Illustrative Mathematics®

LEARN MATH FOR LIFE

Teacher Unit Guide

Unit 4

Grade 6

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Unit 4

Work with fractions in grade 6 draws on earlier work in operations and algebraic thinking, particularly the knowledge of multiplicative situations developed in grades 3 to 5, and making use of the relationship between multiplication and division. Multiplicative situations include three types: equal groups; comparisons of two quantities; dimensions of arrays or rectangles. In the equal groups and comparison situations, there are two subtypes, sometimes called the partitive and the quotitive (or measurement) interpretations of division. Students are not expected to identify the three types of situations or use the terms “partitive” or “quotitive.” However, they should recognize the associated interpretations of division in specific contexts (MP7).

For example, in an equal groups situation when the group size is unknown, division can be used to answer the question, “How many in each group?” If the number of groups is unknown, division answers the question, “How many groups?” For example, if 12 pounds of almonds are equally shared among several bags:

There are 2 bags. How many pounds in each bag? (partitive)

There are 6 pounds in each bag. How many bags? (quotitive)

In a comparison situation that involves division, the size of one object may be unknown or the relative sizes of two objects may be unknown. For example, when comparing two ropes:

A rope is 12 feet long. It is twice as long as another rope. How long is the second rope? (partitive)

One rope is 12 feet long. One rope is 6 feet long. How many times longer than the second rope is the first rope? (quotitive)

In situations that involve arrays or rectangles, division can be used to find an unknown factor. In an array situation, the unknown is the number of entries in a row or a column; in a rectangle, the unknown is a length or a width measurement. For example, “The area of a rectangle is 12 square feet. One side is 6 feet long. How long is the other side?” If the rectangle is viewed as tiled by an array of 12 unit squares with 6 tiles in each row, this question can seen as asking for the number of entries in each column.

At beginning of the unit, students consider how the relative sizes of numerator and denominator affect the size of their quotient. Students first compute quotients of whole numbers, then—without computing—consider the relative magnitudes of quotients that include divisors which are whole numbers, fractions, or decimals, e.g., “Is $800 \div \frac{1}{10}$ larger than or smaller than $800 \div 2.5$?”

The second section of the unit focuses on equal groups and comparison situations. It begins with partitive and quotitive situations that involve whole numbers, represented by tape diagrams and equations. Students interpret the numbers in the two situations (MP2) and consider analogous situations that involve one or more fractions, again accompanied by tape diagrams and equations. Students learn to interpret, represent, and describe these situations, using terminology such as

“What fraction of 6 is 2?,” “How many 3s are in 12?,” “How many fourths are in 3?,” “is one-third as long as,” “is two-thirds as much as,” and “is one-and-one-half times the size of.”

The third section concerns computing quotients of fractions. Students build on their work from the previous section by considering quotients related to products of numbers and unit fractions, e.g., “How many 3s in 12?” and “What is $\frac{1}{3}$ of 12?,” to establish that dividing by a unit fraction $\frac{1}{b}$ is the same as multiplying by its reciprocal b. Building on this and their understanding that $\frac{a}{b} = a \cdot \frac{1}{b}$ (from grade 4), students understand that dividing by a fraction $\frac{a}{b}$ is the same as multiplying by its reciprocal $\frac{b}{a}$.

The fourth section returns to interpretations of division in situations that involve fractions. This time, the focus is on using division to find an unknown area or volume measurement. In grade 3, students connected areas of rectangles with multiplication, viewing a rectangle as tiled by an array of unit squares and understanding that, for whole-number side lengths, multiplying the side lengths yields the number of unit squares that tile the rectangle. In grade 5, students extended the formula for the area of rectangles with whole-number side lengths to rectangles with fractional side lengths. For example, they viewed a $\frac{2}{3}$-by-$\frac{5}{7}$ rectangle as tiled by 10 $\frac{1}{3}$-by-$\frac{1}{7}$ rectangles, reasoning that 21 such rectangles compose 1 square unit, so the area of one such rectangle is $\frac{1}{21}$, thus the area of a shape composed of 10 such rectangles is $\frac{10}{21}$. In a previous grade 6 unit, students used their familiarity with this formula to develop formulas for areas of triangles and parallelograms. In this unit, they return to this formula, using their understanding of it to extend the formula for the volume of a right rectangular prism (developed in grade 5) to right rectangular prisms with fractional side lengths.

The unit ends with two lessons in which students use what they have learned about working with fractions (including the volume formula) to solve problems set in real-world contexts, including a multi-step problem about calculating shipping costs. These require students to formulate appropriate equations that use the four operations or draw diagrams, and to interpret results of calculations in the contexts from which they arose (MP2).

Progression of Disciplinary Language

In this unit, teachers can anticipate students using language for mathematical purposes such as interpreting, representing, justifying, and explaining. Throughout the unit, students will benefit from routines designed to grow robust disciplinary language, both for their own sense-making and for building shared understanding with peers. Teachers can formatively assess how students are using language in these ways, particularly when students are using language to:

Interpret and represent

- situations involving division (Lessons 2, 3, 12, and 16)
- situations involving measurement constraints (Lesson 17)

Justify

- reasoning about division and diagrams (Lessons 4 and 5)
- strategies for dividing numbers (Lesson 11)
- reasoning about volume (Lesson 15)

Explain

- how to create and make sense of division diagrams (Lesson 6)
- how to represent division situations (Lesson 9)
- how to find missing lengths (Lesson 14)
- a plan for optimizing costs (Lesson 17)

In addition, students are expected to critique the reasoning of others about division situations and representations, and make generalizations about division by comparing and connecting across division situations, and across the representations used in reasoning about these situations. The Lesson Syntheses in Lessons 2 and 12 offer specific disciplinary language that may be especially helpful for supporting students in navigating the language of important ideas in this unit.
The table shows lessons where new terminology is first introduced, including when students are expected to understand the word or phrase receptively and when students are expected to produce the word or phrase in their own speaking or writing. Terms from the glossary appear bolded. Teachers should continue to support students' use of a new term in the lessons that follow where it was first introduced.

lesson	new terminology	
	receptive	**productive**
6.4.1	divisor dividend	quotient
6.4.2	equation interpretation	How many groups of ___? How many ___ in each group?
6.4.3	unknown equal-sized	
6.4.4	whole	
6.4.5	relationship	
6.4.6		equal-sized
6.4.7	times as ___ fraction of ___	
6.4.8	container	unknown fraction of ___
6.4.9		whole
6.4.10	**reciprocal** observations	times as ___ numerator denominator
6.4.11	evaluate	
6.4.13		gaps
6.4.14	packed	
6.4.17	assumption	packed

Required Materials

$\frac{1}{2}$-inch cubes
$\frac{1}{4}$-inch graph paper
Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Measuring tapes
Pattern blocks
Pre-printed slips, cut from copies of the blackline master
Rulers
Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Unit 4 Assessments

Assessment : Check Your Readiness (A)

Teacher Instructions

This assesses students' incoming knowledge about concepts that support understanding division of fractions.

Problem 1

The content assessed in this problem is first encountered in Lesson 3: Interpreting Division Situations.

There are a variety of strategies students can use to approach this problem. They may find a helpful fraction equivalent to $\frac{1}{2}$ to use, or they may simply look at whether the numerator is more or less than one-half of the denominator.

If most students struggle with this item, plan to support them in their understanding of fractions by visually representing the fractions in this problem and in problem 3. Students will benefit from placing numbers on the number line, representing fractions, and thinking about each fraction by comparing it to zero, $\frac{1}{2}$, and one.

Statement

Select **all** the fractions that are greater than $\frac{1}{2}$.

A. $\frac{3}{5}$

B. $\frac{7}{13}$

C. $\frac{7}{15}$

D. $\frac{10}{7}$

E. $\frac{100}{200}$

F. $\frac{1000}{2001}$

Solution

["A", "B", "D"]

Aligned Standards

4.NF.A.2

Problem 2

The content assessed in this problem is first encountered in Lesson 2: Meanings of Division.

In this problem, students distinguish between situations involving multiplication and situations involving division. In this unit, students study the meaning of dividing a number by a fraction before developing an algorithm for doing so.

If most students struggle with this item, plan to use Lesson 2 to support students in representing and reasoning about multiplication and division situations. After sharing their responses from Activity 1, ask students to think of a story for this expression and check it against the shared responses.

Statement

Select **all** the situations that can be represented by $\frac{1}{2} \cdot 5$. Explain your choices.

1. Diego lives 5 miles from school, and Elena lives $\frac{1}{2}$ as far away as Diego. How many miles does Elena live from school?
2. Jada has 5 bottles that each contain $\frac{1}{2}$ liter of water. How many liters of water is that in total?
3. Noah has 5 meters of rope. How many pieces of rope of length $\frac{1}{2}$ meter can he cut from it?
4. Lin's goal is to run 5 miles. She ran $\frac{1}{2}$ mile. What fraction of her goal is that?

Solution

Answers vary. Sample responses:

- I chose A and B because if you multiply $\frac{1}{2} \cdot 5$ you get the right answer.
- I chose A because to find half of something, you multiply it by half. Even though it gives the right answer to B, it is 5 halves, not half of 5, so I didn't choose B.

Aligned Standards

5.NF.B.6, 5.NF.B.7

Problem 3

The content assessed in this problem is first encountered in Lesson 3: Interpreting Division Situations.

As in the last problem, students can compare fractions by writing with equivalent denominators. However, this problem is much easier to solve by comparing the numerator of each fraction to the denominator and ballparking the relationship.

If most students struggle with this item, plan to support students in their understanding of fractions by visually representing the fractions in this problem using a number line. Students can then compare the fractions to zero, $\frac{1}{2}$, and one.

Statement

Order these numbers from least to greatest. 0, 1, $\frac{1}{2}$, $\frac{8}{9}$, $\frac{49}{100}$, $\frac{11}{20}$, $\frac{1}{13}$

Solution

0, $\frac{1}{13}$, $\frac{49}{100}$, $\frac{1}{2}$, $\frac{11}{20}$, $\frac{8}{9}$, 1

Aligned Standards

4.NF.A.2

Problem 4

The content assessed in this problem is first encountered in Lesson 1: Size of Divisor and Size of Quotient.

To answer this question, students must in some way explain how to determine the number of times $\frac{1}{2}$ goes into 5. Their explanations will provide insight into how clearly students are thinking about situations that involve division. Students need not use the word “division” or perform the operation of division as part of their answer.

If most students struggle with this item, plan to address any misconceptions while synthesizing Activity 2. These questions offer an opportunity to compare divisors of integers and fractions smaller than one.

Statement

Noah has 5 meters of rope. How many pieces of rope of length $\frac{1}{2}$ meter can he cut from it? Draw a diagram to illustrate your solution.

Solution

He can cut 10 pieces of rope of length $\frac{1}{2}$ meter from 5 meters of rope.

Aligned Standards

5.NF.B.7

Problem 5

The content assessed in this problem is first encountered in Lesson 2: Meanings of Division.

In this problem, students demonstrate their understanding of integer division. Students should have experience drawing diagrams to explain the operation of division from fifth grade. Their diagrams need not match the solution as long as they convey an understanding of division.

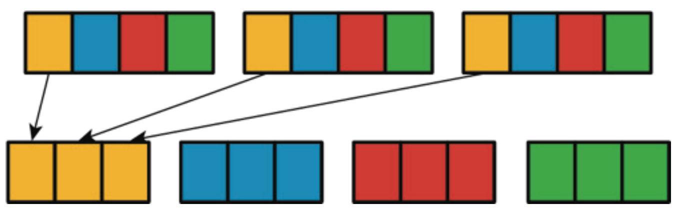
If most students struggle with this item, plan to use the diagrams created in Activity 2, question 2 to support student understanding of the two ways to think about division, labeling the diagrams for clarity. This concept will continue throughout the next few lessons.

Statement

What is $3 \div 4$? Draw a diagram that explains how you know.

Solution

$3 \div 4 = \frac{3}{4}$. If you have 3 wholes and you divide them each into 4 equal pieces, and take one piece from each, you will have 4 shares with $\frac{3}{4}$ of a whole in each.



Aligned Standards

5.NF.B.3

Problem 6

The content assessed in this problem is first encountered in Lesson 4: How Many Groups? (Part 1).

This problem assesses students' comfort with the operations on fractions they learned in fifth grade. Note that the second problem involves dividing a whole number by a unit fraction, while the third one involves dividing a unit fraction by a whole number, both fifth-grade skills. Students will develop a more general algorithm for dividing fractions in this unit.

If most students struggle with this item, plan to use the Lesson 4, Activity 1 Synthesis to engage students in a discussion around the meaning of $3 \div 9 = \frac{1}{3}$ and $1 \div \frac{1}{5} = 5$.

Statement

Compute:

1. $\frac{5}{6} \cdot \frac{10}{11}$
2. $9 \div \frac{1}{4}$
3. $\frac{1}{5} \div 3$

4. $\frac{9}{4} + \frac{2}{3}$

5. $\frac{9}{4} - \frac{2}{3}$

Solution

1. $\frac{50}{66}$ or equivalent
2. 36
3. $\frac{1}{15}$
4. $\frac{35}{12}$ or $2\frac{11}{12}$
5. $\frac{19}{12}$ or $1\frac{7}{12}$

Aligned Standards

5.NF.A.1, 5.NF.B.4, 5.NF.B.7

Problem 7

The content assessed in this problem is first encountered in Lesson 14: Fractional Lengths in Triangles and Prisms.

In the last section of this unit, students apply their understanding of fraction division to find unknown area and volume measurements.

If most students struggle with this item, plan to use this question as part of the launch for Lesson 14 Activity 3, asking students what the answer represents. What is a cubic centimeter?

Statement

A rectangular prism has dimensions of 8 cm by 9 cm by 10 cm. What is its volume?

Solution

720 cubic centimeters or 720 cm^3

Aligned Standards

5.MD.C.5

Assessment : Check Your Readiness (B)

Teacher Instructions

This assesses students' incoming knowledge about concepts that support understanding division of fractions.

Problem 1

The content assessed in this problem is first encountered in Lesson 3: Interpreting Division Situations.

