# MTH 839 Project: 1D NLS equation: periodic solution and Whitham modulation

Project Report


**Sayantan Sarkar**

Mathematics Department
State University of New York at Buffalo


Supervisor: Dr. Gino Biondini

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# 1 Introduction

1 dimensional NLS in semi-classical scaling is

$$i\epsilon\psi_t + \epsilon^2\psi_{xx} - 2|\psi|^2\psi = 0 \tag{1.1}$$

The nonlinear Schrödinger (NLS) equation is a central model for describing weakly nonlinear dispersive waves in systems such as optics, fluid dynamics, and Bose-Einstein condensates. Whitham modulation theory provides a powerful framework to describe the slow evolution of parameters of periodic and soliton solutions by averaging conservation laws over fast oscillations.

We consider the one-dimensional NLS (1DNLS) equation and derive modulation equations using a multiple-scales ansatz. The periodic solutions are expressed in terms of Jacobi elliptic sine functions $\mathrm{sn}(Z, m)$, where $m$ is the elliptic modulus. In the limits $m \to 0$ and $m \to 1$, the solutions reduce to plane waves and solitons, respectively. The slow dynamics of the wave parameters $A$ , $m$ (modulus), $k$, and $\bar{u}$ are governed by Whitham modulation equations, which we systematically derive.

This paper covers:

- Derivation of periodic solutions of the 1DNLS equation.
- Harmonic and soliton limits of the periodic solutions.
- Whitham modulation equations for the slow dynamics of wave parameters.

# 2 Madelung form of the NLS equation

We apply Madelung transformation to the (1.1) to derive dispersive hydrodynamic system of PDEs:

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{i\phi(x,t)} \tag{2.1a}$$

$$u(x,t) = \varepsilon\frac{\partial \phi}{\partial x} \tag{2.1b}$$

## 2.1 Dispersive hydrodynamic system fo 1DNLS

The derivatives of $\psi$ are:

$$\begin{aligned}\psi_t &= \frac{1}{2}\rho^{-1/2}\rho_t e^{i\phi} + i\rho^{1/2}\phi_t e^{i\phi}\\ \psi_{xx} &= \left(\frac{\rho_{xx}}{2\rho^{1/2}} - \frac{\rho_x^2}{4\rho^{3/2}} + i\frac{\rho_x\phi_x}{\rho^{1/2}} - \rho^{1/2}\phi_x^2 + i\rho^{1/2}\phi_{xx}\right)e^{i\phi}.\end{aligned}$$

Substituting into the NLS equation and canceling common term $e^{i\phi(x,t)}$ we find:

$$i\left(\frac{\varepsilon}{2}\rho^{-1/2}\rho_t + \varepsilon^2\phi_x\rho_x\rho^{-\frac{1}{2}} + \varepsilon^2\rho^{\frac{1}{2}}\phi_{xx}\right) + \left(-\varepsilon\rho^{\frac{1}{2}}\phi_t + \frac{\varepsilon^2}{2}\rho^{-\frac{1}{2}}\rho_{xx} - \frac{\varepsilon^2}{4}\rho^{-\frac{3}{2}}\rho_x^2 - \varepsilon^2\rho^{\frac{1}{2}}\phi_x^2\right) - 2\rho^{\frac{3}{2}} = 0$$

- Now, the simplified real part is,

$$-\varepsilon\phi_t + \varepsilon^2\left(\frac{\rho_{xx}}{2\rho} - \frac{\rho_x^2}{4\rho^2}\right) - \varepsilon^2\phi_x^2 - 2\rho = 0$$

Differentiating w.r.t. $x$

$$u_t - \frac{\varepsilon^2}{4}\partial_x\left(\partial_{xx}(\ln\rho) + \frac{1}{\rho}\rho_{xx}\right) + 2uu_x + 2\rho_x = 0 \tag{2.2}$$

- The simplified imaginary part is,

$$\begin{aligned}&\frac{1}{2}\rho_t + \varepsilon\phi_x\rho_x + \varepsilon\rho\phi_{xx} = 0\\ &\rho_t + 2(u\rho_x + \rho u_x) = 0\end{aligned}$$

