1. $x+2y+3z=3$
$2x+3y+8z=4$
$5x+8y+19z=11$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 5 & 8 & 19 & 11 \end{array}\right] \quad R_3'=(-5)R_1+R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 0 & -2 & 4 & -4 \end{array}\right] \quad R_3'=(-5)R_1+R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & -2 & 4 & -4 \end{array}\right] \quad R_2'=(-2)R_1+R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3'=(-2)R_2+R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = (-1)R_2$$

$$\therefore \; x + 2y + 3z = 3$$
$$y - 2z = 2$$

let, $y = 2 + 2z$

$$\therefore \; x = 3 - 2y - 3z$$
$$= 3 - 2(2 + 2z) - 3z$$
$$= 3 - 4 - 4z - 3z$$
$$= -1 - 7z$$

$$\therefore \; x = -1 - 7z$$
$$y = 2 + 2z$$

let, $z = t$

$$\therefore \; x = -1 - 7t$$
$$y = 2 + 2t$$

set, $t=0$

$\therefore x=-1$

$y=2$

$z=0$

set, $t=1$

$\therefore x=-1-7(1)$

$=-1-7$

$=-8$

$y=2+2(1)$

$=2+2$

$=4$

$\therefore x=-8$

$y=4$

$z=1$

2. $x_1 - x_2 + x_3 - x_4 + x_5 = 1$

$2x_1 - x_2 + 3x_3 + 4x_5 = 2$

$3x_1 - 2x_2 + 2x_3 + x_4 + x_5 = 1$

$x_1 + x_3 + 2x_4 + x_5 = 0$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 3 & 0 & 4 & 2 \\ 3 & -2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 3 & 0 & 4 & 2 \\ 3 & -2 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right] \quad R_4' = (-1)R_1 + R_4$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 4 & 2 & -2 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right]$$

$R_2' = (-2)R_1 + R_2$

$R_3'' = (-3)R_1 + R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_3' = (-1)R_2 + R_3$

$R_4' = (-1)R_2 + R_4$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_3' = -\frac{1}{2}R_3$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right] \quad R_4' = R_3 + R_4$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad R_4' = -\frac{1}{2} R_4$$

$$x_1 - x_2 + x_3 - x_4 + x_5 = 1$$

$$x_2 + x_3 + 2x_4 + 2x_5 = 0$$

$$x_3 - x_4 = 1$$

$$x_5 = 0$$

$\therefore x_3 = 1 + x_4$

$\therefore x_2 = -x_3 - 2x_4 - 2x_5$

$= -(1+x_4) - 2x_4 - 2(0)$

$= -1 - x_4 - 2x_4$

$= -1 - 3x_4$

$\therefore x_1 = 1 + x_2 - x_3 + x_4 - x_5$

$= 1 + (-1-3x_4) - (1+x_4) + x_4 - 0$

$= 1 - 1 - 3x_4 - 1 - x_4 + x_4$

$= -3x_4 - 1$

$\therefore x_1 = x_2 = -1 - 3x_4$

$x_3 = 1 + x_4$

$x_5 = 0$

set, $x_4 = t$

$\therefore x_1 = x_2 = -1-3t$

$x_3 = 1+t$

set, $t = 0$

$\therefore x_1 = x_2 = -1$

$x_3 = 1$

$\therefore x_1 = -1, x_2 = -1, x_3 = 1, x_4 = 0, x_5 = 0$

set, $t = 1$

$\therefore x_1 = x_2 = -1-3(1)$

$= -1-3 = -4$

$x_3 = 1+1 = 2$

$\therefore x_1 = -4, x_2 = -4, x_3 = 2, x_4 = 1, x_5 = 0$

3.

