Chittagong University of Engineering & Technology

Department
Of
Electronics & Telecommunication Engineering

Assignment
On
All Numerical Problems

Submitted by-

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Remarks

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Example-15.1: A source produces one of four possible symbols during each interval having possibilities $P(x_1) = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$, $P(x_3) = P(x_4) = \frac{1}{8}$. Obtain the information content of each of these symbols.

Solution:

The information content of each of these symbols are -

$$I(x_1) = -\log_2 P(x_1) = -\log_2\left(\frac{1}{2}\right) = 1 \text{ bit.}$$

$$I(x_2) = -\log_2 P(x_2) = -\log_2\left(\frac{1}{4}\right) = 2 \text{ bits}$$

$$I(x_3) = I(x_4) = -\log_2\left(\frac{1}{8}\right) = 3 \text{ bits.} \quad \text{(Ans)}$$

Example-15.3: Calculate the amount of information (binits) occur with equal likelihood in a binary PCM system.

Solution:

In Binary PCM, there are only two levels 1 or 0. Since they occur with equal likelihood, their probabilities of occurence will be -

$$P_1(0) = \frac{1}{2} \; ; \; P_2(1) = \frac{1}{2}$$

So, the amount of information in these digits are -

$$I_1 = -\log_2[P_1(0)] = -\log_2\left(\frac{1}{2}\right) = 1 \text{ bit}$$

$$I_2 = -\log_2[P_2(1)] = -\log_2\left(\frac{1}{2}\right) = 1 \text{ bit}$$

Hence, $I_1 = I_2 = 1$ bit

Therefore, the correct identification of binary digit in binary PCM carries 1 bit of information.

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Example-15.4: In a Binary PCM if '0' occurs with probability $\frac{1}{4}$ and 1 occurs with probability $\frac{3}{4}$, then calculate the amount of information carried by each limit.

Solution:

Given that,

$$P(0) = \frac{1}{4} \; ; \; P(1) = \frac{3}{4}$$

So, the amount of information carried by each limit is -

$$I(0) = -\log_2[P(0)] = -\log_2\left(\frac{1}{4}\right) = 2 \text{ bits.}$$

$$I(1) = -\log_2[P(1)] = -\log_2\left(\frac{3}{4}\right) = 0.42 \text{ bits.}$$

Example-15.5: ~~In a Binary PCM~~ If there are M equally likely and independent symbols, then prove that amount of information carried by each symbol will be $I(x_i) = N$ bits, where $M = 2^N$ and N is an integer.

Solution:

Since it is given that all the M symbols are equally likely and independent, therefore the probability of occurence of each symbol must be $\frac{1}{M}$.

The amount of information is given as -

$$I(x_i) = \log_2 \frac{1}{P(x_i)} \quad \text{——— (1)}$$

Here, probability of each message is $P(x_i) = \frac{1}{M}$

$\therefore$ From equation (1), $I(x_i) = \log_2 M$

$$= \log_2 2^N \quad [\because M = 2^N]$$

$$= N \text{ bits}$$

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Hence, amount of information carried by each symbol will be N bits. We know that $M = 2^N$. This means that there are N binary digits (binits) in each symbol. This indicates that when the symbols are equally likely and coded with equal number of binary digits (binits) then the information carried by each symbol (measured in bits) is numerically same as the number of binits used for each symbol.

<u>Example-15.7:</u> If $I(x_1)$ is the information carried by symbol x_1 and $I(x_2)$ is the information carried by message x_2, then prove that the amount of information carried compositely due to x_1 & x_2 is $I(x_1, x_2) = I(x_1) + I(x_2)$

<u>Solution:</u>

For independent sources x, y we know that –

$$I(x, y) = I(x) \cdot I(y)$$

So, $I(x_1, x_2) = -\sum_{x_1 x_2} I(x_1, x_2) \log_2 I(x_1, x_2)$

$$= -\sum_{x_1 x_2} I(x_1, x_2) \log_2 [I(x_1) \cdot I(x_2)]$$

$$= -\sum_{x_1 x_2} I(x_1, x_2) \{\log_2 I(x_1) + \log_2 I(x_2)\}$$

$$= -\sum_{x_1 x_2} I(x_1, x_2) \log_2 \{I(x_1)\} - \sum_{x_1 x_2} I(x_1, x_2) \log_2 \{I(x_2)\}$$

$$= -\sum_{x_1} I(x_1) \log_2 I(x_1) \sum_{x_2} I(x_2/x_1) - \sum_{x_2} I(x_2) \log_2 I(x_2) \sum_{x_1} I(x_1/x_2)$$

$$= I(x_1) + I(x_2) \quad \text{(proved)}$$

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Example-15.9: A discrete memoryless source (DMS) X has four symbols x_1, x_2, x_3, x_4 with probabilities $P(x_1) = 0.4$, $P(x_2) = 0.3$, $P(x_3) = 0.2$, $P(x_4) = 0.1$

i) Calculate $H(x)$

ii) Find the amount of information contained in the messages x_1, x_2, x_1, x_3 and $x_4 x_3 x_3 x_2$ and compare with the $H(x)$ obtained in part (i)

Solution:

i) $H(X) = -0.4\log_2(0.4) - 0.3\log_2(0.3) - 0.2\log_2(0.2) - 0.1\log_2(0.1)$

$= 1.846$ bits/symbol.

ii) The information contained in the message $x_1 x_2 x_4 x_3$ is –

$I(x_1 x_2 x_4 x_3) = I(x_1) + I(x_2) + I(x_4) + I(x_3)$

$= -0.4\log_2(0.4) - 0.3\log_2(0.3) - 0.4\log_2(0.4) - 0.2\log_2(0.2)$

$= 2.043$ bits.

Again, the information contained in the message $x_4 x_3 x_3 x_2$ is –

$I(x_4 x_3 x_3 x_2) = I(x_4) + I(x_3) + I(x_3) + I(x_2)$

$= -0.1\log_2(0.1) - 0.2\log_2(0.2) - 0.2\log_2(0.2) - 0.3\log_2(0.3)$

$= 1.782$ bits (Ans.)

