_1) \bigcup day(a_2) \cdots \bigcup day(a_k) \quad for \quad j \in 0, 1, 2, \ldots, 11
\tag{2}
$$

then,

$$
month(j + R) = day(a_1 + P) \bigcup day(a_2 + P) \cdots \bigcup day(a_k + P) \quad for \quad j + R \leq M
\tag{3}
$$

Here, P = 365 and R = 12 will have the meaning of the period of the grouping (day, month) and the number of months in each of these periods.

Generalizing it to any two linear granularities G and H , the formal way of defining the property "groups periodically into" would like the following:

Definition 1 *A granularity H is periodical with respect to a granularity G if (1) $G \trianglelefteq H$, and (2) there exist $R, P \epsilon Z+$, where R is less than the number of granules of H, such that for all $j \epsilon Z$, if $H(j) = \bigcup_{i \in S} G(i)$ and $H(j + R) \neq \phi$ then $H(j + R) = \bigcup_{i \in S} G(i + P)$.*

Another way to represent Equation 1

Month					0					1					2					12				
Day	0	1	...	30	31	32	33	34	...	58	59	60	..	...	89	.....				334	335	..	...	365
Day-of-month	L(0)	L(1)	...	L(30)	L(0)	L(1)	L(2)	L(3)	...	L(28)	L(0)	L(1)	..	...	L(31)	.....				L(0)	...	..	...	L(31)

Figure 1: Quasi-circular granularity day-of-month

$$month(0) = \bigcup_{i \in S_0} day(i), \quad S_0 = 0, 1, 2, \ldots, 30$$

$$month(1) = \bigcup_{i \in S_1} day(i), \quad S_1 = 31, 32, \ldots, 58$$

$$month(2) = \bigcup_{i \in S_2} day(i), \quad S_2 = 59, 60, \ldots, 89 \tag{4}$$

$$\vdots$$

$$month(11) = \bigcup_{i \in S_{11}} day(i), \quad S_{11} = 334, 335, \ldots, 364$$

Here, $S_0, \ldots, S_{11}$ are the sets of indexes of G describing $month(0), \ldots, month(11)$. Then from Definition 1, it also follows that if H is periodical with respect to G, then

$$H(j) = \bigcup_{i \in S_{j \mod R}} G(P * \lfloor j/R \rfloor + i) \tag{5}$$

,

where $S_0, \ldots, S_{R-1}$ are the sets of indexes of G describing $H(0), \ldots, H(R-1)$ respectively. Here $j \mod R$ represents the index among those in $0, 1, \ldots, R-1$ that must be shifted to obtain $H(j)$. The number of periods each granule of G composing $H(j \mod R)$ should be shifted is given by $\lfloor j/R \rfloor$.

Thus, to obtain $month(13)$ in terms of days we can either use 1 or Equation 11

$$
\begin{aligned}
month(13) &= month(1 + 12) \\
&= \bigcup_{i \in S_1} day(i + 365) \quad since \quad month(1) = \bigcup_{i \in S_1} day(i), \quad S_1 = 31, 32, \ldots, 58 \\
&= day(31 + 365) \bigcup day(32 + 365) \bigcup day(58 + 365)
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
month(13) &= \bigcup_{i \in S_{13 \mod 12}} day(365 * \lfloor 13/12 \rfloor + i) \\
&= \bigcup_{i \in S_1} day(365 * 1 + i) \\
&= day(31 + 365) \bigcup day(32 + 365) \bigcup day(58 + 365)
\end{aligned}
\tag{7}
$$

Till now, we saw how the formal definitions of "groups periodically into" can be seen through the example of (day, month). Now, we shall discuss how some quasi-circular granularity can be obtained. **??** shows that index set 34 should correspond to L(3) which is "4th-day-of-month", that of 60 should correspond to L(1), which is "2nd-day-of-month" and 334 should correspond to L(0), which is "1st-day-of-month". Now, quasi-circular granularities can be defined as follows to obtain these labels:
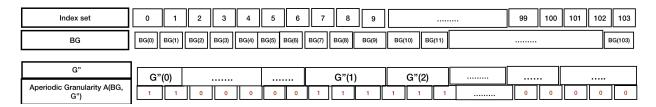
Index set	0	1	2	3	4	5	6	7	8	9	....	99	100	101	102	103
BG	BG(0)	BG(1)	BG(2)	BG(3)	BG(4)	BG(5)	BG(6)	BG(7)	BG(8)	BG(9)	BG(10) BG(11)					BG(103)

G"	G"(0)	....	....	G"(1)	G"(2)	....	......	.....
Aperiodic Granularity A(BG, G")	1 1 0	0 0	0 0	1 1 1	1 1 1	....	0 0	0 0 0

Figure 2: Relating aperiodic granularity and bottom linear granularity

Definition 2 *A quasi-circular granularity $Q_{(}B, G')$ that relates linear granularities G' and bottom granularity B, if*

$$Q_{B,G'}(z) = L(z - \sum_{w=0}^{k-1} |T_{w \mod R'}|), \quad z \in T_k \tag{8}$$

where, z denotes the index set, B denotes a full-integer labelled bottom granularity which groups periodicially into linear granularity G' with irregular mapping, L is a label mapping that defines an unique label for each index $l \in 0, 1, \ldots, (k-1)$, T_w are the sets of indices of B describing $G'(w)$ such that

$$G'(w) = \bigcup_{z \in T_w} B(z), \tag{9}$$

$T_w = 0$ *for* $w \epsilon Z_{<0}$ *and* $|T_w|$ *is the cardinality of set* T_w.

Here, $|T_w| = |T_{w \mod R'}|$ since every wth and $(w+R)^{th}$ granule will have the same number of granules of B. T_w are the sets of indexes of B describing $G'(0), ..., G'(R'-1)$. The term $\sum_{w=0}^{k-1} |T_{w \mod R'}|$ denotes the number of granules of B till the $(k-1)^{th}$ granule of G'.

$$Q_{day,month}(34) = L(35 - 31) \quad since \quad 35 \in \quad month(1)$$
$$Q_{day,month}(60) = L(60 - 31 - 28), \quad since \quad 60 \in \quad month(2) \tag{10}$$
$$Q_{day,month}(334) = L(334 - 334) \quad since \quad 334 \in \quad month(11)$$

.

Definition 3 *An aperiodic granularity $A_{(}B, G'')$ that relates linear granularities G and bottom granularity B, with G'' being aperiodical with respect to B, is given by*

$$A_{B,G''}(z) = L(1), z \in T_q$$
$$= L(0), z \notin T_q \tag{11}$$

where, z denotes the index set (index of B), T_q are the sets of indices of B describing $G''(q)$ such that $G''(q) = \bigcup_{z \in T_q} BG(z), T_q = 0$ for $q \epsilon Z_{<0}$.

Discussion: Aperiodic time granularities map a time index to a categorical variable containing binary values. It takes the value 1 whenever the time index coincides with the events in context (e.g. public or school holiday) and 0 when it does not. Here, 1 and 0 are proxies to denote the binary categorization and can be replaced with a logical vector. An example of aperiodic granularity can be one that relates a bottom granularity (BG) and linear granularity (G'') such that $BG \trianglelefteq G''$, $BG \ntrianglelefteq G$ and BG is not periodic or quasi-periodic with respect to G'', as shown in Figure 2.