NORTH SOUTH UNIVERSITY

DEPARTMENT OF MATHEMATICS & PHYSICS
ASSIGNMENT # 2, FALL 2021
MAT125 INTRODUCTION TO LINEAR ALGEBRA
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Number of solved problems: 20

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Assignment - 02

Answer No - 01

$u = (2, 1, -3, 0, 4)$ and $v = (5, -3, -1, 2, 7)$

Scalar product :

$$(u.v) = (2)(5) + (1)(-3) + (-3)(-1) + (0)(2) + (4)(7)$$
$$= 10 - 3 + 3 + 0 + 28$$
$$= 38$$

Norms :

$$||u|| = \sqrt{(2)^2 + (1)^2 + (-3)^2 + (0)^2 + (4)^2}$$
$$= \sqrt{4 + 1 + 9 + 0 + 16}$$
$$= 5.48$$

$$||v|| = \sqrt{(5)^2 + (-3)^2 + (-1)^2 + (2)^2 + (7)^2}$$
$$= \sqrt{25 + 9 + 1 + 4 + 49}$$
$$= 9.38$$

Distance between u and v :

$$d(u, v) = \|u - v\|$$

$$= \sqrt{(2-5)^2 + (1+3)^2 + (-3+1)^2 + (0-2)^2 + (4-7)^2}$$

$$= \sqrt{(-3)^2 + (4)^2 + (-2)^2 + (-2)^2 + (-3)^2}$$

$$= \sqrt{9 + 16 + 4 + 4 + 9}$$

$$= 6.48$$

Cauchy - Schwartz verification :

$$|u \cdot v| \leq \|u\| \, \|v\|$$

$$38 \leq (5.48) \cdot (9.38)$$

$$38 \leq 51.40$$

$\therefore$ verified.

☐ Minkowski's (triangle) inequality verification:

$$||u+v|| \leq ||u|| + ||v||$$

$$\sqrt{(2+5)^2 + (1-3)^2 + (-3-1)^2 + (0+2)^2 + (4+7)^2}$$

$$\sqrt{(7)^2 + (-2)^2 + (-4)^2 + (2)^2 + (11)^2}$$

$$13.93 \leq 5.48 + 9.38$$

$$13.93 \leq 14.86$$

As inequality is true, triangle inequality is verified.

☐ Pythagorean theorem verification:

As u and v are not orthogonal, Pythagorean theorem does not hold.

$$||u+v||^2 = ||u||^2 + ||v||^2$$

$$13.93 = (5.48)^2 + (9.38)^2$$

$$13.93 \neq 118$$

Answer No - 02

$W = \{ (a, b, c) \mid a \geq b \}$

All vectors in W also belong in V

$\therefore W \in V$

Let, $\vec{u} = (a, b, c)$, $\vec{v} = (a_1, b_1, c_1) \in W$

$\vec{u} + \vec{v} = (a + a_1, b + b_1, c + c_1) \in W$

as $a + a_1 \leq b + b_1$

W is closed under addition

Let, k be any scalar

$k\vec{u} = (ka, kb, kc) \in W$

Since, $a \geq b$

$ka \geq kb$ (multiplying both sides by k)

$\therefore W$ is closed under scalar multiplication.

$\therefore W$ is subspace of V.

Answer No - 03

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a - c = 1 \text{ and } a, b, c \in R \right\}$$

All matrices in W also belong in M_{22}

$\therefore W < M_{22}$

Let, $\vec{u} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \in W$

$$\vec{u} + \vec{v} = \begin{bmatrix} a + a_1 & b + b_1 \\ 0 & c + c_1 \end{bmatrix}$$

$$(a + a_1) - (c + c_1) = a - c + a_1 - c_1$$
$$= 1 + 1$$
$$= 2$$

for $\vec{u} + \vec{v}$, $a - c \neq 1$

As W is not closed under addition

it is not a subspace of M_{22}.

