LU 6 - Individual Assignment: State Estimation using KF

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Question

In this exercise, you must estimate the location of a robot as a one-dimensional scalar number. Assume the robot is staying still in a location. You have the noisy GPS measurements. Assume the standard deviation of the measurement noise to be 0.1 meters $\sigma^2_{\text{obs}} = 0.01$.

- Formulate the problem as a one-dimensional signal problem: write the prediction and correction formulas for it.
- Assume you get the following GPS measurements:

Time (ms)	**Position (m)**
1	0.39
2	0.50
3	0.48
4	0.29

Table 1: GPS Measurements of Robot Position Over Time

Part A: Prediction and Correction Formulae

Prediction

The prediction step of the Kalman Filter provides an estimate of the robot's state before a new measurement is available. The general equations for prediction are as follows:

- **Predicted Mean:**
$$\mu_t^- = A_t\mu_{t-1} + B_t u_t.$$
Since there is no control input ($B_t = 0$) and the robot is stationary ($A_t = 1$), the equation simplifies to:
$$\mu_t^- = \mu_{t-1}.$$
- **Predicted Variance:**
$$\Sigma_t^- = A_t\Sigma_{t-1}A_t^T + Q_t.$$
Since $A_t = 1$ and there is no process noise ($Q_t = 0$), the equation simplifies to:
$$\Sigma_t^- = \Sigma_{t-1}.$$

Correction

The correction step updates the prediction based on the new measurement. The correction formulas are as follows:

- **Kalman Gain:**
$$K_t = \frac{\Sigma_t^- C_t^T}{C_t\Sigma_t^- C_t^T + R_t}.$$
Here:
  - $C_t = 1$: Direct measurement model (no transformation applied to the state).
  - $R_t = \sigma_{\text{obs}}^2$: Measurement noise covariance.

  Substituting $C_t = 1$ into the formula:
$$K_t = \frac{\Sigma_t^-}{\Sigma_t^- + \sigma_{\text{obs}}^2}.$$
This simplified formula calculates the Kalman Gain, which determines the relative weight given to the predicted state (μ_t^-) versus the new measurement (z_t).
- **Updated Mean:**
$$\mu_t = \mu_t^- + K_t(z_t - C_t\mu_t^-).$$
Substituting $C_t = 1$ (direct measurement model), the equation simplifies to:
$$\mu_t = \mu_t^- + K_t(z_t - \mu_t^-).$$
Here:

- z_t: The new measurement.
- μ_t^-: The predicted state mean.
- $K_t(z_t - \mu_t^-)$: The correction term that adjusts the predicted state using the measurement.

- **Updated Variance:**
$$\Sigma_t = (I - K_t C_t)\Sigma_t^-.$$
Substituting $C_t = 1$ and $I = 1$ (since we are in a one-dimensional space), the equation simplifies to:
$$\Sigma_t = (1 - K_t)\Sigma_t^-.$$
This simplified formula adjusts the predicted uncertainty (Σ_t^-) based on the Kalman Gain (K_t).

Part B: Calculation of μ_k and σ_k^2 for 4 Steps

Initialization

We start with the initial belief:

$$\mu_0 = 0, \quad \sigma_{\text{obs}}^2 = 0.1.$$

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Time Step 1:

Measurement: $z_1 = 0.39$
1. Prediction:

$$\mu_1^- = \mu_{\text{t-1}}$$

Substituting value (t=1):

$$\mu_1^- = \mu_{\text{1-1}}$$

$$\mu_1^- = \mu_0$$

$$\mu_1^- = 0$$

2. Prediction:

$$\Sigma_1^- = \Sigma_{\text{t-1}}$$

Substituting value (t=1):

$$\Sigma_1^- = \Sigma_{\text{1-1}}$$

$$\Sigma_1^- = \Sigma_0$$

$$\Sigma_1^- = 1$$

3. Kalman Gain:

$$K_1 = \frac{\Sigma_1^-}{\Sigma_1^- + \sigma_{\text{obs}}^2}$$

Substituting values:

$$K_1 = \frac{1}{1 + 0.1} = 0.9091$$

4. Updated Mean:

$$\mu_1 = \mu_1^- + K_1(z_1 - \mu_1^-)$$

Substituting values:

$$\mu_1 = 0 + 0.9091(0.39 - 0) = 0.3545$$

5. Updated Variance:

$$\Sigma_1 = (1 - K_1)\Sigma_1^-$$

Substituting values:

$$\Sigma_1 = (1 - 0.9091) \cdot 1 = 0.0909$$

$$\boxed{\mu_1 = 0.3545, \quad \Sigma_1 = 0.0909}$$

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Time Step 2:

