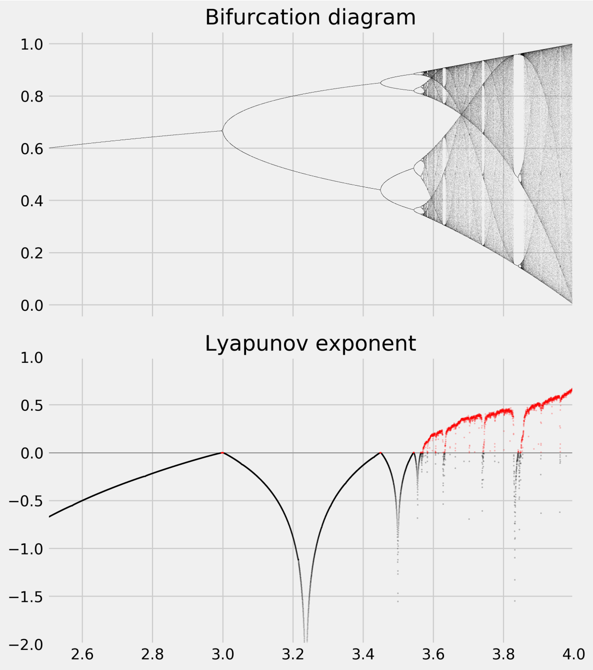
To try out in today's tutorial...

Evaluate Lyapunov Exponents :

$$\lambda \;=\; \lim_{n\to\infty} \{\frac{1}{n} \sum_{i=0}^{n-1} \ln\ |F'(x_i)|\ \}$$

For the Logistic Map:

$$\lambda \;=\; \lim_{n\to\infty} \{\frac{1}{n} \sum_{i=0}^{n-1} \ln\ |\ r\ -\ 2\ r\ x_i\ |\ \}$$

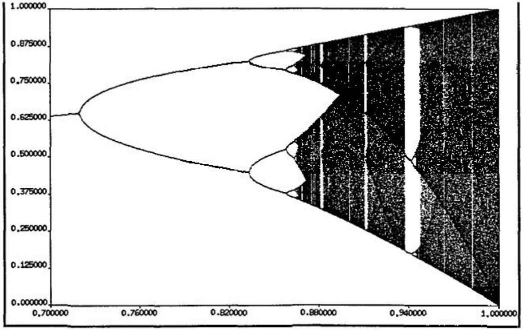


Bifurcation diagram
1.0
0.8
0.6
0.4
0.2
0.0
Lyapunov exponent
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0

Bifurcation Diagram of the Map:

$$x_{n+1} \;=\; r\;\sin(\pi x_n)$$

$x \;\in\; [0:1]$, with $r \leq 1$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0.625000
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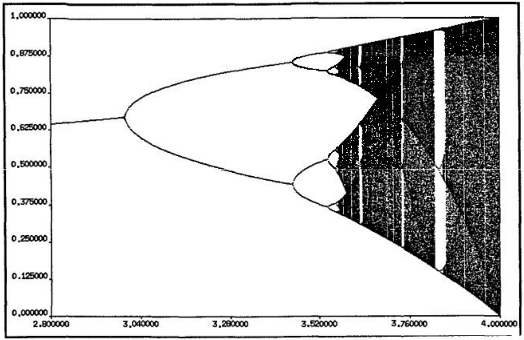
0.820000
0.880000
0.940000
1.000000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0.125000
0.000000
2.800000
3.040000
3.280000
3.520000
3.760000
4.000000

“Renormalization”

Check graphically:

$$f(x, R_0) \;\approx\; \alpha \; f^2(\frac{x}{\alpha}, R_1)$$

For $f(x) = \; r \; - \; x^2$ with $R_0 = 0$ and $R_1 = 1$

Try different α values and see that $\alpha \sim -2.5\ldots$ does best

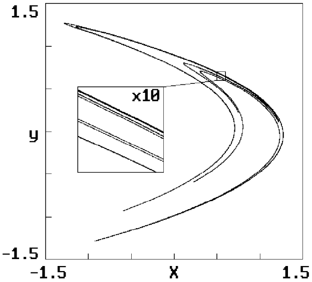
Examine the box $[-1:1]$ as the maximum of this map is at zero

Potential Term Paper Topic

Two-dimensional Invertible Map: Hénon Map

$$x_{n+1} = 1 - a x_n^2 + y_n$$
$$y_{n+1} = b x_n$$

$a = 1.4$ and $b = 0.3$



"CAT MAP" : Two-dimensional Area-preserving Map

$$x_{n+1} = 2x_n + y_n \qquad \mod 1$$
$$y_{n+1} = x_n + y_n \qquad \mod 1$$

To try out as a home assignment

Obtain a histogram of the fraction of time a trajectory of finite length, for the Tent Map and the Logistic Map at $r = 4$, originating from a typical initial point, falls in N bins of equal size along the x axis

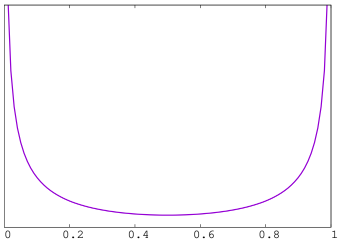
This defines a Natural Invariant Density $\rho(x)$

Fraction of time typical orbits spend in an interval $[a, b]$ is given by $\int_b^a \rho(x)dx$

For the tent map this is Uniform : $\rho(x) = 1$

So the probabilty of finding a point in a bin, when the interval $[0:1]$ is split into N bins, is approximately $\frac{1}{N}$

For the Logistic Map : $\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$



Singularities at $x = 0$ and $x = 1$

Consider $r = 4$ again:

$$x_{n+1} = 4x_n(1 - x_n)$$

Change of variables :

$$x = \sin^2 \frac{\pi y}{2} = \frac{1}{2}[1 - \cos(\pi y)]$$

Substituting :

$$\sin^2(\pi y_{n+1}/2) = 1 - \cos^2(\pi y_n) = \sin^2(\pi y_n)$$

This yields:

$$\frac{\pi y_{n+1}}{2} = \pm\pi y_n + s\pi$$

where s is an integer

Now y has to lie in $[0, 1]$ and this determines the choice of s and sign of (πy_n)

Thus: $y_{n+1} = 2y_n$ for $0 \leq y_n \leq \frac{1}{2}$ ($+$sign and $s = 0$)

$y_{n+1} = 2 - 2y_n$ for $\frac{1}{2} \leq y_n \leq 1$ (-sign and $s = 1$)

Namely its the TENT MAP

One can obtain the invariant density $\rho(x)$ of Logistic Map using the fact that the frequency of visiting an interval $[y, y + dy]$ is the same as visiting the corresponding transformed interval in x : $[x, x + dx]$

So

$$\rho(x) = |\frac{dy(x)}{dx}|\rho(y(x))$$

Since $\rho(y) = 1$ for the tent map :

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$