Term Paper - Lorenz Equations

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Contents

1.0 Introduction

The ***Lorenz Equations*** are given by,

$$\dot{x} = \sigma(y - x) \tag{1}$$
$$\dot{y} = x(\rho - z) - y \tag{2}$$
$$\dot{z} = xy - \beta z \tag{3}$$

Here σ (Prandtl number), ρ (Rayleigh number), $\beta > 0$ are parameters. Lorenz derived this 3-D system from a highly simplified model of convection rolls in the atmosphere. These equations also arise in models of lasers and dynamos.

The interesting feature observed by Lorenz was that the solutions to the equations are irregular and non-repeating, but also bounded. On plotting the trajectory in three dimensions the solution settled onto a set called the strange attractor. A strange attractor is a fractal with fractional dimension between 2 and 3.

2.0 Properties

1. ***Nonlinearity***: The system has only two non-linearities, the quadratic terms xy and xz.
2. ***Symmetry***: There is an important symmetry in the Lorenz equations. If we replace (x, y) with $(-x, -y)$ the equations stay the same. Hence, if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$. In other words, all solutions are either symmetric themselves, or have a symmetric partner.
3. ***Volume Contraction***: The Lorenz system is dissipative i.e. volumes in phase space contract under the flow.
4. ***Fixed Points***: $(x^*, y^*, z^*) = (0, 0, 0)$ is a fixed point for all values of the parameters. For $\rho > 1$ there is also a pair of fixed points $C^{\pm}$ at $x^* = y^* = \pm\sqrt{\beta(\rho - 1)}$, $z^* = \rho - 1$. They merge with the origin as $\rho \to 1^+$ in a pitchfork bifurcation.

2.1 Linear Stability of Origin

Linearisation of the original equations about the origin yields,

$$\dot{x} = \sigma(y - x) \tag{4}$$
$$\dot{y} = x\rho - y \tag{5}$$
$$\dot{z} = -\beta z \tag{6}$$

Thus the z motion decouples and we get,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ \rho & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

with a trace $\tau = -\sigma - 1 < 0$ and determinant $\Delta = \sigma(1-\rho)$. For $\rho > 1$, the origin is a saddle point since $\Delta < 0$. For $\rho < 1$, origin is a sink since $\tau^2 - 4\Delta = (\sigma+1)^2 - 4\sigma(1-\rho) = (\sigma-1)^2 + 4\sigma\tau > 0$ implies a stable node.

Actually, for $\rho < 1$, we can show that every trajectory approaches the origin as $t \to \infty$; the origin is globally stable. Hence there can be no limit cycles or chaos for $\rho < 1$.

2.2 Subcritical Hopf bifurcation

If $\rho > 1$, C^+ and C^- exist. They are linearly stable for,

$$1 < \rho < \rho_H = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1} \tag{8}$$

assuming $\sigma - \beta - 1 > 0$. C^+ and C^- lose stability in a ***Hopf bifurcation*** at $\rho = \rho_H$. But, it is ***subcritical***, the limit cycles are unstable and exist only for $\rho < \rho_H$. I have used $\sigma = 10, \beta = 8/3$ and multiple random initial values for each value of ρ. And, $\rho_H \approx 24.74$.

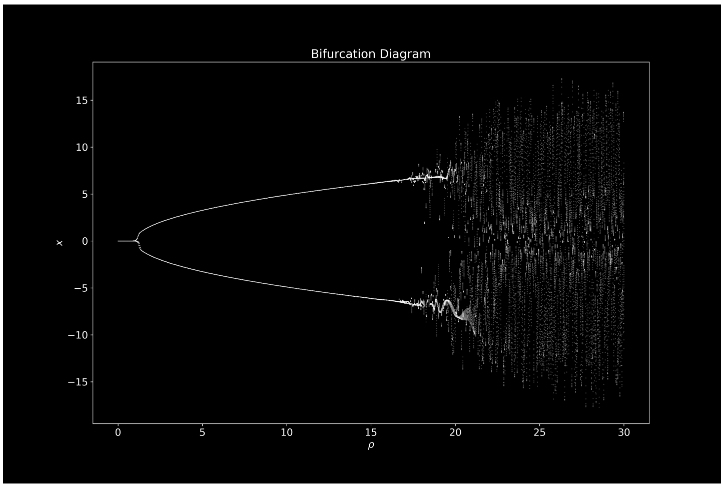

Figure 1: Partial Bifurcation diagram for $t_{max} = 20.0s$ and $0 \leq \rho \leq 30$