There are a variety of strategies students can use to approach this problem. They may find a helpful fraction equivalent to $\frac{1}{2}$, or they may simply look at whether the numerator is more or less than one-half of the denominator. Students who struggle with this problem will need extra support in this unit. They likely do not understand the "part/whole" interpretation of a fraction. They may benefit from spending more time on visual representations of fractions.

Statement

Select **all** the fractions that are greater than $\frac{1}{2}$.

A. $\frac{50}{101}$

B. $\frac{50}{100}$

C. $\frac{12}{8}$

D. $\frac{8}{18}$

E. $\frac{8}{15}$

F. $\frac{5}{8}$

Solution

["C", "E", "F"]

Aligned Standards

4.NF.A.2

Problem 2

The content assessed in this problem is first encountered in Lesson 2: Meanings of Division.

In this problem, students distinguish between situations involving multiplication and situations involving division. In this unit, students study the meaning of dividing a number by a fraction before developing an algorithm for doing so.

Statement

Select **all** the situations that can be represented by $\frac{1}{3} \cdot 6$.

A. Noah has 6 meters of rope. How many pieces of rope of length $\frac{1}{3}$ meter can he cut from it?

B. Jada made 6 free throws. Kiran made $\frac{1}{3}$ as many free throws as Jada. How many free throws did Kiran make?

C. Mai's goal is to mow 6 lawns in a week. She mowed $\frac{1}{3}$ of a lawn. What fraction of the lawns has she cut?

D. Clare swims $\frac{1}{3}$ of a mile each day, for 6 days. How many miles is that in total?

E. Lin has $\frac{1}{3}$ pound of peanuts. She split them into 6 portions. How many pounds of peanuts are in each portion?

Solution

["B", "D"]

Aligned Standards

5.NF.B.6, 5.NF.B.7

Problem 3

The content assessed in this problem is first encountered in Lesson 3: Interpreting Division Situations.

Students can compare fractions by writing them with equivalent denominators. However, this problem is much easier to solve by comparing the numerator of each fraction to the denominator and ballparking the relationship. Students who struggle with this problem need help thinking about whether a fraction is close to zero, close to one, or close to one-half.

Statement

Order these numbers from least to greatest.

0, 1, $\frac{1}{2}$, $\frac{10}{11}$, $\frac{51}{100}$, $\frac{24}{50}$, $\frac{3}{20}$

Solution

0, $\frac{3}{20}$, $\frac{24}{50}$, $\frac{1}{2}$, $\frac{51}{100}$, $\frac{10}{11}$, 1

Aligned Standards

4.NF.A.2

Problem 4

The content assessed in this problem is first encountered in Lesson 1: Size of Divisor and Size of Quotient.

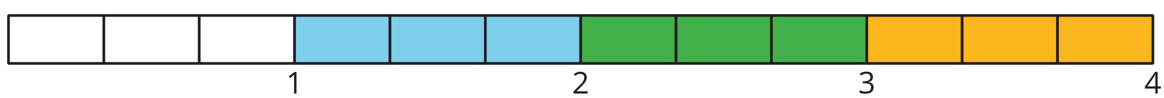
To answer this question, students must in some way explain how to determine the number of times $\frac{1}{3}$ goes into 4. Their explanations will provide insight into how clearly students are thinking about situations that involve division. Students need not use the word “division” or perform the operation of division as part of their answer.

Statement

Jada has 4 meters of ribbon. How many pieces of ribbon of length $\frac{1}{3}$ meter can she cut from it? Draw a diagram to illustrate your solution.

Solution

She can cut 12 pieces of ribbon of length $\frac{1}{3}$ meter from 4 meters of ribbon.



Aligned Standards

5.NF.B.7

Problem 5

The content assessed in this problem is first encountered in Lesson 2: Meanings of Division.

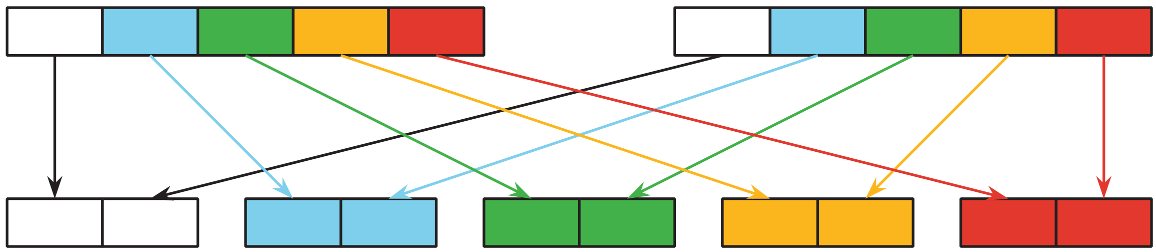
In this problem, students demonstrate their understanding of integer division. Students should have experience drawing diagrams to explain the operation of division from fifth grade. Their diagrams need not match the solution as long as they convey an understanding of division.

Statement

What is $2 \div 5$? Draw a diagram that explains how you know.

Solution

$2 \div 5 = \frac{2}{5}$. If you have 2 wholes and you divide them each into 5 equal pieces, and take one piece from each, you will have 5 shares with $\frac{2}{5}$ of a whole in each.



Aligned Standards

5.NF.B.3

Problem 6

The content assessed in this problem is first encountered in Lesson 4: How Many Groups? (Part 1).

This problem assesses students' comfort with the operations on fractions they learned in fifth grade. The second problem involves dividing a whole number by a unit fraction, while the third one involves dividing a unit fraction by a whole number, which are both fifth-grade skills. Students will develop a more general algorithm for dividing fractions in this unit.

Statement

Compute:

1. $\frac{3}{5} \cdot \frac{8}{12}$
2. $6 \div \frac{1}{3}$
3. $\frac{1}{8} \div 2$
4. $\frac{7}{8} + \frac{3}{5}$
5. $\frac{7}{8} - \frac{3}{5}$

Solution

1. $\frac{24}{60}$ or equivalent
2. 18
3. $\frac{1}{16}$
4. $\frac{59}{40}$ or $1\frac{19}{40}$
5. $\frac{11}{40}$

Aligned Standards

5.NF.A.1, 5.NF.B.4, 5.NF.B.7

Problem 7

In the last section of this unit, students apply their understanding of fraction division to find unknown area and volume measurements.

Statement

A rectangular prism has dimensions of 6 cm by 7 cm by 10 cm. What is its volume?

Solution

420 cubic centimeters or 420 cm^3

Aligned Standards

5.MD.C.5

Assessment : Mid-Unit Assessment (A)

Teacher Instructions

Give this assessment after lesson 9. Calculators should not be used.

Student Instructions

Calculators should not be used.

Problem 1

Students selecting A have reversed the order of division, $4 \div 6$. Students selecting B may be thinking of the greatest integer number of cups that will fit in each container. Students selecting D have incorrectly interpreted this as a multiplication problem.

Statement

Jada made 6 cups of blueberry jam and divided the jam equally among 4 containers. How much jam went in each container?

A. $\frac{2}{3}$ of a cup

B. 1 cup

C. $\frac{3}{2}$ of a cup

D. 24 cups

Solution

C

Aligned Standards

6.NS.A.1

Problem 2

Students selecting B or E, or failing to select A, may be reversing the order of the division. This is notable in A and B where whole numbers are used. Students failing to select C may assume that, because the dividend is less than one, the quotient is automatically less than one. Students failing to select D may have made an arithmetic calculation error, but should have been able to determine the answer without calculation.

Statement

Select **all** the expressions that are greater than 1.

A. $5 \div \frac{9}{10}$

B. $\frac{3}{5} \div 6$

C. $\frac{4}{5} \div \frac{3}{7}$

D. $2\frac{3}{5} \div 1\frac{3}{5}$

E. $\frac{12}{5} \div \frac{13}{5}$

Solution

["A", "C", "D"]

Aligned Standards

6.NS.A.1

Problem 3

Students divide a whole number by a fraction to solve a contextual problem.

Choice A is $75 \cdot \frac{3}{5}$. Students might select A if they misread the question and think that 75 is the total number of pages in the book. Students selecting B may be taking a ballpark guess. Students selecting C have multiplied by $1\frac{3}{5}$.

Statement

Jada has read $\frac{3}{5}$ of a book. She has read 75 pages so far. How many pages are in the whole book?

A. 45 pages

B. 105 pages

C. 120 pages

D. 125 pages

Solution

D

Aligned Standards

6.NS.A.1

Problem 4

Students divide a fraction by a fraction to answer a “how many groups?” question.

Statement

A recipe for cookies calls for $\frac{2}{3}$ of a cup of sugar per batch. Elena used $5\frac{1}{3}$ cups of sugar to make multiple batches of cookies. How many batches did she make?

Solution

8 batches

Aligned Standards

6.NS.A.1

Problem 5

Students write multiplication and division equations for fraction division problems.

Statement

1. Write a multiplication equation that represents the question: how many $\frac{3}{8}$s are in $\frac{5}{4}$?
2. Write a division equation that represents the question: how many $\frac{3}{8}$s are in $\frac{5}{4}$?

Solution

1. $? \cdot \frac{3}{8} = \frac{5}{4}$ or equivalent
2. $\frac{5}{4} \div \frac{3}{8} = ?$ or equivalent

Aligned Standards

6.NS.A.1

Problem 6

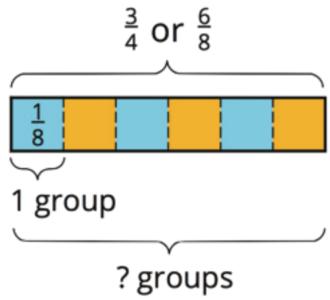
This problem asks students to demonstrate a visual understanding of the division of fractions, rather than just supply the result of a calculation.

Statement

What is $\frac{3}{4} \div \frac{1}{8}$? Explain or show your reasoning.

Solution

Reasoning varies. Students may draw a number line, a tape diagram, or a different representation.



Minimal Tier 1 response:

- Work is complete and correct.
- Sample: $\frac{1}{8}$ fits in $\frac{1}{4}$ twice, and $\frac{3}{4}$ is three $\frac{1}{4}$s. $2 \cdot 3 = 6$.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Acceptable errors: minor errors in representation cause an error in determining the answer.
- Sample errors: representation includes minor errors such as a mislabeled number line or tape diagram; correct answer with no diagram; correct answer based on an equation only.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: incorrect answer with no diagram; incorrect answer with work based on an equation only; major error in representation that shows a lack of understanding of fractions, such as counting on a number line by denominators.

Aligned Standards

6.NS.A.1

Problem 7

The first of these two problems asks students to write a multiplication or division equation to describe the situation, then solve the problem using a method of their choice. The second problem explicitly asks students to draw a diagram.

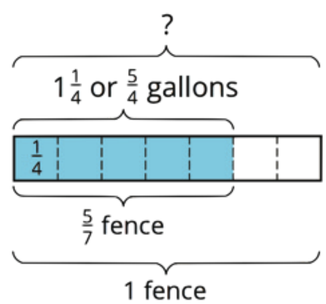
Statement

1. Tyler used $1\frac{1}{4}$ gallons of paint to paint $\frac{5}{7}$ of a fence. How many gallons will it take to paint the whole fence? Write a multiplication equation or a division equation to represent this question, then find the answer. Show your reasoning.

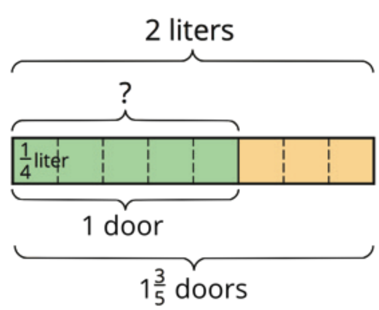
2. Clare is painting some doors that are all the same size. She used 2 liters of paint to cover $1\frac{3}{5}$ doors. How many liters of paint are needed for 1 door? Draw a diagram to represent the situation, then find the answer.

Solution

1. $\frac{7}{4}$ or $1\frac{3}{4}$ gallons. Equations: $\frac{5}{7} \cdot ? = \frac{5}{4}$ (or equivalent) or $\frac{5}{4} \div \frac{5}{7} = ?$. Sample reasoning:
    - Each $\frac{1}{7}$ of the fence requires $\frac{1}{4}$ gallon. The whole fence is $\frac{7}{7}$ of the fence, so multiply $\frac{1}{4}$ by 7 to get $\frac{7}{4}$ or $1\frac{3}{4}$.
    - Using diagram:



2. $1\frac{1}{4}$ or $\frac{5}{4}$ liters. Diagrams vary. Sample diagram:



Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Sample:

1. $? \cdot \frac{5}{7} = \frac{5}{4}$. It takes $\frac{1}{4}$ gallon for $\frac{1}{7}$ of the fence. That means it takes $\frac{7}{4}$ gallons for the whole fence.
2. See diagram. $1\frac{1}{4}$ or $\frac{5}{4}$ liters.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.

- Sample errors: miscounting the number of segments in a tape diagram or including one too many or too few segments, mistakes converting mixed numbers to improper fractions; equation in part a is incorrect but all other work is correct.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: arithmetic errors that show a difficulty interpreting fractions, like $\frac{1}{4} \cdot 7 = \frac{1}{28}$; errors that involve inverting the wrong fraction or reversing the order of division; one diagram/explanation is off the mark but the other is correct or close to correct.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: Each diagram, equation, and/or explanation is badly off; work does not show evidence of framework for understanding division.

Aligned Standards

6.NS.A.1

Assessment : Mid-Unit Assessment (B)

Teacher Instructions

Give this assessment after lesson 9. Calculators should not be used.

Student Instructions

Calculators should not be used.

Problem 1

Students selecting A may be reversing the order of the division. Students selecting B may think that since both the dividend and divisor are less than 1, the quotient must be less than 1. Students selecting D may have made an arithmetic calculation error, but should have been able to determine the answer without calculation.

Statement

Which expression is less than 1?

A. $4 \div \frac{2}{3}$

B. $\frac{3}{4} \div \frac{2}{3}$

C. $\frac{14}{8} \div \frac{15}{8}$

D. $2\frac{4}{8} \div 1\frac{5}{8}$

Solution

C

Aligned Standards

6.NS.A.1

Problem 2

Students divide a whole number by a fraction to solve a contextual problem. Students selecting A may have misread the question and think that 120 is the total number of pieces in the puzzle. Students selecting B may be taking a ballpark guess or had a miscalculation. Students selecting D may have multiplied by $1\frac{4}{5}$.