Example-15.10: Consider a binary memoryless source X with two symbols x_1 and x_2. Prove that $H(x)$ is maximum when both x_1 and x_2 equiprobable.

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Solution: Let us assume that,

$P(x_1) = \alpha$ so that $P(x_2) = 1 - \alpha$

The entropy is, $H(x) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i)$ bits/symbols

Thus, $H(x) = -\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)$ —— ①

Differentiating ① with respect to α, we get

$$\frac{dH(x)}{d\alpha} = \frac{d}{d\alpha}\left[-\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)\right]$$

Using the relation,

$$\frac{d}{dx} \log_b y = \frac{1}{y} \log_b e \frac{dy}{dx}$$

So, $\frac{dH(x)}{d\alpha} = -\log_2(\alpha) + \log_2(1-\alpha) = \log_2 \frac{1-\alpha}{\alpha}$

$\left[\because I(x_i) = \log_2 \frac{1}{P(x_i)}\right]$

The maximum value of $H(x)$ requires that,

$$\frac{dH(x)}{d\alpha} = 0$$

$$\therefore \frac{1-\alpha}{\alpha} = 1 \quad \text{or } \alpha = \frac{1}{2}$$

when $\alpha = 0$ or 1, $H(x) = 0$

i.e when $P(x_1) = P(x_2) = \frac{1}{2}$, $H(x)$ is maximum and

$H(x) = \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 2 = 1$ bit/symbol (Proved)

Example-15.11: Verify the following expression: $0 \le H(x) \le \log_2 m$ where m is the size of the alphabet of x

Solution:

Since $0 \le P(x_i) \le 1$

$\frac{1}{P(x_i)} \ge 1$ and $\log_2 \frac{1}{P(x_i)} \ge 0$

Then it follows that, $P(x_i) \log_2 \frac{1}{P(x_i)} \ge 0$

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Thus, $H(x) = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)} \geq 0$

again, $P(x_i) \log_2 \frac{1}{P(x_i)} = 0$

if and only if $P(x_i) = 0$ or 1. Since $\sum_{i=1}^{m} P(x_i) = 1$

when $P(x_i) = 1$ then $P(x_j) = 0$ for $i \neq j$. Thus only $H(x) = 0$

<u>Proof of the upper bound:</u>

Let's consider, two probability distribution $[P(x_i) = P_i]$ and $[Q(x_i) = Q_i]$ on the alphabet (x_i), $i = 1, 2, \ldots, m$

such that $\sum_{i=1}^{m} P_i = 1$ and $\sum_{i=1}^{m} Q_i = 1$

We know that, $\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} = \frac{1}{\ln 2} \sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i}$

Using the inequality,

$$\ln \alpha \leq \alpha - 1 \; ; \; \alpha \geq 0$$

The equality holds only if $\alpha = 1$, we obtain

$$\sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i} \leq \sum_{i=1}^{m} P_i \left(\frac{Q_i}{P_i} - 1 \right) = \sum_{i=1}^{m} (Q_i - P_i)$$

$$= \sum_{i=1}^{m} Q_i - \sum_{i=1}^{m} P_i = 0$$

where the equality holds only if $Q_i = P_i$ for all

Setting, $Q_i = \frac{1}{m}$, $i = 1, 2, 3, \ldots, m$

We get, $\sum_{i=1}^{m} P_i \log_2 \frac{1}{P_i \cdot m} = -\sum_{i=1}^{m} P_i \log_2 P_i - \sum_{i=1}^{m} P_i \log_2 m$

$$= H(x) - \log_2 m \sum_{i=1}^{m} P_i$$

Therefore, we get $H(x) \leq \log_2 m$ and the equality holds only if the symbols in x are equiprobable

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Example-15.12: A high resolution black and white TV picture consists of about 2×10^6 picture elements and 16 different brightness levels. Pictures are repeated at the rate of 32 per second. All picture elements are assumed to be independent and all levels have equal likelihood of occurence. Calculate the average rate of information conveyed by this TV picture source.

Solution:

We know that, the entropy,

$$H(x) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ bits/symbols} \quad \text{——} (1)$$

Here, $m = 16$, $P(x_i) = \frac{1}{16}$

$\therefore$ From the equation (1),

$$H(x) = -\sum_{i=1}^{16} \frac{1}{16} \log_2 \frac{1}{16} = 4 \text{ bits/element}$$

The rate, r, at which symbols are generated is given by-

$$r = 2(10^6)(32) = 64(10^6) \text{ elements/second.}$$

Hence, the average rate of information is—

$$R = rH(x) = 64(10^6) \cdot 4 = 256(10^6) \text{ b/s}$$

$$= 256 \text{ Mb/s (Ans.)}$$

Example-15.16: A discrete source emits one of five symbols once every milliseconds with probabilities $1/2$, $1/4$, $1/8$, $1/16$ and $1/16$ respectively. Determine the source entropy and information rate.

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Solution:

The source entropy, $H(S) = \sum_{i=1}^{5} P_i \log_2 \frac{1}{P_i}$

$$= -\sum_{i=1}^{5} P_i \log_2 P_i$$

$$= -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) - \frac{1}{8}\log_2\left(\frac{1}{8}\right) - \frac{1}{16}\log_2\frac{1}{16} - \frac{1}{16}\log_2\frac{1}{16}$$

$$= 1.875 \text{ bits/symbol}$$

Information rate, $R = r.H(S) = (1000 \times 1.875)$ bits/sec

$= 1875$ bits/sec (Ans.)

Example-15.18: An analog signal bandlimited to 10KHz is quantized in 3 levels of a PCM system with probabilities of 1/4, 1/5, 1/5, 1/10, 1/10, 1/20, 1/20 and 1/20 respectively. Find the entropy and the rate of information.