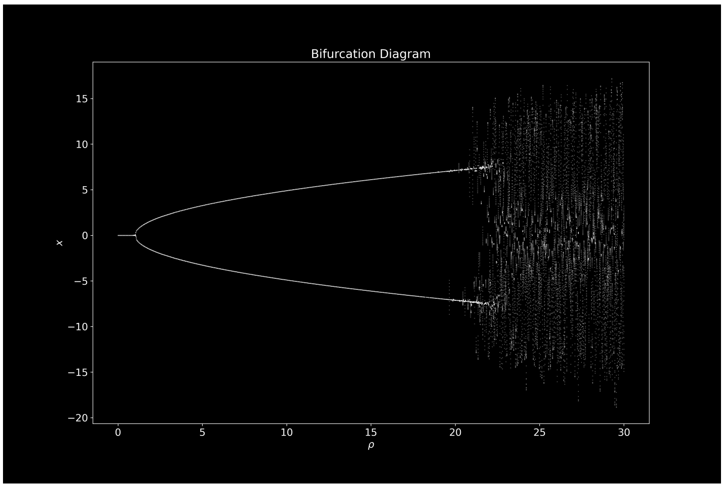

Figure 2: Partial Bifurcation diagram for $t_{max} = 60.0s$ and $0 \leq \rho \leq 30$

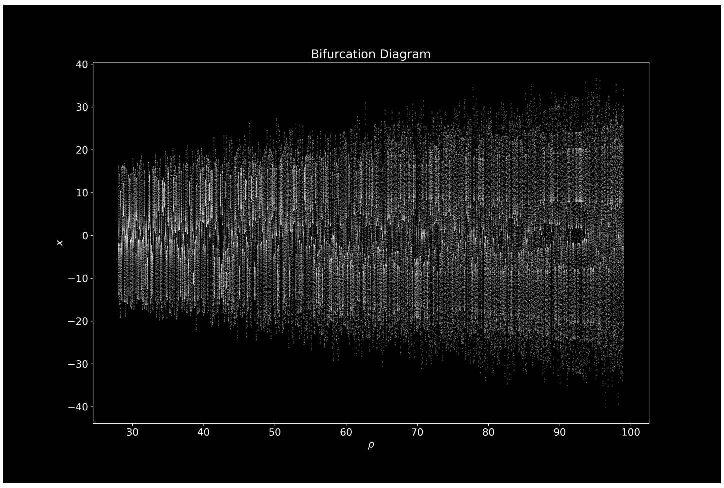

Figure 3: Partial Bifurcation diagram for $t_{max} = 60.0s$ and $28 \leq \rho \leq 99$

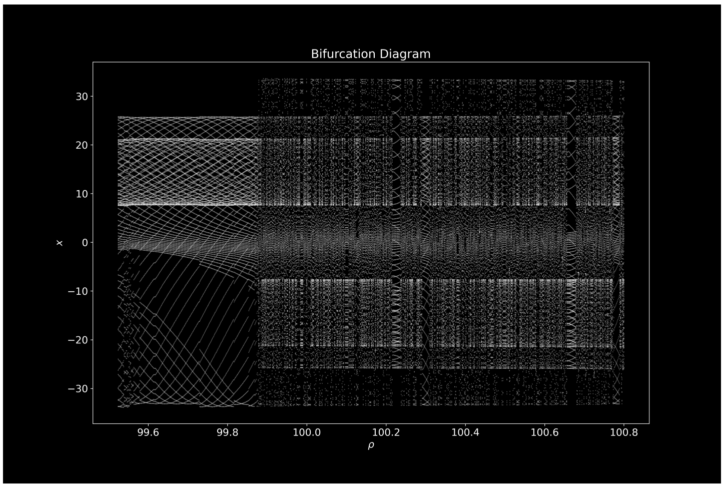

Figure 4: Partial Bifurcation diagram for $t_{max} = 60.0s$ and $99.524 \leq \rho \leq 100.8$