Answer No - 04

$$W = \{(a_0 + a_1x + a_2x^2 + a_3x^3) \mid a_0 + a_2 = a_1a_3\}$$

All polynomials in W also belong in P_3.

$\therefore W < P_3$

Let, $\vec{u} = a_0 + a_1x + a_2x^2 + a_3x^3$

$\vec{v} = b_0 + b_1x + b_2x^2 + b_3x^3 \in W$

$\vec{u} + \vec{v} = a_0 + b_0 + a_1x + b_1x + a_2x^2 + b_2x^2 + a_3x^3 + b_3x^3$

$= a_0 + b_0 + a_2 + b_2$

$= a_1a_3 + b_1b_3 \neq a_1a_3$

$\therefore W$ is not closed under addition

So, it is not a subspace of P_3.

Answer No - 05

$$x = k_1 u_1 + k_2 u_2 + k_3 u_3$$

$$(a, b, c) = k_1(1, 2, -3) + k_2(2, 6, -11) + k_3(2, -4, 14)$$

Equating components:

$$k_1 + 2k_2 + 2k_3 = a$$

$$2k_1 + 6k_2 - 4k_3 = b$$

$$-3k_1 - 11k_2 + 14k_3 = c$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 2 & 6 & -4 & b \\ -3 & -11 & 14 & c \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 2 & -8 & b-2a \\ 0 & -5 & 20 & 3a+c \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 + (-2)R_1 \\ R_3' = R_3 + (3)R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 1 & -4 & \frac{b-2a}{2} \\ 0 & -5 & 20 & 3a+c \end{array}\right] \qquad R_2' = \left(\tfrac{1}{2}\right) R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 1 & -4 & \frac{b-2a}{2} \\ 0 & 0 & 0 & \frac{5b-7a+c}{2} \end{array}\right] \quad R_3' = R_3 + (5) R_2$$

$$\frac{5b-7a+c}{2} = 0$$

when x belongs to $W = \text{Span}(u_1, u_2, u_3)$

Answer.

Answer No - 06

Let, $\vec{x} = (3,9,-4,-2)$, $\vec{u} = (1,-2,0,3)$,

$\vec{v} = (2,3,0,-1)$, $\vec{\omega} = (2,-1,2,1)$

$\vec{x}$ is linear combination of $\vec{u}, \vec{v}$ and $\vec{\omega}$

$$\vec{x} = k_1\vec{u} + k_2\vec{v} + k_3\vec{w}$$

$$(3,9,-4,-2) = k_1(1,-2,0,3) + k_2(2,3,0,-1) + k_3(2,-1,2,1)$$

writing as system of equation:

$$k_1 + 2k_2 + 2k_3 = 3$$
$$-2k_1 + 3k_2 - k_3 = 9$$
$$+2k_3 = -4$$
$$3k_1 - k_2 + k_3 = -2$$

Solving using Gauss-~~Jordan~~ elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ -2 & 3 & -1 & 9 \\ 0 & 0 & 2 & -4 \\ 3 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & -5 & -11 \end{bmatrix} \qquad \begin{aligned} R_2' &= R_2 + (2) R_1 \\ R_4' &= R_4 + (-3) R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 1 & -2 \\ 0 & -7 & -5 & -11 \end{bmatrix} \qquad R_3' = \left(\frac{1}{2}\right) R_3$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{3}{7} & \frac{15}{7} \\ 0 & 0 & 1 & -2 \\ 0 & -7 & -5 & -11 \end{bmatrix} \qquad R_2' = \left(\frac{1}{7}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{3}{7} & \frac{15}{7} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \qquad R_4' = R_4 + (7) R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{3}{7} & \frac{15}{7} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4' = R_4 + (2) R_3$$