$$x+2y-3z=4$$
$$x+3y+z=11$$
$$2x+5y-4z=13$$
$$2x+6y+2z=22$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 1 & 3 & 1 & 11 \\ 2 & 5 & -4 & 13 \\ 2 & 6 & 2 & 22 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 1 & 3 & 1 & 11 \\ 1 & 2 & -3 & 4 \\ 2 & 6 & 2 & 22 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & \frac{1}{2} & 3 & \frac{9}{2} \\ 1 & 2 & -3 & 4 \\ 2 & 6 & 2 & 22 \end{array}\right] \quad R_2' = R_2 - \frac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & \frac{1}{2} & 3 & \frac{9}{2} \\ 0 & -\frac{1}{2} & -1 & -\frac{5}{2} \\ 2 & 6 & 2 & 22 \end{array}\right] \quad R_3' = R_3 - \frac{1}{2}R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & \frac{1}{2} & 3 & \frac{9}{2} \\ 0 & -\frac{1}{2} & -1 & -\frac{5}{2} \\ 0 & 1 & 6 & 9 \end{array}\right] \quad R_4' = R_4 - R_1$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & 1 & 6 & 9 \\ 0 & -\frac{1}{2} & -1 & -\frac{5}{2} \\ 0 & \frac{1}{2} & 3 & \frac{9}{2} \end{array}\right] \quad R_2 \leftrightarrow R_4$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 2 & 2 \\ 0 & \frac{1}{2} & 3 & \frac{9}{2} \end{array}\right] \quad R_3' = R_3 + \frac{1}{2}R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_4' = R_4 - \frac{1}{2}R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = \frac{1}{2}R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & -4 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = R_2 - 6R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 5 & 0 & 17 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = R_1 + 4R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = R_1 - 5R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = \frac{1}{2} R_1$$

$\therefore$ $X = 1$

$Y = 3$

$Z = 1$

4. $10x_2 - 4x_3 + x_4 = 1$

$x_1 + 4x_2 - x_3 + x_4 = 2$

$3x_1 + 2x_2 + x_3 + 2x_4 = 5$

$-3x_1 - 8x_2 + 2x_3 - 2x_4 = -4$

$x_1 - 6x_2 + 3x_3 = 1$

$$\left[\begin{array}{cccc|c} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ -3 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ -3 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 0 & \frac{10}{3} & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 10 & -4 & 1 & 1 \\ 0 & -6 & 3 & 0 & 1 \\ 0 & -\frac{20}{3} & \frac{8}{3} & -\frac{2}{3} & -\frac{2}{3} \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - \frac{1}{3}R_1 \\ R_4' = R_4 + R_1 \\ R_5' = R_5 - \frac{1}{3}R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & \frac{10}{3} & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -6 & 3 & 0 & 1 \\ 0 & -\frac{20}{3} & \frac{8}{3} & -\frac{2}{3} & -\frac{2}{3} \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3' = R_3 - \frac{1}{3}R_2 \\ R_4' = R_4 + \frac{3}{5}R_2 \\ R_5' = R_5 + \frac{2}{3}R_2 \end{array}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & \frac{3}{5} & \frac{3}{5} & \frac{8}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 1 & 2 & 5 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = \frac{5}{3} R_3$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & \frac{7}{3} \\ 0 & 10 & 0 & 5 & \frac{35}{3} \\ 0 & 0 & 1 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + 4R_3 \\ R_1' = R_1 - R_3 \end{array}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 0 & 1 & \frac{7}{3} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2' = \frac{1}{10} \cdot R_2$$

$$= \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = R_1 - 2R_2$$

$\cancel{0}\ \cancel{0}$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 1 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1' = \frac{1}{3} R_1$$

$$\therefore\ x_1 = 0$$

$$x_2 + \frac{1}{2} x_4 = \frac{7}{6}$$

$$x_3 + x_4 = \frac{8}{3}$$

$$\therefore x_1 = 0$$

$$x_2 = \frac{7}{6} - \frac{1}{2}x_4$$

$$x_3 = \frac{8}{3} - x_4$$

now set $x_4 = r$

$$\therefore x_1 = 0$$

$$x_2 = \frac{7}{6} - \frac{1}{2}r$$

$$x_3 = \frac{8}{3} - r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{7}{6} - \frac{1}{2}r \\ \frac{8}{3} - r \\ r \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{7}{6} \\ \frac{8}{3} \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ \cancel{0}1 \end{bmatrix}$$

5.

Since, R is in reduced row echelon form,

so, $d = [4\ 0\ 0]^T$

The other two vectors provide special solution for R, showing that R has rank 1: again, since it is in reduced row echelon form, the bottom two rows must be all 0, and the top row is $[1\ {-2}\ {-5}]^T$

