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HW1

1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}$ $D = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$

a) $2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix}$

$(2A)^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \\ 6 & 8 \end{bmatrix}$

b) $(A-B)^T$

$= (A-B) \rightarrow$ The difference between two matrices is defined only when both the matrices are of the same size. Since A is 2×3 and B is 3×2, subtraction between A and B is not possible as they are not ordered the same way. Therefore, we cannot compute $(A-B)^T$.

c) $(3B^T - A)^T$

$B^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ $3B^T = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \end{bmatrix}$

$3B^T - A = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ -2 & 2 & 2 \end{bmatrix}$

$(3B^T - A)^T = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ 6 & 2 \end{bmatrix}$

d) $(-A)^T E$

$(-A)^T = \begin{bmatrix} -1 & -2 \\ -2 & -1 \\ -3 & -4 \end{bmatrix}$

$(-A)^T \cdot E = \begin{bmatrix} -1 & -2 \\ -2 & -1 \\ -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$

$= \begin{pmatrix} -1\times3+(-2)\times2 & -1\times(-2)+(-2)\times4 \\ -2\times3+(-1)\times2 & -2\times-2+(-1)\times4 \\ -3\times3+(-4)\times2 & -3\times-2+(-4)\times4 \end{pmatrix}$

$= \begin{pmatrix} -7 & -6 \\ -8 & 0 \\ -17 & -10 \end{pmatrix}$

e) $(C + 2D^T + E)^T$

$C + 2D^T + E$

$D^T = \begin{bmatrix} 2 & 0 & 3 \\ -4 & 1 & 2 \\ 5 & 4 & 1 \end{bmatrix}$ $2D^T = \begin{bmatrix} 4 & 0 & 6 \\ -8 & 2 & 4 \\ 10 & 8 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 6 \\ -8 & 2 & 4 \\ 10 & 8 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 7 & -1 & 9 \\ -4 & 3 & 9 \\ 12 & 9 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$ → The addition between these two matrices is defined only when both the metrices are of the same size. Since $C + 2D^T$ is 3×3 and E is 2×2, addition between them is not possible as they are not ordered the same way. Therefore, we cannot compute $C + 2D^T + E$ and therefore $(C + 2D^T + E)^T$ cannot be computed also.

2) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

$= \begin{pmatrix} 1\times 2 + 4\times -3 & 1\times -1 + 4\times 4 \\ 3\times 2 + 2\times -3 & 3\times -1 + 2\times 4 \end{pmatrix}$

$= \begin{pmatrix} -10 & 15 \\ 0 & 5 \end{pmatrix}$

$BA = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$= \begin{pmatrix} 2\times 1 + (-1)\times 3 & 2\times 4 + (-1)\times 2 \\ -3\times 1 + 4\times 3 & -3\times 4 + 4\times 2 \end{pmatrix}$

$= \begin{pmatrix} -1 & 6 \\ 9 & -4 \end{pmatrix}$

$AB = \begin{pmatrix} -10 & 15 \\ 0 & 5 \end{pmatrix} \neq \begin{pmatrix} -1 & 6 \\ 9 & -4 \end{pmatrix} = BA.$

Two matrices are equal if the corresponding entrices are equal, but in this case it can be seen that is not the case, so the two matrices are not equal.

3) Given three vectors, $V_1 = (-2, 0, 1)$ $V_2 = (0, 1, 0)$ $V_3 = (2, 0, 4)$

Showing orthogonal set ⇒

$V_1 \cdot V_2 = -2\cdot 0 + 0\cdot 1 + 1\cdot 0 = 0$

$V_1 \cdot V_3 = -2\cdot 2 + 0\cdot 0 + 1\cdot 4 = 0$

$V_2 \cdot V_3 = 0\cdot 2 + 1\cdot 0 + 0\cdot 4 = 0$

} Proving $\{V_1, V_2, V_3\}$ is an orthogonal set.

Showing that they are not an orthonormal set $\Rightarrow$

$$\|v_1\| = \sqrt{4+0+1} = \sqrt{5} \neq 1$$

$$\|v_2\| = \sqrt{0+1+0} = 1 = 1$$

$$\|v_3\| = \sqrt{4+0+16} = \sqrt{20} \neq 1$$

As $\|v_1\|$ & $\|v_3\|$ does not equal 1. Hence $\{v_1, v_2, v_3\}$ is not an orthonormal set.