$\Rightarrow k_3 = -2$

$\Rightarrow k_2 + \frac{3}{7}k_3 = \frac{15}{7}$

$k_2 = \frac{15}{7} + \frac{6}{7} = \frac{21}{7} = 3$

$\Rightarrow k_1 + 2k_2 + 2k_3 = 3$

$k_1 = 4 + 3 - 6 = \underline{1}$

x can be written as linear combination of u, v and w

$x = \vec{u} + 3\vec{v} - 2\vec{w}$

<u>verify</u>:

$(3, 9, -4, -2) = (1, -2, 0, 3) + (6, 9, 0, -3) + (-4, -2, -4, -2)$

$(3, 9, -4, -2) = (1+6+4, -2+9+(-2), 0+0-4, 3-3-2)$

$(3, 9, -4, -2) = (3, 9, -4, -2)$

Therefore, verified.

Answer No - 07

Let, $p = 2t^2 - 3t + 1$

$$p = k_1 p_1 + k_2 p_2 + k_3 p_3$$

$$2t^2 - 3t + 1 = k_1(t^2 - 2t + 1) + k_2(t-1) + k_3(3)$$

$$2t^2 - 3t + 1 = (k_1 t^2 - 2k_1 t + k_1) + (k_2 t - k_2) + 3k_3$$

Equating Co-efficients:

$$k_1 = 2$$

$$-2k_1 + k_2 = -3$$

$$k_1 - k_2 + 3k_3 = 1$$

Solving using Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & -3 \\ 1 & -1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 3 & -1 \end{bmatrix} \qquad \begin{aligned} R_2' &= R_2 + (2)R_1 \\ R_3' &= R_3 + (-1)R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad R_3' = R_3 + (1) R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3' = \left(\frac{1}{3}\right) R_3$$

Here, $k_1 = 2,\ k_2 = 1,\ k_3 = 0$

Verifying :

$$2t^2 - 3t + 1 = k_1(t^2 - 2t + 1) + k_2(t - 1) + k_3(3)$$

$$2t^2 - 3t + 1 = 2t^2 - 4t + 2 + t - 1 + 0$$

$$= 2t^2 - 3t + 1$$

Therefore, Verified.

Answer No - 08

Here, $A = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Set A as linear combination of B, C and D

$$\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 & -k_2 + k_3 \\ k_1 - k_3 & k_2 \end{bmatrix}$$

Equating components :

$$k_1 + k_2 = 5$$

$$-k_2 + k_3 = 1$$

$$k_1 - k_3 = -2$$

$$k_2 = 3$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & -7 \\ 0 & 1 & 0 & 3 \end{bmatrix} \qquad R_3' = R_3 + (-1)R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & -7 \\ 0 & 1 & 0 & 3 \end{bmatrix} \qquad R_2' = (R_2) \times (-1)$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{aligned} R_3' &= R_3 + (1)R_2 \\ R_4' &= R_4 + (-1)R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad R_3' = \left(\frac{-1}{2}\right) R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4' = R_4 + (-1)R_3$$

$\Rightarrow k_3 = 4$

$\Rightarrow k_2 - k_3 = -1$

$k_2 = 4 - 1 = 3$

$\Rightarrow k_1 + k_2 = 5$

$k_1 = 5 - 3 = 2$

$k_1 = 2, \quad k_2 = 3, \quad k_3 = 4$

Verifying:

$$\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 + k_3 \\ k_1 - k_3 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & -3+4 \\ 2-4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$$

Therefore, verified.

Answer No - 09

Let, $\vec{b} = (b_1, b_2, b_3)$ be in arbitary vector in $\mathbb{R}^3$

$$\vec{b} = k_1 u_1 + k_2 u_2 + k_3 u_3$$

$$(b_1, b_2, b_3) = k_1(-1,1,0) + k_2(-1,0,1) + k_3(1,1,1)$$

$$(b_1, b_2, b_3) = (-k_1 - k_2 + k_3, k_1 + 0 + k_3, 0 + k_2 + k_3)$$

Equating components:

$$-k_1 - k_2 + k_3 = b_1$$

$$k_1 + k_3 = b_2$$

$$k_2 + k_3 = b_3$$

The co-efficient matrix, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\det(A) = 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$\det(A) = [(-1)(1) - (1)(1)] + [(-1)(1) - (-1)(0)]$$

$$= -3$$

$$\det(A) \neq 0$$

A is singular system is consistant.