Statement

Kiran has used $\frac{4}{5}$ of the pieces in his jigsaw puzzle. He has used 120 pieces. How many pieces are in the whole puzzle?

A. 96

B. 125

C. 150

D. 216

Solution

C

Aligned Standards

6.NS.A.1

Problem 3

Students divide a fraction by a fraction to answer a "how many groups?" question. Students selecting B may have assumed that each batch used 1 stick of butter. Students selecting C multiplied the numbers in the problem, perhaps misunderstanding what information was given and what they needed to find. Students selecting D reversed the numbers when dividing.

Statement

Andre used $4\frac{1}{2}$ sticks of butter to make multiple batches of brownies. The recipe calls for $\frac{3}{4}$ of a stick of butter for each batch. How many batches did he make?

A. 6 batches

B. $4\frac{1}{2}$ batches

C. $3\frac{3}{8}$ batches

D. $\frac{1}{6}$ of a batch

Solution

A

Aligned Standards

6.NS.A.1

Problem 4

Students answering 48 have incorrectly interpreted this as a multiplication problem. Students who answer that it is impossible may not understand that it is possible for each container to contain less than 1 cup. Students answering $\frac{4}{3}$ have reversed the order of division.

Statement

Mai made 6 cups of vegetable soup and divided the soup equally among 8 containers. How much soup went in each container?

Solution

$\frac{3}{4}$ cup or equivalent

Aligned Standards

6.NS.A.1

Problem 5

Students write multiplication and division equations for fraction division problems.

Statement

1. Write a multiplication equation that represents the question "How many $\frac{2}{3}$'s are in $\frac{9}{8}$?"
2. Write a division equation that represents the question "How many $\frac{2}{3}$'s are in $\frac{9}{8}$?"

Solution

1. $? \cdot \frac{2}{3} = \frac{9}{8}$ or equivalent
2. $\frac{9}{8} \div \frac{2}{3} =?$ or equivalent

Aligned Standards

6.NS.A.1

Problem 6

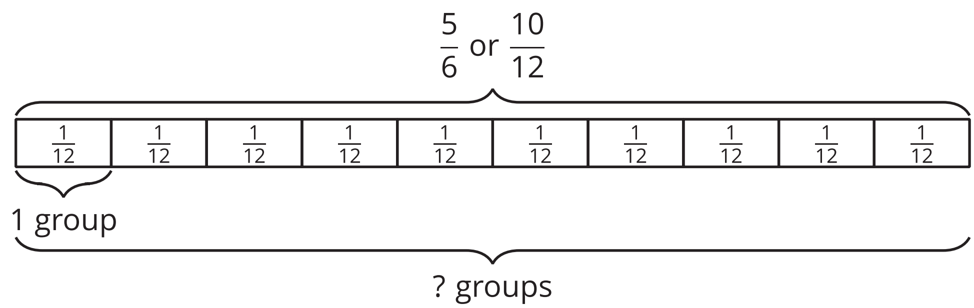
This problem asks students to demonstrate a visual understanding of the division of fractions, rather than just supply the result of a calculation.

Statement

What is $\frac{5}{6} \div \frac{1}{12}$? Explain or show your reasoning.

Solution

Reasoning varies. Students may draw a number line, a tape diagram, or a different representation.



Minimal Tier 1 response:

- Work is complete and correct.
- Sample: $\frac{1}{12}$ fits in $\frac{1}{6}$ twice, and $\frac{5}{6}$ is five $\frac{1}{6}$'s. $2 \cdot 5 = 10$.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Acceptable errors: minor errors in representation cause an error in determining the answer.
- Sample errors: representation includes minor errors such as a mislabeled number line or tape diagram; correct answer with no diagram; correct answer based on an equation only.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: incorrect answer with no diagram; incorrect answer with work based on an equation only; major error in representation that shows a lack of understanding of fractions, such as counting on a number line by denominators.

Aligned Standards

6.NS.A.1

Problem 7

The first of these two problems asks students to write a multiplication or division equation to describe the situation, then solve the problem using a method of their choice. The second problem explicitly asks students to draw a diagram.

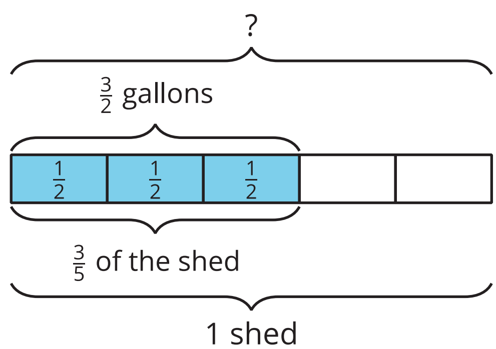
Statement

1. Han used $1\frac{1}{2}$ gallons of paint to paint $\frac{3}{5}$ of a shed. How many gallons of paint will it take to paint the whole shed? Write a multiplication equation or a division equation to represent this situation, then find the answer. Show your reasoning.

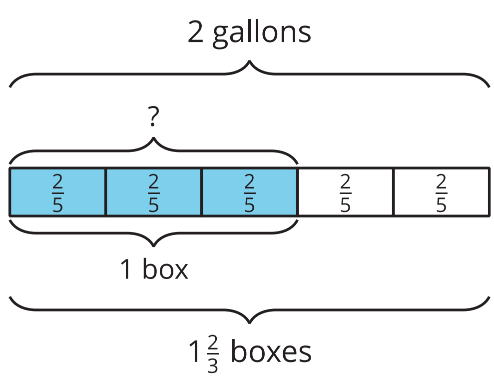
2. Noah is painting boxes that are all the same size. He used 2 ounces of paint to cover $1\frac{2}{3}$ boxes. How many ounces of paint are needed for 1 box? Draw a diagram to represent the situation, then find the answer.

Solution

1. $\frac{15}{6}$ or $2\frac{1}{2}$ gallons Equations: $\frac{3}{5} \cdot ? = \frac{3}{2}$ (or equivalent) or $\frac{3}{2} \div \frac{3}{5} = ?$. Sample reasoning:
    - Each $\frac{1}{5}$ of the shed requires $\frac{1}{2}$ gallon of paint, and the whole shed is $\frac{5}{5}$. You multiply $\frac{1}{2}$ by 5 to get $2\frac{1}{2}$ gallons.
    - Using a diagram:



2. $\frac{6}{5}$ or $1\frac{1}{5}$ ounces of paint Equations: $2 \div \frac{5}{3} = ?$ or $\frac{5}{3} \cdot ? = 2$



Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Sample:

1. $\frac{15}{6}$ or $2\frac{1}{2}$ gallons Equations: $\frac{3}{5} \cdot ? = \frac{3}{2}$ (or equivalent) or $\frac{3}{2} \div \frac{3}{5} = ?$. Sample reasoning:
    - Each $\frac{1}{5}$ of the shed requires $\frac{1}{2}$ gallon of paint, and the whole shed is $\frac{5}{5}$. You multiply $\frac{1}{2}$ by 5 to get $2\frac{1}{2}$ gallons.

2. See diagram. $\frac{6}{5}$ or equivalent ounces of paint

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: miscounting the number of segments in a tape diagram or including one too many or too few segments, mistakes converting mixed numbers to improper fractions; equation in part a is incorrect but all other work is correct.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: arithmetic errors that show a difficulty interpreting fractions (e.g. $\frac{3}{2} \cdot \frac{3}{5} = \frac{9}{10}$ or $\frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$); errors that involve inverting the wrong fraction or reversing the order of division; one diagram or explanation is off the mark but the other is correct or close to correct.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: Each diagram, equation, or explanation is badly off; work does not show evidence of framework for understanding division.

Aligned Standards

6.NS.A.1

Assessment : End-of-Unit Assessment (A)

Teacher Instructions

Calculators should not be used.

Student Instructions

Calculators should not be used.

Problem 1

Students selecting A have reversed the order of division. Students selecting C have answered the question, “how many miles longer was Mai’s ride?” Students selecting D have multiplied the lengths of the two rides together.

Statement

Mai biked $6\frac{3}{4}$ miles today, and Noah biked $4\frac{1}{2}$ miles. How many times the length of Noah’s bike ride was Mai’s bike ride?

A. $\frac{2}{3}$ times as far

B. $\frac{3}{2}$ times as far

C. $\frac{9}{4}$ times as far

D. $\frac{243}{8}$ times as far

Solution

B

Aligned Standards

6.NS.A.1

Problem 2

Students selecting B reversed the order of the division. Students selecting C may have treated $\frac{5}{8}$ as a height rather than a fraction of a height. Students selecting D are representing the calculations needed to find the height of a tower $\frac{5}{8}$ the size of the one Priya has already built. Students selecting E and F but not A might think that the equations must be in “solved” form. Likewise, students selecting A only might think that they are looking for an equation that has yet to be solved.

Statement

Select **all** equations that represent this question:

Priya is stacking building blocks to make a tower. She takes a break when the tower is $2\frac{1}{2}$ feet tall, which is $\frac{5}{8}$ of the height of the tower she wants to build. How tall is the tower when finished?

A. $\frac{5}{8} \cdot ? = 2\frac{1}{2}$

B. $\frac{5}{8} \div 2\frac{1}{2} = ?$

C. $2\frac{1}{2} \cdot ? = \frac{5}{8}$

D. $2\frac{1}{2} \cdot \frac{5}{8} = ?$

E. $2\frac{1}{2} \cdot \frac{8}{5} = ?$

F. $2\frac{1}{2} \div \frac{5}{8} = ?$

Solution

["A", "E", "F"]

Aligned Standards

6.NS.A.1

Problem 3

Students selecting A or C have picked a statement that is equivalent to "multiply by $\frac{2}{9}$" instead of "multiply by $\frac{9}{2}$".

Statement

Select **all** statements that show correct reasoning for finding $15 \div \frac{2}{9}$.

A. Multiply 15 by 2, then divide by 9.

B. Multiply 15 by 9, then divide by 2.

C. Multiply 15 by $\frac{1}{9}$, then multiply by 2.

D. Multiply 15 by 9, then multiply by $\frac{1}{2}$.

Solution

["B", "D"]

Aligned Standards

6.NS.A.1

Problem 4

Students apply the algorithm for dividing fractions.

Statement

Divide.

1. $\frac{3}{4} \div \frac{1}{5}$
2. $\frac{9}{2} \div \frac{3}{4}$

1. $\frac{4}{9} \div \frac{8}{15}$
2. $5\frac{2}{3} \div \frac{3}{2}$

Solution

1. $\frac{15}{4}$ or $3\frac{3}{4}$ (or equivalent)
2. 6 (or equivalent)
3. $\frac{60}{72}$ or $\frac{5}{6}$ (or equivalent)
4. $\frac{34}{9}$ or $3\frac{7}{9}$ (or equivalent)

Aligned Standards

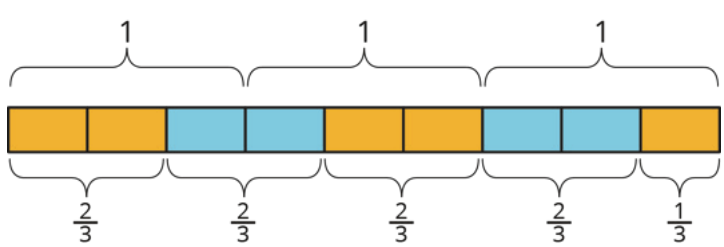
6.NS.A.1

Problem 5

Students who agree with Andre's statement may have a fundamental misunderstanding of division, one that may come from working with remainders of whole number division.

Statement

Andre draws this tape diagram for $3 \div \frac{2}{3}$:



Andre says that $3 \div \frac{2}{3} = 4\frac{1}{3}$ because there are 4 groups of $\frac{2}{3}$ and $\frac{1}{3}$ left. Do you agree with Andre? Explain your reasoning.

Solution

Answers vary. Sample response: No, I disagree. There are 4 groups of $\frac{2}{3}$ in $2\frac{2}{3}$. Then there is $\frac{1}{3}$ left and this makes $\frac{1}{2}$ of another group. There are $4\frac{1}{2}$ groups of $\frac{2}{3}$ in 3.

Minimal Tier 1 response:

- Work is complete and correct.
- Sample: No, because $3 \div \frac{2}{3} = 4\frac{1}{2}$, not $4\frac{1}{3}$.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: disagreement with Andre but a minor error in logic or calculation leads to a different result other than $4\frac{1}{2}$; agreement with Andre with a reasonable but flawed argument.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: disagreement with Andre based on a major flaw in logic or calculation; agreement with Andre with a badly flawed argument; agreement or disagreement without any stated justification.

Aligned Standards

6.NS.A.1

Problem 6

Students will need to use fraction division to calculate how many cubes will fit along each side of the box. Rather than using the standard algorithm, some students may visualize or otherwise reason conceptually about how many cubes with side length $\frac{1}{3}$ inch it will take to reach a length of $2\frac{2}{3}$ inches, etc.

Statement

How many $\frac{1}{3}$ inch cubes does it take to fill a box with width $2\frac{2}{3}$ inches, length $3\frac{1}{3}$ inches, and height $2\frac{1}{3}$ inches?

Solution

560 (8 cubes fit along the width of the box, 10 cubes fit along the length, and 7 cubes fit vertically.)

Aligned Standards

6.G.A.2

Problem 7

While we expect students to understand the context of the problem, some may still have difficulty understanding without a diagram. The first problem is about area, even though the pan is described as a rectangular prism.

Statement

Lin has two small baking pans, each shaped like a rectangular prism. For each question, explain or show your reasoning.

1. Lin lines the bottom of her first pan with aluminum foil. The area of the rectangular piece of foil is $11\frac{1}{4}$ square inches. Its length is $4\frac{1}{2}$ inches. What is the width of the foil?

2. Lin's second pan has a length of $\frac{8}{3}$ inches, a width of $\frac{15}{4}$ inches, and a height of $\frac{3}{2}$ inches. What is the volume of the second pan?