Therefore, the final form is,

$$\rho_t + 2(\rho u)_x = 0 \tag{2.3}$$

## 2.2 Madelung form the conservation laws of 1DNLS equation

The conservation laws in differential form are

$$\frac{\partial E}{\partial t} = 0 \quad \text{where, } E = \int_{\mathbb{R}} |\psi(x,t)|^2 dx. \tag{2.4a}$$

$$\frac{\partial P}{\partial t} = 0 \quad \text{where, } P = \frac{i\varepsilon}{2}\int_{\mathbb{R}} (\psi\psi_x^* - \psi^*\psi_x)\, dx \tag{2.4b}$$

$$\frac{\partial H}{\partial t} = 0 \quad \text{where, } H = \int_{\mathbb{R}} \left(\varepsilon^2\|\psi_x|^2 + |\psi|^4\right) dx \tag{2.4c}$$

### 2.2.1 Derivation of the momentum conservation law in terms of Madelung variables

We start from differentiating momentum $\rho u$ w.r.t. time,

$$
\begin{aligned}
(\rho u)_t &= \rho_t u + \rho u_t \\
&= -2u\frac{\partial}{\partial x}(\rho u) + \rho\left(-2u\frac{\partial u}{\partial x} - 2\frac{\partial \rho}{\partial x} + \frac{\epsilon^2}{4}\frac{\partial}{\partial x}\left(\frac{\partial^2 \ln\rho}{\partial x^2} + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2\right)\right) \\
\implies (\rho u)_t + 2\left(u(\rho u)_x + \rho u u_x\right) + 2\rho\rho_x &= \frac{\varepsilon^2}{4}\rho\frac{\partial}{\partial x}\left(\frac{\rho_{xx}}{\rho} - \frac{\rho_x^2}{\rho^2} + \frac{\rho_{xx}}{\rho}\right) \\
&= \frac{\varepsilon^2}{4}\rho\left(\frac{2\rho_{xxx}}{\rho} - \frac{4\rho_x\rho_{xx}}{\rho^2} + \frac{2\rho_x^3}{\rho^3}\right) \\
&= \frac{\varepsilon^2}{2}\left(\rho_{xxx} - \frac{2\rho_x\rho_{xx}}{\rho} + \frac{\rho_x^3}{\rho^2}\right) \\
\implies (\rho u)_t + 2(\rho u^2)_x + (\rho^2)_x + \frac{\varepsilon^2}{2}\left(\rho_{xxx} - \left(\frac{\rho_x^2}{\rho}\right)_x\right) &= 0.
\end{aligned}
$$

Therefore, the momentum conservation law in differential form and in terms of Madelung variables is:

$$
(\rho u)_t + 2(\rho u^2)_x + (\rho^2)_x + \frac{\varepsilon^2}{2}\left(\rho_{xxx} - \left(\frac{\rho_x^2}{\rho}\right)_x\right) = 0. \tag{2.5}
$$

### 2.2.2 Derivation of the energy conservation law in terms of Madelung variables

Energy density is defined by,

$$
h = \rho|u|^2 + \rho^2 + \frac{\epsilon^2}{4\rho}\rho_x^2
$$

Differentiating $h$ w.r.t. $t$ we get,

$$
\begin{aligned}
h_t &= (\rho u^2)_t + (\rho^2)_t + \frac{\varepsilon^2}{4\rho}\rho_x^2 \\
&= -2u^3\rho_x - 2^2 u_x + \varepsilon^2\left(u\rho_{xxx} - \frac{2\rho_x\rho_{xx}u}{\rho} + \frac{\rho_x^3 u}{\rho^2}\right) - 4\rho u^2 u_x - 4\rho\rho_x u \\
&+ -4\rho_x u - 4\rho^2 u_x \\
&+ \varepsilon^2\left[\frac{\rho_x\rho_{xx}u}{\rho} - \frac{2\rho_x^2 u_x}{\rho} - \rho_x u_{xx} + \frac{2\rho_x^3 u}{\rho^2} + \frac{2\rho_x^2 u_x}{\rho}\right] \\
&= -2\left((h+\rho^2)u\right) + \left(u\rho_{xx} - \frac{(\rho u)_x\rho_x}{\rho}\right)
\end{aligned}
$$