Therefore, $S = \{u_1, u_2, u_3\}$ span $\mathbb{R}^3$

Answer No - 10

Here, $P_1 = (1-t)^3$

$= 1^3 - 3(1)^2 t + 3(1)t^2 - t^3$

$= 1 - 3t + 3t^2 - t^3$

$= t^3 + 3t^2 - 3t + 1$

$P_2 = (1-t)^2$

$= 1 - 2t + t^2$

$= t^2 - 2t + 1$

$P_3 = 1 - t$

$P_4 = 1$

let, $p = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ be any arbitrary polynomial in P_3

$p = K_1 P_1 + K_2 P_2 + K_3 P_3 + K_4 P_4$

$a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$= K_1(t^3 + 3t^2 - 3t + 1) + K_2(t^2 - 2t + 1) + K_3(1-t) + K_4$

Equaling co-efficients :

$K_1 + K_2 + K_3 + K_4 = a_0$

$-3K_1 - 2K_2 - K_3 = a_1$

$3K_1 + K_2 = a_2$

$K_1 = a_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -2 & -1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = 1 \begin{bmatrix} -3 & -2 & -1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= 1 \left\{ 1 \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$= 1 \left\{ 1 (0 + 1) \right\}$$

$$= 1$$

So, $\det(A) \neq 0$

Therefore, A is invertible. So the Polynomials span P_3.

Answer No - 11

$$x = K_1 u + K_2 v + K_3 w$$

$$(a,b,c) = K_1 (2,1,0) + K_2 (1,-1,2) + K_3 (0,3,-4)$$

Equating components:

$$2K_1 + K_2 = a$$
$$K_1 - K_2 + 3K_3 = b$$
$$2K_2 - 4K_3 = c$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 1 & -1 & 3 & b \\ 0 & 2 & -4 & c \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & b \\ 2 & 1 & 0 & a \\ 0 & 2 & -4 & c \end{array}\right] \qquad R_1 \rightleftarrows R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & b \\ 0 & 3 & -6 & a-2b \\ 0 & 2 & -4 & c \end{array}\right] \qquad R_2' = R_2 + (-2)R_1$$

$$\begin{bmatrix} 1 & -1 & 3 & b \\ 0 & 1 & -2 & \frac{a-2b}{2} \\ 0 & 2 & -4 & c \end{bmatrix} \qquad R_2' = (R_2)\frac{1}{3}$$

$$\begin{bmatrix} 1 & -1 & 3 & b \\ 0 & 1 & -2 & \frac{a-2b}{c} \\ 0 & 0 & 0 & 2b+c-a \end{bmatrix} \qquad R_3' = R_3 + (-2)R_2$$

Therefore, $2b+c-a = 0$

when x belongs to space generated

by u, v and w.

Answer No :- 12

$$\vec{W} = k_1\vec{u} + k_2\vec{v} + k_3\vec{\omega}$$

$$(0,b,c) = k_1(0,1,2) + k_2(0,2,3) + k_3(0,3,1)$$

$$\underset{x\ \ y\ \ z}{(0,b,c)} = (0+0+0) + (k_1+2k_2+3k_3) + (2k_1+3k_2+k_3)$$

Equating Components :

$$0 + 0 + 0 = 0$$

$$k_1 + 2k_2 + 3k_3 = b$$

$$2k_1 + 3k_2 + k_3 = c$$

Solving using Gaussian elimination :

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & b \\ 2 & 3 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & c \\ 1 & 2 & 3 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 \rightleftarrows R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & b \\ 2 & 3 & 1 & c \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 \rightleftarrows R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & b \\ 0 & -1 & -5 & c-2b \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2' = R_2 + (-2)R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & b \\ 0 & 1 & 5 & 2b-c \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2' = (-1)R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & y \\ 0 & 1 & 5 & 2y-z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As there isn't x in the equation

It is proved that yz plane w in R^3

is generated by u, v and w.