$$\therefore R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. $-2x+y+z=a$

$x-2y+z=b$

$x+y-3z=c$

$$\therefore \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -3 & c \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{2b+a}{2} \\ 0 & \frac{3}{2} & -\frac{5}{2} & \frac{2c+a}{2} \end{array}\right] \begin{array}{l} \\ R_2' = R_2 + \frac{1}{2}R_1 \\ R_3' = R_3 + \frac{1}{2}R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{2b+a}{2} \\ 0 & 0 & -1 & a+b+c \end{array}\right] \begin{array}{l} \\ \\ R_3' = R_3 + R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -\frac{3}{2} & 0 & \frac{4a+5b+3c}{2} \\ 0 & 0 & -1 & a+b+c \end{array}\right] \quad R_2' = R_2 + \frac{3}{2} R_3$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -\frac{3}{2} & 0 & \frac{4a+5b+3c}{2} \\ 0 & 0 & 1 & -a-b-c \end{array}\right] \quad R_3' = (-1) R_3$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 0 & 2a+b+c \\ 0 & -\frac{3}{2} & 0 & \frac{4a+5b+3c}{2} \\ 0 & 0 & 1 & -a-b-c \end{array}\right] \quad R_1' = R_1 - R_3$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 0 & 2a+b+c \\ 0 & 1 & 0 & -\frac{4a+5b+3c}{3} \\ 0 & 0 & 1 & -a-b-c \end{array}\right] \quad R_2' = -\frac{2}{3} R_2$$

$$=\left[\begin{array}{ccc|c} -2 & 0 & 0 & \frac{10a+8b+6c}{3} \\ 0 & 1 & 0 & -\frac{4a+5b+3c}{3} \\ 0 & 0 & 1 & -a-b-c \end{array}\right] \quad R_1' = R_1 - R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{5a+4b+3c}{3} \\ 0 & 1 & 0 & -\frac{4a+5b+3c}{3} \\ 0 & 0 & 1 & -a-b-c \end{array}\right] \quad R_1' = -\frac{1}{2}R_1$$

$\therefore$

$$X = -\frac{5a+4b+3c}{3}$$

$$Y = -\frac{4a+5b+3c}{3}$$

$$Z = -a-b-c$$

are the condition for which given equations has a solution.

7. $x+2y-3z=a$

$2x+6y-11z=b$

$2x-4y+14z=2c$

$$\therefore \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 2 & -4 & 14 & 2c \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -11 & b \\ 1 & 2 & -3 & a \\ 2 & -4 & 14 & 2c \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -11 & b \\ 0 & -1 & \frac{5}{2} & \frac{2a-b}{2} \\ 0 & -10 & 25 & 2c-b \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - \frac{1}{2}R_1 \\ R_3' = R_2 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -11 & b \\ 0 & -10 & 25 & 2c-b \\ 0 & -1 & \frac{5}{2} & \frac{2a-b}{2} \end{array}\right] R_2 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 2 & 6 & -11 & b \\ 0 & -10 & 25 & 2c-b \\ 0 & 0 & 0 & \frac{-2b+5a-c}{5} \end{array}\right] R_3' = R_3 - \frac{1}{10} R_2$$

here, last row state that

$$0x + 0y + 0z = \frac{-2b+5a-c}{5}$$

0 cannot equal $\frac{-2b+5a-c}{5}$. So this system of linear equations has no solution.

9. $A = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 0 & 8 \\ 1 & 2 & 3 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_1' = R_1 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & -3 & 8 & -1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2' = R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & -3 & 8 & -1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 0 & -2 \end{array}\right] \quad R_3' = R_1 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & -3 & 8 & -1 & 1 & 1 \\ 0 & 0 & -1 & 2 & 1 & -5 \end{array}\right] \quad R_3' = 3R_3 + R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & -3 & 0 & 15 & 9 & -39 \\ 0 & 0 & -1 & 2 & 1 & -5 \end{array}\right] \quad R_2' = 8R_3 + R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 16 & 9 & -40 \\ 0 & -3 & 0 & 15 & 9 & -39 \\ 0 & 0 & -1 & 2 & 1 & -5 \end{array}\right] \quad R_1' = R_1 + R_2$$

$$[I|A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 16 & 9 & -40 \\ 0 & 1 & 0 & -5 & -3 & 13 \\ 0 & 0 & 1 & -2 & -1 & 5 \end{array}\right] \quad \begin{array}{l} R_2' = -\frac{1}{3}R_2 \\ R_3' = (-1)R_3 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 16 & 9 & -40 \\ -5 & -3 & 13 \\ -2 & -1 & 5 \end{bmatrix}$$

10. $B = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & -4 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} \text{Adj}(B)$; $|B| \neq 0$

$|B| = 3(0-3) - 4(-4-6) - 1(1-0)$

$= -9 + 40 - 1$

$= 30$

$|B| \neq 0$; so inverse of B exist.

