# HW-2

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## 1 Two Country RBC with Incomplete Markets

We have two countries: Home and Foreign. There is a representative agent in both countries producing the same good that are traded across the border. There is a one-period bond that is traded at price $P_t^B$ and pays one unit of final goods in the next period. The size of the Home economy is $\pi$ and that of the foreign economy is $1-\pi$.

### 1.1 Home Economy Problem

The representative household in the Home economy solves:

$$\max_{\{C_t, K_{t+1}, B_{t+1}, H_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} - \kappa \frac{H_t^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \right\}$$

$$\begin{aligned} \text{s.t.:} \quad C_t + K_{t+1} + P_t^B B_{t+1} &= Z_t K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t + B_t \\ &- \frac{\phi_k}{2}(K_{t+1} - K_t)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1} - \bar{B})^2 \end{aligned}$$

The Lagrangian is:

$$\begin{aligned} \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{C_t^{1-\gamma}}{1-\gamma} - \kappa \frac{H_t^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} + \lambda_t \Big( Z_t K_t^\alpha H_t^{1-\alpha} + (1-\delta)K_t \\ & + B_t - C_t - K_{t+1} - P_t^B B_{t+1} - \frac{\phi_k}{2}(K_{t+1} - K_t)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1} - \bar{B})^2 \Big) \Bigg\} \end{aligned}$$

We have the following first-order conditions
1. With respect to $C_t$:

$$\begin{aligned}\partial C_t: \quad \beta^t C_t^{-\gamma} - \beta^t \lambda_t &= 0 \\ C_t^{-\gamma} - \lambda_t &= 0 \end{aligned} \tag{1}$$

2. With respect to $H_t$:

$$\begin{aligned}\partial H_t: \quad \beta^t \kappa H_t^{\frac{1}{\psi}} - \beta^t \lambda_t (1-\alpha) Z_t K_t^{\alpha} H_t^{-\alpha} &= 0 \\ \implies \kappa H_t^{\frac{1}{\psi}} &= \lambda_t (1-\alpha) Z_t K_t^{\alpha} H_t^{-\alpha} \end{aligned} \tag{2}$$

3.With respect to $K_{t+1}$:

$$\begin{aligned}\partial K_{t+1}: \quad & \beta^t \lambda_t \left[-1 - \phi_k (K_{t+1} - K_t)\right] \\ & + E_t \beta^{t+1} \lambda_{t+1} \Big[ (1-\delta) + \alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} \\ & + \phi_k (K_{t+2} - K_{t+1}) \Big] = 0 \end{aligned}$$

$$\implies \lambda_t \left[1 + \phi_k (K_{t+1} - K_t)\right] = \beta E_t \lambda_{t+1} \left[ (1-\delta) + \alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + \phi_k (K_{t+2} - K_{t+1}) \right] \tag{3}$$

4. With respect to $B_{t+1}$:

$$\begin{aligned}\partial B_{t+1}: \quad \beta^t \lambda_t \left[ P_t^B + \phi_b P_t^B (B_{t+1} - \bar{B}) \right] - \beta^{t+1} E_t \lambda_{t+1} &= 0 \\ \beta E_t \lambda_{t+1} &= \lambda_t \left[ P_t^B + \phi_b P_t^B (B_{t+1} - \bar{B}) \right] \end{aligned} \tag{4}$$

5. With respect to $\lambda_t$:

$$\begin{aligned}\partial \lambda_t: \quad & Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t + B_t - C_t - K_{t+1} \\ & - P_t^B B_{t+1} - \frac{\phi_k}{2} (K_{t+1} - K_t)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1} - \bar{B})^2 = 0 \end{aligned} \tag{5}$$

## 1.2 Foreign Country

The foreign country solves the exact problem . The variables have a $*$ superscript

$$\max_{\{C_t^*, K_{t+1}^*, B_{t+1}^*, H_t^*\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{*1-\gamma}}{1-\gamma} - \kappa^* \frac{H_t^{*1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \right\}$$

$$\begin{aligned} \text{s.t.:} \quad & C_t^* + K_{t+1}^* + P_t^B B_{t+1}^* = Z_t^* K_t^{*\alpha} H_t^{*1-\alpha} + (1-\delta)K_t^* + B_t^* \\ & - \frac{\phi_k}{2}(K_{t+1}^* - K_t^*)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1}^* - \bar{B}^*)^2 \end{aligned}$$

The same set of first order conditions:

1. With respect to $C_t^*$:

$$\partial C_t^* : C_t^{*-\gamma} - \lambda_t^* = 0 \tag{6}$$

2. With respect to $H_t^*$:

$$\partial H_t^* : \kappa^* H_t^{*\frac{1}{\psi}} = \lambda_t^* (1-\alpha) Z_t^* K_t^{*\alpha} H_t^{*-\alpha} \tag{7}$$