Answer No - 13

$$K_1 u + K_2 v + K_3 w = 0.$$

$$K_1 (1, -3, 7) + K_2 (3, -1, -1) + K_3 (2, 4, -5) = 0$$

$$K_1 + 3K_2 + 2K_3 = 0$$

$$-3K_1 - K_2 + 4K_3 = 0$$

$$7K_1 - K_2 - 5K_3 = 0$$

Coefficient matrix, $A = \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & 4 \\ 7 & -1 & -5 \end{bmatrix}$

$$\det(A) = 1 \begin{vmatrix} -1 & 4 \\ -1 & 5 \end{vmatrix} - 3 \begin{vmatrix} -3 & 4 \\ 7 & -5 \end{vmatrix} + 2 \begin{vmatrix} -3 & -1 \\ 7 & -1 \end{vmatrix}$$

$$= \{1(5+4) - 3(15-28) + 2(3+7)\}$$

$$= 9 + 39 + 20$$

$$= 68 \neq 0$$

Therefore, A is invertible. Hence the system is Consistant and has unique solution.

$$K_1 = K_2 = K_3 = 0$$

Therefore, given vectors are linearly dependent.

Answer No - 14

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}$$

$$x \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + y \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} + z \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x+3y+z & 2x-y-5z \\ 3x+2y-4z & x+2y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Forming linear equations :

$x + 3y + z = 0$, $2x - y - 5z = 0$, $3x + 2y - 4z = 0$

$x + 2y = 0$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & -5 & 0 \\ 3 & 2 & -4 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

$R_2' = R_2 + (-2) R_1$

$R_3' = R_3 + (-3) R_1$

$R_4' = R_4 + (-1) R_1$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -7 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix} \qquad R_2' = R_2\left(-\frac{1}{7}\right)$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} R_3' &= R_3 + (7)R_2 \\ R_4' &= R_4 + (1)R_2 \end{aligned}$$

Here, x, y are leading variable and z is free variable.

Let, $z = t$

$y = -t$

$x = 3t - t$

so, the system has nontrivial solution and hence that the vectors are linearly dependent.

Answer NO - 15

$$k_1 P_1 + k_2 P_2 + k_3 P_3 = 0$$

$$k_1 u + k_2 v + k_3 w = 0$$

$$k_1 (3 - 2t + 4t^2 + t^3) + k_2 (4 - t + 6t^2 + t^3)$$
$$+ k_3 (7 - 8t + 8t^2 + 3t^3) = (0, 0, 0, 0).$$

Forming linear equation,

$$3k_1 + 4k_2 + 7k_3 = 0$$
$$-2k_1 - k_2 - 8k_3 = 0$$
$$4k_1 + 6k_2 + 8k_3 = 0$$
$$k_1 + k_2 + 3k_3 = 0$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 7 & 0 \\ -2 & -1 & -8 & 0 \\ 4 & 6 & 8 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & -1 & -8 & 0 \\ 4 & 6 & 8 & 0 \\ 3 & 4 & 7 & 0 \end{array}\right] \quad R_1 \rightleftarrows R_4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$R_2' = R_2 + 2R_1$

$R_3' = R_3 + (-4)R_1$

$R_4' = R_3 + (-3)R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right]$$

$R_2 \rightleftarrows R_4$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -9 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

here, $x_3 = 0$, $x_2 = 0$, $x_1 = 0$

one solution & system is independent.