Solution

1. $2\frac{1}{2}$ inches or equivalent. The width is the solution to $4\frac{1}{2} \cdot w = 11\frac{1}{4}$. By writing each mixed number as a fraction, the problem is made simpler: $\frac{9}{2}w = \frac{45}{4}$. Then $w = \frac{5}{2}$.

2. 15 cubic inches or equivalent. The volume is the product of the pan's length, width, and height: $\frac{3}{2} \cdot \frac{8}{3} \cdot \frac{15}{4} = 15$.

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Sample:

1. $\frac{5}{2}$ inches, because $11\frac{1}{4} \div 4\frac{1}{2} = \frac{5}{2}$.
2. 15 cubic inches, because $\frac{3}{2} \cdot \frac{8}{3} \cdot \frac{15}{4} = 15$.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: correct answers without justification; one or two errors in calculation, such as incorrect rewriting of mixed numbers, but correct equations or representations used.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: one incorrect answer with invalid work or no work shown; any incorrect choice of multiplication or division; invalid method used to multiply or divide fractions or mixed numbers; more than two errors in calculation.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: two incorrect answers with invalid work or no work shown; consistently incorrect choices of multiplication or division; repeated use of invalid methods to multiply or divide.

Aligned Standards

6.G.A.2, 6.NS.A.1

Assessment : End-of-Unit Assessment (B)

Teacher Instructions

Calculators should not be used.

Student Instructions

Calculators should not be used.

Problem 1

Students selecting A have multiplied the lengths of the two runs together. Students selecting B have answered the question, “How many miles longer was Tyler’s run?” Students selecting D have reversed the order of division.

Statement

Priya ran $1\frac{1}{2}$ miles today, and Tyler ran $3\frac{3}{4}$ miles. How many times the length of Priya’s run was Tyler’s run?

A. $\frac{45}{8}$ times as far

B. $\frac{9}{4}$ times as far

C. $\frac{5}{2}$ times as far

D. $\frac{2}{5}$ times as far

Solution

C

Aligned Standards

6.NS.A.1

Problem 2

Students selecting B reversed the order of the division. Students selecting D may have treated $\frac{2}{3}$ as the distance rather than a fraction of a distance. Students selecting F have multiplied, not realizing $8\frac{1}{2} \cdot \frac{2}{3} = 5\frac{2}{3}$ (which is less distance than Diego has already run). Students who select A and E but not C might think the equations must be in “solved” form.

Statement

Select **all** equations that represent this question:

Diego has run $\frac{2}{3}$ of a race’s distance. He has run $8\frac{1}{2}$ kilometers. How long is the whole race?

A. $8\frac{1}{2} \div \frac{2}{3} =?$

B. $\frac{2}{3} \div 8\frac{1}{2} =?$

C. $\frac{2}{3} \cdot ? = 8\frac{1}{2}$

D. $8\frac{1}{2} \cdot ? = \frac{2}{3}$

E. $8\frac{1}{2} \cdot \frac{3}{2} =?$

F. $8\frac{1}{2} \cdot \frac{2}{3} =?$

Solution

["A", "C", "E"]

Aligned Standards

6.NS.A.1

Problem 3

Students selecting B or D have picked a statement that is equivalent to “multiply by $\frac{3}{8}$” instead of “multiply by $\frac{8}{3}$”.

Statement

Which statement shows correct reasoning for finding $12 \div \frac{3}{8}$?

A. Multiply 12 by 8, then divide by 3.

B. Multiply 12 by 3, then divide by 8.

C. Multiply 3 by 8, then divide by 12.

D. Multiply 12 by $\frac{1}{8}$, then multiply by 3.

Solution

A

Aligned Standards

6.NS.A.1

Problem 4

Students apply the algorithm for dividing fractions.

Statement

Divide.

1. $\frac{2}{3} \div \frac{1}{4}$

2. $\frac{10}{3} \div \frac{2}{6}$

3. $\frac{5}{6} \div \frac{7}{12}$

4. $2\frac{3}{4} \div \frac{4}{3}$

Solution

1. $\frac{8}{3}$ (or equivalent)

2. $\frac{60}{6}$ (or equivalent)

3. $\frac{60}{42}$ (or equivalent)

4. $\frac{33}{16}$ (or equivalent)

Aligned Standards

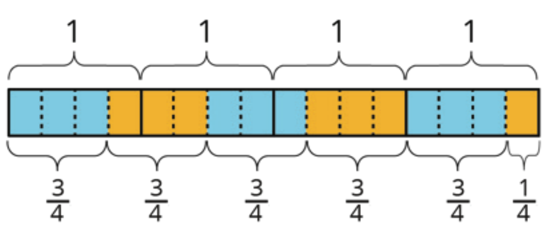
6.NS.A.1

Problem 5

Students who agree with Lin's statement may have a fundamental misunderstanding of division that may come from working with remainders of whole number division.

Statement

Lin draws this tape diagram for $4 \div \frac{3}{4}$:



Lin says that $4 \div \frac{3}{4} = 5\frac{1}{4}$ because there are 5 groups of $\frac{3}{4}$ and $\frac{1}{4}$ left. Do you agree with Lin? Explain your reasoning.

Solution

Answers vary. Sample response: No, I disagree. There are 5 groups of $\frac{3}{4}$ in $3\frac{3}{4}$. Then there is $\frac{1}{4}$ left, and this makes $\frac{1}{3}$ of another group. There are $5\frac{1}{3}$ groups of $\frac{3}{4}$ in 4.

Minimal Tier 1 response:

- Work is complete and correct.

- Sample: No, because $4 \div \frac{3}{4} = 5\frac{1}{3}$, not $5\frac{1}{4}$.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: disagreement with Lin, but a minor error in logic or calculation leads to a different result other than $5\frac{1}{3}$; agreement with Lin, with a reasonable but flawed argument.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: disagreement with Lin based on a major flaw in logic or calculation; agreement with Lin with a badly flawed argument; agreement or disagreement without any stated justification.

Aligned Standards

6.NS.A.1

Problem 6

Students will need to use fraction division to calculate how many cubes will fit along each side of the box. Rather than using the standard algorithm, some students may visualize or otherwise reason conceptually about how many cubes with side length $\frac{1}{4}$ inch it will take to reach a length of $2\frac{1}{4}$ inches, etc.

> ### Statement
>
> How many $\frac{1}{4}$-inch cubes does it take to fill a box with width $2\frac{1}{4}$ inches, length $2\frac{1}{2}$ inches, and height $1\frac{3}{4}$ inches?

Solution

630 cubes (9 cubes fit along the width of the box, 10 cubes fit along the length, and 7 cubes fit vertically.) $9 \cdot 10 \cdot 7 = 630$

Aligned Standards

6.G.A.2

Problem 7

While most students should understand the context of the problem, some may still have difficulty understanding without a diagram. The second problem is about area, even though the aquarium is described as a rectangular prism.

Statement

Elena has two aquariums, each shaped like a rectangular prism. For each question, explain or show your reasoning.

1. One aquarium has a length of $\frac{7}{2}$ feet, a width of $\frac{4}{3}$ feet, and a height of $\frac{3}{2}$ feet. What is the volume of the aquarium?
2. Elena paints the back of the second aquarium. It has a height of $1\frac{3}{4}$ feet. The painted area is $5\frac{5}{6}$ square feet. What is its length?

Solution

1. 7 cubic feet or equivalent. The volume is the product of the aquarium's length, width, and height: $\frac{7}{2} \cdot \frac{4}{3} \cdot \frac{3}{2} = 7$.
2. $3\frac{1}{3}$ feet or equivalent. The length is the solution to $\ell \cdot 1\frac{3}{4} = 5\frac{5}{6}$. By writing each mixed number as a fraction, the problem is made simpler: $\frac{7}{4}\ell = \frac{35}{6}$. Then $\ell = \frac{10}{3}$.

Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Sample:
- 7 cubic feet, because $\frac{7}{2} \cdot \frac{4}{3} \cdot \frac{3}{2} = 7$.
- $\frac{10}{3}$ feet, because $5\frac{5}{6} \div 1\frac{3}{4} = \frac{10}{3}$.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: correct answers without justification; one or two errors in calculation, such as incorrect rewriting of mixed numbers, but correct equations or representations used.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: one incorrect answer with invalid work or no work shown; any incorrect choice of multiplication or division; invalid method used to multiply or divide fractions or mixed numbers; more than two errors in calculation.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: two incorrect answers with invalid work or no work shown; consistently incorrect choices of multiplication or division; repeated use of invalid methods to multiply or divide.

Aligned Standards

6.G.A.2, 6.NS.A.1

Section: Making Sense of Division

Lesson 1: Size of Divisor and Size of Quotient

Goals

- Comprehend the terms “dividend” and “divisor” (in spoken language) to refer to the numbers in a division problem.
- Explain (orally) how to estimate quotients, by comparing the size of the dividend and divisor.
- Generalize about the size of a quotient, i.e., predicting whether it is a very large number, a very small number, or close to 1.

Learning Targets

- When dividing, I know how the size of a divisor affects the quotient.

Lesson Narrative

The first three lessons of this unit help students make sense of division situations. In this opening lesson, students begin thinking about the relationships between the numbers in a division equation. They see that they can estimate the size of the quotient by reasoning about the relative sizes of the *divisor* and the *dividend*.

Students begin exploring these relationships in concrete situations. For example, they estimate how many thinner and thicker objects are needed to make a stack of a given height, and how many segments of a certain size make a particular length.

Later, they generalize their observations to division expressions (MP7). Students become aware that dividing by a number that is much smaller than the dividend results in a quotient that is larger than 1, that dividing by a number that is much larger than the dividend gives a quotient that is close to 0, and that dividing by a number that is close to the dividend results in a quotient that is close to 1.

Alignments

Building On

- 5.NBT.B.6: Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Building Towards

- 6.NS.A: Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Number Talk
- Poll the Class

Required Materials

Pre-printed slips, cut from copies of the blackline master

Required Preparation

Print and cut up slips containing expressions from the blackline master. Consider copying Set 1 and Set 2 on paper of different colors. Prepare 1 copy (8 slips of Set 1 and 8 slips of Set 2) for every 3 students.

> **Student Learning Goals**
>
> Let's explore quotients of different sizes.

1.1 Number Talk: Size of Dividend and Divisor

Warm Up: 5 minutes

This number talk prompts students to notice how the values of the dividend and divisor affect the size of the quotient. To mentally evaluate the series of expressions, students think carefully about the numbers and rely on what they know about division, structure, and properties of operations.

The focus of the activity is less on the strategies for evaluating quotients and more on what happens when we divide the same number by different-sized numbers. Students should notice that dividing by a number much smaller than the dividend results in a large quotient and that dividing by a number much larger than the quotient results in a small quotient.

Building On

- 5.NBT.B.6

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Number Talk

Launch

Reveal one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the task.

Support for Students with Disabilities

Representation: Internalize Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organization

Anticipated Misconceptions

Some students may cross out the zeros without being aware of what doing so means. They may or may not reach the correct quotients. If students cross out zeros, ask the class during discussion to explain what they believe is happening mathematically when zeros are crossed out. Clarify any confusion accordingly.

Student Task Statement

Find the value of each expression mentally.

$5{,}000 \div 5$

$5{,}000 \div 2{,}500$

$5{,}000 \div 10{,}000$

$5{,}000 \div 500{,}000$

Student Response

- 1,000. Possible reasoning: 5 thousands divided by 5 is 1 thousand.
- 2. Possible reasoning: There are 2 groups of 2,500 in 5,000.
- $\frac{1}{2}$ (or 0.5). Possible reasoning: 5,000 is half of 10,000, and 5,000 divided into 10,000 groups means 0.5 in each group.
- $\frac{1}{100}$ (or 0.01). Possible reasoning: $5{,}000 \div 1{,}000 = 5$ and $500{,}000 \div 1{,}000 = 500$, and $5 \div 500 = \frac{5}{500}$, which is $\frac{1}{100}$.

Activity Synthesis

Invite a couple of students to share their answer and strategies for each problem. Record and display their explanations for all to see. Refer to MLR 2 (Collect and Display). After evaluating all four expressions, ask students:

- "What do you notice about the value of each expression as the *divisor* (the number we use to divide) gets larger?"
- "Why is the value of the expression getting smaller each time?"

Highlight explanations that support two ways of thinking about division:

1. Dividing means breaking the *dividend* into a certain number of equal parts, and when there are more parts, the size of each part gets smaller.
2. Dividing means breaking the dividend into parts of a particular size, and when the size of each part gets larger, the number of parts gets smaller.

To involve more students in the conversation, consider asking as the students share their ideas:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone solve the problem the same way but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

Support for English Language Learners

Speaking: MLR8 Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimize output (for explanation)

1.2 All Stacked Up

15 minutes
This lesson aims to give students a concrete setting for thinking about division. Students estimate how many of each given object are needed to make a stack of a given height. To do so, they use what they know about the sizes of familiar objects (boxes, bricks, notebooks, and coins) and their intuition that it takes more of a thinner object and fewer of a thicker object to reach the same height. Later, they will use this idea to think about division more generally.

We often refer to certain objects (coins, books, etc.) as having a thickness rather than a height. Clarify that "thickness" and “height” refer to the same dimension in these examples. The images of the boxes and the bricks show stacks with more items at the base. Clarify that we are concerned only with a stack with one item per layer.

As students discuss in groups, monitor for those who:

- Can explain clearly why the relationship between the height of the object being stacked and the height of a stack.

- Can explain clearly why the situation can be represented with a division expression.

Building On

- 5.NBT.B.6

Building Towards

- 6.NS.A

Instructional Routines

- MLR8: Discussion Supports
- Poll the Class

Launch

Arrange students in groups of 3–4. Give students 4–5 minutes quiet think time and then time to discuss their solutions with their group.

Support for Students with Disabilities

Representation: Internalize Comprehension. Begin the activity with concrete or familiar contexts. Provide students with access to some cardboard boxes, bricks, notebooks, or coins (whatever is available) for them to explore, but not measure.
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Some students may compare the height of objects in inches to the height of a stack in feet. Remind them to attend to units of measurement when making their estimates.

If students struggle to make estimates for the first set of questions, ask them which object we would need the most of and the least of to reach 5 feet. Then, prompt them to reason about the relative heights of those objects.

Student Task Statement

1. Here are several types of objects. For each type of object, estimate how many are in a stack that is 5 feet high. Be prepared to explain your reasoning.