3. With respect to $K_{t+1}^*$:

$$\partial K_{t+1}^* : \lambda_t^* \left[1 + \phi_k (K_{t+1}^* - K_t^*)\right] = \beta E_t \lambda_{t+1}^* \left[(1-\delta) + \alpha Z_{t+1}^* {K_{t+1}^*}^{\alpha-1} {H_{t+1}^*}^{1-\alpha} + \phi_k (K_{t+2}^* - K_{t+1}^*)\right] \tag{8}$$

4. With respect to $B_{t+1}^*$:

$$\partial B_{t+1}^* : \beta E_t \lambda_{t+1}^* = \lambda_t^* \left[P_t^B + \phi_b P_t^B (B_{t+1}^* - \bar{B}^*)\right] \tag{9}$$

5. With respect to $\lambda_t^*$

$$\begin{aligned} \partial \lambda_t^* : \quad & Z_t^* K_t^{*\alpha} H_t^{*1-\alpha} + (1-\delta)K_t^* + B_t^* - C_t^* - K_{t+1}^* \\ & - P_t^B B_{t+1}^* - \frac{\phi_k}{2}(K_{t+1}^* - K_t^*)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1}^* - \bar{B}^*)^2 = 0 \end{aligned} \tag{10}$$

## 1.3 Bond Market Clearing

We have

$$\pi B_t + (1-\pi){B_t}^* = 0 \tag{11}$$

Together with the budget constraints the bond market clearing condition implies clearing of the aggregate goods market. So do not need another equation.

Thus the complete set of equations are :

- Eqs (1-5) for Home country
- Eqs (6-10) for Foreign country
- Bond Market clearing : Eq 11
- Process governing $Z_t, {Z_t}^*$

$$
\begin{aligned}
& \text{Home Economy} \\
\partial C_t: \quad & C_t^{-\gamma} - \lambda_t = 0 \\
\partial H_t: \quad & \kappa H_t^{\frac{1}{\psi}} = \lambda_t (1-\alpha) Z_t K_t^{\alpha} H_t^{-\alpha} \\
\partial K_{t+1}: \quad & \lambda_t \left[1 + \phi_k (K_{t+1} - K_t)\right] = \beta E_t \lambda_{t+1} \left[(1-\delta) + \alpha Z_{t+1} K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + \phi_k (K_{t+2} - K_{t+1})\right] \\
\partial B_{t+1}: \quad & \beta E_t \lambda_{t+1} = \lambda_t \left[P_t^B + \phi_b P_t^B (B_{t+1} - \bar{B})\right] \\
\partial \lambda_t: \quad & Z_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t + B_t - C_t - K_{t+1} - P_t^B B_{t+1} - \frac{\phi_k}{2} (K_{t+1} - K_t)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1} - \bar{B})^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
& \text{Foreign Economy} \\
\partial C_t^*: \quad & (C_t^*)^{-\gamma} - \lambda_t^* = 0 \\
\partial H_t^*: \quad & \kappa^* (H_t^*)^{\frac{1}{\psi}} = \lambda_t^* (1-\alpha) Z_t^* (K_t^*)^{\alpha} (H_t^*)^{-\alpha} \\
\partial K_{t+1}^*: \quad & \lambda_t^* \left[1 + \phi_k (K_{t+1}^* - K_t^*)\right] = \beta E_t \lambda_{t+1}^* \left[(1-\delta) + \alpha Z_{t+1}^* (K_{t+1}^*)^{\alpha-1} (H_{t+1}^*)^{1-\alpha} + \phi_k (K_{t+2}^* - K_{t+1}^*)\right] \\
\partial B_{t+1}^*: \quad & \beta E_t \lambda_{t+1}^* = \lambda_t^* \left[P_t^B + \phi_b P_t^B (B_{t+1}^* - \bar{B}^*)\right] \\
\partial \lambda_t^*: \quad & Z_t^* K_t^{*\alpha} H_t^{*1-\alpha} + (1-\delta) K_t^* + B_t^* - C_t^* - K_{t+1}^* - P_t^B B_{t+1}^* - \frac{\phi_k}{2} (K_{t+1}^* - K_t^*)^2 - \frac{\phi_b}{2} P_t^B (B_{t+1}^* - \bar{B}^*)^2 =
\end{aligned}
$$

Bond Market Clearing
$\pi B_t + (1-\pi){B_t}^* = 0$
Process for $Z_t, {Z_t}^*$
$\begin{pmatrix} \ln Z_{t+1} \\ \ln Z^*_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & \tilde{\rho} \\ \tilde{\rho} & \rho^* \end{pmatrix} \begin{pmatrix} \ln Z_t \\ \ln Z^*_t \end{pmatrix} + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, V)$

We have 13 unknowns and 13 equations. We also can add the investment and output equations to the above set :

$$Y_t = Z_t K_t^\alpha H_t^{1-\alpha} \tag{12}$$
$$Y_t^* = Z_t^* (K_t^*)^\alpha (H_t^*)^{1-\alpha} \tag{13}$$
$$I_t = K_{t+1} - (1-\delta)K_t \tag{14}$$
$${I_t}^* = {K_{t+1}}^* - (1-\delta){K_t}^* \tag{15}$$