Solution:

The number of levels = 8

So, number of messages = 8. $\therefore M = 8$

$f_s = 10$ KHz

The entropy, $H(x) = \sum_{i=1}^{8} P_i \log_2 \frac{1}{P_i} = -\sum_{i=1}^{8} P_i \log_2 P_i$

$$= -\frac{1}{4}\log_2\left(\frac{1}{4}\right) - \frac{1}{5}\log_2\left(\frac{1}{5}\right) - \frac{1}{5}\log_2\left(\frac{1}{5}\right) - \frac{1}{10}\log_2\left(\frac{1}{10}\right) - \frac{1}{10}\log_2\left(\frac{1}{10}\right) - \frac{1}{20}\log_2\left(\frac{1}{20}\right) - \frac{1}{20}\log_2\left(\frac{1}{20}\right) - \frac{1}{20}\log_2\left(\frac{1}{20}\right)$$

$$= 2.743 \text{ bits/message}$$

The minimum rate of sampling is Nyquist rate,

$\therefore f = 2f_s = (2 \times 10)$ KHz $= 20$ KHz $= 2000$ symbols/sec.

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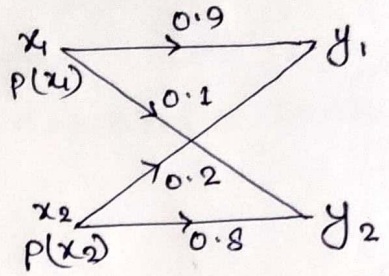
Example-15.19: Given a binary channel shown in the figure

i) Find the channel matrix of the channel.

ii) Find $P(y_1)$ and $P(y_2)$ when $P(x_1) = P(x_2) = 0.5$

iii) Find the joint probabilities $P(x_1, y_2)$ and $P(x_2, y_1)$ when $P(x_1) = P(x_2) = 0.5$



Solution:

i) The channel matrix of the channel is –

$$P(Y|X) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

ii) $P(X) = [0.5 \quad 0.5]$

$$y_1 = P(x_1) \cdot P(y_1/x_1) + P(x_2) + P(y_2/x_2)$$

$$P(Y) = P(X) \cdot P(Y/X) = [0.5 \quad 0.5] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= [0.55 \quad 0.45]$$

$$= [P(y_1) \quad P(y_2)]$$

$P(y_1) = 0.55$ and $P(y_2) = 0.45$

iii) $P(X, Y) = P(X)_d \cdot P(Y/X) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

$$= \begin{bmatrix} 0.45 & 0.05 \\ 0.1 & 0.4 \end{bmatrix}$$

Hence, $P(x_1, y_2) = 0.05$ and $P(x_2, y_1) = 0.1$ (Ans.)

Example-4: The international Morse code uses a sequence of symbols of dots and dashes to transmit letters of the English alphabet. The dash is represented by a current pulse of duration 2ms and dot by duration of 1 ms. The probability of dash is half as that of dot.

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Consider 1 ms duration of gap is given in between the symbols. Calculate -

a) Self information of a dot and a dash

b) Average information content of a dot and a dash

c) Average rate of information.

Solution:

Let, P_{dot} and P_{dash} be the probabilities of dot and dash, respectively,

Given, $P_{dash} = \frac{1}{2} P_{dot}$ and $P_{dot} + P_{dash} = 1$

Or, $P_{dot} + \frac{1}{2} P_{dot} = 1$

Or, $P_{dot} = \frac{2}{3}$

$\therefore P_{dash} = \frac{1}{2} P_{dot}$

$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

Now, $I(dot) = -\log_2 P_{dot} = -\log_2 (2/3) = 0.5849$ bits.

$I(dash) = -\log_2 P_{dash} = -\log_2 (1/3) = 1.5849$ bits

b) $H(S) = -(P_{dot} \log_2 P_{dot} + P_{dash} \log_2 P_{dash})$

$= -\left[\frac{2}{3} \log_2 (2/3) + \frac{1}{3} \log_2 (1/3)\right] = 0.9182$ bits/symbol.

c) From the probabilities of dot and dash, it is clear that for every three symbols transmitted, there will be one symbol of type dash and two of type dot. Also the duration of dash is 2 ms and that of dot is 1 ms and the 1 ms gap is left in between the symbols. Therefore, a total of $(1+1+1+1+2+1) = 7$ ms time is required to transmit an average of 3 symbols (...). So, the symbol rate is given by, $r_s = \frac{\frac{3}{7} \text{ sym}}{\text{ms}} = \frac{\frac{3000}{7} \text{ sym}}{\text{s}}$

Or, $R_s = H(S) \times r_s = 0.9182 \times \frac{3000}{7} = 393.51$ bits/sec

(Ans)

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Exercise-2: Given JPM is

$$P(X,Y) = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0 \\ 0 & 0 & 0.05 & 0.1 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}$$; Find all entropies.

Solution:

$P(x_1) = 0.25$

$P(x_2) = 0.1 + 0.3 = 0.4$

$P(x_3) = 0.05 + 0.10 = 0.15$

$P(x_4) = 0.05 + 0.1 = 0.15$

$P(x_5) = 0.05$

The marginal entropy $H(x)$

$$H(x) = P(x_i) \log_2 \left[\frac{1}{P(x_i)} \right]$$

$$= -P(x_1)\log_2[P(x_1)] - P(x_2)\log_2[P(x_2)] - P(x_3)\log_2[P(x_3)]$$
$$- P(x_4)\log_2[P(x_4)] - P(x_5)\log_2[P(x_5)]$$

$$= -0.25\log_2(0.25) - 0.4\log_2(0.4) - 0.15\log_2(0.15) - \cancel{0.15\log_2}$$
$$0.15\log_2(0.15) - 0.05\log_2(0.05)$$

$$= \frac{1}{2} + 0.529 + 0.4105 + 0.4105 + 0.2161$$

$= 2.066$ bits/symbol.

$P(y_1) = 0.25 + 1 = 0.35$

$P(y_2) = 0.3 + 0.05 = 0.35$

$P(y_3) = 0.10 + 0.05 + 0.05 = 0.2$

$P(y_4) = 0.1$.