Answer No - 16

$$W(x) = \begin{bmatrix} e^{2x} & x^2 & x \\ 2e^{2x} & 2x & 1 \\ 4e^{2x} & 2 & 0 \end{bmatrix}$$

$$= x \begin{vmatrix} 2e^{2x} & 2x \\ 4e^{2x} & 2 \end{vmatrix} - 1 \begin{vmatrix} e^{2x} & x^2 \\ 4e^{2x} & 2 \end{vmatrix}$$

$$= x(4e^{2x} - 8xe^{2x}) - (2e^{2x} - 4x^2e^{2x})$$

$$= 4xe^{2x} - 8x^2e^{2x} - 2e^{2x} + 4x^2e^{2x}$$

$$= 4xe^{2x} - 2e^{2x} - 4x^2e^{2x}$$

$$= -2e^{2x}(2x^2 - 2x + 1)$$

$$2x^2 - 2x + 1 \neq 0$$

$$-2e^{2x} \neq 0$$

This function does not have value zero for all x in the interval $(-\infty, +\infty)$

So, $f(x)$, $g(x)$ and $h(x)$ form a linearly independent set.

Answer No - $\cancel{17}$ 17

$$W(t) = \begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$$

$$= e^t \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} - e^t \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix}$$

$$+ e^t \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= e^t(\sin^2 t + \cos^2 t) - e^t(-\cos t \sin t + \sin t \cos t)$$

$$+ e^t(\cos^2 t + \sin^2 t)$$

$$= e^t(1) - e^t(0) + e^t(1)$$

$$= 2e^t \neq 0$$

The function $2e^t$ does not have value zero for all x in the interval $(-\infty, +\infty)$

so, the functions are linearly independent.

Answer No - 18

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -3 & 3 & 7 & 1 & 3 \\ -1 & 3 & 9 & 3 & 1 \\ -5 & 3 & 5 & -1 & 4 \end{bmatrix}$$

Making it REF form,

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -3 & 3 & 7 & 1 & 3 \\ -1 & 3 & 9 & 3 & 1 \\ -5 & 3 & 5 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & 10 & 4 & 9 \\ 0 & 3 & 10 & 4 & 3 \\ 0 & 3 & 10 & 4 & 14 \end{bmatrix} \qquad \begin{aligned} R_2' &= R_2 + 3(R_1) \\ R_3' &= R_3 + 1(R_1) \\ R_4' &= R_4 + 5(R_1) \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} & 3 \\ 0 & 3 & 10 & 4 & 3 \\ 0 & 3 & 10 & 4 & 14 \end{bmatrix} \qquad R_2' = R_2\left(\frac{1}{3}\right)$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3 + (-3) R_2 \\ R_4' &= R_4 + (-3) R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightleftarrows R_4$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & \frac{10}{3} & \frac{4}{3} & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3' = \left(\frac{1}{5}\right) R_3$$

Since the matrix is in the REF, we know, row with leading 1 and column with leading 1 will be row space and column space respectively.

The vectors for row space

$$r_1 = [1\ 0\ 1\ 1\ 2]$$

$$r_2 = [0\ 1\ \tfrac{10}{3}\ \tfrac{4}{3}\ 3]$$

$$r_3 = [0\ 0\ 0\ 0\ 1]$$

the vectors for column space

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; c_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

Since this matrix has three leading 1's its row space and colum spaces are three dimensional and rank(A) = 3

& nullity(A) = 2 as there are two free variables.

