CSE 404: Perceptron Pop-Quiz Solution

1. Why does PLA Work?

   We can show that the weight update rule above has the nice interpretation that it moves in the direction of classifying x_* correctly.

   - Show that $y_*\mathbf{w}(t)^T\mathbf{x}_* < 0$;
     - **Solution:** Since data point $(y_*, \mathbf{x}_*)$ is misclassified by $\mathbf{w}(t)$, then the prediction $\text{sign}(\mathbf{w}(t)^T\mathbf{x}_*)$ is different sign from y_*, i.e., when $y_* = 1$, $\text{sign}(\mathbf{w}(t)^T\mathbf{x}_*) = -1$, and $\mathbf{w}(t)^T\mathbf{x}_*) < 0$, and therefore $y_*\mathbf{w}(t)^T\mathbf{x}_* < 0$. We can draw the same concolusion when $y_* = -1$.
   - Show that $y_*\mathbf{w}(t+1)^T\mathbf{x}_* > y_*\mathbf{w}(t)^T\mathbf{x}_*$;
     - **Solution:** Note that $\mathbf{w}(t+1) = \mathbf{w}(t) + y_*\mathbf{x}_*$, therefore

$$\begin{aligned} y_*\mathbf{w}(t+1)^T\mathbf{x}_* &= y_*(\mathbf{w}(t) + y_*\mathbf{x}_*)^T\mathbf{x}_* \\ &= y_*(\mathbf{w}(t))^T\mathbf{x}_* + y_*(y_*\mathbf{x}_*)^T\mathbf{x}_* \\ &= y_*(\mathbf{w}(t))^T\mathbf{x}_* + y_*^2\mathbf{x}_*^T\mathbf{x}_* > y_*(\mathbf{w}(t))^T\mathbf{x}_* \end{aligned}$$

       where the last inequality is because $y_*^2\mathbf{x}_*^T\mathbf{x}_* > 0$ (given that $\|\mathbf{x}_*\| \neq 0$, i.e., $\mathbf{x}_*$ is not an empty data point).
   - Move from $\mathbf{w}(t)$ to $\mathbf{w}(t+1)$ is a move 'in the right direction' in terms of classifying $\mathbf{x}_*$
     - **Solution:** Given $y_*\mathbf{w}(t+1)^T\mathbf{x}_* > y_*\mathbf{w}(t)^T\mathbf{x}_*$, we are pushing forward the classifier $\mathbf{w}(t+1)$ in a way that has a higher chance to make $y_*\mathbf{w}(t+1)^T\mathbf{x}_*$ positive, and thus a higher chance to classify $(\mathbf{x}_*, y_*)$ correctly.