The marginal entropy $H(Y) = P(y_i)\log_2\left[\frac{1}{P(y_i)}\right]$

$$= -P(y_1)\log_2[P(y_1)] - P(y_2)\log_2[P(y_2)] -$$
$$P(y_3)\log_2[P(y_3)] - P(y_4)\log_2[P(y_4)]$$

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$= -0.35\log_2(0.35) - 0.35\log_2(0.35) - 0.2\log_2(0.2) - 0.1\log_2(0.1)$

$= 1.857$ bits/symbol.

Joint Entropy, $H(X,Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)\log_2 P(x_i, y_j)$

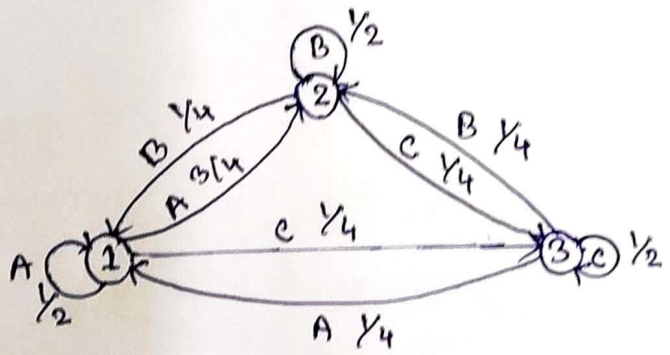
$= -0.25\log_2(0.25) - 0.1\log_2(0.1) - 0.3\log_2(0.3)$
$- 0.05\log_2(0.05) - 0.10\log_2(0.10) - 0.05\log_2(0.05)$
$- 0.1\log_2(0.1) - 0.05\log_2(0.05)$

$= 2.665$ bits/symbol

The conditional entropy, $H(X|Y) = H(X,Y) - H(Y)$
$= (2.665 - 1.857)$
$= 0.809$ bits/symbol

$H(Y|X) = H(X,Y) - H(X) = (2.665 - 2.066)$
$= 0.6$ bits/symbol. (Ans.)

1]



For the 1st order Markov source, draw the tree diagram representing the states of the end of second symbol interval and find the corresponding probabilities.

Solution:

Let, $P(1) = P(2) = P(3) = \frac{1}{3}$

$P(AA) \rightarrow P[(1\rightarrow1\rightarrow1) \text{ or } (2\rightarrow1\rightarrow1) \text{ or } (3\rightarrow1\rightarrow1)]$

$= \left(\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4}\,\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{2}\right) = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} = \frac{1}{6}$

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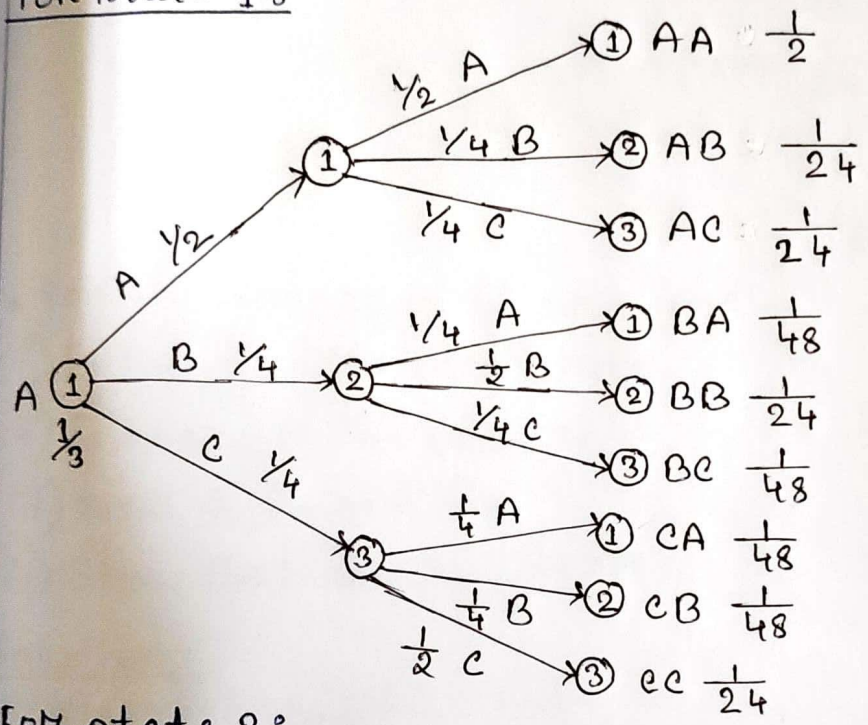
$P(AC) \rightarrow \frac{1}{24} + \frac{1}{48} + \frac{1}{48} = \frac{1}{12}$; $P(BB) = \frac{1}{24} + \frac{1}{12} + \frac{1}{24} = \frac{1}{6}$

$P(AB) = \frac{1}{24} + \frac{1}{48} + \frac{1}{48} = \frac{1}{12}$; $P(BC) = \frac{1}{48} + \frac{1}{24} + \frac{1}{48} = \frac{1}{12}$