Cardboard boxes

Notebooks

Bricks

Coins

2. A stack of books is 72 inches tall. Each book is 2 inches thick. Which expression tells us how many books are in the stack? Be prepared to explain your reasoning.

   ○ $72 \cdot 2$　　○ $72 - 2$　　○ $2 \div 72$　　○ $72 \div 2$

3. Another stack of books is 43 inches tall. Each book is $\frac{1}{2}$-inch thick. Write an expression that represents the number of books in the stack.

Student Response

1. Estimates vary. Sample responses and reasoning:
   a. About 5 boxes. I estimated that each box is about 12 inches or 1 foot tall, so it would take 5 of them to reach 5 feet.

   b. About 25 bricks. A brick is about 2.5 inches thick, so it would take about 5 to reach 1 foot, or $5 \cdot 5$ to reach 5 feet.

   c. About 60 books, assuming the notebooks are about 1 inch thick. $60 \cdot 1 = 60$.

   d. About 960 pennies. I estimated each penny to be about $\frac{1}{16}$ inch, so it would take 16 pennies to make 1 inch, and $60 \cdot 16$, or 960 pennies, to make 60 inches or 5 feet.

2. $72 \div 2$

3. $43 \div \frac{1}{2}$ (or 86)

Activity Synthesis

Ask a few students to share their estimates and explanations for the first set of questions. After a student gives an estimate, poll the class to see if their estimate is less, greater, or about the same.

Record for all to see the range of estimates of each object. After all four objects are discussed, ask students:

- “How did your estimates for the number of objects change as the object got thinner?” (The estimates got larger.)
- “Why might that be?” (As the thickness decreased, more objects were needed to make a height of 5 feet.)

Select previously identified students to share their responses and reasoning to the last two questions. Highlight that all situations in this activity involve division. Point out that:

- In the case of the 5-foot (60-inch) stack, the relationship can be represented as: $60 \div \text{ height} = \text{number of objects}$.
- The height of each object is the *divisor* (the number we use to divide) and the number of objects is the *quotient* (or result of division).
- The greater the divisor, the smaller the quotient, and vice versa.

Consider using this applet: https://ggbm.at/RJQyS6av to further illustrate the relationship between the size of the divisor and the size of the quotient.

Support for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. Use this routine to amplify mathematical uses of language to communicate about divisors, dividends, quotients, and expressions. Remind students to use these words when stating their ideas. Ask students to chorally repeat the phrases that include these words in context.
Design Principle(s): Support sense-making; Optimize output (for explanation)

1.3 All in Order

20 minutes
In this sorting activity, students continue to explore the relationship between dividends, divisors, and quotients.

First, they study two sets of division expressions and arrange them in order—from the largest to smallest—based on the size of the quotients. The first set of quotients has the same dividend (800). The second set has the same divisor (25).

Next, students estimate the size of quotients relative to 0 and 1. They approximate the size of divisors such that the quotients are close to 0, close to 1, or much greater than 1.

As students work, monitor how they think about placing the expressions. Select students who could explain their rationale clearly so that they could share later.

Also notice the expressions students find difficult to put in order. Expressions with fractional or decimal divisors, or expressions in which the dividends and divisors are very close to each other (e.g. $800 \div 800.1$ and $800 \div 799.5$), may be particularly challenging. This is an opportunity to make use of the structure in the relationship between the three parts in a division (MP7), i.e. to see that as the divisor gets larger, even if only by a very small amount, the quotient necessarily gets smaller.

In the extension, students have the option to work in groups of 2. Instead of writing a list of expressions, partners may take turns writing expressions that have values increasingly closer to 1 without equaling 1.

Building Towards

- 6.NS.A

Instructional Routines

- MLR2: Collect and Display

Launch

If not already done in the preceding activity, review the terms “quotient” and “divisor.” Display a division equation labeled with these terms so students can refer to it as needed.

Arrange students in groups of 3–4 and pair each group with another group so that they could check each other's work. Give each group 2 sets of pre-cut slips from the blackline master. Give students 6–8 minutes to sort the two sets of expressions, a couple of minutes to review another group's work, and another 5 minutes to record their sorted lists and complete the last question.

If desired, collect the slips after students record their lists so that they could be reused.

Support for Students with Disabilities

Representation: Internalize Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Provide select students fewer cards to start with and introduce the remaining cards once students have completed their initial sort. Some students may benefit from an additional set of cards with easier values to get them started.
Supports accessibility for: Conceptual processing; Organization

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Support for English Language Learners

Conversing, Representing, Writing: MLR2 Collect and Display. Listen for and collect the mathematical language students use as they discuss each expression. Record words and phrases such as “the quotient is close to 1” and “the divisor is larger than the dividend,” on a display for all to see. Remind students to borrow words, phrases or expressions from the display as needed. This will help students use mathematical language during paired and group discussions.
Design Principle(s): Optimize output (for explanation); Maximize meta-awareness

Anticipated Misconceptions

Students may try to compute the value of each expression because they are unsure how to begin otherwise. Suggest that they compare two quotients at a time, starting with those that have very different divisors. Ask, for instance, "Which is greater, $800 \div 250$ or $800 \div 2.5$?" Urge them to use the patterns they saw earlier about how the size of a divisor affects the quotient.

Student Task Statement

Your teacher will give you two sets of papers with division expressions.

1. Without computing, estimate the quotients in each set and order them from greatest to least. Be prepared to explain your reasoning.

   Pause here for a class discussion.

   Record the expressions in each set in order from the greatest value to the least.

   a. Set 1

   b. Set 2

2. Without computing, estimate the quotients and sort them into the following three groups. Be prepared to explain your reasoning.

$30 \div \frac{1}{2}$	$9 \div 10$	$18 \div 19$	$15{,}000 \div 1{,}500{,}000$
$30 \div 0.45$	$9 \div 10{,}000$	$18 \div 0.18$	$15{,}000 \div 14{,}500$

   - Close to 0
   - Close to 1
   - Much larger than 1

Student Response

1. a. Set 1: $800 \div 0.0001$, $800 \div \frac{1}{10}$, $800 \div 2.5$, $800 \div 250$, $800 \div 799.5$, $800 \div 801$, $800 \div 1{,}250$, $800 \div 10{,}000$

b. Set 2: $5{,}000{,}000 \div 25$, $1{,}000 \div 25$, $625 \div 25$, $75 \div 25$, $25.25 \div 25$, $24 \div 25$, $6.25 \div 25$, $0.0625 \div 25$

2.
   - Close to 0: $15{,}000 \div 1{,}500{,}000$ and $9 \div 10{,}000$
   - Close to 1: $9 \div 10$, $18 \div 19$, and $15{,}000 \div 14{,}500$
   - Much larger than 1: $30 \div 0.45$, $18 \div 0.18$, and $30 \div \frac{1}{2}$

Are You Ready for More?

Write 10 expressions of the form $12 \div ?$ in a list ordered from least to greatest. Can you list expressions that have value near 1 without equaling 1? How close can you get to the value 1?

Student Response

Answers vary. Sample response: $12 \div 13,\ 12 \div 12.5,\ 12 \div 12.1,\ 12 \div 12.05,\ 12 \div 12.01,\ 12 \div 12.001$. I can get closer and closer to the value of 1 by dividing 12 by a number that is closer and closer to 12 but is not exactly 12.

Activity Synthesis

Ask a couple of groups to share how they sorted the two sets of expressions. Discuss expressions that students found hard to place (e.g., those involving decimal, fractional, or very large divisors), and select other students to share how they made their decisions.

Next, discuss ways to reason about the size of quotients relative to 0 and 1. The terms "dividend," "divisor," and "quotient" may not be intuitive to students. Support them by keeping a reference (a division equation labeled with these terms) displayed or referring to numerical examples. Encourage students to use these terms as they respond to the following questions, which focus on reasoning about the size of quotients relative to 1 and 0:

- “How did you decide that a quotient is close to 0?” (The divisor is much larger than the dividend.)
- “How did you decide that a quotient is close to 1?” (If the divisor is close to the dividend, then it would be closer to 1.)
- “Is there a way to tell if a quotient is less than 1 or more than 1?” (If the divisor is smaller than the dividend, then the quotient is more than 1. Otherwise, it is less than 1.)
- “Suppose a divisor is less than the dividend. How can we tell if the quotient is just a little larger than 1 or much larger than 1?” (If the dividend and divisor are very far apart in size, then the quotient is much larger than 1.)

Lesson Synthesis

In this lesson, we explored situations that involve division and the numbers in divisions. We noticed that the size of the *dividend* (the number that we are dividing) and the size of the *divisor* (the number we use to divide) both affect the quotient(the result of dividing). Let's recall our observations.

- “What happens to the quotient when we divide by smaller and smaller numbers?” (The quotient gets increasingly larger.)
- “Which would result in a smaller quotient: dividing a number by 0.5 or dividing it by 5? Why?” (5, because we would need fewer 5s than 0.5s to reach the size of the dividend.)

We also compared the size of a divisor to that of a dividend and saw how they affect the quotient. Ask students:

- “What can we say about the quotient when the divisor is much smaller than the dividend?” (The quotient will be much greater than 1.)
- “What about when we divide a number by another number that is much larger?” (The quotient will be close to 0.)
- “What can we say about the quotient when the divisor and dividend are about the same size?” (The quotient will be close to 1.)

1.4 Result of Division

Cool Down: 5 minutes

Building Towards

- 6.NS.A

Student Task Statement

Without computing, decide whether the value of each expression is much smaller than 1, close to 1, or much larger than 1.

1. $1{,}000{,}001 \div 99$
2. $3.7 \div 4.2$
3. $1 \div 835$
4. $100 \div \frac{1}{100}$
5. $0.006 \div 6{,}000$
6. $50 \div 50\frac{1}{4}$

Student Response

1. Much larger than 1
2. Close to 1
3. Much smaller than 1
4. Much larger than 1
5. Much smaller than 1

6. Close to 1

Student Lesson Summary

Here is a division expression: $60 \div 4$. In this division, we call 60 the *dividend* and 4 the *divisor*. The result of the division is the quotient. In this example, the quotient is 15, because $60 \div 4 = 15$.

We don't always have to make calculations to have a sense of what a quotient will be. We can reason about it by looking at the size of the dividend and the divisor. Let's look at some examples.

- In $100 \div 11$ and in $18 \div 2.9$ the dividend is larger than the divisor. $100 \div 11$ is very close to $99 \div 11$, which is 9. The quotient $18 \div 2.9$ is close to $18 \div 3$ or 6.

  In general, when a larger number is divided by a smaller number, the quotient is greater than 1.

- In $99 \div 101$ and in $7.5 \div 7.4$ the dividend and divisor are very close to each other. $99 \div 101$ is very close to $99 \div 100$, which is $\frac{99}{100}$ or 0.99. The quotient $7.5 \div 7.4$ is close to $7.5 \div 7.5$, which is 1.

  In general, when we divide two numbers that are nearly equal to each other, the quotient is close to 1.

- In $10 \div 101$ and in $50 \div 198$ the dividend is smaller than the divisor. $10 \div 101$ is very close to $10 \div 100$, which is $\frac{10}{100}$ or 0.1. The division $50 \div 198$ is close to $50 \div 200$, which is $\frac{1}{4}$ or 0.25.

  In general, when a smaller number is divided by a larger number, the quotient is less than 1.

Lesson 1 Practice Problems

Problem 1

Statement

Order from smallest to largest:

- Number of pennies in a stack that is 1 ft high
- Number of books in a stack that is 1 ft high
- Number of dollar bills in a stack that is 1 ft high
- Number of slices of bread in a stack that is 1 ft high

Solution

Number of books, number of slices of bread, number of pennies, number of dollar bills

Problem 2

Statement

Use each of the numbers 4, 40, and 4000 once to complete the sentences.

a. The value of ________________ $\div 40.01$ is close to 1.

b. The value of ________________ $\div 40.01$ is much less than 1.

c. The value of ________________ $\div 40.01$ is much greater than 1.

Solution

a. 40

b. 4

c. 4000

Problem 3

Statement

Without computing, decide whether the value of each expression is much smaller than 1, close to 1, or much greater than 1.

a. $100 \div \frac{1}{1000}$

b. $50\frac{1}{3} \div 50\frac{1}{4}$

c. $4.7 \div 5.2$

d. $2 \div 7335$

e. $2{,}000{,}001 \div 9$

f. $0.002 \div 2{,}000$

Solution

a. Much greater than 1

b. Close to 1

c. Close to 1

d. Much smaller than 1

e. Much greater than 1

f. Much smaller than 1

Problem 4

Statement

A rocking horse has a weight limit of 60 pounds.

a. What percentage of the weight limit is 33 pounds?

b. What percentage of the weight limit is 114 pounds?

c. What weight is 95% of the limit?

Solution

a. 55%

b. 190%

c. 57 pounds

(From Unit 3, Lesson 16.)

Problem 5

Statement

Compare using >, =, or <.

a. 0.7 _____ 0.70

b. $0.03 + \frac{6}{10}$ _____ $0.30 + \frac{6}{100}$

c. 0.9 _____ 0.12

Solution

a. 0.7 = 0.70 because $\frac{7}{10} = \frac{70}{100}$.

b. $0.03 + \frac{6}{10} > 0.30 + \frac{6}{100}$ because $0.63 > 0.36$.

c. 0.9 > 0.12 because $\frac{9}{10} = \frac{90}{100}$.

(From Unit 3, Lesson 15.)

Problem 6

Statement

Diego has 90 songs on his playlist. How many songs are there for each genre?

a. 40% rock

b. 10% country

c. 30% hip-hop

d. The rest is electronica

Solution

a. 36, because $(0.4) \cdot 90 = 36$.

b. 9, because $(0.1) \cdot 90 = 9$.

c. 27, because $(0.3) \cdot 90 = 27$.

d. 18, because $(0.2) \cdot 90 = 18$.

(From Unit 3, Lesson 14.)

Problem 7

Statement

A garden hose emits 9 quarts of water in 6 seconds. At this rate:

a. How long will it take the hose to emit 12 quarts?

b. How much water does the hose emit in 10 seconds?

Solution

a. 8 seconds

b. 15 quarts

(From Unit 3, Lesson 8.)