Therefore, we find the final conservation law in Madelung form,

$$
h_t + 2\left((h+\rho^2)u\right) = \left(u\rho_{xx} - \frac{(\rho u)_x\rho_x}{\rho}\right) \tag{2.6}
$$

# 3 Periodic solutions of 1DNLS using a two-phase ansatz

To study the slow dynamics of periodic solutions, we first derive their structure for the hydrodynamic system (2.2), (2.3). These periodic solutions form the basis for constructing modulation equations, capturing key nonlinear wave interactions. We now employ a two-phase ansatz to parameterize the solution, facilitating the derivation of the modulation dynamics:

$$\rho(x,t) = \rho(Z)\,, \quad \Phi(x,t) = \phi(Z) + S, \tag{3.1a}$$

where,

$$Z(x,t) = \frac{kx - \omega t}{\varepsilon}\,, \quad S(x,t) = \frac{vx - \mu t}{\varepsilon}, \tag{3.1b}$$

$\rho(Z)$ and $\phi(Z)$ are periodic functions of $Z$ with period one. It is important to note that, since the NLS equation is complex-valued, employing a two-phase ansatz is essential. A one-phase ansatz, like the one used for the KdV equation (real-valued), would only capture a subset of all possible periodic solutions ( in that case to obtain the most general family of periodic solutions for the NLS equation, a Galilean boost must be applied a posteriori).

## 3.1 Hydrodynamic equations in two-phase ansatz

### 3.1.1 Analysis on $\rho$:

With (3.1) now the (2.1b) becomes,

$$u(Z) = \varepsilon\left(\frac{\partial}{\partial Z}(\phi)\frac{\partial Z}{\partial x} + \frac{\partial S}{\partial x}\right) = k\phi' + v. \tag{3.2}$$

Where,

$$\mathbf{v} = \bar{\mathbf{u}}, \tag{3.3}$$

since $\phi(Z)$ is periodic with unit period $\implies \int_a^{(a+1)} \phi' dz = [\phi(a+1) - \phi(a)] = 0$
Next, we substitute the two-phase ansatz into the hydrodynamic equations (2.2) and (2.3) to find

$$-u'\frac{\omega}{\varepsilon} + 2uu'\frac{k}{\varepsilon} + 2\rho'\frac{k}{\varepsilon} - \frac{\varepsilon^2}{4}\left((\ln\rho)''\frac{k^2}{\varepsilon^2} + \frac{\rho''}{\rho}\frac{k^2}{\varepsilon^2}\right)'\frac{k}{\varepsilon} = 0$$

$$\boxed{-\omega u' + 2kuu' + 2k\rho' - \frac{k^3}{4}\left((\ln\rho)'' + \frac{\rho''}{\rho}\right)' = 0} \tag{3.4a}$$

and,

$$\frac{\partial\rho}{\partial Z}\frac{\partial Z}{\partial t} + \frac{\partial}{\partial Z}(\rho u)\frac{\partial Z}{\partial x} = 0$$

$$\boxed{\implies -\omega\rho' + 2k(\rho u)' = 0} \tag{3.4b}$$

Immediately, we can see that (3.4b) can be integrated to find,

$$-\omega\rho + 2k\rho u = J \tag{3.5}$$

$J$ is an integral constant which will be determined later. Now, applying (3.2) we find

$$\begin{aligned} &-\omega\rho + 2k\rho(k\phi' + \bar{u}) = 2Jk \\ &-\frac{\omega}{2k} + k\phi' + \bar{u} = \frac{J}{\rho}. \end{aligned}$$