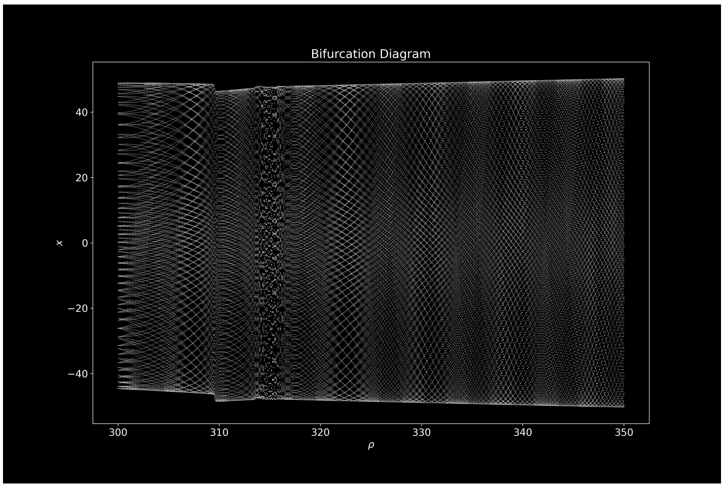

Figure 5: Partial Bifurcation diagram for $t_{max} = 60.0s$ and $300 \leq \rho \leq 350$

I have observed that as the value of ρ increases it takes much longer for convergence to the fixed point. The bifurcation diagrams for $\rho > 28$ ranges were obtained with $\sigma = 10, \beta = 8/3$ and initial value $(-7, 8, 26)$.

At a subcritical Hopf bifurcation trajectories must fly off to a distant attractor. This 2nd attractor must have some strange properties, since any limit cycle for $\rho > \rho_H$ are unstable. The trajectories for $\rho > \rho_H$ are therefore continually being repelled from one unstable object to another. At the same time, they are confined to a bounded set of zero volume, yet manage to move in this set forever without intersecting - ***a strange attractor***!

3.0 Chaos on a Strange Attractor

The parameters $\sigma = 10, \beta = 8/3, \rho = 28$ were used to generate the following time series.