Let $v_1' = \dfrac{v_1}{\|v_1\|} = \left(\dfrac{-2}{\sqrt{5}}, \dfrac{0}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right) = \left(\dfrac{-2}{\sqrt{5}}, 0, \dfrac{1}{\sqrt{5}}\right)$

& $v_2' = v_2 = (0, 1, 0)$

& $v_3' = \dfrac{v_3}{\|v_3\|} = \left(\dfrac{2}{\sqrt{20}}, \dfrac{0}{\sqrt{20}}, \dfrac{4}{\sqrt{20}}\right) = \left(\dfrac{2}{\sqrt{20}}, 0, \dfrac{4}{\sqrt{20}}\right)$

$$\|v_1'\| = \sqrt{\left(\frac{-2}{\sqrt{5}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

$\|v_2'\| = \|v_2\|$ (as $v_2' = v_1$) then $\|v_2'\| = 1$

$$\|v_3'\| = \sqrt{\left(\frac{2}{\sqrt{20}}\right)^2 + 0^2 + \left(\frac{4}{\sqrt{20}}\right)^2} = 1$$

So, $\|v_1'\| = \|v_2'\| = \|v_3'\| = 1$

And now, $v_1' \cdot v_2' = 0 \times \dfrac{-2}{\sqrt{5}} + 0 \times 1 + 0 \times \dfrac{1}{\sqrt{5}} = 0$

$$v_1' \cdot v_3' = \frac{2}{\sqrt{20}} \times \frac{-2}{\sqrt{5}} + 0 \times 0 + \frac{1}{\sqrt{5}} \times \frac{4}{\sqrt{20}} = -\frac{2}{5} + \frac{2}{5} = 0$$

$$v_2' \cdot v_3' = 0 \times \frac{2}{\sqrt{20}} + 0 \times 0 + \frac{4}{\sqrt{20}} \times 0 = 0.$$

So, $v_1' \cdot v_2' = v_1' \cdot v_3' = v_2' \cdot v_3' = 0$. There fo

Putting all together, hence $\{v_1', v_2', v_3'\}$ is an orthonormal set.

4) Given $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$:

Let,

$$[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1} \quad \text{and} \quad [y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} \quad \text{then,}$$

$[y]^T = [y_1 \; y_2 \; y_3 \; \cdots \; y_n]_{1\times n}$ So, now showing xy^T

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_2 & x_2y_2 & x_3y_3 & \cdots & x_2y_n \\ \vdots & & & & \vdots \\ x_my_1 & \cdots & & \cdots & x_my_n \end{bmatrix}_{m\times n}$$

When both the real numbers contain only one dimentional Space then $m=n=1$.

$xy^T = [x_1y_1]_{1\times 1}$ in this case the rank of matrix xy^T is 1.

5) Given $X = [x_1, x_2, \cdots, x_n] \in \mathbb{R}^{m\times n}$ where $x_i \in \mathbb{R}^m$ $\forall i$

$Y = [y^1, y^2, \cdots, y^n] \in \mathbb{R}^{p\times n}$ where $y^i \in \mathbb{R}^p$ $\forall i$

So from the above information, the following can be inferred $\Longrightarrow$
each x_i is a row vector $\Rightarrow$ contcining 'n' columns
and y^i is also a row vector $\Rightarrow$ containing 'n' columns $\Longrightarrow$
$(y^i)^T$ is a column vector $\Rightarrow$ containing 'n' rows. Therefore,

$x_i(y^i)^T$ is the ith element in the Summation, thus the product of XY is possible and can be defined by

$$XY = \sum_{i=1}^{n} x_i (y^i)^T.$$

6) Given $X \in \mathbb{R}^{m \times n}$.

We know that a matrix, for instance a matrix A is symmetric if $A^T = A$.
So, taking $(X^T X)^T = X^T (X^T)^T$ into consideration,

$(X^T X)^T = X^T (X^T)^T \rightarrow$ By algebric Rule 4 for Matrix Transpose
$\Rightarrow (X^T X)^T = X^T X \rightarrow$ By algebric Rule 1 for Matrix Transpose

Hence by definition of symmetry $(X^T X)^T$ is symmetric to $X^T X$.

Positive definite definition $\Rightarrow$ A positive definite matrix $\Longleftrightarrow$

$$x^T A x > 0 \quad \forall x \in \mathbb{R}^n \setminus 0$$

Positive semi-definite definition $\Rightarrow$ B positive semi-definite matrix $\Longleftrightarrow$

$$x^T B x \geq 0 \quad \forall x \in \mathbb{R}^n$$

Lets take $(y, X^T X y) = (X^T y, X^T y) = \|X^T y\|$.

The outcome from the above equation is always greater than or equal to zero. Hence it cam be implied that $(y, X^T X y) \geq 0$ so from the definition of Positive semi-definite it can be implied that $y, X^T X y$ is positive semi-definite $\Longrightarrow$ $X^T X$ is positive semi-definite.

The reason it is not a positive definite matrix because the eigen-values for a positive definite matrix are non-zero and by the definition $x^T A x > 0 \; \forall x \in \mathbb{R}^n \setminus 0$ but in this case it is $x^T A x \geq 0 \; \forall x \in \mathbb{R}^n$, so it is not positive definite. It can be positive definite when all its eigen-values are nonzero & that it follows the positive definite definition.

