Problem Set - 3
Compact Binary Evolution, Rates and Population Modelling

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8 June 2022

Problem 1

Unstable particles are emitted from a source and decay at a distance x, a real number that has an exponential probability distribution with characteristic length λ. Decay events can be observed only if they occur in a window extending from $x = 1$ cm to $x = 20$ cm. N decays are observed at locations $x_1, ..., x_N$. What is λ?

i) Write down a few possible ways to solve this problem. What are the pitfalls of these methods?
ii) Use Bayes Theorem to solve this problem. Define the likelihood, prior etc. and put it into Bayes theorem. Assuming a flat prior on λ, show the distribution of the posterior.
iii) [Bonus] Use MCMC sampler like emcee to solve this problem.
iv) [Bonus] Write your own MCMC algorithm.

Solution

The probability of one data point, given λ:

$$P(x|\lambda) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} / Z(\lambda) & \text{if } 1 \leq x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

where

$$Z(\lambda) = \int_1^{20} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx = (e^{-\frac{1}{\lambda}} - e^{-\frac{20}{\lambda}})$$

Now from *Bayes' theorem*,

$$P(\lambda|\{x_1, ..., x_N\} = \frac{P(\{x\}|\lambda)P(\lambda)}{P(\{x\})}$$
$$\propto \frac{1}{(\lambda Z(\lambda))^N} \exp(-\sum_{n=1}^{N} \frac{x_n}{\lambda}) P(\lambda)$$

The straightforward distribution $P(\{x_1, ..., x_N\}|\lambda)$, defining the probability of the data given the hypothesis λ, is turned to the probability of a hypothesis given the data.

What we know about λ after the data arrive is what we knew before $[P(\lambda)]$, *and what the data told us* $[P(x|\lambda)]$.

Below we show the posterior/likelihood distribution of λ given a six-point dataset, $\{x\} = \{1.5, 2, 3, 4, 5, 12\}$.

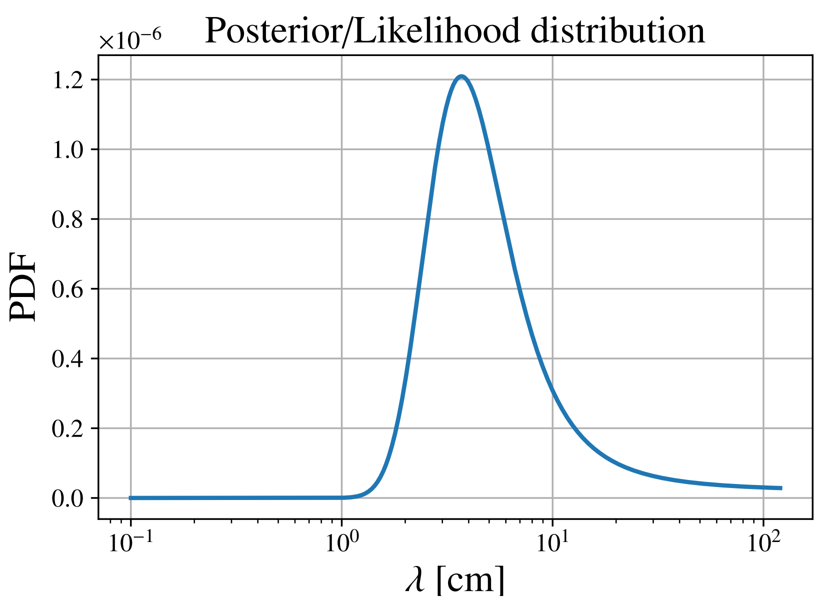

Figure 1: Posterior distribution of λ given 6 data points

Problem 2

On a game show, a contestant is told the rules as follows:

There are three doors, labelled 1, 2, 3. A single prize has been hidden behind one of them. You get to select one door. Initially your chosen door will not be opened. Instead, the gameshow host will open one of the other two doors, and he will do so in such a way as not to reveal the prize. For example, if you first choose door 1, he will then open one of doors 2 and 3, and it is guaranteed that he will choose which one to open so that the prize will not be revealed. At this point, you will be given a fresh choice of door: you can either stick with your first choice, or you can switch to the other closed door. All the doors will then be opened and you will receive whatever is behind your final choice of door.

Imagine that the contestant chooses door 1 first; then the gameshow host opens door 3, revealing nothing behind the door, as promised. Should the contestant (a) stick with door 1, or (b) switch to door 2, or (c) does it make no difference?

Solution

Let $\mathcal{H}_i$ denote the hypothesis that the prize is behind door i. We make the following assumptions: the three hypotheses $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are equiprobable *a priori*, i.e., $P(\mathcal{H}_1) = P(\mathcal{H}_2) = P(\mathcal{H}_3) = \frac{1}{3}$.

If the prize is behind door 1, then the host has a free choice. In this case, we assume that the host selects at random between $D = 2$ and $D = 3$. Otherwise, the choice of the host is forced, and the probabilities are 0 and 1.

$$\left| \begin{array}{c|c|c} P(D=2|\mathcal{H}_1) = \frac{1}{2} & P(D=2|\mathcal{H}_2) = 0 & P(D=2|\mathcal{H}_3) = 1 \\ P(D=3|\mathcal{H}_1) = \frac{1}{2} & P(D=3|\mathcal{H}_2) = 1 & P(D=3|\mathcal{H}_3) = 0 \end{array} \right|$$

Bayes' theorem gives the posterior probabilities of the hypotheses:

$$\boxed{P(\mathcal{H}_i|D=3) = \frac{P(D=3|\mathcal{H}_i)P(\mathcal{H}_i)}{P(D=3)}}$$

$$\left| \begin{array}{c|c|c} P(\mathcal{H}_1|D=3) = \frac{(\frac{1}{2})(\frac{1}{3})}{P(D=3)} & P(\mathcal{H}_2|D=3) = \frac{(1)(\frac{1}{3})}{P(D=3)} & P(\mathcal{H}_3|D=3) = \frac{(0)(\frac{1}{3})}{P(D=3)} \end{array} \right|$$

Since the sum of the posterior probabilities is 1 (normalisation condition), the term $P(\mathcal{H}_1|D = 3) = \frac{1}{2}$. And so,

$$\left| \begin{array}{c|c|c} P(\mathcal{H}_1|D=3) = \frac{1}{3} & P(\mathcal{H}_2|D=3) = \frac{2}{3} & P(\mathcal{H}_3|D=3) = 0 \end{array} \right|$$

So the contestant should switch to door 2 in order to have the biggest chance of getting the prize.