Lesson 2: Meanings of Division

Goals

- Identify or generate a multiplication equation that represents the same relationship as a division expression, and explain (orally) the reasoning.
- Interpret and create tape diagrams that represent situations involving equal-sized groups.
- Recognize there are two different ways to interpret a division expression, i.e., asking "how many groups?" or "how many in each group?"

Learning Targets

- I can explain how multiplication and division are related.
- I can explain two ways of interpreting a division expression such as $27 \div 3$.
- When given a division equation, I can write a multiplication equation that represents the same situation.

Lesson Narrative

In this lesson, students revisit the relationship between multiplication and division that they learned in prior grades. Specifically, students recall that we can think of multiplication as expressing the number of equal-size groups, and that we can find a product if we know thenumber of groups and thesize of each group. They interpret division as a way of finding a missing factor, which can either be the number of groups, or the size of one group. They do so in the context of concrete situations and by using diagrams and equations to support their reasoning.

As they represent division situations with diagrams and equations and interpret division equations in context, students reason quantitatively and abstractly (MP2).

Alignments

Building On

- 3.OA.A.2: Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as $56 \div 8$.

Building Towards

- 6.NS.A: Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

Instructional Routines

- MLR8: Discussion Supports

- Think Pair Share

Student Learning Goals

Let's explore ways to think about division.

2.1 A Division Expression

Warm Up: 5 minutes
The purpose of this warm-up is to review students' prior understanding of division and elicit the ways in which they interpret a division expression. This review prepares them to explore the meanings of division in the lesson.

Some students may simply write the value of the expression because they struggle to put into words how they think about the problem. Encourage them to think of a story with a question, in which the expression could be used to answer the question.

Building On

- 3.OA.A.2

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Ask students to write a list of all of the ways they think about $20 \div 4$. Explain that they can write what the expression means to them, how they think about it when evaluating the expression, or a situation that matches the expression.

Give students 1 minute of quiet think time, followed by 1 minute of partner discussion. During discussion, ask students to share their responses and notice what they have in common.

Student Task Statement

Here is an expression: $20 \div 4$.

What are some ways to think about this expression? Describe at least two meanings you think it could have.

Student Response

Answers vary. Possible responses:

- How many groups of 4 are in 20?
- How many are in each group if we split 20 into 4 groups?
- How many 4's are in 20?
- What times 4 equals 20?

- What is the other side length of a rectangle with a side length of 4 and an area of 20?
- What is $\frac{1}{4}$ of 20?
- What is $\frac{20}{4}$?

Activity Synthesis

Invite partners to share the interpretations of $20 \div 4$ that they had in common. Record and display these responses for all to see. Ask students to notice any themes or trends in the range of responses.

Highlight the two ways students will be thinking about division in this unit:

- Division means partitioning a number or a quantity into equal groups and finding out *how many groups can be made*.
- Division means partitioning a number or a quantity into equal groups, and finding out *how much is in each group*.

2.2 Bags of Almonds

25 minutes
This activity prompts students to explore two ways of thinking about division by connecting it to multiplication, thinking about what it means in the context of a situation, and drawing visual representations.

Building On

- 3.OA.A.2

Building Towards

- 6.NS.A

Instructional Routines

- MLR8: Discussion Supports

Launch

Keep students in groups of 2. Ask students to keep their materials closed. Display the following question for all to see:

A baker has 12 pounds of almonds. She puts them in bags, so that each bag has the same weight. In terms of pounds and bags of almonds, what could $12 \div 6$ mean?

Give students a minute of quiet think time and 1–2 minutes to explain their thinking to their partner. Ask a few students who interpreted the expression differently to share their interpretations. If students do not bring up one of the two ways to interpret the 6, ask them about it: Could the 6 represent the number of bags (or the amount in each bag)?

Once students see that the divisor could be interpreted in two ways, ask students to open the materials and give students 4–5 minutes to complete the first question.

Reconvene as a class afterwards. Select a couple of students to explain Clare and Tyler's diagrams and equations. Highlight that, in this context, $12 \div 6$ could mean 12 pounds of almonds being divided equally into 6 bags, or 12 pounds of almonds being divided so that each bag has 6 pounds.

Give students quiet time to complete the rest of the activity.

Support for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to blank tape diagrams. Encourage students to annotate diagrams with details to show how each value is represented—for example number of pounds of almonds in total, number of pounds in one bag, or number of bags of almonds.
Supports accessibility for: Visual-spatial processing; Organization

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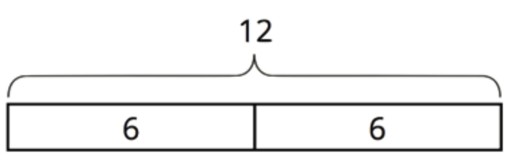
Student Task Statement

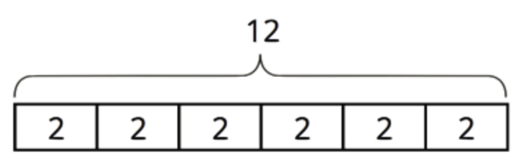
A baker has 12 pounds of almonds. She puts them in bags, so that each bag has the same weight.

Clare and Tyler drew diagrams and wrote equations to show how they were thinking about $12 \div 6$.



$\_\_\_ \cdot 6 = 12$

Clare's diagram and equation



$6 \cdot \_\_\_ = 12$

Tyler's diagram and equation

1. How do you think Clare and Tyler thought about $12 \div 6$? Explain what each diagram and the parts of each equation could mean about the situation with the bags of almonds. Make sure to include the meaning of the missing number.

   Pause here for a class discussion.

2. Explain what each division expression could mean about the situation with the bags of almonds. Then draw a diagram and write a multiplication equation to show how you are thinking about the expression.

   a. $12 \div 4$

b. $12 \div 2$

c. $12 \div \frac{1}{2}$

Student Response

1. Clare might have seen 6 as the number of pounds in a bag. The missing factor is then how many bags the baker would have. Tyler might have seen 6 as the number of bags. The missing factor would then mean how many pounds of almonds are in each bag.

2. Diagrams and interpretations vary.
    a. $12 \div 4$ could mean, "How many pounds in 1 bag if there were 12 pounds in 4 bags?" ($4 \cdot$ ____ $= 12$) or "How many bags can we get if we put 4 pounds in each bag?" (____ $\cdot\ 4 = 12$).

    b. $12 \div 2$ could mean, "How many pounds in 1 bag if there were 12 pounds in 2 bags?" ($2 \cdot$ ____ $= 12$) or "How many bags can we get if we put 2 pounds in each bag?" (____ $\cdot\ 2 = 12$).

    c. $12 \div \frac{1}{2}$ could mean, "How many pounds in 1 bag if there were 12 pounds in $\frac{1}{2}$ bag?" ($\frac{1}{2} \cdot$ ____ $= 12$) or "How many bags can we get if we put $\frac{1}{2}$ pound in each bag?" (____ $\cdot\ \frac{1}{2} = 12$).

Are You Ready for More?

A loaf of bread is cut into slices.

1. If each slice is $\frac{1}{2}$ of a loaf, how many slices are there?
2. If each slice is $\frac{1}{5}$ of a loaf, how many slices are there?
3. What happens to the number of slices as each slice gets smaller?
4. What would dividing by 0 mean in this situation about slicing bread?

Student Response

1. 2
2. 5
3. The number of slices grows as the slices get smaller. This process is limited by how small the bread can get physically.
4. Dividing the bread into slices of length 0, which doesn't make sense.

Activity Synthesis

Select a few students to share their diagrams and equations for the problems in the last question. After each explanation, highlight the connections between the expression, the diagram, and the context. Make sure students understand that the division expression $12 \div 6$ can be interpreted as the answer to the question “6 times what number equals 12?” or the question, “What number times 6 equals 12?" (or "How many 6s are in 12?"). More generally, division can be interpreted as a way to find two values:

- The size of each group when we know the number of groups and a total amount
- How many groups are in a total amount given the size of one group

Note that students may write either $\_\_\_\_ \cdot 6 = 12$ or $6 \cdot \_\_\_\_ = 12$ for each interpretation as long as they understand what each factor represents. Because we tend to say “___ groups of ___” in these materials, we follow that order in writing the multiplication:

$$\text{(number of groups)} \cdot \text{(size of each group)} = \text{total amount}$$

When discussing $12 \div 2$, make explicit how its multiplication equations and diagram connect to those of $12 \div 6$ in the first question. Students may see that the diagrams for $2 \cdot \_\_\_\_ = 12$ and $\_\_\_\_ \cdot 6 = 12$ are partitioned the same way. Point out that:

- In $2 \cdot \_\_\_\_ = 12$, the size of each group (each bag) was unknown, but because there are 2 equal groups in 12, we concluded that there were 6 pounds in each group.
- In $\_\_\_\_ \cdot 6 = 12$, we know each group (each bag) has 6 pounds of almonds, so there must be 2 groups of 6 in 12 pounds.

This discussion will be helpful in upcoming work, as students use their understanding of representations of division to divide fractions.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each diagram and expression that is shared, ask students to restate what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

Lesson Synthesis

In this lesson, we explored the relationship between multiplication and division in order to understand the meanings of division. We know that multiplication can represent the number of equal-size groups. For instance, $3 \cdot 5 = 15$ can mean 3 groups of 5 make 15. Let's review how we can use the same idea of equal-size groups to think about division.

- "How can we interpret $20 \div 8$?" (We can think of it as "how many groups of 8 are in 20?" or "how much is in each group if there are 20 in 8 groups?)
- "Suppose we interpret it as 'how many groups of 8 are in 20?'. How might we draw a diagram to show this?" (A bar that represents 20 divided into equal parts of 8.) "What multiplication equation can we write?" ($? \cdot 8 = 20$ or $8 \cdot ? = 20$, as long as we are clear what each factor represents.)
- "If we think of it as 'how much is in each group if there are 20 in 8 groups?', how would the diagram be different?" (A bar that represents 20 divided into 8 equal parts.) "What multiplication equation can we write?" ($8 \cdot ? = 20$ or $? \cdot 8 = 20$, as long as we know what each factor represents.)

2.3 Groups on A Field Trip

Cool Down: 5 minutes

Building Towards

- 6.NS.A

Student Task Statement

1. During a field trip, 60 students are put into equal-sized groups.
   a. Describe two ways to interpret $60 \div 5$ in this context.
   b. Find the quotient.
   c. Explain what the quotient would mean in each of the two interpretations you described.
2. Consider the division expression $7\frac{1}{2} \div 2$. Select **all** multiplication equations that correspond to this division expression.
   a. $2 \cdot ? = 7\frac{1}{2}$
   b. $7\frac{1}{2} \cdot ? = 2$
   c. $2 \cdot 7\frac{1}{2} = ?$
   d. $? \cdot 7\frac{1}{2} = 2$
   e. $? \cdot 2 = 7\frac{1}{2}$

Student Response

1. a. $60 \div 5$ could represent, "How many students are in each group if there are 5 groups?" (or "How many groups can be formed if there are 5 students per group?")
   b. $60 \div 5 = 12$

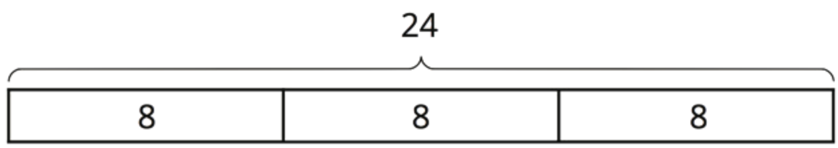
c. The quotient could mean there are 12 students in each of the 5 groups (or there are 12 groups with 5 students in each group).

2. A ($2 \cdot ? = 7\frac{1}{2}$) and E ($? \cdot 2 = 7\frac{1}{2}$)

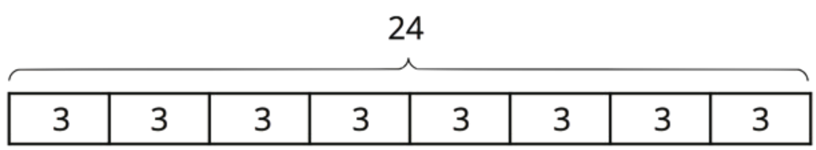
Student Lesson Summary

Suppose 24 bagels are being distributed into boxes. The expression $24 \div 3$ could be understood in two ways:

- 24 bagels are distributed equally into 3 boxes, as represented by this diagram:



- 24 bagels are distributed into boxes, 3 bagels in each box, as represented by this diagram:



In both interpretations, the quotient is the same ($24 \div 3 = 8$), but it has different meanings in each case. In the first case, the 8 represents the number of bagels in each of the 3 boxes. In the second, it represents the number of boxes that were formed with 3 bagels in each box.

These two ways of seeing division are related to how 3, 8, and 24 are related in a multiplication. Both $3 \cdot 8$ and $8 \cdot 3$ equal 24.

- $3 \cdot 8 = 24$ can be read as "3 groups of 8 make 24."
- $8 \cdot 3 = 24$ can be read as "8 groups of 3 make 24."

If 3 and 24 are the only numbers given, the multiplication equations would be:

$$3 \cdot ? = 24$$
$$? \cdot 3 = 24$$

In both cases, the division $24 \div 3$ can be used to find the value of the "?" But now we see that it can be interpreted in more than one way, because the "?" can refer to *the size of a group* (as in "3 groups of what number make 24?"), or to *the number of groups* (as in "How many groups of 3 make 24?").

Lesson 2 Practice Problems

Problem 1

Statement

Twenty pounds of strawberries are being shared equally by a group of friends. The equation $20 \div 5 = 4$ represents the division of strawberries.

a. If the 5 represents the number of people, what does the 4 represent?

b. If the 5 represents the pounds of strawberries per person, what does the 4 represent?

Solution

a. The number of pounds of strawberry each person got.

b. The number of friends who were sharing the strawberries.

Problem 2

Statement

A sixth-grade science club needs $180 to pay for the tickets to a science museum. All tickets cost the same amount.

What could $180 \div 15$ mean in this situation? Describe two different possible meanings of this expression. Then, find the quotient and explain what it means in each case.