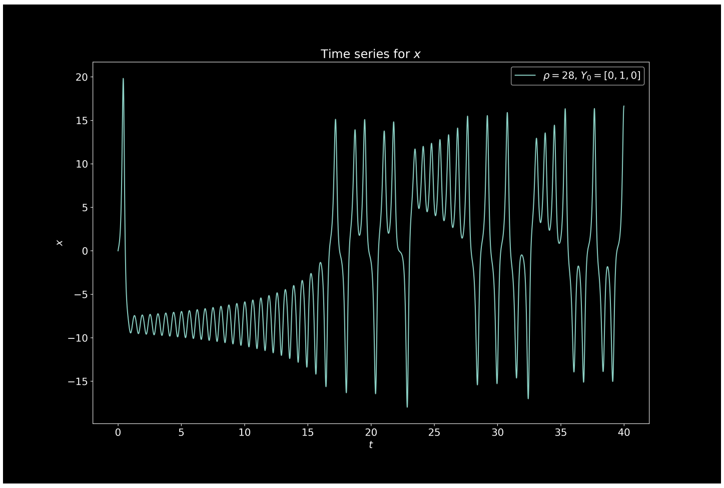

Figure 6: Time series showing chaos in x

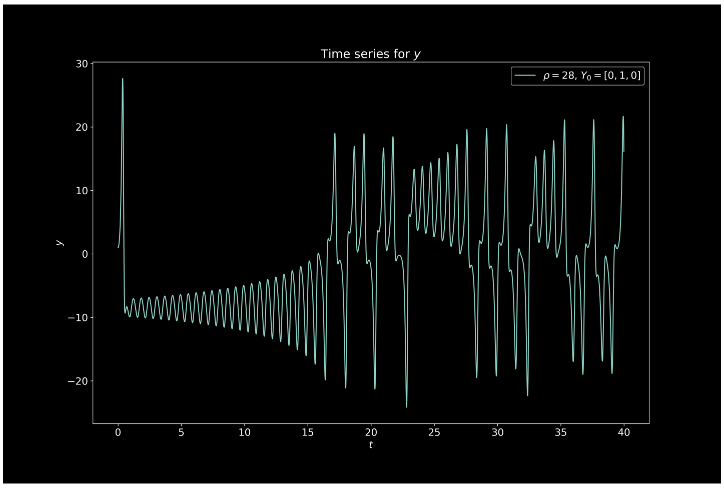

Figure 7: Time series showing chaos in y

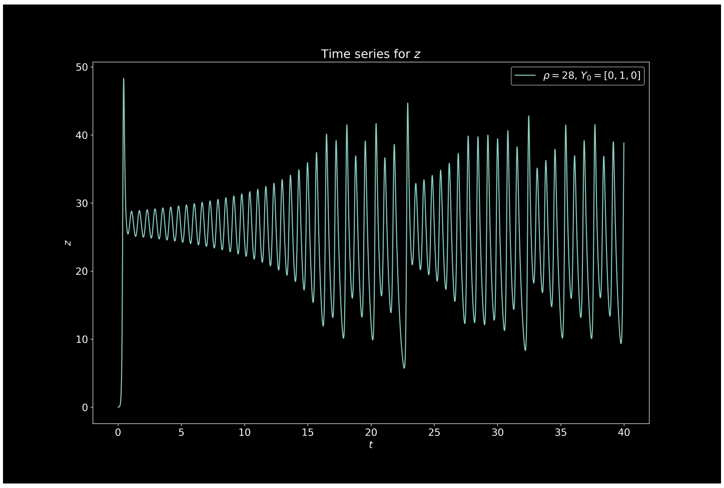

Figure 8: Time series showing chaos in z

After an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$ but never repeats exactly. The motion is ***aperiodic***.

Lorenz discovered that a wonderful structure emerges if the solution is visualized as a trajectory in phase space.

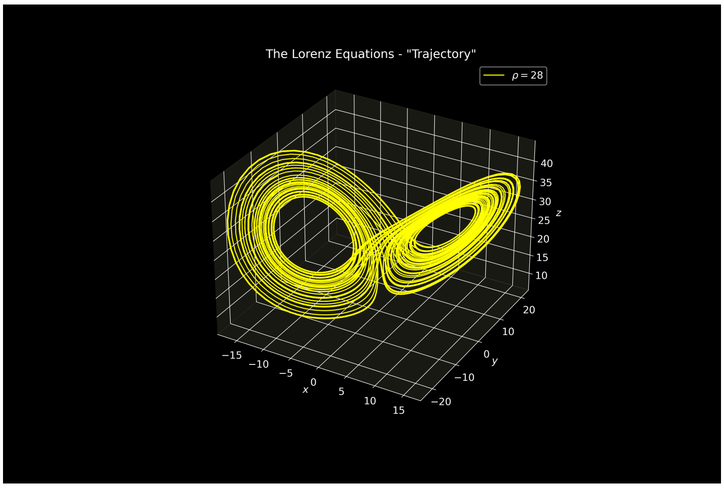

Figure 9: Famous Butterfly Wing Pattern

I have used $\sigma = 10, \beta = 8/3, \rho = 28$ and initial value $(-7, 8, 26)$ to generate Figure 9. The trajectory appears to cross itself repeatedly because I am inserting a 2-D image of the 3-D trajectory. But there is no intersection in 3-D. I have made a simulation that illustrates the non intersecting nature better.

The number of circuits made on either side varies unpredictably from one cycle to the next. The sequence of the number of circuits in each lobe has many of the characteristics of a random sequence! The uniqueness theorem means that trajectories cannot cross or merge, hence the two surfaces of the strange attractor can only appear to merge.

It would seem, then, that the two surfaces merely appear to merge, and remain distinct surfaces. Following these surfaces along a path parallel to a trajectory, and circling C^+ and C^-, we see that each surface is really a pair of surfaces, so that, where they appear to merge, there are really four surfaces. Continuing this process for another circuit, we see that there are really eight surfaces, etc., and we finally conclude that there is an infinite complex of surfaces, each extremely close to one or the other of two merging surfaces.

