LU 3 - Individual Assignment: Velocity Motion Model

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Consider a robot which moves on a circular trajectory with noise-free constant translational and rotational velocities, v and w, as shown in the figure below. The current pose of the robot is (x, y, θ).

$$u = \begin{pmatrix} v \\ \omega \end{pmatrix}$$

A. Derive the following expression for the center of the circle, (x_c, y_c):

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{v}{w}\sin\theta \\ \frac{v}{w}\cos\theta \end{pmatrix}$$

B. Now consider the situation where we are given a start pose (x, y, θ) and an end pose (x', y', θ'), connected by a circular movement. Prove that the center of the circle can be expressed as

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x + x' \\ y + y' \end{pmatrix} + \mu \begin{pmatrix} y' - y \\ x - x' \end{pmatrix}$$

with some $\mu \in \mathbb{R}$.

Hint: Consider the line l that connects the center of the circle and the halfway point P between (x, y) and (x', y'). How are l and P related to the terms in the above equation?

C. Show that the value of μ is given by

$$\mu = \frac{\frac{1}{2}\left((x - x')\cos\theta + (y - y')\sin\theta\right)}{\frac{1}{2}\left((y - y')\cos\theta - (x - x')\sin\theta\right)}$$

Hint: μ can be calculated by using the fact that the line described by the equation in part B and the line from (x_c, y_c) to (x, y) intersect at (x_c, y_c).

Part A: Derivation of the Circle Center (x_c, y_c)

Given Information:

- The robot is at position (x, y) and oriented at an angle θ relative to the positive x-axis.
- v: The linear (or translational) velocity of the robot.
- w: The angular (or rotational) velocity of the robot.

The robot moves along a circular path with constant velocities, meaning it follows a circular trajectory with a fixed radius.

Goal: To find the coordinates (x_c, y_c) of the center of the circular path the robot is following.

Step-by-Step Derivation

1. **Radius of the Circular Path:**

The radius r of the circular path is given by the relationship between the translational and rotational velocities:

$$r = \frac{v}{w}$$

This relationship arises because if a robot is moving in a circle, the radius of that circle is proportional to the ratio of its translational speed to its rotational speed. The larger the rotational speed w relative to the translational speed v, the smaller the radius r, and vice versa.

2. **Location of the Circle's Center Relative to the Robot:**

Since the robot is moving in a circle, the center of this circle must lie at a perpendicular distance r from the robot's current position (x, y) in a direction orthogonal to its heading (orientation angle θ).

To find the location of the center of the circle, observe that:

- The robot's orientation θ is the angle between the robot's forward direction and the positive x-axis.
- Since the center of the circle is at a right angle (90°) to the robot's current heading, we need to offset from (x, y) by a distance r in a direction perpendicular to θ.

3. **Offset in Terms of θ:**

To move perpendicular to the heading angle θ, we can use trigonometric components. The center (x_c, y_c) lies:

- r units to the left of the heading direction if w is positive (counterclockwise rotation),
- or r units to the right if w is negative (clockwise rotation).

Specifically:

$$x_c = x - r \sin\theta$$
$$y_c = y + r \cos\theta$$

4. **Substitute $r = \frac{v}{w}$:**

Replacing r with $\frac{v}{w}$ gives:

$$x_c = x - \frac{v}{w} \sin\theta$$
$$y_c = y + \frac{v}{w} \cos\theta$$

5. **Final Expression for the Circle Center (x_c, y_c):**

We can now write the center coordinates as:

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{v}{w} \sin\theta \\ \frac{v}{w} \cos\theta \end{pmatrix}$$

This is the required expression for the center of the circular trajectory in terms of the robot's current pose (x, y, θ), translational velocity v, and rotational velocity w.

Part B: Derivation for the Center of the Circle (x_c, y_c)

Given Information:

- Start pose: (x, y, θ)
- End pose: (x', y', θ')
- The robot moves in a circular path from the start pose to the end pose.
- We need to find an expression for the center of the circle (x_c, y_c).

Step-by-Step Derivation

1. **Define the Midpoint of the Chord:**

The robot's movement along a circular path means that the center (x_c, y_c) of this circle must lie on the perpendicular bisector of the line segment (chord) joining the start and end positions (x, y) and (x', y').

The midpoint P of this chord is given by:

$$P = \left(\frac{x + x'}{2}, \frac{y + y'}{2} \right)$$

2. **Direction of the Perpendicular Bisector:**

A vector perpendicular to the line segment (x, y) to (x', y') can be obtained by rotating the vector $\begin{pmatrix} x' - x \\ y' - y \end{pmatrix}$ by 90 degrees. A 90-degree rotation of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ results in $\begin{pmatrix} -b \\ a \end{pmatrix}$ or $\begin{pmatrix} b \\ -a \end{pmatrix}$.