$$\text{Adj}(B) = \begin{bmatrix} \begin{vmatrix} 0 & 3 \\ 1 & -4 \end{vmatrix} & -1\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ -1\begin{vmatrix} 4 & -1 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} & -1\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (0-3) & -(-4-6) & (1-0) \\ (-16+1) & (-12+2) & -(3-8) \\ (12-0) & -(9+1) & (0-4) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 10 & 1 \\ 15 & -10 & 5 \\ 12 & -10 & -4 \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 15 & 12 \\ 10 & -10 & -10 \\ 1 & 5 & -4 \end{bmatrix}$$

$$B^{-1} = \frac{1}{30}\begin{bmatrix} -3 & 15 & 12 \\ 10 & -10 & -10 \\ 1 & 5 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{30} & \frac{1}{6} & -\frac{2}{15} \end{bmatrix}$$

12. $x+y+z=6$

$2x+3y+4z=20$

$3x-2y+z=2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 3 & -2 & 1 & 2 \end{array}\right]$$

first find the inverse of main matrix.

$\det = 1(3+8)-1(2-12)+1(-4-9)$

$= 11+10-13$

$= 8$

So, inverse exist.

Now find the Adj.

$$
\begin{bmatrix}
\begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \\
-\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \\
\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(3+8) & -(2-12) & (-4-9) \\
-(1+2) & (1-3) & -(-2-3) \\
(4-3) & -(4-2) & (3-2)
\end{bmatrix}
$$

$$
= \begin{bmatrix}
11 & 10 & -13 \\
-3 & -2 & 5 \\
1 & -2 & 1
\end{bmatrix}^T
$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ 1 & -2 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

$$\therefore \text{inverse} = \frac{1}{8}\begin{bmatrix} 11 & -3 & 1 \\ 10 & -2 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{8} & -\frac{3}{8} & \frac{1}{8} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{13}{8} & \frac{5}{8} & \frac{1}{8} \end{bmatrix}$$

Now multiply the inverse matrix by the solution vector.

$$\begin{bmatrix} \frac{11}{8} & -\frac{3}{8} & \frac{1}{8} \\ \frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{13}{8} & \frac{5}{8} & \frac{1}{8} \end{bmatrix} \times \begin{bmatrix} 6 \\ 20 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{8}\times 6 - \frac{3}{8}\times 20 + \frac{1}{8}\times 2 \\ \frac{5}{4}\times 6 - \frac{1}{4}\times 20 - \frac{1}{4}\times 2 \\ -\frac{13}{8}\times 6 + \frac{5}{8}\times 20 + \frac{1}{8}\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ $X = 1$