Today this “infinite complex of surfaces” would be called a fractal. It is a set of points with zero volume but infinite surface area.

4.0 Exponential Divergence of nearby trajectories

The motion on the attractor exhibits sensitive dependence on initial conditions. Two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures. The practical implication is that long-term prediction becomes impossible in a system like this, where small uncertainties are amplified enormously fast. Suppose we let transients decay so that the trajectory is “on” the attractor. Suppose $x(t)$ is a point on the attractor at time t, and $x(t) + \delta(t)$ is a nearby point, where δ is a tiny separation vector r of initial length $||\delta_0|| = 10^{-15}$.

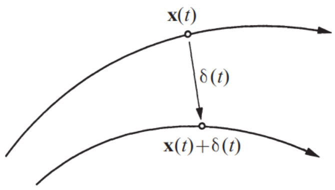

Figure 10: Distance between initial conditions

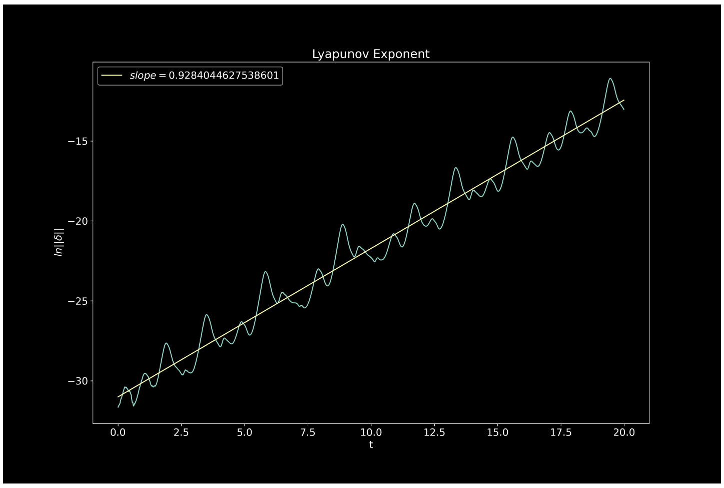

Figure 11: Linear Regression of $ln||\delta||$

In numerical studies of the Lorenz attractor one finds that $||\delta(t)|| = ||\delta_0||e^{\lambda t}$, where $\lambda = 0.93$. Hence neighbouring trajectories separate exponentially fast!

I have made a simulation to demonstrate the behaviour of the trajectories separated by a small separation.

5.0 Lorenz Map

Lorenz wrote that "the trajectory apparently leaves one spiral only after exceeding some critical distance from the centre... It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit." The "single feature" he focused on was z_n, the nth local maximum of $z(t)$.

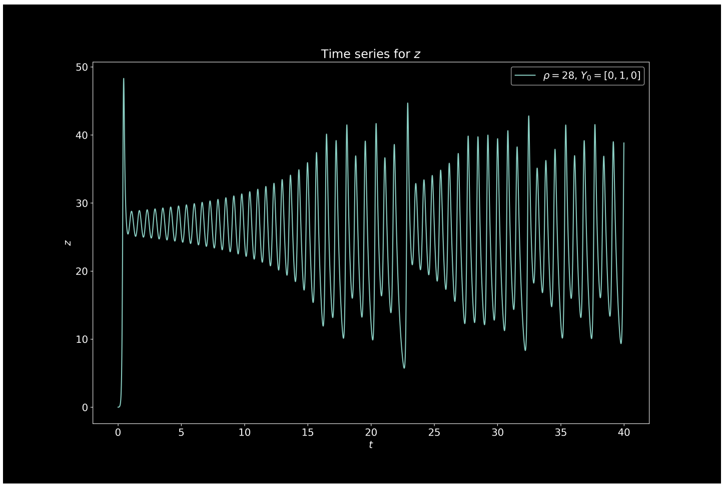

Figure 12: Time series showing chaos in z

Lorenz's idea is that z_n should predict z_{n+1}. He checked this by numerical integration. The plot of z_{n+1} vs z_n looks like....