Writing $U = \frac{\omega}{2k}$ and rearranging we finally get,

$$\phi' = \frac{1}{k}\left(U + \frac{J}{\rho} - \bar{u}\right). \tag{3.6}$$

Then with this $\phi'$ now, (3.2) becomes,

$$u(Z) = U + \frac{J}{\rho} \tag{3.7}$$

Averaging $u$ over one period we get,

$$\bar{u} = U + \overline{J\rho^{-1}} \implies \boxed{U = \bar{u} - \overline{J\rho^{-1}}} \tag{3.8}$$

Similarly, using (3.6) into (3.4a) (actually substituting $\omega = 2kU = 2k(u - \frac{J}{\rho})$) we obtain,

$$\begin{aligned}
2k\frac{J}{\rho}\left(-\frac{J}{\rho^2}\right) + 2k\rho' - \frac{k^3}{4}\left((\ln\rho)'' + \frac{\rho''}{\rho}\right)' &= 0\\
\implies -\frac{4J^2\rho'}{\rho^2} - 4\rho\rho' - k^2\left(\rho''' - \frac{2\rho'\rho''}{\rho} + \frac{(\rho')^3}{\rho^2}\right) &= 0\\
\implies k^2\rho''' + k^2\frac{(\rho')^3 - 2\rho\rho'\rho''}{\rho^2} - 4\rho\rho' + \frac{4J^2\rho'}{\rho^2} &= 0\\
\implies k^2\rho''' - \left(\frac{(\rho')^2}{\rho}\right)' - 4\rho\rho' + \frac{4J^2\rho'}{\rho^2} &= 0\\
\text{Intergating with resect to } Z,\\
k^2\rho'' - k^2\frac{(\rho')^2}{\rho} - 2\rho^2 - \frac{4J^2}{\rho} + 2c_1 &= 0\\
\text{Multiplying the above expression by } \frac{2\rho'}{\rho^2}\\
\implies k^2\left(\frac{2\rho'\rho''\rho^2 - 2\rho(\rho')^3}{\rho^4}\right) - 4\rho' - 8J^2\frac{\rho'}{\rho^3} + 4c_1\frac{\rho'}{\rho^2} &= 0\\
\implies k^2\left(\frac{(\rho')^2}{\rho^2}\right)' - 4\rho' - 8J^2\frac{\rho'}{\rho^3} + 4c_1\frac{\rho'}{\rho^2} &= 0\\
\text{Finally, integrating again with respect to } Z,
\end{aligned}$$

$$k^2(\rho')^2 = 4\rho^3 - 4c_2\rho^2 + 4c_1\rho - 4J^2 \tag{3.9}$$

Substituting $\rho(Z) = A + By^2(Z)$ in(3.9) we find the following ODE,

$$(y')^2 = \frac{1}{B^2k^2}\left(A^3 - A^2c_2 + Ac_1 - J^2\right)\frac{1}{y^2} + \frac{1}{Bk^2}\left(3A^2 - 2Ac_2 + c_1\right) + \left(\frac{3A}{k^2} - \frac{c_2}{k^2}\right)y^2 + \frac{B}{k^2}y^4. \tag{3.10}$$

Now, Jacobi elliptic sine ($y(Z) = sn(cZ; m)$) satisfies the ODE,

$$(y')^2 = c^2(1 - y^2)(1 - my^2). \tag{3.11}$$

Requiring that (3.10) has solution in terms of elliptic sine with period 1, i.e. $sn(2K_mZ; m)$ we compare the coefficients of (3.10) with (3.11).

*[$sn(Z; m)$ has real period as $2K_m$. In order to make sure the period is 1 we scale by $c = 2K_m$].*
**Coefficient of $y^4$:**

$$\frac{B}{k^2} = 4K_m^2 m \implies \boxed{B = 4mk^2K_m^2.} \tag{3.12}$$

**Coefficient of $y^2$:**

$$\begin{aligned}\frac{3A}{k^2}-\frac{c_2}{k^2}&=-4K_m^2(1+m)\\ 3A-c_2&=-4K_m^2k^2(1+m)\end{aligned}$$