$Y = 2$

$Z = 3$

13. $x+y+z-2t=-4$

$x-2y+3z+4t=10$

$2x+3y-z+2t=9$

$4x-y+2z-t=-7$

Lets,

$$A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & -2 & 3 & 4 \\ 2 & 3 & -1 & 2 \\ 4 & -1 & 2 & -1 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_4$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & -\frac{7}{4} & \frac{5}{2} & \frac{17}{4} & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & \frac{7}{2} & -2 & \frac{5}{2} & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & \frac{5}{4} & \frac{1}{2} & -\frac{7}{4} & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right] \begin{array}{l} R_2' = R_2 - \frac{1}{4}R_1 \\ R_3' = R_3 - \frac{1}{2}R_1 \\ R_4' = R_4 - \frac{1}{4}R_1 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & \frac{7}{2} & -2 & \frac{5}{2} & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{7}{4} & \frac{5}{2} & \frac{17}{4} & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & \frac{5}{4} & \frac{1}{2} & -\frac{7}{4} & 1 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & \frac{7}{2} & -2 & \frac{5}{2} & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} & \frac{11}{2} & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{17}{14} & -\frac{37}{14} & 1 & 0 & -\frac{5}{14} & \frac{7}{14} \end{array}\right] \begin{array}{l} \\ R_3' = R_3 + \frac{1}{2}R_2 \\ R_4' = R_4 - \frac{5}{14}R_2 \\ \\ \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & \frac{7}{2} & -2 & \frac{5}{2} & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} & \frac{11}{2} & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{149}{21} & 1 & -\frac{17}{21} & -\frac{16}{21} & \frac{1}{3} \end{array}\right] \begin{array}{l} \\ R_4' = R_4 - \frac{17}{21}R_3 \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & \frac{7}{2} & -2 & \frac{5}{2} & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & \frac{11}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] \quad R_4' = -\frac{21}{149} R_4$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & 0 & -\frac{21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{142}{149} \\ 0 & \frac{7}{2} & -2 & 0 & \frac{105}{298} & \frac{-85}{298} & \frac{109}{149} & \frac{-57}{149} \\ 0 & 0 & \frac{3}{2} & 0 & \frac{231}{298} & \frac{111}{298} & \frac{-27}{298} & \frac{-36}{298} \\ 0 & 0 & 0 & 1 & \frac{-21}{298} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] \quad \begin{array}{l} R_1' = R_1 + R_4 \\ R_2' = R_2 - \frac{5}{2} R_4 \\ R_3' = R_3 - \frac{11}{2} R_4 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 2 & 0 & -\frac{21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{142}{149} \\ 0 & \frac{7}{2} & -2 & 0 & \frac{105}{298} & \frac{-85}{298} & \frac{109}{149} & \frac{-57}{149} \\ 0 & 0 & 1 & 0 & \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{-29}{149} \\ 0 & 0 & 0 & 1 & -\frac{21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] \quad R_3' = \frac{2}{3} R_3$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 0 & 0 & \frac{-175}{149} & \frac{-57}{149} & \frac{34}{149} & \frac{190}{149} \\ 0 & \frac{7}{2} & 0 & 0 & \frac{413}{298} & \frac{63}{298} & \frac{91}{149} & \frac{-105}{149} \\ 0 & 0 & 1 & 0 & \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{-24}{149} \\ 0 & 0 & 0 & 1 & \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] \begin{array}{l} R_1' = R_1 - 2R_3 \\ R_2' = R_2 + 2R_3 \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & -1 & 0 & 0 & \frac{-175}{149} & \frac{-57}{149} & \frac{34}{149} & \frac{190}{149} \\ 0 & 1 & 0 & 0 & \frac{59}{149} & \frac{9}{149} & \frac{26}{149} & \frac{-30}{149} \\ 0 & 0 & 1 & 0 & \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{-24}{149} \\ 0 & 0 & 0 & 1 & \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] \begin{array}{l} \\ R_2' = \frac{2}{7}R_2 \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & -\frac{116}{149} & -\frac{48}{149} & \frac{60}{149} & \frac{160}{149} \\ 0 & 1 & 0 & 0 & \frac{59}{149} & \frac{9}{149} & \frac{26}{149} & \frac{-30}{149} \\ 0 & 0 & 1 & 0 & \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{24}{149} \\ 0 & 0 & 0 & 1 & \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] R_1' = R_1 + R_2$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{29}{149} & \frac{-12}{149} & \frac{15}{149} & \frac{40}{149} \\ 0 & 1 & 0 & 0 & \frac{59}{149} & \frac{9}{149} & \frac{26}{149} & \frac{-30}{149} \\ 0 & 0 & 1 & 0 & \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{24}{149} \\ 0 & 0 & 0 & 1 & \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{array}\right] R_1' = \frac{1}{4} R_1$$

$$= \left[I \mid A^{-1} \right]$$

Now multiply the A^{-1} with result vector.

$$\begin{bmatrix} -\frac{29}{149} & -\frac{12}{149} & \frac{15}{149} & \frac{40}{149} \\ \frac{59}{149} & \frac{9}{149} & \frac{26}{149} & \frac{-30}{149} \\ \frac{77}{149} & \frac{37}{149} & \frac{-9}{149} & \frac{-29}{149} \\ \frac{-21}{149} & \frac{17}{149} & \frac{16}{149} & \frac{-7}{149} \end{bmatrix} \begin{bmatrix} 4 \\ 10 \\ 9 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{29}{149}\right)4 + \left(-\frac{12}{149}\right)\cdot 10 + \frac{15}{149}\cdot 9 + \frac{40}{149}(-7) \\ \frac{59}{149}\cdot 4 + \frac{9}{149}\cdot 10 + \frac{26}{149}\cdot 9 + \left(-\frac{30}{149}\right)(-7) \\ \frac{77}{149}\cdot 4 + \frac{37}{149}\cdot 10 + \left(-\frac{9}{149}\right)9 + \left(-\frac{24}{149}\right)(-7) \\ \left(-\frac{21}{149}\right)\cdot 4 + \frac{17}{149}\cdot 10 + \frac{16}{149}\cdot 9 + \left(-\frac{7}{149}\right)(-7) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{381}{149} \\ \frac{770}{149} \\ \frac{765}{149} \\ \frac{279}{149} \end{bmatrix}$$