To find the nulity of A. we need to find the dimension of null space of $Ax = 0$

$$x_1 + x_3 + x_4 + 2x_5 = 0$$

$$x_2 + \frac{10}{3}x_3 + \frac{4}{3}x_4 + 3x_5 = 0$$

$$x_5 = 0$$

We obtain general solution,

$$x_1 = -s - t$$

$$x_2 = -\frac{10}{3}s - \frac{4}{3}t$$

$$x_3 = s$$

$$x_4 = t$$

$$x_5 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ -\frac{10}{3}s - \frac{4}{3}t \\ s \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ -\frac{10}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -\frac{4}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

dimension of null space $= 2$

nulity (A) $= 2$

Answer No - 19

$$2x_1 + x_2 + 3x_4 = 0$$
$$x_1 + 2x_2 - 2x_3 + x_4 = 0$$
$$-3x_1 + x_2 + x_3 - x_4 = 0$$

Augmented matrix for the system

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 0 \\ 1 & 2 & -2 & 1 & 0 \\ -3 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 2 & 1 & 0 & 3 & 0 \\ -3 & 1 & 1 & -1 & 0 \end{array}\right] \quad R_1 \rightleftarrows R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & -3 & 4 & 1 & 0 \\ 0 & 7 & -5 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + (-2) R_1 \\ R_3' = R_3 + (3) R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 7 & -5 & 2 & 0 \end{array}\right] \quad R_2' = \left(-\frac{1}{3}\right) R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{13}{3} & \frac{13}{3} & 0 \end{array}\right] \quad R_3' = R_3 + (-7) R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \quad R_3' = \left(-\frac{3}{13}\right) R_3$$

Corresponding system of equations is

$$x_1 + 2x_2 - 2x_3 + x_4 = 0$$

$$x_2 - \frac{4}{3}x_3 - \frac{1}{3}x_4 = 0$$

$$x_3 - x_4 = 0$$

General solution is (let, $x_4 = t$)

$$\cancel{x_1 = -2x_2 + 2x_3 - x_4}$$

$$x_3 = x_4 = t$$

$$x_2 = \frac{4}{3}x_3 + \frac{1}{3}x_4 = \frac{5t}{3}$$

$$x_1 = -2x_2 + 2x_3 - x_4 = -3t$$

Therefore, the solution vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3t \\ \frac{5}{3}t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ \frac{5}{3} \\ 1 \\ 1 \end{bmatrix}$$

Vector $v_1 = \begin{bmatrix} -3 \\ \frac{5}{3} \\ 1 \\ 1 \end{bmatrix}$ is the solution space.

$S = \{v_1\}$ is a basis for the solution space and the solution space is one dimensional.

Answer No - 20

$$A = \begin{bmatrix} 3 & 4 & -2 \\ 1 & 4 & -1 \\ 2 & 6 & -1 \end{bmatrix}$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 & -2 \\ 1 & 4 & -1 \\ 2 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-3 & -4 & 2 \\ -1 & \lambda-4 & 1 \\ -2 & -6 & \lambda+1 \end{bmatrix}$$

characteristic polynomial of A

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-3 & -4 & 2 \\ -1 & \lambda-4 & 1 \\ -2 & -6 & \lambda+1 \end{bmatrix}$$

$$= (\lambda-3)\{(\lambda-4)(\lambda+1) - (-6)\}$$
$$-4\{-(\lambda+1) + 2\} + 2\{6 - (\lambda-4)(-2)\}$$
$$= (\lambda-3)(\lambda^2-3\lambda+2) + 4(-\lambda+1) + 2(2\lambda-2)$$

$= \lambda^3 - 6\lambda^2 + 11\lambda - 6$

$= f(\lambda)$

if $f(\lambda) = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$f(1) = 1^3 - 6.1^2 + 11.1 - 6$

$= 0$

$(\lambda - 1)$ must be a factor of $f(\lambda)$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda^2 + 5\lambda + 6\lambda - 6 = 0$

$\Rightarrow \lambda^2(\lambda - 1) - 5\lambda(\lambda - 1) + 6(\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\lambda - 1 = 0$ and, $\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 1$ $\qquad \lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda - 3 = 0$ and $\lambda - 2 = 0$

$\lambda = 3$ $\qquad \lambda = 2$

Therefore, the eigen values of A

are $\lambda = 1, 2, 3$

So, the corresponding eigen space

$\lambda = 3$