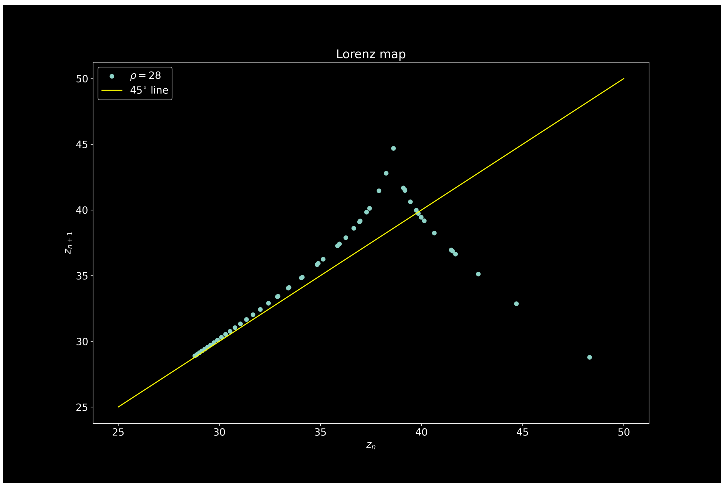

Figure 13: Lorenz Map

The data from the chaotic time series appears to fall neatly on a curve - there is no "thickness" to the graph. Hence Lorenz was able to extract order from chaos! The function $z_{n+1} = f(z_n)$ is now called the Lorenz Map. The graph is not actually a curve - it does have some thickness so, strictly speaking, $f(z)$ is not a well defined function.

6.0 Spiral

If $\rho = 21$, with $\sigma = 10$ and $\beta = 8/3$ the solution spirals towards C^+ or C^-. The initial value used to generate the following figure was $(-7, 8, 26)$.