$$\boxed{c_2=3A+4(1+m)k^2K_m^2.} \tag{3.13}$$

**Constant Term:**
Substituting $B=4mk^2K_m^2$ and $c_2=3A+4(1+m)k^2K_m^2$:

$$3A^2-2A\left(3A+4(1+m)k^2K_m^2\right)+c_1=16mK_m^4k^4,$$

$$\boxed{c_1=3A^2+8A(1+m)k^2K_m^2+16mk^4K_m^4.} \tag{3.14}$$

**Expression for $J^2$:**

$$J^2=A^3-A^2c_2+Ac_1. \tag{3.15}$$

Substitute $c_2=3A+4K_m^2k^2(1+m)$ and $c_1=3A^2+8AK_m^2k^2(1+m)+16K_m^4mk^4$.

$$\begin{aligned}J^2&=A^3-A^2\left(3A+4K_m^2k^2(1+m)\right)+A\left(3A^2+8AK_m^2k^2(1+m)+16K_m^4mk^4\right)\\ &=A^3-3A^3-4A^2K_m^2k^2(1+m)+3A^3+8A^2K_m^2k^2(1+m)+16AK_m^4mk^4.\end{aligned}$$

Simplifying the terms:

$$\begin{aligned}J^2&=A^3+\left(-3A^3+3A^3\right)+\left(-4A^2K_m^2k^2(1+m)+8A^2K_m^2k^2(1+m)\right)+16AK_m^4mk^4\\ &=A^3+4A^2K_m^2k^2(1+m)+16AK_m^4mk^4.\end{aligned}$$

Factorizing:

$$\boxed{J^2=A\left(A+4K_m^2k^2m\right)\left(A+4K_m^2k^2\right).} \tag{3.16}$$

In other words the solution of (3.9) is,

$$\rho(Z)=A+4mk^2K_m^2sn(2K_mZ;m). \tag{3.17}$$

Alternatively, RHS of (3.9) is a cubic polynomial, therefore it is trivial from the 'fundamental theorem of algebra' that,

$$(\rho')^2=\frac{4}{k^2}\prod_{i=1}^{3}(\rho-\lambda_i) \tag{3.18}$$

Now, from the relation between the roots and coefficients of a cubic equation we can find,

$$\sum_{i=1}^{3}\lambda_i=c_2=3A+4K_m^2k^2(1+m) \tag{3.19a}$$

$$\sum_{\substack{i=1\\ j\neq i}}^{3}\lambda_i\lambda_j=c_1=3A^2+8AK_m^2k^2(1+m)+16K_m^4mk^4 \tag{3.19b}$$

$$\lambda_1\lambda_2\lambda_3=J^2=A\left(A+4K_m^2k^2m\right)\left(A+4K_m^2k^2\right) \tag{3.19c}$$

With simple steps of routine algebra it can be found

$$(\lambda_1,\lambda_2,\lambda_3)=\left(A\,,A+4mk^2K_m^2\,,A+4k^2K_m^2\right) \tag{3.20}$$

Also, if $\lambda_i$'s are known then one can find reverse relations as follows,

$$\left(A\,,k^2\,,m\right)=\left(\lambda_1\,,\frac{\lambda_3-\lambda_1}{4K_m^2}\,,\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right) \tag{3.21}$$

### 3.1.2 Analysis on $\mu$:

If we substitute (2.1) and (3.1) into (1.1) and simplify, we find the ODE,

$$-\omega\phi' + 2k\bar{u}\phi' + k^2(\phi')^2 + 2\rho - \mu + \bar{u}^2 - \frac{k^2}{4}\left((\ln\rho)'' + \frac{\rho''}{\rho}\right) = 0. \tag{3.22}$$

Substituting $\omega = 2kU$ and $\phi' = \frac{1}{k}\left(U + \frac{J}{\rho} - \bar{u}\right)$ (3.22) becomes:

$$-2U\left(U + \frac{J}{\rho} - \bar{u}\right) + 2\bar{u}\left(U + \frac{J}{\rho} - \bar{u}\right) + \left(U + \frac{J}{\rho} - \bar{u}\right)^2 + 2\rho - \mu + \bar{u}^2 - \frac{k^2}{4}\left(\frac{2\rho''}{\rho} - \frac{(\rho')^2}{\rho^2}\right) = 0.$$