$\therefore X = -\frac{381}{149}$

$Y = \frac{770}{149}$

$Z = \frac{765}{149}$

$t = \frac{279}{149}$

14. $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 7 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & -3 & 0 & 0 & 1 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & -3 & 0 & 0 & 1 \\ 0 & -\frac{19}{7} & \frac{26}{7} & 0 & 1 & -\frac{4}{7} \\ 0 & \frac{5}{7} & \frac{16}{7} & 1 & 0 & -\frac{3}{7} \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - \frac{4}{7}R_1 \\ R_3' = R_3 - \frac{3}{7}R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & -3 & 0 & 0 & 1 \\ 0 & -\frac{19}{7} & \frac{26}{7} & 0 & 1 & -\frac{4}{7} \\ 0 & 0 & \frac{62}{19} & 1 & \frac{5}{19} & -\frac{11}{19} \end{array}\right] \quad R_3' = R_3 + \frac{5}{19}R_2$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & -3 & 0 & 0 & 1 \\ 0 & -\frac{19}{7} & \frac{26}{7} & 0 & 1 & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{array}\right] \quad R_3' = \frac{19}{62} R_3$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & 0 & \frac{57}{62} & \frac{15}{62} & \frac{29}{62} \\ 0 & -\frac{19}{7} & 0 & -\frac{247}{217} & \frac{152}{217} & \frac{19}{217} \\ 0 & 0 & 1 & \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{array}\right] \quad \begin{array}{l} R_1' = R_1 + 3R_3 \\ R_2' = R_2 - \frac{26}{7} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 7 & 3 & 0 & \frac{57}{62} & \frac{15}{62} & \frac{29}{62} \\ 0 & 1 & 0 & \frac{13}{31} & -\frac{8}{31} & -\frac{1}{31} \\ 0 & 0 & 1 & \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{array}\right] \quad R_2' = -\frac{7}{19} R_2$$

$$= \left[\begin{array}{ccc|ccc} 7 & 0 & 0 & -\frac{21}{62} & \frac{63}{62} & \frac{35}{62} \\ 0 & 1 & 0 & \frac{13}{31} & -\frac{8}{31} & -\frac{1}{31} \\ 0 & 0 & 1 & \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{array}\right] \quad R_1' = R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{62} & \frac{9}{62} & \frac{5}{62} \\ 0 & 1 & 0 & \frac{13}{31} & -\frac{8}{31} & -\frac{1}{31} \\ 0 & 0 & 1 & \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{array}\right] \quad R_1' = \frac{1}{7} R_1$$

$$= [I \mid A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{3}{62} & \frac{9}{62} & \frac{5}{62} \\ \frac{13}{31} & -\frac{8}{31} & -\frac{1}{31} \\ \frac{19}{62} & \frac{5}{62} & -\frac{11}{62} \end{bmatrix}$$

$3x+4y+7z=14$

$2x-y+3z=4$

$x+2y-3z=0$

$$\therefore \left[\begin{array}{ccc|c} 3 & 4 & 7 & 14 \\ 2 & -1 & 3 & 4 \\ 1 & 2 & -3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & 7 & 14 \\ 0 & -\frac{11}{3} & -\frac{5}{3} & -\frac{16}{3} \\ 0 & \frac{2}{3} & -\frac{16}{3} & -\frac{14}{3} \end{array}\right] \begin{array}{l} R_2' = R_2 - \frac{2}{3}R_1 \\ R_3' = R_3 - \frac{1}{3}R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & 7 & 14 \\ 0 & -\frac{11}{3} & -\frac{5}{3} & -\frac{16}{3} \\ 0 & 0 & -\frac{62}{11} & -\frac{62}{11} \end{array}\right] \quad R_3' = R_3 + \frac{2}{11} R_2$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & 7 & 14 \\ 0 & -\frac{11}{3} & -\frac{5}{3} & -\frac{16}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3' = -\frac{11}{62} R_3$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & 0 & 7 \\ 0 & -\frac{11}{3} & 0 & -\frac{11}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + \frac{5}{3} R_3 \\ R_1' = R_1 - 7R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 3 & 4 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_2' = -\frac{3}{11} R_2$$

$$= \left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1' = R_1 - 4R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1' = \frac{1}{3} R_1$$