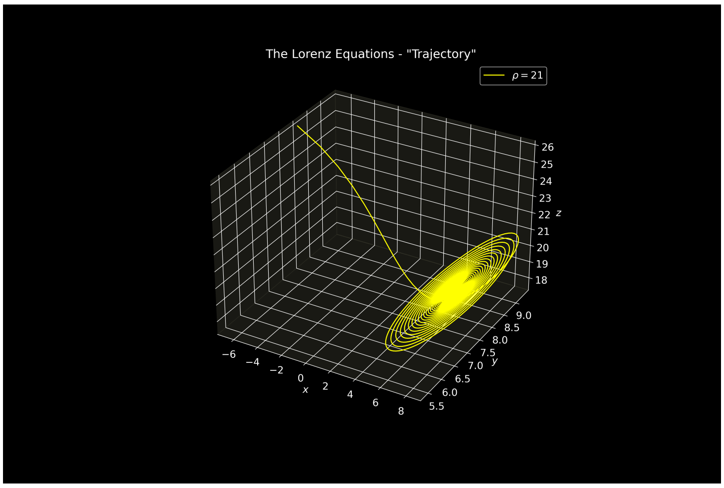

Figure 14: Spiral

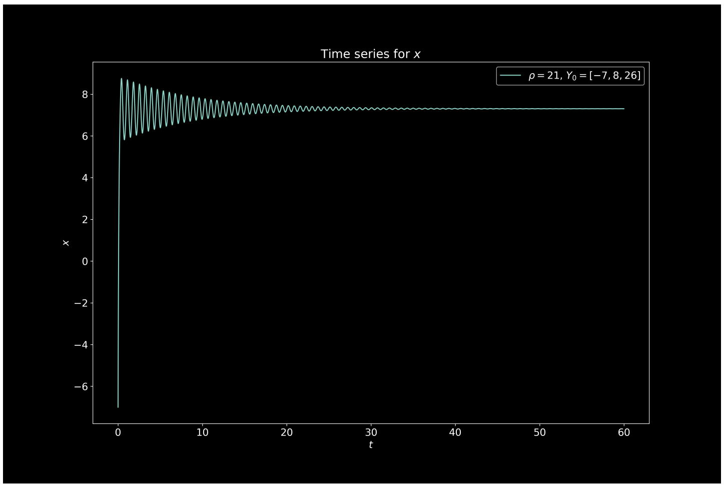

Figure 15: Time Series for spiral in x

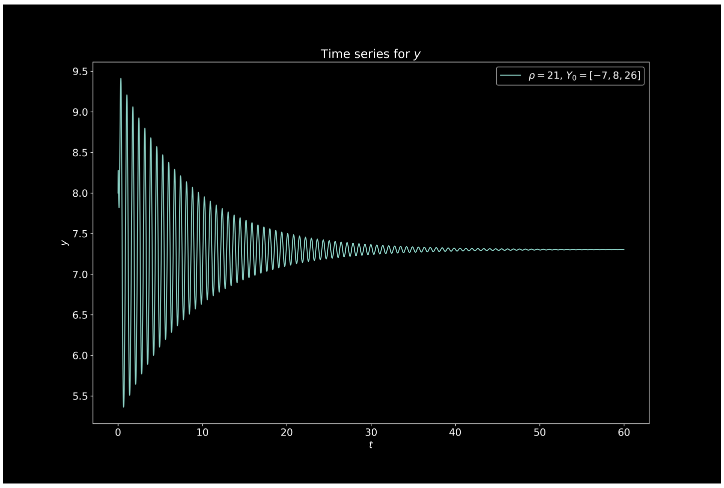

Figure 16: Time Series for spiral in y

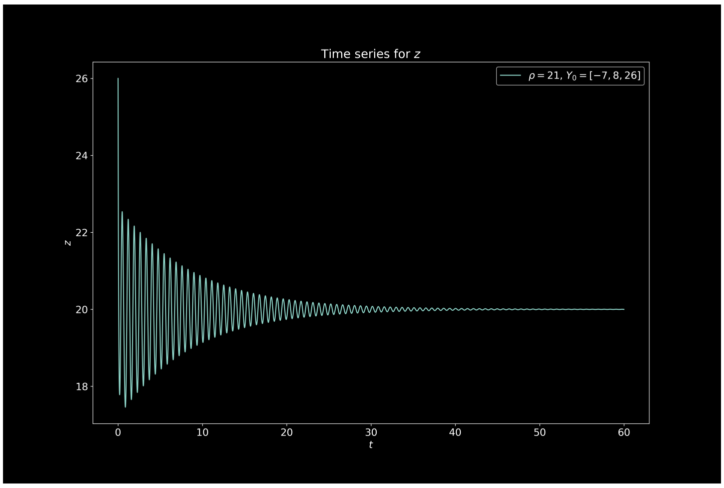

Figure 17: Time Series for spiral in z

The long-term behaviour is not aperiodic - hence the dynamics are not chaotic. However, the dynamics do exhibit sensitive dependence on initial conditions - the trajectory could end up on either C^+ or C^-1 - hence the system's behaviour is unpredictable. Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple. This is familiar from everyday experience - many games of “chance” used in gambling are essentially demonstrations of transient chaos....e.g. a rolling dice...

7.0 Limit Cycles

It turns out that for large ρ the dynamics become simple again - the system has a global attracting limit cycle for $\rho > 313$. The following images were generated using $\rho = 350, \sigma = 10, \beta = 8/3$ and initial value was $(-7, 8, 26)$.