Solution

$180 \div 15$ could mean: "How many tickets could the club buy with $180 if each ticket costs $15?" or "How much does each ticket cost if $180 buys 15 tickets?" The answer is $180 \div 15 = 12$. In the first case, it means the club could buy 12 tickets. In the second, it means each ticket costs $12.

Problem 3

Statement

Write a multiplication equation that corresponds to each division equation.

a. $10 \div 5 = ?$

b. $4.5 \div 3 = ?$

c. $\frac{1}{2} \div 4 = ?$

Solution

Answers vary. Sample responses:

a. $? \cdot 5 = 10$ or $5 \cdot ? = 10$

b. $? \cdot 3 = 4.5$ or $3 \cdot ? = 4.5$

c. $? \cdot 4 = \frac{1}{2}$ or $4 \cdot ? = \frac{1}{2}$

Problem 4

Statement

Write a division or multiplication equation that represents each situation. Use a "?" for the unknown quantity.

a. 2.5 gallons of water are poured into 5 equally sized bottles. How much water is in each bottle?

b. A large bucket of 200 golf balls is divided into 4 smaller buckets. How many golf balls are in each small bucket?

c. Sixteen socks are put into pairs. How many pairs are there?

Solution

a. $2.5 \div 5 = ?$ or $5 \cdot ? = 2.5$

b. $200 \div 4 = ?$ or $4 \cdot ? = 200$

c. $16 \div 2 = ?$ or $? \cdot 2 = 16$

Problem 5

Statement

Find a value for a that makes each statement true.

a. $a \div 6$ is greater than 1

b. $a \div 6$ is equal to 1

c. $a \div 6$ is less than 1

d. $a \div 6$ is equal to a whole number

Solution

a. Answers vary. (Any number $a > 6$, for example $a = 7$)

b. $a = 6$

c. Answers vary. (Any positive number $a < 6$, for example $a = 3$)

d. If a is a multiple of 6, then $a \div 6$ is a whole number.

(From Unit 4, Lesson 1.)

Problem 6

Statement

Complete the table. Write each percentage as a percent of 1.

fraction	decimal	percentage
$\frac{1}{4}$	0.25	25% of 1
	0.1	
		75% of 1
$\frac{1}{5}$		
	1.5	
		140% of 1

Solution

fraction	decimal	percentage
$\frac{1}{4}$	0.25	25% of 1
$\frac{1}{10}$	0.1	10% of 1
$\frac{3}{4}$	0.75	75% of 1
$\frac{1}{5}$	0.2	20% of 1
$\frac{3}{2}$	1.5	150% of 1
$\frac{7}{5}$	1.4	140% of 1

(From Unit 3, Lesson 14.)

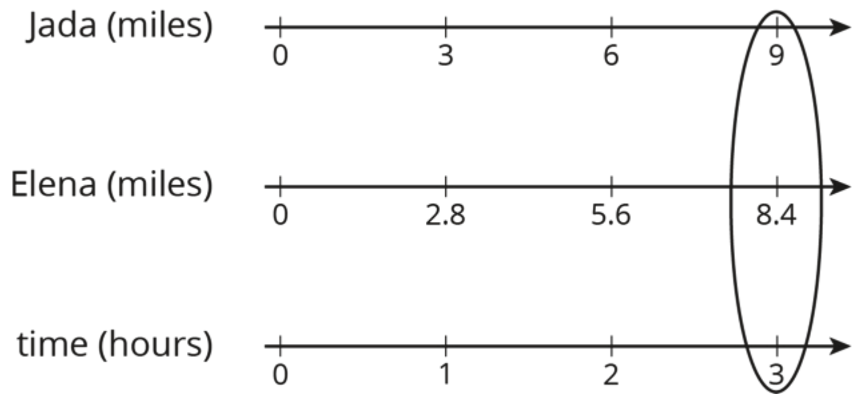
Problem 7

Statement

Jada walks at a speed of 3 miles per hour. Elena walks at a speed of 2.8 miles per hour. If they both begin walking along a walking trail at the same time, how much farther will Jada walk after 3 hours? Explain your reasoning.

Solution

Jada will have walked 0.6 miles farther. Sample reasoning:



After 3 hours Jada will have walked 9 miles, and Elena will have walked 8.4 miles. $9 - 8.4 = 0.6$.

(From Unit 3, Lesson 8.)

Lesson 3: Interpreting Division Situations

Goals

- Create an equation and a diagram to represent a multiplication or division situation involving fractions, and coordinate these representations (orally).
- Explain (using words and other representations) how to find the unknown quantity in a multiplication or division situation involving fractions.
- Interpret a verbal description of a multiplication situation (in spoken or written language), and identify which quantity is unknown, i.e., the number of groups, the amount in one group, or the total amount.

Learning Targets

- I can create a diagram or write an equation that represents division and multiplication questions.
- I can decide whether a division question is asking “how many groups?” or “how many in each group?”.

Lesson Narrative

In an earlier lesson, students were reminded of the connection between multiplication and division. They revisited the idea of division as a way to find a missing factor, which can either be the number of groups, or the size of one group.

In this lesson, students interpret division situations in story problems that involve equal-size groups. They draw diagrams and write division and multiplication equations to make sense of the relationship between known and unknown quantities (MP2).

Alignments

Building On

- 3.OA.A: Represent and solve problems involving multiplication and division.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Building Towards

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports
- Notice and Wonder
- Think Pair Share

> **Student Learning Goals**
>
> Let’s explore situations that involve division.

3.1 Dot Image: Properties of Multiplication

Warm Up: 5 minutes

In this warm-up, students determine the number of dots in an image without counting and explain how they arrived at that answer. The goal is to prompt them to visualize and articulate different ways to decompose the dots, using what they know about arrays, multiplication, and area to arrive at the total number. To discourage counting and encourage students to focus on the structure of the dots, the image will be flashed and hidden a couple of times, rather than displayed the entire time.

Building On

- 3.OA.A

Launch

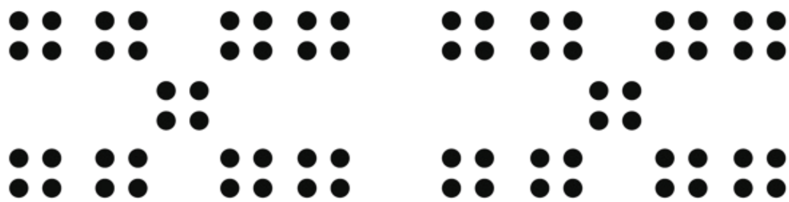
Tell students to keep their materials closed. Explain that you will show an image that contains dots for 3 seconds. Their job is to determine how many dots there are and explain how they saw them.

Display the image for 3 seconds and then hide it. Do this twice. Give students up to 1 minute of quiet think time after each flash. Encourage students who have one way of seeing the dots to think about another way while they wait.

Support for Students with Disabilities

Representation: Internalize Comprehension. Guide information processing and visualization. To support working memory, show the image for a longer period of time or repeat the image flash as needed. Students may also benefit from being explicitly told not to count the dots, but instead to look for helpful structure within the image.
Supports accessibility for: Memory; Organization

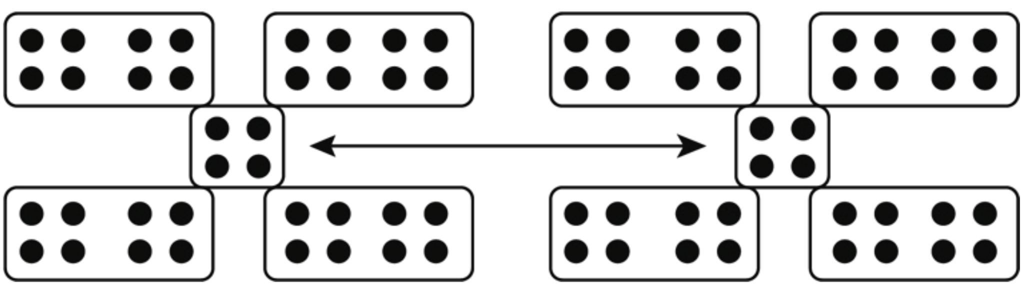
Student Task Statement



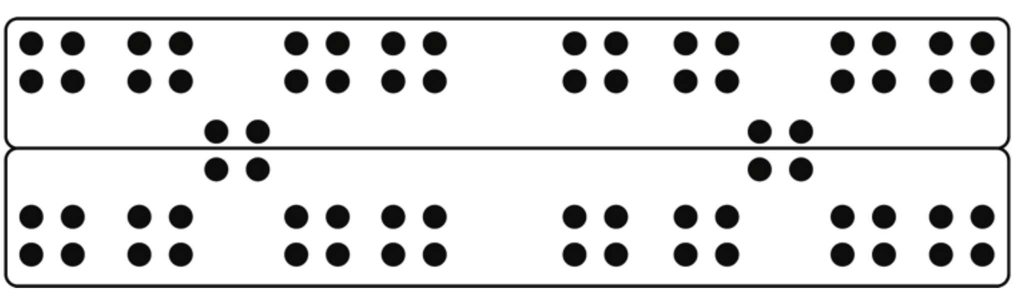
Student Response

72 dots. Possible strategies:

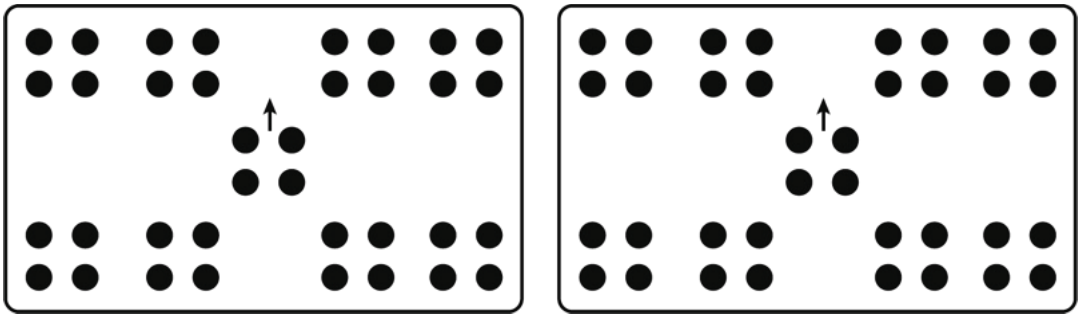
- $9 \cdot 8 = 72$



- $[(2 \cdot 4 \cdot 4) + 4] \cdot 2 = 72$



- $[(5 \cdot 4) + (4 \cdot 4)] \cdot 2 = 72$



Activity Synthesis

Select a couple of students to share the number of dots they saw. To focus students' reasoning on the structure of the dots, ask how they *saw* them, instead of how they *found* them. Record and display student explanations for all to see. Consider re-displaying the image to support students in their explanation.

To involve more students in the conversation, consider asking:

- Who can restate the way ___ saw the dots in different words?
- Did anyone see the dots the same way but would explain it differently?
- Does anyone want to add an observation to the way ____ saw the dots?
- Who saw the dots differently?
- Do you agree or disagree? Why?

3.2 Homemade Jams

20 minutes (there is a digital version of this activity)

This activity allows students to draw diagrams and write equations to represent simple division situations. Some students may draw concrete diagrams; others may draw abstract ones. Any diagrammatic representation is fine as long as it enables students to make sense of the relationship between the number of groups, the size of a group, and a total amount.

The last question is likely more challenging to represent with a diagram. Because the question asks for the number of jars, and because the amount per jar is a fraction, students will not initially know how many jars to draw (unless they know what $6\frac{3}{4} \div \frac{3}{4}$ is). Suggest that they start with an estimate, and as they reason about the problem, add jars to (or remove jars from) their diagram as needed.

As students work, monitor for the range of diagrams that students create. Select a few students whose work represent the range of diagrams to share later.

Students in digital classrooms can use an applet to make sense of the problems, but it is still preferable that they create their own diagrams.

Building Towards

- 6.NS.A.1

Instructional Routines

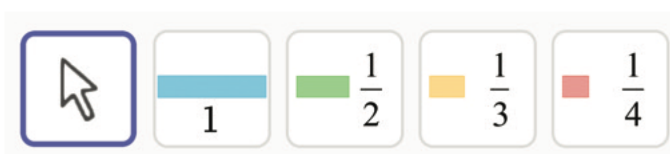
- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports
- Notice and Wonder

Launch

Arrange students in groups of 2. Tell the class that you will read the three story problems, and ask them to be prepared to share at least one thing they notice and one thing they wonder. After reading, give them a minute to share their observation and question with their partner.

Clarify that their job is to draw a diagram and write a multiplication equation to express the relationship in each story and then answer the question. Give students 7–8 minutes of quiet work time, followed by 2–3 minutes to share their work with their partner.

If the applet is used to complete the activity or for class discussion, note that the toolbar includes colored rectangles that represent fractional parts. Encourage students to drop the fractional parts in the work space on the left of the window, and then use the Move tool (the arrow) to drag them into the jars. The blocks do snap, but the grid is very fine so this may be challenging for some students. Troubleshooting tip: if two blocks get stuck together, delete them. Try not to overlap blocks when adding them to the work space.



Student Task Statement

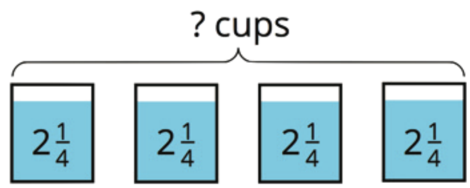
Draw a diagram, and write a multiplication equation to represent each situation. Then answer the question.

1. Mai had 4 jars. In each jar, she put $2\frac{1}{4}$ cups of homemade blueberry jam. Altogether, how many cups of jam are in the jars?
2. Priya filled 5 jars, using a total of $7\frac{1}{2}$ cups of strawberry jam. How many cups of jam are in each jar?
3. Han had some jars. He put $\frac{3}{4}$ cup of grape jam in each jar, using a total of $6\frac{3}{4}$ cups. How many jars did he fill?

Student Response

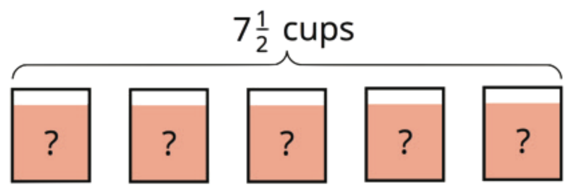
Equations and diagrams vary.