Using $U = \bar{u} - \overline{J\rho^{-1}}$ and further simplifying, the equation reduces to:

$$2k^2\rho'' - k^2\frac{(\rho')^2}{\rho} - 8\rho^2 + C\rho - 4\frac{J^2}{\rho} = 0, \tag{3.23}$$

where:

$$C = 4\mu - 4\left(\bar{u}^2 - \left(\overline{J\rho^{-1}}\right)^2\right). \tag{3.24}$$

Next, multiplying through by $\frac{\rho'}{\rho}$:

$$\begin{aligned}
\left(2k^2\rho''\right)\frac{\rho'}{\rho} - \left(k^2\frac{(\rho')^2}{\rho}\right)\frac{\rho'}{\rho} - \left(8\rho^2\right)\frac{\rho'}{\rho} + (C\rho)\frac{\rho'}{\rho} - \left(4\frac{J^2}{\rho}\right)\frac{\rho'}{\rho} &= 0\\
\implies k^2\frac{d}{dz}\left(\frac{(\rho')^2}{\rho}\right) - 8\rho\rho' + C\rho' - 4J^2\frac{\rho'}{\rho^2} &= 0
\end{aligned}$$

$$\implies k^2(\rho')^2 = 4\rho^3 - C\rho^2 - 4J^2 + 4c_3\rho. \tag{3.25}$$

Comparing (3.25) with (3.9) we find, $c_1 = c_3$ and $4c_2 = C$. Therefore, finally we find,

$$\begin{aligned}
4\mu - 4\bar{u}^2 + 4\overline{J\rho^{-1}}^2 &= 4\left(3A + 4(1+m)k^2K_m^2\right)\\
\implies &\boxed{\mu = 3A + 4(1+m)k^2K_m^2 + \bar{u}^2 - \overline{J\rho^{-1}}.}
\end{aligned} \tag{3.26}$$

## 3.2 Harmonic and soliton limits of the periodic solutions:

Harmonic ($m \to 0$) and soliton ($m \to 1$) limits of the Whitham equations are of great importance, so it is wise to look for the respective limits of the periodic solutions.

### 3.2.1 Harmonic limit($m \to 0$)

When $m \to 0$, corresponding to $\lambda_2 \to \lambda_1^+$, the solution (2.1) simplifies to a plane wave. In this regime, the solution parameters take the form:

$$\rho(Z) = A, \quad B = 0, \quad \mu = 2A + \bar{u}^2, \quad J^2 = A^2\left(\pi^2k^2 + A\right), \tag{3.27}$$

where the wave function reduces to:

$$\psi(x,t) = \sqrt{A}\, e^{i\left(kx - \bar{u}^2t + 2At\right)}. \tag{3.28}$$

In this limit, the independent solution parameters reduce to $A$ and $\bar{u}$.

#### 3.2.2 Soliton Limit ($m \to 1$)

In the opposite limit, where $m \to 1$ (or equivalently $\lambda_2 \to \lambda_3^-$), the solution reduces to a soliton profile. From (3.17) and (3.20),

$$\rho(Z) = \lambda_1 + (\lambda_3 - \lambda_1)\tanh^2\left(\sqrt{\lambda_3 - \lambda_1}\,\frac{kx - \omega t}{k}\right). \tag{3.29}$$

Here, the parameters satisfy:

$$B = \lambda_3 - \lambda_1, \quad J^2 = \lambda_1\lambda_3^2, \quad U = k \cdot \bar{u} - \sigma\sqrt{\lambda_1}, \quad \mu = 2\lambda_3 + \bar{u}^2. \tag{3.30}$$