Measurement: $z_2 = 0.5$

1. Prediction:

$$\mu_t^- = \mu_{\text{t-1}}$$

Substituting value (t=2):

$$\mu_2^- = \mu_{\text{2-1}}$$

$$\mu_2^- = \mu_1$$

$$\mu_2^- = 0.3545$$

2. Prediction:

$$\Sigma_t^- = \Sigma_{\text{t-1}}$$

Substituting value (t=2):

$$\Sigma_2^- = \Sigma_{\text{2-1}}$$

$$\Sigma_2^- = \Sigma_1$$

$$\Sigma_2^- = 0.0909$$

3. Kalman Gain:

$$K_2 = \frac{\Sigma_2^-}{\Sigma_2^- + \sigma_{\text{obs}}^2}$$

Substituting values:

$$K_2 = \frac{0.0909}{0.0909 + 0.1} = 0.4762$$

4. Updated Mean:

$$\mu_2 = \mu_2^- + K_2(z_2 - \mu_2^-)$$

Substituting values:

$$\mu_2 = 0.3545 + 0.4762(0.5 - 0.3545) = 0.4238$$

5. Updated Variance:

$$\Sigma_2 = (1 - K_2)\Sigma_2^-$$

Substituting values:

$$\Sigma_2 = (1 - 0.4762) \cdot 0.0909 = 0.0476$$

$$\boxed{\mu_2 = 0.4238, \quad \Sigma_2 = 0.0476}$$

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Time Step 3:

Measurement: $z_3 = 0.48$

1. Prediction:

$$\mu_t^- = \mu_{\text{t-1}}$$

Substituting value (t=3):

$$\mu_3^- = \mu_{\text{3-1}}$$

$$\mu_3^- = \mu_2$$

$$\mu_3^- = 0.4238$$

2. Prediction:

$$\Sigma_t^- = \Sigma_{\text{t-1}}$$

Substituting value (t=3):

$$\Sigma_3^- = \Sigma_{\text{3-1}}$$

$$\Sigma_3^- = \Sigma_2$$

$$\Sigma_3^- = 0.0476$$

3. Kalman Gain:

$$K_3 = \frac{\Sigma_3^-}{\Sigma_3^- + \sigma_{\text{obs}}^2}$$

Substituting values:

$$K_3 = \frac{0.0476}{0.0476 + 0.1} = 0.3225$$

4. Updated Mean:

$$\mu_3 = \mu_3^- + K_3(z_3 - \mu_3^-)$$

Substituting values:

$$\mu_3 = 0.4238 + 0.3225(0.48 - 0.4238) = 0.4419$$

5. Updated Variance:

$$\Sigma_3 = (1 - K_3)\Sigma_3^-$$

Substituting values:

$$\Sigma_3 = (1 - 0.3225) \cdot 0.0476 = 0.0322$$

$$\boxed{\mu_3 = 0.4419, \quad \Sigma_3 = 0.0322}$$

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Time Step 4:

Measurement: $z_4 = 0.29$

1. Prediction:

$$\mu_t^- = \mu_{\text{t-1}}$$

Substituting value (t=4):

$$\mu_4^- = \mu_{\text{4-1}}$$

$$\mu_4^- = \mu_3$$

$$\mu_4^- = 0.4419$$

2. Prediction:

$$\Sigma_t^- = \Sigma_{\text{t-1}}$$

Substituting value (t=3):

$$\Sigma_4^- = \Sigma_{\text{4-1}}$$

$$\Sigma_4^- = \Sigma_3$$

$$\Sigma_4^- = 0.0322$$

3. Kalman Gain:

$$K_4 = \frac{\Sigma_4^-}{\Sigma_4^- + \sigma_{\text{obs}}^2}$$

Substituting values:

$$K_4 = \frac{0.0322}{0.0322 + 0.1} = 0.2436$$

4. Updated Mean:

$$\mu_4 = \mu_4^- + K_4(z_4 - \mu_4^-)$$

Substituting values:

$$\mu_4 = 0.4419 + 0.2436(0.29 - 0.4419) = 0.4049$$

5. Updated Variance:

$$\Sigma_4 = (1 - K_4)\Sigma_4^-$$

Substituting values:

$$\Sigma_4 = (1 - 0.2436) \cdot 0.0322 = 0.0243$$

$$\boxed{\mu_4 = 0.4049, \quad \Sigma_4 = 0.0243}$$

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Final answers:

$$\mu_1 = 0.3545, \quad \Sigma_1 = 0.0909$$

$$\mu_2 = 0.4238, \quad \Sigma_2 = 0.0476$$

$$\mu_3 = 0.4419, \quad \Sigma_3 = 0.0322$$

$$\mu_4 = 0.4049, \quad \Sigma_4 = 0.0243$$

Reference: Canvas week 6 Kalman Filter slide no: 21