7) $g(x,y) = e^x + e^{y^2} + e^{2xy}$

$$\frac{dg}{dy}\left(e^x + e^{y^2} + e^{2xy}\right)$$

$$= \frac{dg}{dy}(e^x) + \frac{d}{dy}\left(e^{y^2}\right) + \frac{d}{dy}\left(e^{2xy}\right)$$

$$= 0 + \frac{d}{dy}\left(e^{y^2}\right) + \frac{d}{dy}\left(e^{2xy}\right)$$

$\frac{d}{dy}\left(e^{y^2}\right) \Rightarrow u = y^2$ then $\frac{df(u)}{dx} = \frac{df}{du} \times \frac{du}{dx} = \frac{d}{du}(e^u)\frac{d}{dy}(y^2)$

$$\Rightarrow e^u \times 2y = 2ye^u = 2ye^{y^2}$$

$\frac{d}{dy}\left(e^{2xy}\right) \Rightarrow u = 2xy$ then $\frac{df(u)}{dx} = \frac{df}{du} \times \frac{du}{dx} = \frac{d}{du}(e^u)\frac{d}{dy}(2xy)$

$$= e^u \times 2x = 2xe^u = 2xe^{2xy}.$$

Then $\frac{dg}{dy}\left(e^x + e^{y^2} + e^{2xy}\right) = 2y\,e^{y^2} + 2xe^{2xy}$

8) a) Computing eigenvalue of A.

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\lambda I = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Formulas $\Rightarrow$ determinant of $A \Rightarrow$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei - afh - bdi\,(-)(-)\,bgf + cdh - ceg$$

$$= aei - afh - bdi + bgf + cdh - ceg$$

For eigenvalue $\Rightarrow$

$$\det(A - \lambda I) = 0$$

$A - \lambda I$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & 3 \\ 1 & 1-\lambda & 2 \\ 3 & 2 & 5-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)(1-\lambda)(5-\lambda) - 4(2-\lambda) - (5-\lambda) + 6 + 6 - 9(1-\lambda)$$
$$= (2-\lambda)(1-\lambda)(5-\lambda) - 8 + 4\lambda - 5 + \lambda + 12 - 9 + 9\lambda$$
$$= 2((1-\lambda)(5-\lambda)) - \lambda((1-\lambda)(5-\lambda)) + 14\lambda - 10$$
$$= 2(5-\lambda-5\lambda+\lambda^2) - \lambda(5-\lambda-5\lambda+\lambda^2) + 14\lambda - 10$$
$$= 10 - 2\lambda - 10\lambda + 2\lambda^2 - 5\lambda + \lambda^2 + 5\lambda^2 - \lambda^3 + 14\lambda - 10$$
$$= -\lambda^3 + 8\lambda^2 - 3\lambda$$

Eigenvectors ⇒ Are to be found in the code attached.

Then $-\lambda^3 + 8\lambda^2 - 3\lambda = 0$

$-\lambda^3 + 8\lambda^2 - 3\lambda = 0$
$\Rightarrow \lambda^3 - 8\lambda^2 + 3\lambda = 0$
$\Rightarrow \lambda(\lambda^2 - 8\lambda + 3) = 0$
$\Rightarrow \lambda_1 = 0$

$\lambda^2 - 8\lambda + 3 = 0$
$\Rightarrow \lambda = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4\times1\times3}}{2\times1}$
$= -\sqrt{13} + 4, \sqrt{13} + 4.$

$\left.\begin{array}{l}\lambda_1 = 0 \\ \lambda_2 = -\sqrt{13} + 4 \\ \lambda_3 = \sqrt{13} + 4\end{array}\right\}$ eigen values of vector A

b) The eigen decomposition of A is $A = QDQ^{-1}$; here Q is a 3x3 matrix with v_1, v_2, v_3 as its column and D is 3x3 diagonal matrix with $\lambda_1, \lambda_2, \lambda_3$ as the entries on its leading diagonal.

c) Rank of the matrix ⇒

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 5 \end{pmatrix}$$

Swapping matrix rows ⇒ $R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

then $R_2 \leftarrow R_2 - \frac{1}{3} \cdot R_1$

$$\begin{pmatrix} 3 & 2 & 5 \\ 0 & 1/3 & 1/3 \\ 2 & 1 & 3 \end{pmatrix}$$

then $R_3 \leftarrow R_3 - 2/3 \cdot R_1$

$$= \begin{pmatrix} 3 & 2 & 5 \\ 0 & 1/3 & 1/3 \\ 0 & -1/3 & -1/3 \end{pmatrix}$$

Cancel leading coefficient in row R_3 by performing $R_3 \leftarrow R_3 + 1 \cdot R_2$

$$= \begin{pmatrix} 3 & 2 & 5 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply matrix row $\Rightarrow R_2 \leftarrow 3 \cdot R_2$

$$\begin{pmatrix} 3 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then $R_1 \leftarrow R_1 - 2 \cdot R_2$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then $R_1 \leftarrow 1/3\, R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix is the number of non-all-zero rows. Here it is 2, so rank=2.

d) A positive definite matrix is a symmetric matrix with all positive eigenvalues. Here A has two positive eigenvalues, but the other eigenvalue is a zero. Therefore A is not positive definite.

e) A positive semi-definite matrix is a symmetric matrix with non-negative eigenvalues. As stated, A has 2 positive eigenvalues and 1 zero eigenvalue, therefore it is a positive semi-definite matrix.

f) Since A has a zero eigenvalue, it is singular.