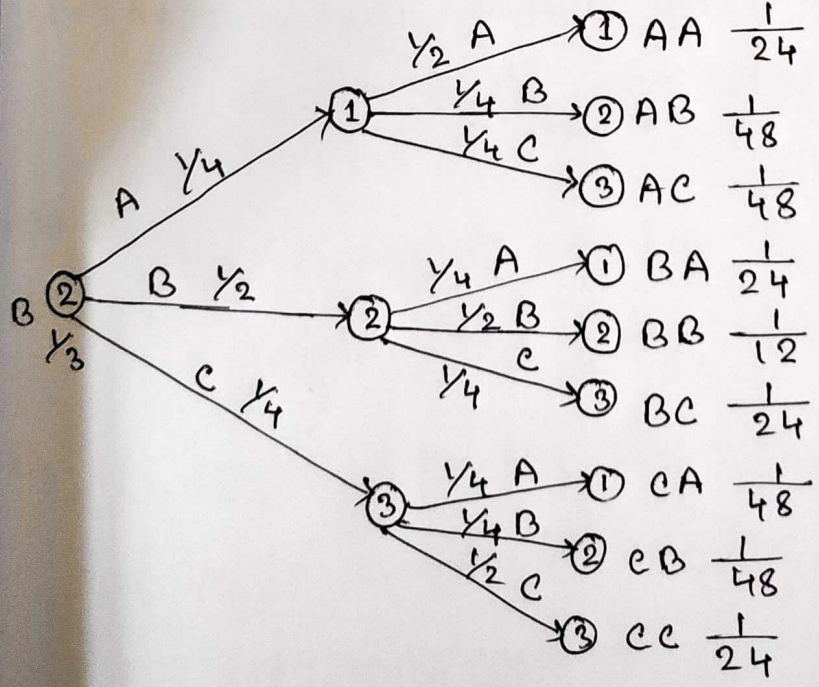
$P(BA) = \frac{1}{48} + \frac{1}{24} + \frac{1}{48} = \frac{1}{12}$; $P(CB) = \frac{1}{48} + \frac{1}{48} + \frac{1}{24} = \frac{1}{12}$

$P(CA) = \frac{1}{48} + \frac{1}{48} + \frac{1}{24} = \frac{1}{12}$; $P(CC) = \frac{1}{24} + \frac{1}{24} + \frac{1}{12} = \frac{1}{6}$

For state 1:



For state 2:



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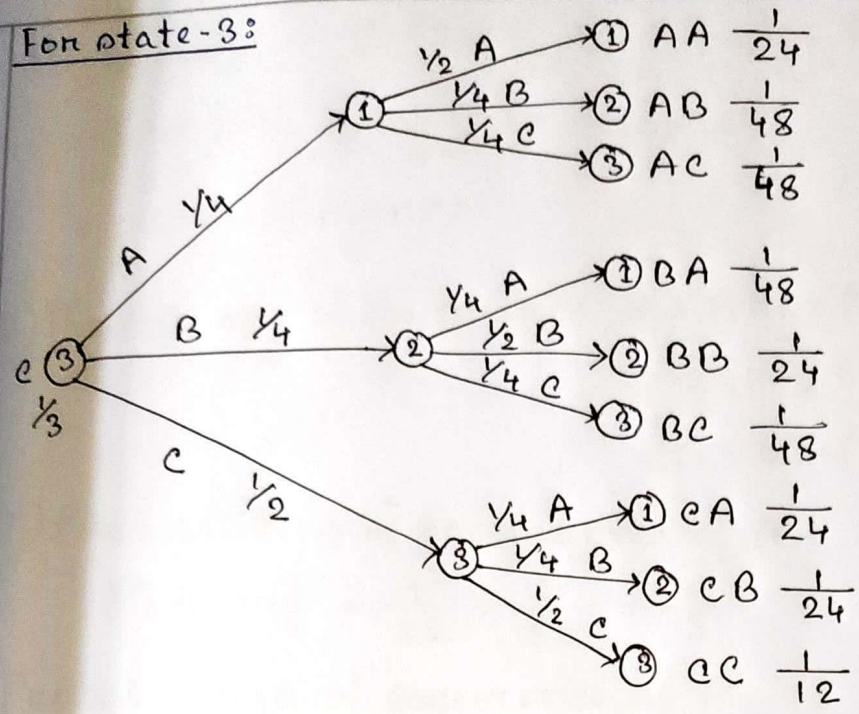
For state-3:



- For the source of figure find -

i) Entropy of the each state

ii) Entropy of the source

iii) Find G_1, G_2 and G_3

iv) Show that $G_1 > G_2 > G_3 > H$

Solution:

State probabilities $= \frac{1}{3}$

i) Entropy of each state is given by,

$$H_i = \sum_{j=1}^{n} P_{ij} \log_2 \frac{1}{P_{ij}} \text{ bits/msg symbol}$$

For H_1,

$$H_1 = P_{11} \log_2 \frac{1}{P_{11}} + P_{12} \log_2 \frac{1}{P_{12}} + P_{13} \log_2 \frac{1}{P_{13}}$$

$$= \frac{1}{2} \log_2(2) + \frac{1}{4} \log_2(4) + \frac{1}{4} \log_2(4) = 1.5 \text{ bits/msg symbol.}$$

$$H_2 = P_{21} \log_2 \frac{1}{P_{21}} + P_{22} \log_2 \frac{1}{P_{22}} + P_{23} \log_2 \frac{1}{P_{23}}$$

$$= \frac{1}{4} \log_2(4) + \frac{1}{2} \log_2(2) + \frac{1}{4} \log_2(4)$$

$$= 1.5 \text{ bits/msg symbol.}$$

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$$H_3 = P_{31}\log_2\frac{1}{P_{31}} + P_{32}\log_2\frac{1}{P_{32}} + P_{33}\log_2\frac{1}{P_{33}}$$

$$= \frac{1}{4}\log_2(4) + \frac{1}{4}\log_2(4) + \frac{1}{2}\log_2(2)$$

$$= 1.5 \text{ bits/symbol.}$$

ii) The entropy of the source, $H = P_1H_1 + P_2H_2 + P_3H_3$

$$= \frac{1}{3}(1.5) + \frac{1}{3}(1.5) + \frac{1}{3}(1.5)$$

$$= 1.5 \text{ bits/msg-symbol.}$$

iii) Probabilities of occurrence of A,

$$P(A) = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$$

Probabilities of occurrence of B,

$$P(B) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$$

Probabilities of occurrence of C,

$$P(C) = \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{3}$$

For G_1, $P(A) = P(B) = P(C) = \frac{1}{3}$

$$G_1 = \frac{1}{1}\sum_i P(m_i)\log_2\frac{1}{P(m_i)}$$

$$= 1\left[\frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(3) + \frac{1}{3}\log_2(3)\right]$$

$$= 1.585 \text{ bits/msg symbol.}$$

and, $H = 1.5$ bits/msg-symbol.