We also need to add auxiliary variables to fit ino the framework in SGU.

$$\begin{aligned} K_{fu,t} &= K_{t+1} \\ \Longrightarrow \; K_{fu,t+1} &= K_{t+2} \qquad (16) \\ K^*_{fu,t} &= K^*_{t+1} \\ \Longrightarrow \; K^*_{fu,t+1} &= K^*_{t+2} \qquad (17) \end{aligned}$$

## 1.4 Steady State Calculations

In steady state:

- All variables are constant (no time subscripts)
  - $X_t = X_{ss}$ for $C, H, Y, I, K, \lambda$ , $C^*, H^*, Y^*, I^*, K^*, \lambda^*$ and $P, B, B^*$
- $Z = Z^* = 1$
- Since countries are symmetric we have symmetric steady states
- We will solve in terms of $H_{ss}$, then set $H_{ss} = H^*_{ss} = 1/3$

1. From FOCs:

$$\begin{aligned} {C_{ss}^{-\gamma}} &= \lambda_{ss} \\ \kappa H_{ss}^{1/\psi} &= \lambda_{ss}(1-\alpha)K_{ss}^\alpha H_{ss}^{-\alpha} \\ 1 &= \beta[1-\delta+\alpha K_{ss}^{\alpha-1} H_{ss}^{1-\alpha}] \\ 1 &= \beta / P_{ss}^B \\ K_{ss}^\alpha H_{ss}^{1-\alpha} &= C_{ss} + \delta K_{ss} \end{aligned}$$

2. Solve for $K_{ss}$ from the capital Euler equation:

$$
\begin{aligned}
1 &= \beta[1-\delta+\alpha K_{ss}^{\alpha-1}H_{ss}^{1-\alpha}] \\
\frac{1}{\beta} &= 1-\delta+\alpha K_{ss}^{\alpha-1}H_{ss}^{1-\alpha} \\
\frac{1}{\beta}-(1-\delta) &= \alpha K_{ss}^{\alpha-1}H_{ss}^{1-\alpha} \\
K_{ss}^{1-\alpha} &= \frac{\alpha H_{ss}^{1-\alpha}}{\frac{1}{\beta}-(1-\delta)} \\
K_{ss} &= \left(\frac{\alpha}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{1}{1-\alpha}} \cdot H_{ss}
\end{aligned}
$$

3. Then output:

$$Y_{ss} = K_{ss}^{\alpha}H_{ss}^{1-\alpha}$$

4. Consumption from resource constraint:

$$
\begin{aligned}
C_{ss} &= Y_{ss}-\delta K_{ss} \\
&= K_{ss}^{\alpha}H_{ss}^{1-\alpha}-\delta K_{ss}
\end{aligned}
$$

5. Lagrange multiplier:

$$\lambda_{ss} = C_{ss}^{-\gamma}$$

6. Bond market:

$$
\begin{aligned}
P_{ss}^{B} &= \beta \\
B_{ss} &= 0 \text{ (given)} \\
B_{ss}^{*} &= 0
\end{aligned}
$$

7. Solve for $\kappa$ from labor FOC:

$$
\begin{aligned}
\kappa H_{ss}^{1/\psi} &= \lambda_{ss}(1-\alpha)K_{ss}^{\alpha}H_{ss}^{-\alpha} \\
\kappa &= \lambda_{ss}(1-\alpha)K_{ss}^{\alpha}H_{ss}^{-\alpha-1/\psi}
\end{aligned}
$$

Plugging in all expressions:

$$\kappa = \left[K_{ss}^{\alpha}H_{ss}^{1-\alpha}-\delta K_{ss}\right]^{-\gamma} \cdot (1-\alpha) \cdot K_{ss}^{\alpha} \cdot H_{ss}^{-\alpha-1/\psi}$$

where $K_{ss}$ is:

$$K_{ss} = \left(\frac{\alpha}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{1}{1-\alpha}} \cdot H_{ss}$$

Finally, setting $H_{ss} = H_{ss}^{*} = 1/3$ gives us the required values of $\kappa$ and $\kappa^{*}$ (which are identical due to symmetry). The value ensures that $H = 1/3$ is optimal in steady state.

**Q1 What are the state variables? What are the control variables?**

**State Variables:**

$$\{K, K^*, B, Z, Z^*\}$$

**Control Variables:**

$$\{C, C^*, H, H^*, Y, Y^*, IV, IV^*, \lambda, \lambda^*, B^*, PB, K_{fu}, K^*_{fu}\}$$

**Q2 What does the bond market clearing condition imply about the goods market clearing condition?**

As discussed earlier together with the budget constraints the bond market clearing condition implies clearing of the aggregate goods market.