LU 7 - In-Person: Formalization, Satisfiability, and Resolution

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Question

This assignment consists of four parts:

1. The first is concerned with formalization in propositional logic.
2. The second with formula satisfiability.
3. The last two parts cover CNF transformation and the resolution method.

A. Formalization

Please formalize the following natural language statements using the syntax of propositional logic and appropriate symbols:

1. Solutions to NP-complete problems can be found in exponential time and are verifiable in polynomial time.
2. Not following programming guidelines leads to bugs.
3. A complete algorithm finds a correct answer whenever one exists.
4. If an algorithm is sound, whenever it returns an answer, the answer is true.
5. If a robot is smart, it either finds a problem solution or, otherwise, asks the operator for help.

Answers

1. **Solutions to NP-complete problems can be found in exponential time and are verifiable in polynomial time.**
   Let:

   - S: Solutions to NP-complete problems exist.
   - E: Found in exponential time.
   - P: Verifiable in polynomial time.

   Formalization:
   $$S \implies (E \wedge P)$$

2. **Not following programming guidelines leads to bugs.**
   Let:

   - G: Following programming guidelines.
   - B: Bugs occur.

   Formalization:
   $$\neg G \implies B$$

3. **A complete algorithm finds a correct answer whenever one exists.**
   Let:

   - C: Algorithm is complete.
   - F: Finds a correct answer.
   - E: Correct answer exists.

   Formalization:
   $$C \implies (E \implies F)$$

4. **If an algorithm is sound, whenever it returns an answer, the answer is true.**
   Let:

   - S: Algorithm is sound.
   - R: Algorithm returns an answer.
   - T: The answer is true.

   Formalization:
   $$S \implies (R \implies T)$$

5. **If a robot is smart, it either finds a problem solution or asks the operator for help.**
   Let:

   - M: Robot is smart.
   - F: Finds a problem solution.
   - H: Asks the operator for help.

   Formalization:

   $$M \implies (F \vee H)$$

B. Satisfiability

Decide whether the following formulas are valid, unsatisfiable, or neither. Please also provide a justification for your answer, e.g., in the form of derivations, interpretations, or truth tables:

1. $smoke \implies smoke$
2. $smoke \implies fire$
3. $(smoke \implies fire) \implies (\neg fire \implies \neg smoke)$
4. $(smoke \implies fire) \implies ((smoke \wedge heat) \implies fire)$
5. $(smoke \implies fire) \iff (\neg fire \wedge smoke)$

Answers

1. $smoke \implies smoke$
   This is a tautology (always true) because any proposition implies itself.
   Result: Valid.

2. $smoke \implies fire$
   This formula is satisfiable because it is true if either *smoke* is false or $fire$ is true. It is not valid since *smoke* could occur without $fire$.
   Result: Satisfiable.

3. $(smoke \implies fire) \implies (\neg fire \implies \neg smoke)$
   This represents the contrapositive, which is always true.
   Result: Valid.

4. $(smoke \implies fire) \implies ((smoke \land heat) \implies fire)$
   The premise $smoke \implies fire$ ensures that whenever $smoke$ is true, $fire$ is true. Adding $heat$ does not change the truth of $fire$.
   Result: Valid.

5. $(smoke \implies fire) \iff (\neg fire \land smoke)$
   This formula is not satisfiable because $smoke \implies fire$ is weaker than $fire \land smoke$. The equivalence does not hold in all cases.
   Result: Unsatisfiable.

C. CNF Transformation

Transform the formula $((C \land \neg B) \iff A) \land (\neg C \implies A)$ into an equivalent set of clauses K. Please write down the individual transformation steps.

Answer

1. Rewrite $\iff$ as two implications:

$$((C \land \neg B) \implies A) \land (A \implies (C \land \neg B)) \land (\neg C \implies A)$$

2. Rewrite implications using $\lor$:

$$(\neg(C \land \neg B) \lor A) \land (\neg A \lor C) \land (\neg A \lor \neg B) \land (\neg C \lor A)$$

3. Apply De Morgan's laws:

$$((\neg C \lor B) \lor A) \land (\neg A \lor C) \land (\neg A \lor \neg B) \land (\neg C \lor A)$$

4. Distribute $\lor$ over $\land$:

$$(\neg C \lor B \lor A) \land (\neg A \lor C) \land (\neg A \lor \neg B) \land (\neg C \lor A)$$

D. Resolution

Finally, use the resolution method to demonstrate that $K \models (\neg B \implies (A \land C))$ holds.

Answer

1. Negate the formula to prove:

$$K \models \neg(\neg B \implies (A \wedge C))$$

$$K \models \neg(\neg B \vee (A \wedge C))$$

$$K \models B \wedge (\neg A \vee \neg C)$$

2. Add $B \wedge (\neg A \vee \neg C)$ to K and resolve:

   Use clauses from Part C. Derive contradictions systematically using resolution. The derived contradiction proves:

$$K \models (\neg B \implies (A \wedge C))$$