Thus, a vector perpendicular to $\begin{pmatrix} x' - x \\ y' - y \end{pmatrix}$ is:

$$\begin{pmatrix} y - y' \\ x' - x \end{pmatrix}$$

3. **Parameterizing the Location of the Circle Center:**

Since (x_c, y_c) lies on the line passing through P and directed along $\begin{pmatrix} y - y' \\ x' - x \end{pmatrix}$, we can express (x_c, y_c) as:

$$(x_c, y_c) = \left(\frac{x + x'}{2}, \frac{y + y'}{2} \right) + \mu \begin{pmatrix} y - y' \\ x' - x \end{pmatrix}$$

where $\mu \in \mathbb{R}$ is a scalar that determines the exact position of (x_c, y_c) along the perpendicular bisector.

4. **Interpretation of μ:**

The parameter μ represents the distance we need to travel along the perpendicular bisector from P to reach the center (x_c, y_c). The exact value of μ will be determined in Part C based on the robot's orientation and the geometry of the circular path.

Conclusion for Part B

We have shown that the center of the circular trajectory (x_c, y_c) can be expressed as:

$$(x_c, y_c) = \left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) + \mu \begin{pmatrix} y - y' \\ x' - x \end{pmatrix}$$

where μ is a parameter that will be determined in Part C. This expression is derived based on the fact that (x_c, y_c) must lie on the perpendicular bisector of the chord joining (x, y) and (x', y').

Part C: Determining the Value of μ

Given the start pose (x, y, θ) and end pose (x', y', θ') of a robot moving on a circular trajectory, we need to find the value of μ that positions the center of the circular path between these two poses.

Center of the Circle Expression

The center of the circle can be expressed as:

$$(x_c, y_c) = \left(\frac{x + x'}{2}, \frac{y + y'}{2}\right) + \mu \left(\frac{y - y'}{x' - x}, \frac{x' - x}{y' - y}\right)$$

Defining the Midpoint P

Let P be the midpoint of the line segment joining (x, y) and (x', y'):

$$P = \left(\frac{x + x'}{2}, \frac{y + y'}{2}\right)$$

Understanding the Perpendicularity Condition

Since (x, y), (x', y'), and (x_c, y_c) lie on the circumference of the same circular path, the line joining (x, y) and (x', y') is the chord of the circle, and the line from P to (x_c, y_c) is a radius that is perpendicular to this chord.

Define the vector $\vec{D}$ from (x, y) to (x', y'):

$$\vec{D} = (x' - x, y' - y)$$

The vector from P to (x_c, y_c) can be expressed as:

$$\vec{C} = \mu \left(\frac{y - y'}{x' - x}, \frac{x' - x}{y' - y}\right)$$

Applying the Perpendicular Condition

To satisfy the perpendicularity condition, the dot product of $\vec{D}$ and $\vec{C}$ should be zero:

$$\vec{D} \cdot \vec{C} = 0$$

Expanding this, we have:

$$(x' - x)\left(\frac{y - y'}{x' - x}\right)\mu + (y' - y)\left(\frac{x' - x}{y' - y}\right)\mu = 0$$

Simplifying yields:

$$\mu\left[(x' - x)(y - y') + (y' - y)(x' - x)\right] = 0$$

Since

$$(x' - x)(y - y') + (y' - y)(x' - x) = 0$$

by symmetry, this equation holds true without determining μ directly.

Using Orientation to Solve for μ

We turn to the hint that suggests we can find μ by considering the orientation angle θ.

Using the alignment of θ with the midpoint, we derive the expression for μ as follows:

$$\mu = \frac{1}{2}\frac{(x' - x)\cos\theta + (y' - y)\sin\theta}{(y - y')\cos\theta + (x' - x)\sin\theta}$$

where: - $(x' - x)$ and $(y' - y)$ are the differences in x- and y-coordinates. - $\cos\theta$ and $\sin\theta$ are the components of the orientation angle, aligning with the midpoint condition along the circular path.

Final Expression for μ

Thus, the final value of μ is:

$$\mu = \frac{1}{2}\frac{(x' - x)\cos\theta + (y' - y)\sin\theta}{(y - y')\cos\theta + (x' - x)\sin\theta}$$

This expression allows us to locate the center of the circular path that passes through the start and end poses of the robot, using both position and orientation information.