For G_2, $P(AA) = \frac{1}{6}$, $P(BB) = \frac{1}{6}$, $P(CC) = \frac{1}{6}$

$P(AB) = \frac{1}{12}$, $P(BC) = \frac{1}{12}$, $P(CB) = \frac{1}{12}$

$P(AC) = \frac{1}{12}$, $P(BA) = \frac{1}{12}$, $P(CA) = \frac{1}{12}$

$$G_3 = \frac{1}{3}\sum_i P(m_i)\log_2\frac{1}{P(m_i)}$$

$$= \frac{1}{3}\left[3 \times \frac{1}{12}\log_2(12) + 12\cdot\frac{1}{24}\log_2(24) + 12\cdot\frac{1}{48}\log_2(48)\right]$$

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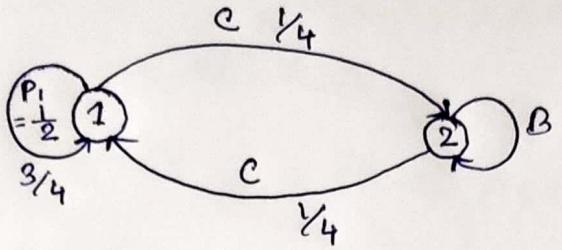
$= 1.528$ bits/msg-symbol.

2.



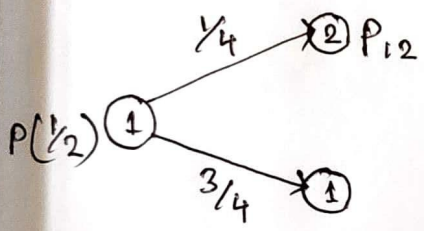
Find the followings:

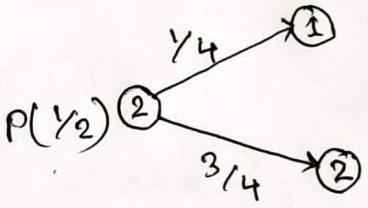
i) Source Entropy

ii) Average information content per symbol [G_{11}, G_{12}, G_{13}].

<u>Solution:</u>

Source Entropy, $H = \sum_{i=1}^{m} P_i H_i$; $H_i = \sum_{j=1}^{n} P_{i,j} \log_2 \frac{1}{P_{i,j}}$





$$H_1 = P_{11} \log_2 \frac{1}{P_{11}} + P_{12} \log_2 \frac{1}{P_{12}}$$

$$= \frac{3}{4} \log_2 \frac{4}{3} + \frac{1}{4} \log_2 4 = 0.8112 \text{ bits/symbol.}$$

$$H_2 = P_{21} \log_2 \frac{1}{P_{21}} + P_{22} \log_2 \frac{1}{P_{22}}$$

$$= \frac{1}{4} \log_2 (4) + \frac{3}{4} \log_2 \left(\frac{4}{3}\right) = 0.8112 \text{ bits/symbol.}$$

Source Entropy, $H = \sum_{i=1}^{n} P_i H_i$

Entropy of source, $H = \left(\frac{1}{2} \times 0.8112\right) + \left(\frac{1}{2} \times 0.8112\right)$

$= 0.8112$ bits/symbol.

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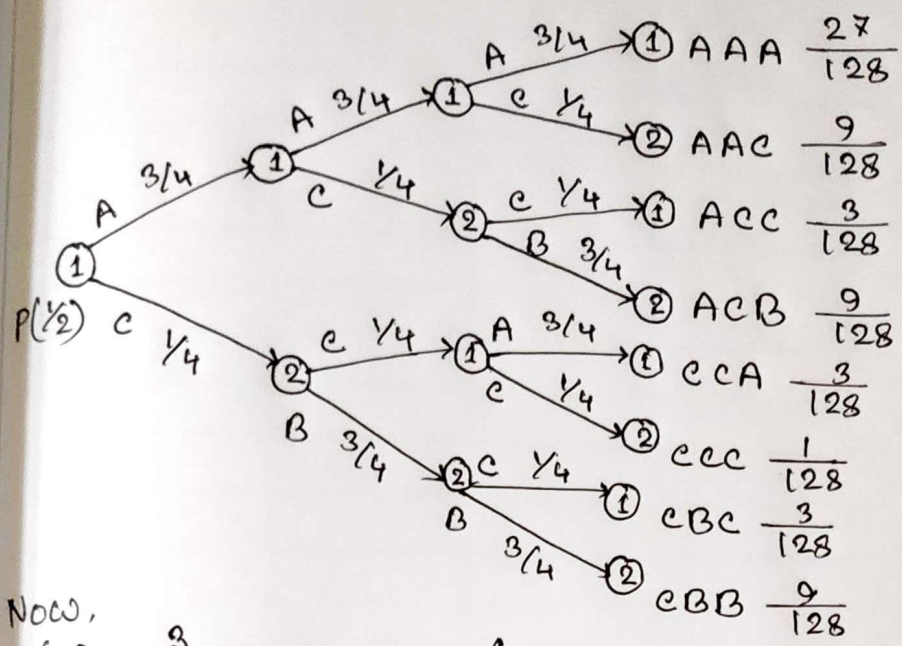
For state-1:

