Solutions:Problem Set - 1

Forming merging binaries

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Problem 1:

Binary evolution: Consider two point masses inspiraling around each other under gravitational radiation reaction. The average energy emission rate for such a system is given by,

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4 m_1^2 m_2^2 (m_1+m_2)}{c^5 a^5 (1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{1}$$

The average angular momentum emission rate is,

$$\frac{dJ}{dt} = -\frac{32}{5}\frac{G^{\frac{7}{2}} m_1^2 m_2^2 (m_1+m_2)^{\frac{1}{2}}}{c^5 a^{\frac{7}{2}} (1-e^2)^2}\left(1+\frac{7}{8}e^2\right) \tag{2}$$

1. Use $\frac{\mathrm{dE}}{\mathrm{dt}}$ or $\frac{\mathrm{dJ}}{\mathrm{dt}}$ to calculate the change in the separation between the two bodies. Assume circular case for simplification.

   Solution: The separation between the two bodies, m_1 and m_2 is given by,

$$\begin{aligned} a &= -\frac{Gm_1m_2}{2E} && (3)\\ \implies \frac{da}{dt} &= \frac{Gm_1m_2}{2E^2}\frac{dE}{dt} && (4)\end{aligned}$$

   For a circular orbit, $e=0$. Substituting the values of E and $\frac{\mathrm{dE}}{\mathrm{dt}}$ from Eq. 3 and Eq. 1,

$$\frac{da}{dt} = -\frac{\beta}{a^3} \tag{5}$$

   where,

$$\beta = \frac{64}{5}\frac{G^3 m_1 m_2 (m_1+m_2)}{c^5} \tag{6}$$

2. Use the expression of the change in the separation between the two bodies to obtain lifetime of the binary as a function of the initial separation, a_0.

   Solution: On integrating Eq. 5 from $t=0$, when the initial separation between the components is a_0, to t when the separation is a,

$$\begin{aligned} \frac{da}{dt} &= -\frac{\beta}{a^3} \\ \implies \int_{a_0}^{a} a^3 da &= -\int_0^t \beta dt \\ \implies a(t) &= \left(a_0^4 - 4\beta t\right)^{\frac{1}{4}} && (7)\end{aligned}$$

   The system will inspiral towards each other over time, under gravitational radiation reaction and will decay in finite time, T_c. So, from Eq. 7 we get the lifetime of the binary to be,

$$T_c(a_0) = \frac{a_0^4}{4\beta} \tag{8}$$

3. Calculate the lifetime for a binary star with equal component masses of $1M_\odot$ for a range of initial separation from 0.01 AU to 1 AU and plot it. Vary the mass ratio ($\mathrm{q} = \mathrm{m}_2/\mathrm{m}_1$ with $\mathrm{m}_1 > \mathrm{m}_2$) and see how the lifetime changes for the same values of initial separation. [$\mathrm{G} = 6.67 \times 10^{-8}\,\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2}$, $\mathrm{c} = 3.0 \times 10^10\,\mathrm{cms}^{-1}$, $\mathrm{M}_\odot = 1.99 \times 10^33\,\mathrm{g}$, $\mathrm{s} = 3.17098 \times 10^{-8}\,\mathrm{yr}$, $\mathrm{H}_0 = 14.4 \times 10^9\,\mathrm{yr}$, $\mathrm{AU} = 1.496 \times 10^13\,\mathrm{cm}$]

   Solution: Refer to this jupyter notebook.

4. Calculate the initial separation of the binary required for it to merge within the Hubble time. Compare this value with the typical radius of the star in its giant phase.

   Hint: Typical radius of a star in its giant phase is, $\mathrm{R_G} = 1\mathrm{AU}$. *If* $\mathrm{a}_0 < \mathrm{R_G}$*, the stars will engulf each other in this phase and will not form a binary. But, if* $\mathrm{a}_0 > \mathrm{R_G}$*, the stars will not merge in Hubble time. This tells that some mechanism is needed to shrink the orbits, for the binaries to be close enough to merge through gravitational-wave emission, after they have evolved through their giant phase.*

   Solution: Refer to this jupyter notebook.

5. An approximate estimation of the radius of the Roche lobe is given by Eggleton's approximate formula,

$$\frac{r_1}{a} = \frac{0.49q^{\frac{2}{3}}}{0.6q^{\frac{2}{3}} + ln(1 + q^{\frac{1}{3}})}, \tag{9}$$

   where, r_1 is the radius of the Roche lobe around m_1.

   Calculate the Roche lobe radius, for a range of separation values. See at what separation the stars would undergo mass transfer.

   Hint: Mass transfer will happen if m_1 *expands and fills its Roche lobe:* $\mathrm{R_G} > \mathrm{R_{Roche}}$.

   Solution: Refer to this jupyter notebook.

Problem 2:

Interpret multi-channel model data: In this problem we will investigate some possible mechanisms of compact binary formation. Overplot posterior probability distributions of the instrinsic parameters from different models and explain the key differences.

1. Use H5PY python package to read the .h5 file.
2. Data for each model can be extracted using the *read_hdf* function from PANDAS python package. Note that you will have to call each model by passing the corresponding key to *read_hdf* function.
3. Extract samples for chirp mass ($\mathcal{M}_{\rm c}$) from every model by keeping natal spin ($\chi_{\rm b}$) and common envelop efficiency ($\alpha_{\rm CE}$) fixed. Overplot them as posterior probability distributions (histograms) and explain the key differences.
4. Repeat the above for other intrinsic parameters like mass ratio (q), effective spin parameter, ($\chi_{\rm eff}$) and redshift (z).
5. For some cases, you may not be able to distinguish clearly between the features for different models, when plotted as a histogram. For such cases do a kernel density estimate using *gaussian_kde* function from SCIPY.STATS.

   Solution: Refer to this jupyter notebook