$\therefore X = 1$

$Y = 1$

$Z = 1$

15. $A = \begin{bmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{bmatrix}$

$$|A| = (x+y)(x-y) - (y+z)(z-y) + (z+x)(z-x)$$

$$= x^2 - y^2 - z^2 + y^2 + z^2 - x^2$$

$$= 0$$

17.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -2 & 3 \\ 1 & 2 & 1 & 2 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -2 & 3 \\ 0 & \frac{5}{3} & \frac{5}{3} & 1 \\ 0 & \frac{1}{3} & \frac{7}{3} & 2 \\ 0 & \frac{7}{3} & \frac{10}{3} & -3 \end{bmatrix} \quad \begin{aligned} R_2' &= R_2 - \tfrac{1}{3}R_1 \\ R_3' &= R_3 + \tfrac{1}{3}R_1 \\ R_4' &= R_4 - \tfrac{2}{3}R_1 \end{aligned}$$

$$= \begin{bmatrix} 3 & 1 & -2 & 3 \\ 0 & \frac{7}{3} & \frac{10}{3} & -3 \\ 0 & \frac{1}{3} & \frac{7}{3} & 2 \\ 0 & \frac{5}{3} & \frac{5}{3} & 1 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$= \begin{bmatrix} 3 & 1 & -2 & 3 \\ 0 & \frac{7}{3} & \frac{10}{3} & -3 \\ 0 & 0 & \frac{13}{7} & \frac{17}{7} \\ 0 & 0 & -\frac{5}{7} & \frac{22}{7} \end{bmatrix} \begin{matrix} R_3' = R_3 - \frac{1}{7} R_2 \\ R_4' = R_4 - \frac{5}{7} R_2 \end{matrix}$$

$$= \begin{bmatrix} 3 & 1 & -2 & 3 \\ 0 & \frac{7}{3} & \frac{10}{3} & -3 \\ 0 & 0 & \frac{13}{7} & \frac{17}{7} \\ 0 & 0 & 0 & \frac{53}{13} \end{bmatrix} R_4' = R_4 + \frac{5}{13} \cdot R_3$$

$$\therefore \det = 3 \cdot \frac{7}{3} \cdot \frac{13}{7} \cdot \frac{53}{13}$$

$$= 53$$

18. $x+y+z-2t=-4$

$x-2y+3z+4t=10$

$2x+3y-z+2t=9$

$4x-y+2z-t=-7$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & -2 & 3 & 4 \\ 2 & 3 & -1 & 2 \\ 4 & -1 & 2 & -1 \end{bmatrix}$$

now find the detenminent.

$$\Delta = 1\{-2(1-4)-3(-3+2)+4(6-1)\}-1\{1(1-4)$$
$$-3(-2-8)+4(4+4)\}+1\{1(-3+2)$$
$$+2(-2-8)+4(-2-12)\}+2\{(6-1)+2(6+4)$$
$$+3(-2-12)\}$$

$$= (6+3+20) - (-3+30+32) + (-1-20-56)$$
$$+2(5+16-42)$$

$$= 29 - 59 - 77 - 42$$

$$\Delta = -149$$

Now replace first column with solution vector.

$$\begin{bmatrix} -4 & 1 & 1 & -2 \\ 10 & -2 & 3 & 4 \\ 9 & 3 & -1 & 2 \\ -7 & -1 & 2 & -1 \end{bmatrix}$$

$$\Delta_1 = -4\{-2(1-4) - 3(-3+2) + 4(6-1)\} - 1\{10(1-4)$$
$$-3(-9+14) + 4(18-7)\} + 1\{10(-3+2) + 2(-9+14)$$
$$+4(-9+21)\} + 2\{10(6-1) + 2(18-7) + 3(-9+21)\}$$

$$= -4(6+3+20) - (-30-15+44) + (-10+10+48)$$
$$+2(50+22+36)$$

$$= -116 + 1 + 48 + 216$$

$$\Delta_1 = 149$$

Now replace second column with solution vector.

$$\begin{bmatrix} 1 & -4 & 1 & -2 \\ 1 & 10 & 3 & 4 \\ 2 & 9 & -1 & 2 \\ 4 & -7 & 2 & -1 \end{bmatrix}$$

$$\therefore \Delta_2 = 1\{10(1-4)-3(-9+14)+4(18\ 7)\}$$
$$-4\{1(1-4)-3(-2-8)+4(4+4)\}+1\{1(-9+14)$$
$$-10(-2-8)+4(-14-36)\}+2\{1(18-7)$$
$$-10(4+4)+3(-14-36)\}$$

$$=(-30-15+44)+4(-3+30+32)+(5+100-200)$$
$$+2(11-80-150)$$

$$= -1+236-95-438$$

$$\Delta_2 = -298$$

Now replace third column with solution vector.