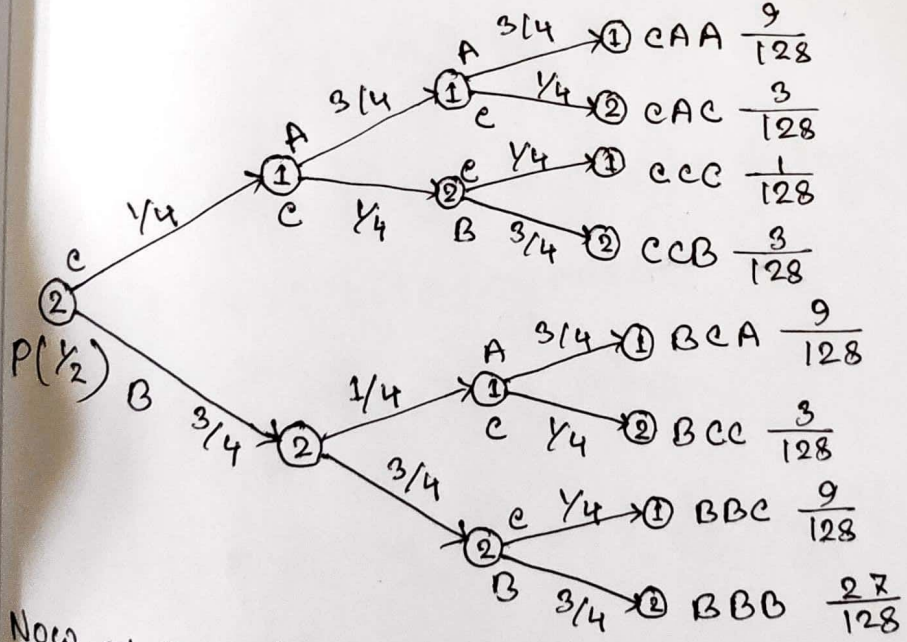


Now,

$P(A) = \frac{3}{8}$; $P(C) = \frac{1}{8}$

$P(AA) = \frac{9}{32}$; $P(CC) = \frac{1}{32}$

$P(AC) = \frac{3}{32}$; $P(CB) = \frac{3}{32}$



Now, $P(C) = \frac{1}{8}$; $P(B) = \frac{3}{8}$

$P(CA) = \frac{3}{32}$; $P(BC) = \frac{3}{32}$

$P(CC) = \frac{1}{32}$; $P(BB) = \frac{9}{32}$

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Message of length 1 (m_1)	P	m_2	P	m_3	P
A	3/8	AA	9/32	AAA	27/128
B	3/8	AC	3/32	AAC	9/128
C	1/4	CC	1/16	ACC	3/128
		CB	3/32	ACB	9/128
		CA	3/32	CCA	3/128
		BC	3/2	CCC	1/64
		BB	9/32	CBC	3/128
				CBB	9/128
				CAA	9/128
				CAC	3/128
				CCB	3/128
				BCA	9/128
				BCC	3/128
				BBC	9/128
				BBB	27/128

$$G_{1N} = \frac{1}{N}\sum_{i=1}^{n} P(m_i)\log_2 \frac{1}{P(m_i)}$$

$$G_{11} = \frac{1}{1}\sum_{i=1}^{n} P(m_1)\log_2 \frac{1}{P(m_1)}$$

$$= \frac{3}{8}\log_2\left(\frac{8}{3}\right) + \frac{3}{8}\log_2\left(\frac{8}{3}\right) + \frac{1}{4}\log_2(4)$$

$$= 1.5613 \text{ bits/symbol.}$$

$$G_{12} = \frac{1}{2}\sum_{i=1}^{n} P(m_2)\log_2 \frac{1}{P(m_2)}$$

$$= \frac{1}{2}\left[2\times\frac{9}{32}\log_2\frac{32}{9} + 4\times\frac{3}{32}\log_2\frac{32}{3} + \frac{1}{16}\log_2(16)\right]$$

$$= 1.28 \text{ bits/symbol.}$$

$$G_{13} = \frac{1}{3}\sum_{i=1}^{n} P(m_3)\log_2 \frac{1}{P(m_3)}$$

$$= \frac{1}{3}\left[2\times\frac{27}{128}\log_2\left(\frac{128}{27}\right) + 6\times\frac{9}{28}\log_2\left(\frac{128}{9}\right) + 6\times\frac{3}{128}\log_2\left(\frac{128}{3}\right) + \frac{1}{64}\log_2(64)\right]$$

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$= 1.139$ bits/symbol.

Example-1: The data is to be transmitted at the rate of 10000 bits/sec. over a channel having bandwidth $B = 3000\,Hz$. Determine the signal to noise ratio required. If the bandwidth is increased to 10000 Hz, then determine the signal to noise ratio.

Solution:

Given that,

Data rate = 10,000 bits/sec

$BW = 3000\,Hz$

So, channel capacity, $C = B\log_2\left(1+\frac{S}{N}\right)$

$$\text{Or, } 10000 = 3000\log_2\left(1+\frac{S}{N}\right)$$

$$\text{Or, } \log_2\left(1+\frac{S}{N}\right) = 3.333$$

$$\text{Or, } 1+\frac{S}{N} = 2^{3.333} = 10.08$$

$$\text{Or, } SNR = 9.08$$

for $BW = 10,000\,Hz$

Channel capacity, $C = B\log_2(1+SNR)$

$$\text{Or, } 10,000 = 10,000\log_2(1+SNR)$$

$$\text{Or, } 2 = 1+SNR$$

$$\text{Or, } SNR = 1$$

Hence it is seen that when bandwidth is increased to 10,000 Hz, then the signal to noise ratio is reduced by nine times. This means, if bandwidth is increased, the required signal power is reduced.

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Example-2: An analog signal having 4 KHz bandwidth is sampled at 1.25 times the Nyquist rate and each sample is quantized into one of 256 equally likely levels. Assuming the samples to be statistically independent:

a) What is the information rate of the source?

b) Can the output of this source transmitted without error over AWGN channel with a bandwidth of 10KHz and SNR of 20dB.

c) Find SNR required for error free transmission of part (b).

d) Find the BW required for an AWGN channel for an error free transmission of the output of this source if SNR is 20dB.