As $m \to 1$, $k \to 0$, but the product $kK_m$ remains finite, approaching:

$$kK_m \to \sqrt{\lambda_3 - \lambda_1}/2.$$

Using equation (3.6), the phase variable becomes:

$$\phi(Z) = \tan^{-1}\left[\sqrt{\lambda_3 - \lambda_1}\,\tanh\left(\sqrt{\lambda_3 - \lambda_1}\,\frac{kx - \omega t}{k}\right)/\sqrt{\lambda_1}\right]. \tag{3.31}$$

This results in the following wave function:

$$e^{i\phi + iS} = e^{i\left[\sqrt{\lambda_1} + i\sqrt{\lambda_3 - \lambda_1}\,\tanh\left(\sqrt{\lambda_3 - \lambda_1}\,\frac{kx - \omega t}{k}\right)\right]}/\sqrt{\rho(Z)}. \tag{3.32}$$

Here, $S$ represents the phase shift defined by:

$$S = \bar{u}x - \mu t.$$

Combining these results, the wave function takes the form:

$$\psi(x,t) = A_0 e^{-2iA_0^2 t} e^{i\bar{u}x - i\bar{u}^2 t}\left\{(\cos\theta + i\sin\theta)\tanh\left[A_0\sin\theta\,(k\bar{u} - A_0\cos\theta)\,t\right]\right\}, \tag{3.33}$$

where:

$$\theta = \arctan\left[\sqrt{\frac{\lambda_3 - \lambda_1}{\lambda_1}}\right], \quad A_0 = \sqrt{\lambda_3}.$$

In this limit, the solution parameters simplify to $\lambda_1, \lambda_3$ (or equivalently $A_0, \theta$), k , and $\bar{u}$.

# 4 Nonlinear Modulations and Averaged Conservation Laws

We introduce the following multiple scales ansatz to solve the NLS equation (1.1):

$$\rho(x,t) = \rho(Z,X,T), \quad \Phi(x,t) = \phi(Z,X,T) + S, \tag{4.1}$$

where $X = x$ and $T = t$, and $\rho$ and $\phi$ are periodic in $Z$ with period 1. The fast phases are:

$$Z_x = \frac{k(X,T)}{\varepsilon}, \quad Z_t = -\frac{\omega(X,T)}{\varepsilon}, \quad S_x = \frac{v(X,T)}{\varepsilon}, \quad S_t = -\frac{\mu(X,T)}{\varepsilon}. \tag{4.2}$$

Following the results of previous section, we set $v = \bar{u}$. The ansatz implies:

$$\partial_x \mapsto \frac{k}{\varepsilon}\partial_Z + \frac{v}{\varepsilon}\partial_S + \partial_X, \quad \partial_t \mapsto -\frac{\omega}{\varepsilon}\partial_Z - \frac{\mu}{\varepsilon}\partial_S + \partial_T. \tag{4.3}$$

Substituting this ansatz into (1.1) and simplifying to leading order gives the periodic solution (2.1). Here, the parameters $A, m, k, \bar{u}$ vary slowly in $X$ and $T$. Differentiating equations in terms of $X$ and $T$ yields the conservation laws:

$$k_t + \partial_x \omega = 0, \tag{4.4}$$
$$\bar{u}_t + \partial_x \mu = 0, \tag{4.5}$$

Above equations- (4.4) and (4.5) form the first two modulation equations.

To derive the remaining modulation equations, we average the conservation laws (2.3),(2.5)(2.6) over the fast phase $Z$. Substituting derivatives using (4.3), expanding terms in powers of $\varepsilon$, and averaging at order $O(\varepsilon^0)$, we obtain:

$$\partial_T \bar{\rho} + 2\partial_x(\bar{\rho}\bar{u}) = 0, \tag{4.6}$$
$$\partial_T(\bar{\rho}\bar{u}) + 2\partial_x(\bar{\rho}\bar{u}^2) + \partial_x(\bar{\rho}^2) + \partial_x\left(\frac{(\rho')^2}{2\bar{\rho}}k^2\right) = 0, \tag{4.7}$$
$$\partial_T \bar{h} + \partial_x\left(2\bar{h}\bar{u} + 2\bar{\rho}\bar{u}^2 + \left(k\frac{\rho'}{\rho}(\bar{\rho}\bar{u})'\right)k - k^2\rho''\bar{u}\right) = 0, \tag{4.8}$$

where $\bar{h}$ is the averaged energy density, given by:

$$\bar{h} = \rho\bar{u}^2 + \rho^2 + \frac{1}{4}k^2\frac{(\rho')^2}{\rho}. \tag{4.9}$$

Together, equations (4.4), (4.5), and (4.6)–(4.8) form a system of 5 scalar PDEs for the 4 parameters $A, m, k$, and $\bar{u}$.