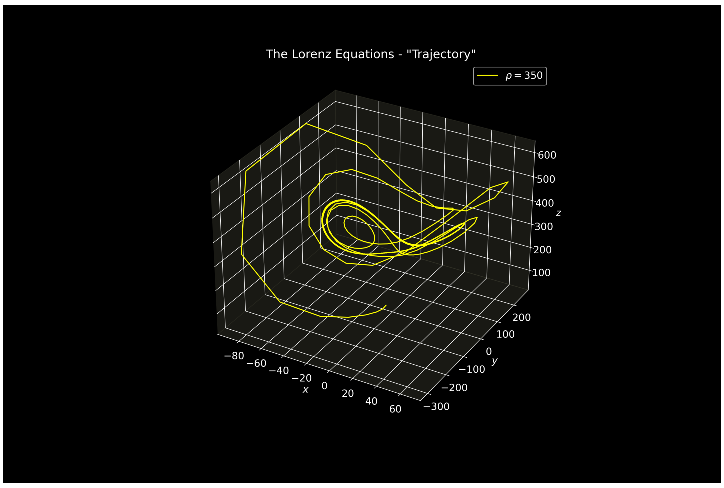

Figure 18: Limit Cycle

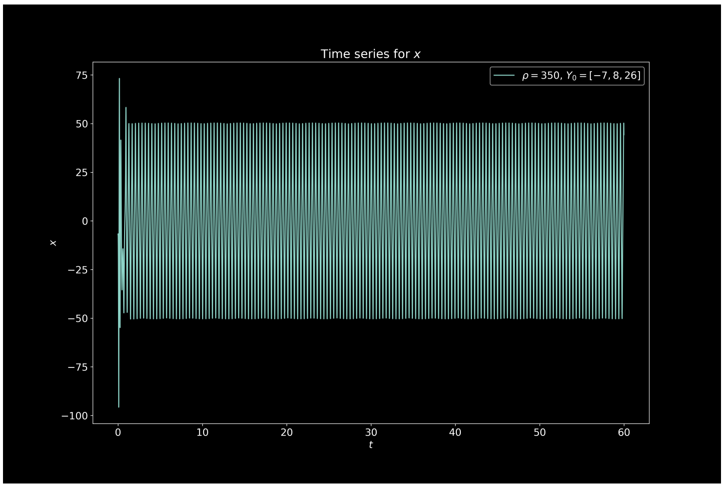

Figure 19: Time Series for limit cycle in x

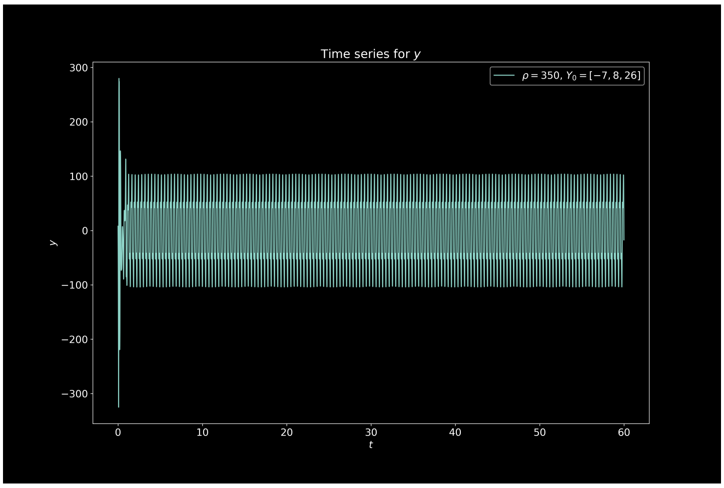

Figure 20: Time Series for limit cycle in y

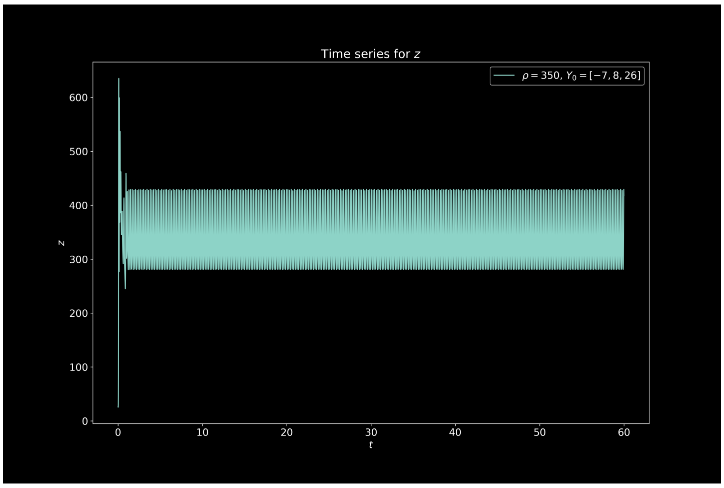

Figure 21: Time Series for limit cycle in z

8.0 Code and Simulation

https://drive.google.com/drive/folders/1IW5GX3S0zAL9rNrXE6uf7QxTCxU2fQgw?usp=sharing

9.0 References

1. Nonlinear Dynamics and Chaos by Steven H. Strogatz
2. https://medium.com/geekculture/visualizing-the-lorenz-chaotic-differential-equations-in-python-329d8ae43f33