Solution:

a) We know, Information rate, $R = rH$

Given that, $BW = 4$ KHz

Nyquist rate $= (4 \times 2) = 8$ KHz

Sampling rate $= (8 \times 1.25)$

$= (8000 \times 1.25) = 10{,}000$ messages/sec.

again, Entropy, $H = \log_2\left(\frac{1}{P}\right) = \log_2 M$

$= \log_2 256 = 8$ bit/sample

$\therefore$ Information rate, $R = (10000 \times 8)$

$= 80{,}000$ bits/sec.

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b) The condition of error free transmission is $R \leq C$

Now, $\left(\frac{S}{N}\right)dB = 10\log_{10}\left(\frac{S}{N}\right)$

Or, $20 = 10\log_{10}\left(\frac{S}{N}\right)$

Or, $\frac{S}{N} = 100$

The channel capacity, $C = B\log_2(1+SNR)$

$= 10{,}000\log_2(1+100)$

$= 66.582$ kbits/sec.

If $R \leq C$, then error free transmission is possible. Hence, $R = 80{,}000$ and $C = 66.582$ kbits/sec. But $R > C$. So, error free transmission is not possible

c) For error free transmission, $R \leq C$

Or, $R \leq B\log_2(1+SNR)$

Or, $B \leq \log_2(1+SNR)$

Or, $(1+SNR) \geq 2^B$

Or, $\frac{S}{N} \geq 255$

Or, $\left(\frac{S}{N}\right)dB = 24dB$

i.e the minimum SNR required for error free transmission is $= 24dB$.

d) Given, $(SNR)dB = 20dB$ $\therefore SNR = 100$

For error free transmission, $R \leq C$

Or, $R \leq B\log_2(1+SNR)$

Or, $80{,}000 \leq B\log_2(1+100)$

Or, $B \geq 11.974$ KHz.

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Example-3: In a facsimile transmission of a picture there are about 2.25×10^6 picture elements per frame. For good reception, twelve brighten levels are necessary. Assuming all these levels to be equiprobable, calculate the channel BW required to transmit on picture in every three minutes for a SNR ratio of 30dB. If SNR requirement increases by 40dB, calculate the new BW. Explain the trade-off between bandwidth and SNR by comparing two results.

Solution:

Message rate, $r = \dfrac{2.25 \times 10^6}{3 \text{ min}} = \dfrac{2.25 \times 10^6}{180}$

$= 12500$ picture elements/sec.

Here, $M = 12$. All these levels are equally probable

$\therefore H = \log_2 M = \log_2 12 = 3.585$ bits/symbol.

The transmission rate, $R = rH = (12500 \times 3.585)$

$= 44812$ bits/sec

Given, $SNR = 30dB = 10\log_{10}(SNR)$

$\therefore SNR = 1000$

For, error free transmission, $R \leq C$.

Or $R \leq B\log_2(1 + SNR)$

Or, $44812 \leq B\log_2(1+1000)$

$\therefore B \geq 4495.93$ Hz or, 4.5 KHz

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Again, for SNR = 40dB $\therefore$ SNR = 10,000

Now, $R \leq C$

Or, $R \leq B\log_2(1+SNR)$

Or, $44812 \leq B\log_2(1+10,000)$

$\therefore B \geq 3372.4\,Hz$

Or, $B \geq 3.37\,KHz$

Example-4: A voice grade telephone channel has a BW of 3400 Hz. If the SNR on the channel is 30dB, determine the capacity of the channel. If the above channel is to be used to transmit 4.8 kbps of data determine the minimum SNR required on the channel.

Solution: Given that, BW = 3400 Hz and $\left(\frac{S}{N}\right)dB = 30dB$

$(SNR)dB = 10\log_{10}(SNR)$

Or, $30 = 10\log_{10}(SNR)$ $\therefore$ SNR = 1000

$\therefore$ The channel capacity, $C = B\log_2(1+SNR)$

$= 3400\log_2(1+1000)$

$= 33.888\,kbps$

Again, Given transmission rate, $R = 4.8\,kbps$

For error free transmission, $R \leq C$

Or, $R \leq B\log_2(1+SNR)$

Or, $4.8\times10^3 \leq 3400\log_2(1+SNR)$

Or, $SNR \geq 1.66$ or $2.2\,dB$

So, the minimum 1.66 SNR or 2.2dB SNR will be required to transmit data at the rate of 4.8kbbps.

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2016 (Self Study) - 2c : Prove that for CRC code, $V = T + E$.

Solution:

For a k-bit block of bits the (n, k) CRC encoder generates $(n-k)$ bit long frame check sequence (FCS).

Let's consider the followings :-

T = n-bit frame to be transmitted

D = k-bit message block (information bits)

F = $(n-k)$ bit FCS, the last $(n-k)$ bits of T

P = the predetermined divisor, a pattern of $(n-k+1)$ bits.

Since the predetermined divisor, P, should be able to divide the codeword T. Thus T/P has no remainder. Now, D is the k-bit message block. Therefore $2^{n-k}D$ amounts to shifting the k-bits to the left by $(n-k)$ bits, and padding the result with zeros. The codeword, T, can then be represented as -

$$T = 2^{n-k}D + F$$

[$\because$ Left shift by 1 bit of a binary sequence is equavalent to multiplying the number represented by the binary sequence by two]

Adding F in the above equation yields the concatenation of D and F. If we divide $2^{n-k}D$ by P, we obtain

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$$\frac{2^{n-k}D}{P} = Q + \frac{R}{P}$$

where, Q is the quotient and R/p is the remainder

Suppose we use R as the FCS, then,

$$T = 2^{n-k}D + R$$

In this case, upon dividing T by p, we obtain

$$\frac{T}{P} = \frac{2^{n-k}D + R}{P} = \frac{2^{n-k}D}{P} + \frac{R}{P}$$

$$= Q + \frac{R}{P} + \frac{R}{P} = Q$$

Thus there is no remainder, i.e. T is exactly divis-
-ible by P. To generate such an FCS, we simply divi-
-de $2^{n-k}D$ by P and use the (n-k) bit remainder as the
FCS.

Now, if an error E occurs when T is transmitted
over a noisy channel. The received word is then
given by -

$$V = T + E$$

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