CSE 404: Introduction to Machine Learning (Fall 2019)

Homework #1 Solution

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, E = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}.$$

If possible, compute the following:

(a) $(2A)^T$

(b) $(A - B)^T$

(c) $(3B^T - A)^T$

(d) $(-A)^T E$

(e) $(C + 2D^T + E)^T$

Solution:

(a) $(2A)^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \\ 6 & 8 \end{bmatrix}$

(b) Not possible, dimensions do not agree.

(c) $(3B^T - A)^T = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ 6 & 2 \end{bmatrix}$

(d) $(-A)^T E = \begin{bmatrix} -7 & -6 \\ -8 & 0 \\ -17 & -10 \end{bmatrix}$

(e) Not possible, dimensions do not agree.

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$

Is $AB = BA$? Justify your answer.

Solution: We can verify

$$AB = \begin{bmatrix} -10 & 15 \\ 0 & 5 \end{bmatrix}, BA = \begin{bmatrix} -1 & 6 \\ 9 & -4 \end{bmatrix}.$$

Therefore $AB \neq BA$.

3. (10 points) Given the three vectors $v_1 = (-2, 0, 1), v_2 = (0, 1, 0)$ and $v_3 = (2, 0, 4)$ in $\mathbb{R}^3$.

- Show that they form an orthogonal set under the standard Euclidean inner product for $\mathbb{R}^3$ but not an orthonormal set.
- Turn them into a set of vectors that will form an orthonormal set of vectors under the standard Euclidean inner product for $\mathbb{R}^3$.

Solution:

- Show that they form an orthogonal set is equivalent to showing that $v_i^T v_j = 0, \forall i \neq j$.

$$\begin{aligned} v_1^T v_2 &= -2 * 0 + 0 * 1 + 1 * 0 = 0 \\ v_1^T v_3 &= -2 * 2 + 0 + 1 * 4 = 0 \\ v_2^T v_3 &= 0 * 2 + 0 * 1 + 0 * 4 = 0. \end{aligned}$$

  Hence, $\{v_1, v_2, v_3\}$ forms an orthogonal set under the standard Euclidean inner product for $\mathbb{R}^3$. However, note that $\|v_1\|_2 = \sqrt{5} \neq 1$ and $\|v_3\|_2 = 2\sqrt{5} \neq 1$, which means that $\{v_1, v_2, v_3\}$ are not an orthonormal set.

- By normalizing v_1 and v_3, we can obtain the orthonormal set:

$$\left\{ \left(\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right), (0, 1, 0), \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right) \right\}$$

4. (10 points) Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, show that the rank of matrix xy^T is one.

   Solution: Outer product generates the matrix whose first row is $x_1(y_1, y_2, \ldots, y_n)$, and the i^{th} row is $x_i(y_1, y_2, \ldots, y_n)$. So the rows are the vector $(y_1, y_2, \ldots, y_n)$ multiplied by scalars. So this itself is the basis. Hence the rank is 1.

5. (10 points) Given $X = [x_1, x_2, \cdots, x_n] \in \mathbb{R}^{m \times n}$ where $x_i \in \mathbb{R}^m$ for all i, and $Y^T = [y^1, y^2, \cdots, y^n] \in \mathbb{R}^{p \times n}$ where $y^i \in \mathbb{R}^p$ for all i. Show that

$$XY = \sum_{i=1}^{n} x_i (y^i)^T.$$

   Solution: Since $(XY)_{i,j} = \sum_{k=1}^n x_{ki} y_j^k$ and $(\sum_{k=1}^n x_{k,i} (y_j^k)^T)_{ij} = \sum_{k=1}^n x_{ki} y_j^k$, $XY = \sum_{k=1}^n x_{k,i} (y_j^k)^T$

6. (10 points) Given $X \in \mathbb{R}^{m \times n}$, show that the matrix $X^T X$ is symmetric and positive semi-definite. When is it positive definite?

   Solution: $(X^T X)^T = X^T X$. Hence $X^T X$ is symmetric.

   $\forall y \in R^n, y^T X^T X y = (Xy)^T (Xy) \geq 0$. So it is symmetric and positive semidefinite.

   When $\text{rank}(X) = n$, X is column full rank, it is positive semidefinite.

7. (10 points) Given $g(x, y) = e^x + e^{y^2} + e^{2xy}$, compute $\frac{\partial g}{\partial y}$.

   Solution: $\frac{\partial g}{\partial y} = 2ye^{y^2} + 2xe^{2xy}$

8. (30 points) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix},$$

   (a) Compute the eigenvalues and corresponding eigenvectors of A. You are allowed to use Matlab/Python to compute the eigenvectors (but not the eigenvalues).

   Solution:

$$\begin{bmatrix} 2-\lambda & 1 & 3 \\ 1 & 1-\lambda & 2 \\ 3 & 2 & 5-\lambda \end{bmatrix} = -\lambda^3 + 8\lambda^2 - 3\lambda = 0$$

Therefore

$$\lambda = 0, \lambda = \frac{8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = 4 \pm \sqrt{13}$$

Eigenvalues: $\lambda = 0, 4 - \sqrt{13}, 4 + \sqrt{13}$
Eigenvectors:

$$\begin{pmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{pmatrix}$$

$$\begin{pmatrix} 0.6526 \\ -0.7513 \\ -0.0988 \end{pmatrix}$$

$$\begin{pmatrix} 0.4908 \\ 0.3197 \\ 0.8105 \end{pmatrix}$$

(b) What is the eigen-decomposition of A?
Solution:

$$U = \begin{pmatrix} 0.5774 & 0.6526 & 0.4908 \\ 0.5774 & -0.7513 & 0.3197 \\ -0.5774 & -0.0988 & 0.8105 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4-\sqrt{13} & 0 \\ 0 & 0 & 4+\sqrt{13} \end{pmatrix}$$

Eigen-decomposition: $A = UDU^T$

(c) What is the rank of A?
Solution: 2

(d) Is A positive definite?
Solution: No.

(e) Is A positive semi-definite?
Solution: Yes.

(f) Is A singular?
Solution: Yes.