CS 325 - Analysis of Algorithms

Prof. Julianne Schutford

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Homework 5

NP-Completeness

by

Abdullah Saydemir

saydemia@oregonstate.edu

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Q1)

a) This is not always true. X can be in P or NP.
b) This is not always true. Y can be NP-hard.
c) This is not always true. X can be P or NP.
d) This is correct. This follows from the definition of NP-completeness.
e) This is not always true. Y can be in NP or NP-complete.
f) This is correct. If X is no harder than Y we can also solve X in polynomial time.
g) False. X can be NP and Y can be NP-Complete. X reduces to Y in polynomial time.

Q2)

a) 3-SAT $\leq_P$ TSP

   Both 3-SAT and TSP are in NP-Complete. Since every NP-Complete problem reduces to every other NP-Complete problem in polynomial time this statement is correct.

b) If P $\neq$ NP, then 3-SAT $\leq_P$ 2-SAT

   This is not correct because 2-SAT can be solved in polynomial time which means it is in P. 3-SAT is known to be NP-Complete. If we are able to reduce 3-SAT to 2-SAT that means either P = NP or 3-SAT is in P, both contradicts the statement.

c) If TSP $\leq_P$ 2-SAT, then P = NP

   This is true. TSP is in NP-complete and 2-SAT is in P. If TSP $\leq_P$ 2-SAT this means P = NP-Complete = NP

Q3)

1) Show that HAM-PATH ∈ NP

Given a graph G(V,E) and a certificate path, we can traverse the path and make sure that each vertex is visited exactly once. This can be done in O(V+E) time.

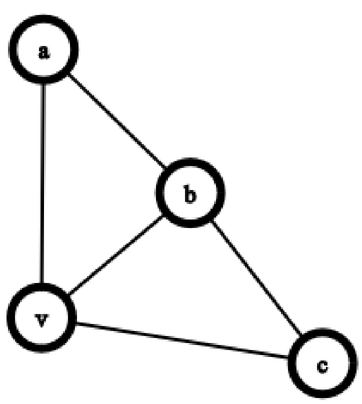
2) Show that HAM-CYCLE $\leq_P$ HAM-PATH

a) Show a polynomial algorithm to transform an arbitrary instance G of HAM-CYCLE into an instance of G' of HAM-PATH.

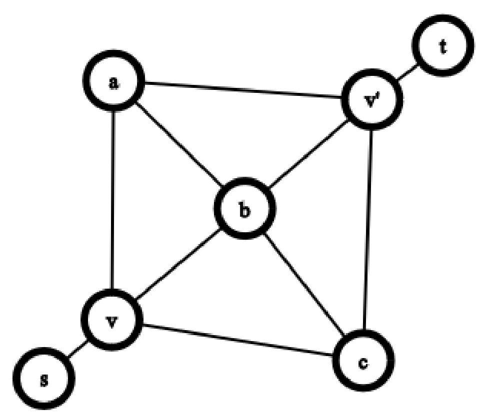
Let G be a graph that has HAM-CYCLE. Choose an arbitrary vertex v and create graph G' that has HAM-PATH by adding three other vertices s, t, and v' such that:

- v' is adjacent to all vertices that v is adjacent
- s is adjacent to v
- t is adjacent to v'

Example



Graph G



Graph G'

b) Show that (HAM-CYCLE(G) = **YES**) ⇔ (HAM-PATH(G') = **YES**)

i) (HAM-CYCLE(G) = **YES**) ⇒ (HAM-PATH(G') = **YES**)

Assume that G has HAM-CYCLE. We can list the vertices that form HAM-CYCLE starting and ending at v as

$$\{ v, a_1, a_2, \dots, a_{n-1}, v \}$$

Then, G' has the following HAM-PATH,

$$\{ s, v, a_1, a_2, \dots, a_{n-1}, v', t \}$$

ii) (HAM-CYCLE(G) = **YES**) ⇐ (HAM-PATH(G') = **YES**)

If G' has HAM-PATH, starting and ending vertices must be s and t since they are degree 1 vertices. We can list the vertices in the HAM-PATH as

$$\{ s, v, b_1, b_2, \dots, b_{n-1}, v', t \} \quad \textit{(or reverse)}$$

Then, G has the following HAM-CYCLE

$\{ v, b_1, b_2, \dots, b_{n-1}, v \}$

Conclusion[1]

We have proven that HAM-PATH ∈ NP and HAM-CYCLE $\leq_P$ HAM-PATH. Then, HAM-PATH ∈ NP-Complete

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[1] I benefited from two different proofs on the internet. Still, I used the provided proof template on Canvas and expressed the proof in my own words.

1) Answer from @Rotenberg on https://math.stackexchange.com/questions/7130/reduction-from-hamiltonian-cycle-to-hamiltonian-path
2) An answer sheet to a Quiz on https://www.usna.edu/Users/cs/wcbrown/courses/S04SI333/quizzes/Q03/Solution.html

I drew the graph myself using https://csacademy.com/app/graph_editor/

Q4)

1) Show that 4-COLOR ∈ NP

   Given a graph G(V,E) and corresponding color assignments, for each vertex v ∈ G, check color assignments of all adjacent vertices and make sure that all vertices have different colors. While going through the graph take note of the different colors used and make sure that only 4 unique colors are used. With an adjacency matrix this can be done in O(V+E)

2) Show that 3-COLOR $\leq_P$ 4-COLOR

   a) Show a polynomial algorithm to transform an arbitrary instance G of 3-COLOR into an instance of G' of 4-COLOR.

      Let G be a 3-colorable graph. Then, create the 4-colorable G' by adding a vertex *v* that is connected to every other vertex in the original graph G. This can be done in O(V+1) time.

   b) Show that (3-COLOR(G) = YES) ⇔ (4-COLOR(G') = YES)

      i) (3-COLOR(G) = YES) ⇒ (4-COLOR(G') = YES)
      Assume that G is 3-colorable. Then, G' is 4-colorable since the new vertex can be colored by a 4th color and it is adjacent to all other vertices that are colored by 3 other colors.

      ii) (3-COLOR(G) = YES) ⇐ (4-COLOR(G') = YES)
      If G' is 4-colorable, the new vertex must be colored by a unique color since it is adjacent to all other vertices in the graph. That means all other vertices are colored by one of the other three colors. Therefore, G is 3-colorable.

Conclusion

We have proven that 4-COLOR $\in$ NP and 3-COLOR $\leq_P$ 4-COLOR. It follows from that 4-COLOR $\in$ NP-Complete