# Appendix

## Direct derivation of the periodic solutions of the NLS equation

We derive the periodic solutions of the NLS equation in an arbitrary number of dimensions without relying on the hydrodynamic form. Consider the one-phase ansatz:

$$\psi(x,t) = \sqrt{\rho(Z)}e^{i\Phi(Z,t)}, \quad Z = k\cdot x - \omega t, \tag{4.10}$$

where $k$ is the wave vector and $\omega$ is the frequency. Substituting (4.10) into the NLS equation and separating real and imaginary parts, we obtain:

$$(\sqrt{\rho})'' - \sqrt{\rho}(\Phi')^2 + \frac{\omega}{k^2}\rho^{3/2} - \frac{2}{k^2}\rho^{3/2} = 0, \tag{4.11}$$
$$\sqrt{\rho}\Phi'' + \left(2\Phi' - \frac{\omega}{k^2}\right)(\sqrt{\rho})' = 0. \tag{4.12}$$

Integrating (4.12), we get:

$$\Phi' = \frac{J}{k\rho} + \frac{\omega}{2k}, \tag{4.13}$$

where $J$ is an integration constant. Substituting (4.13) into (4.11) and simplifying, we obtain:

$$(\sqrt{\rho})'' = \frac{J^2}{a^2\rho^3} + \left(\frac{\omega}{2a}\right)^2 \rho - \frac{2}{a}\rho^2. \tag{4.14}$$

Multiplying (4.14) by $2\rho'/\rho$ and integrating with respect to $Z$, we let $f = \rho$ and find:

$$(f')^2 = \frac{4}{a}f^3 - 4\left(\frac{\omega}{2a}\right)^2 f^2 + 4cf - \frac{4J^2}{a}, \tag{4.15}$$

where $c$ is an integration constant. Substituting $f(Z) = A + By^2(Z)$, where $y$ is a new variable, into (4.15), we get:

$$(y')^2 = \frac{1}{B^2 a}\left[A^3 - A^2 c_2 + Ac_1 - J^2\right]\frac{1}{y^2} + \frac{1}{Ba}\left(3A^2 - 2Ac_2 + c_1\right) + \left(\frac{3A}{a} - \frac{c_2}{a}\right)y^2 + \frac{B}{a}y^4. \tag{4.16}$$

The Jacobi elliptic sine function $y(Z) = \mathrm{sn}(cZ|m)$ solves the ODE $(y')^2 = (1-y^2)(1-my^2)$. Matching (4.16) to this form yields:

$$J^2 = 4aK_m^2 A\left(1 + \frac{A}{4K_m^2 a}\right)\left(A + 4mK_m^2 a\right), \tag{4.17}$$

where $B = 4mk^2$. The symmetric polynomials of the roots $\lambda_1, \lambda_2, \lambda_3$ are:

$$e_1 = \lambda_1 + \lambda_2 + \lambda_3 = \frac{\omega^2}{4a}, \quad e_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = c_1 a, \quad e_3 = \lambda_1\lambda_2\lambda_3 = J^2. \tag{4.18}$$

Finally, the general solution can be extended via Galilean invariance:

$$\psi(x,t) = \psi(X - 2\bar{v}t)e^{i(k\cdot\bar{v} - \|\bar{v}\|^2)t}, \tag{4.19}$$

$$\Phi(Z,x,t) = \Phi(Z) + (x - \bar{v}t)^2/\varepsilon. \tag{4.20}$$