$$\begin{bmatrix} 1 & 1 & -4 & -2 \\ 1 & -2 & 10 & 4 \\ 2 & 3 & 9 & 2 \\ 4 & -1 & -7 & -1 \end{bmatrix}$$

$$\therefore \Delta_3 = 1\{-2(-9+14)-10(-3+2)+4(-2+9)\}$$

$$-1\{1(-9+14)-10(-2-8)+4(-14-36)\}$$

$$-4\{1(-3+2)+2(-2-8)+4(-2-12)\}+2\{1(-2+9)$$

$$+2(-14-36)+10(-2-12)\}$$

$$=(-10+10-48)-(5+100-200)$$

$$-4(-1-20-56)+2(-12-100-140)$$

$$=-48+95+308-504=-149$$

Now replace column four with solution vector

$$\begin{bmatrix} 1 & 1 & 1 & -4 \\ 1 & -2 & 3 & 10 \\ 2 & 3 & -1 & 9 \\ 4 & -1 & 2 & -7 \end{bmatrix}$$

$$\therefore \Delta_4 = 1\{-2(7-18)-3(-21+9)+10(6-1)\}$$
$$-1\{1(7-18)-3(-14-36)+10(4+4)\}$$
$$+1\{1(-21+9)+2(-14-36)+10(-2-12)\}$$
$$+4\{1(6-1)+2(4+4)+3(-2-12)\}$$
$$=(22+36+50)-(-11+150+80)+(-12-100-140)$$
$$+4(5+16-42)$$
$$=108-219-252-84$$

$$\Delta_4 = -447$$

$$x = \frac{\Delta_1}{\Delta} = \frac{149}{-149} = -1$$

$$Y = \frac{\Delta_2}{\Delta} = \frac{-298}{-149} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-149}{-149} = 1$$

$$t = \frac{\Delta_4}{\Delta} = \frac{-447}{-149} = 3$$

20.

Let, $x = (x_1, x_2, x_3, x_4)$ be a vector which is orthogonal to u, v and w.

Then we have that $(x,u) = 2x_1 + x_2 + 4x_3 = 0$ and

$(x,v) = -x_1 - x_2 + 2x_3 + 2x_4 = 0$ and

$(x,w) = 3x_1 + 2x_2 + 5x_3 + 4x_4 = 0$

since, $2x_1 + x_2 - 4x_3 = 0$, $-x_1 - x_2 + 2x_3 + 2x_4 = 0$

we have that $2x_1 + x_2 - 4x_3 + 2(-x_1 - x_2 + 2x_3 + 2x_4)$ $= 0 + 2 \cdot 0$ and $3(-x_1 - x_2 + 2x_3 + 2x_4) + 3x_1 + 2x_2$ $+ 5x_3 + 4x_4 = 3 \cdot 0 + 0$ which gives us that $-x_2 + 4x_4 = 0$ and then $-x_2 + 11x_3$ $+ 10x_4 = 0$

Thus $x_2 = 4x_4$ and $x_3 = -\frac{6x_4}{11}$ and therefore since $2x_1 + x_2 - 4x_3 = 0$ we have that $x_1 = -\frac{34x_4}{11}$

$$\therefore x = \left(-\frac{34x_4}{11}, 4x_4, -\frac{6x_4}{11}, x_4\right)$$

Now in order for x to be unitary we must have $\|x\| = 1$ and then we must have $\sqrt{\left(-\frac{34x_4}{11}\right)^2 + (4x_4)^2 + \left(-\frac{6x_4}{11}\right)^2 + x_4^2} = 1$

since $\sqrt{\left(-\frac{34x_4}{11}\right)^2 + (4x_4)^2 + \left(\frac{6x_4}{11}\right)^2 + x_4^2}$

$$= \sqrt{\frac{1156}{121}x_4^2 + 16x_4^2 + \frac{36}{121}x_4^2 + x_4^2}$$

$$= \sqrt{\left(\frac{1156}{121} + 16 + \frac{36}{121} + 1\right)x_4^2}$$

$$= \sqrt{\frac{3249}{121}x_4^2} = \frac{57}{11}x_4 = 1$$

which gives us $|x_4| = \frac{11}{57}$

$\therefore x_4 = \pm \frac{11}{57}$

Thus $x = \left(\frac{34}{57}, -\frac{44}{57}, \frac{6}{57}, -\frac{11}{57}\right)$

and $x' = \left(-\frac{34}{57}, \frac{44}{57}, -\frac{6}{57}, \frac{11}{57}\right)$

are two unit vectors which are orthogonal to u, v and w.