1. Multiplication equation: $4 \cdot \left(2\frac{1}{4}\right) = ?$ Sample diagram:



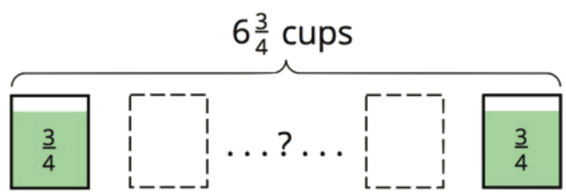
   Answer: 9 cups

2. Multiplication equation: $5 \cdot ? = 7\frac{1}{2}$. Sample diagram:



   Answer: $1\frac{1}{2}$ cups per jar

3. Multiplication equation: $? \cdot \frac{3}{4} = 6\frac{3}{4}$. Sample diagram:



   Answer: 9 jars

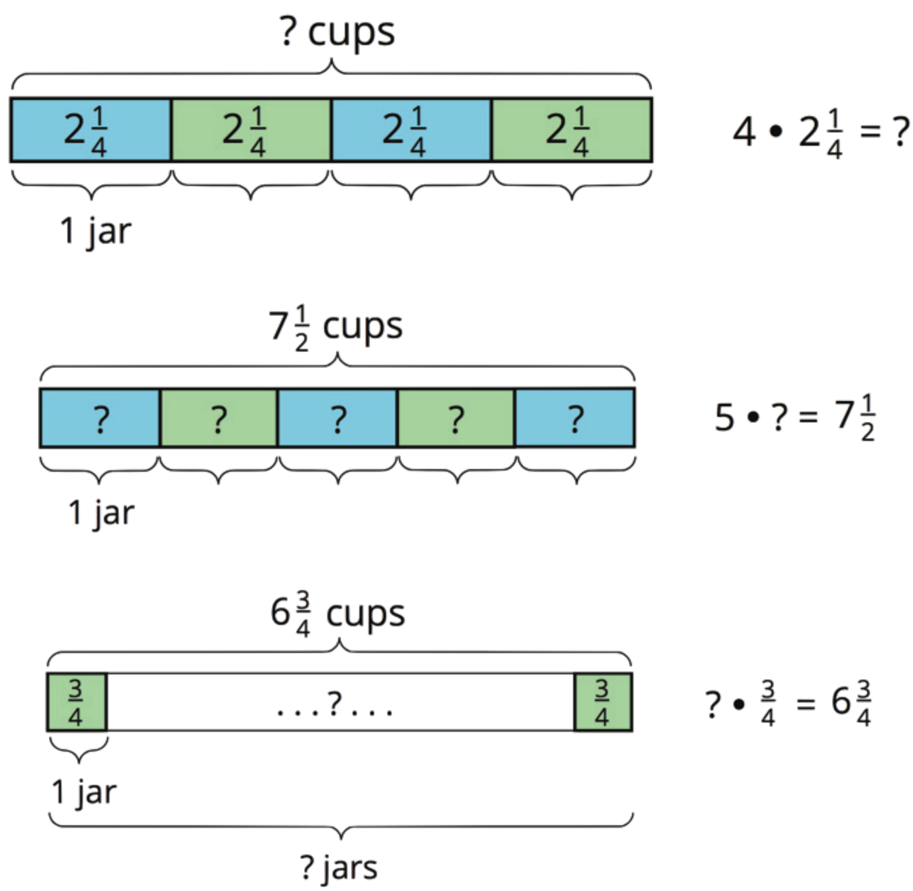
Activity Synthesis

Select previously identified students to share their diagrams, sequenced from the more concrete (e.g., pictures of jars and cups) to the more abstract (e.g, rectangles, tape diagrams). Display the diagrams and equations for all to see. Ask them how they used the diagrams to answer the questions (if at all).

If tape diagrams such as the ones here are not already shown and explained by a student, display them for all to see. Help students make sense of the diagrams and connecting them to multiplication and division by discussing questions such as these:

- “In each diagram, what does the ‘?’ represent?” (The unknown amount)
- “What does the length of the entire tape represent?” (The total amount, which is sometimes known.)
- “What does each rectangular part represent?” (One jar)
- “What does the number in each rectangle represent?” (The amount in each jar)

- “How do the three parts of each multiplication equation relate to the diagram?” (The first factor refers to the number of rectangles. The second factor refers to the amount in each rectangle. The product is the total amount.)
- “The last diagram doesn't represent all the jars and shows a question mark in the middle of the tape. Why might that be?” (The diagram shows an unknown number of jar, which was the question to be answered.)



Highlight that the last two situations can be described with division: $7\frac{1}{2} \div 5$ and $6\frac{3}{4} \div \frac{3}{4}$.

Support for English Language Learners

Representing: MLR8 Discussion Supports. Use this routine to support whole-class discussion. As students discuss the questions listed in the Activity Synthesis, label the display of the diagrams and equations accordingly. Annotate the display to illustrate connections between equivalent parts of each representation. For example, next to each question mark, write “unknown amount.”
Design Principle(s): Support sense-making

3.3 Making Granola

15 minutes

In this activity, students continue to investigate division problems in context, think of them in terms of equal-size groups, and represent them using diagrams and equations.

As students work, monitor the different diagrams and interpretations for each problem. Select students whose diagrams are especially clear in showing the meanings of division to share later.

Both division problems result in quotients that are not whole numbers. As students work, encourage them to use their multiplication equations to check the answers to the division problems.

Building Towards

- 6.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Keep students in the same groups. Some students may not be familiar with granola and oats, so show or explain what they are.

Give students 7–8 minutes of quiet work time, and 2–3 minutes to discuss their responses with a partner. During partner discussion, ask them to compare their equations and diagrams in the first question, and their interpretations of division in the second question.

Support for Students with Disabilities

Representation: Internalize Comprehension. Represent the same information through different modalities using diagrams. If students are unsure where to begin, suggest that they draw a diagram to help organize the information provided. Some students may benefit from support to be able to draw abstract diagrams. For example, demonstrate how 4 ounces of oats can be represented with 1 scoop in a drawing.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

Some students may round their answers to the nearest whole number rather than including the fractions of a scoop or a batch. Ask students to consider if it is possible to have a part of a scoop, a part of a batch, or a part of a unit of weight. Encourage them to think about how to show a part of a whole unit on a diagram.

Student Task Statement

1. Consider the problem: To make 1 batch of granola, Kiran needs 26 ounces of oats. The only measuring tool he has is a 4-ounce scoop. How many scoops will it take to measure 26 ounces of oats?

    a. Will the answer be more than 1 or less than 1?

    b. Write a multiplication equation and a division equation that represent this situation. Use "?" to represent the unknown quantity.

    c. Find the unknown quantity. If you get stuck, consider drawing a diagram.

2. The recipe calls for 14 ounces of mixed nuts. To get that amount, Kiran uses 4 bags of mixed nuts.

    a. Write a mathematical question that might be asked about this situation.

    b. What might the equation $14 \div 4 = \ ?$ represent in Kiran's situation?

    c. Find the quotient. Show your reasoning. If you get stuck, consider drawing a diagram.

Student Response

1. a. More than 1

    b. $? \cdot 4 = 26$ (or $4 \cdot ? = 26$) and $26 \div 4 = ?$

    c. The missing quantity is $6\frac{1}{2}$ scoops. Reasoning varies. Sample reasoning: I know $6 \cdot 4$ is 24, and $\frac{1}{2}$ of 4 is 2, so it would take $6 + \frac{1}{2}$ or $6\frac{1}{2}$ scoops to make 26 ounces.

2. a. Answers vary. Sample responses:
    - How many ounces of mixed nuts are in each bag?
    - Did he use up all 4 bags, or are there leftover mixed nuts?
    - How many bags will he need if he is doubling the recipe?

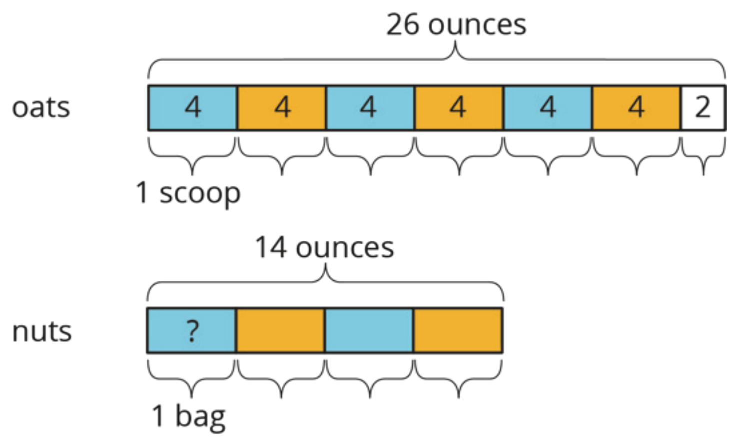
    b. How many ounces of mixed nuts are in each bag?

    c. $3\frac{1}{2}$ ounces. Reasoning varies. Sample reasoning: $14 \div 2 = 7$, so there are 7 ounces in 2 bags of mixed nuts, or $3\frac{1}{2}$ ounces in 1 bag of mixed nuts. $7 \div 2 = 3\frac{1}{2}$.

Activity Synthesis

The aim of the discussion is to solidify students' understanding of the two interpretations of division ("how many groups?" and "how much in a group?"). Ask students who drew effective diagrams to display and explain them. For each interpretation, write a multiplication equation and discuss what each factor represents in the context (e.g., the number of batches, scoops, or bags, vs. how much is in each batch, scoop, or bag).

If no students drew tape diagrams to represent the situations, show the following. Reiterate that each rectangle represents one group (one scoop or 1 bag), the number inside represents the amount in one group, and the number of rectangles tells us how many groups there are.



Lesson Synthesis

In this lesson, we solved problems that involved multiplication and division. Reiterate to students that in division situations that involve equal-size groups, we are not always looking for the same unknown. There are typically three pieces of information involved: the number of groups, the size of each group, and the total amount. Knowing what information we have and what is missing can help us answer questions.

Present a few more story problems. Ask students: "What information is unknown in each situation?"

- Flour is sold in 3-pound bags. How many pounds are in 7 bags? (The total amount is unknown.)
- Five tickets to a play cost $38. What does each ticket cost if they all cost the same? (The amount for one person is unknown.)
- One quart is equal to 32 ounces. How many quarts are in 128 ounces? (The number of groups is unknown.)

For each problem, discuss: “What multiplication equation can we write? What diagram can we draw to represent the quantities?”

3.4 Rice and Beans

Cool Down: 5 minutes
Addressing

- 6.NS.A.1

Student Task Statement

1. Here are three problems. Select **all** problems that can be solved using division.

a. Jada cut 4 pieces of ribbon that were equal in length. She used a total of 5 feet of ribbon. How long, in feet, was each piece of ribbon she cut?

b. A chef bought 3 bags of beans. Each bag contains $1\frac{2}{5}$ kilograms of beans. How many kilograms of beans did she buy?

c. A printer takes $2\frac{1}{2}$ seconds to print a flyer. It took 75 seconds to print a batch of flyers without stopping. How many flyers were in the batch?

2. Consider the problem: Andre poured 27 ounces of rice into 6 containers. If all containers have the same amount of rice, how many ounces are in each container?

   a. Write an equation to represent the situation. Use a "?" to represent the unknown quantity.

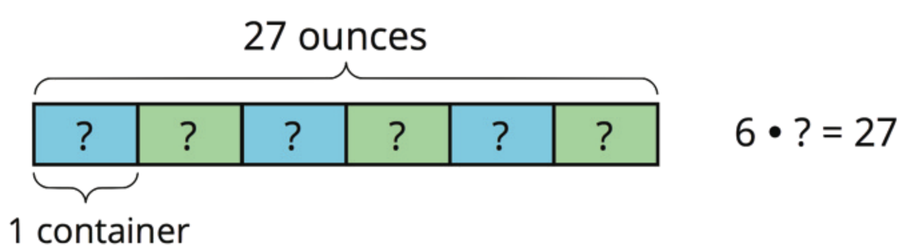
   b. Find the unknown quantity. Show your reasoning.

Student Response

1. A and C

2. a. $6 \cdot ? = 27$ or $27 \div 6 = ?$

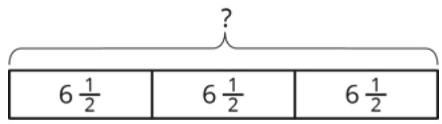
   b. $4\frac{1}{2}$ ounces. Reasoning varies. Sample reasoning:

   - 3 groups of 9 make 27. Splitting each of the 3 groups into 2 gives us 6 groups. Each group is half of 9, which is $4\frac{1}{2}$.
   - 27 divided by 6 is 4 with a remainder of 3. That means 27 is made of 6 groups of size 4 and a remainder of 3. Splitting the remainder of 3 into 6 groups makes 6 groups of size $\frac{1}{2}$. That means that 27 is made of 6 groups of size $4\frac{1}{2}$.
   - 

Student Lesson Summary

If a situation involves equal-sized groups, it is helpful to make sense of it in terms of the number of groups, the size of each group, and the total amount. Here are three examples to help us better understand such situations.

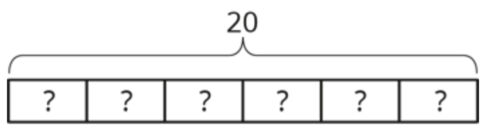
- Suppose we have 3 bottles with $6\frac{1}{2}$ ounces of water in each, and the total amount of water is not given. Here we have 3 groups, $6\frac{1}{2}$ ounces in each group, and an unknown total, as shown in this diagram:



We can express this situation as a multiplication problem. The unknown is the product, so we can simply multiply the 2 known numbers to find it.

$$3 \cdot 6\frac{1}{2} = ?$$

- Next, suppose we have 20 ounces of water to fill 6 equal-sized bottles, and the amount in each bottle is not given. Here we have 6 groups, an unknown amount in each, and a total of 20. We can represent it like this:



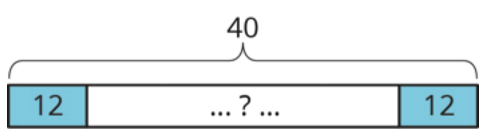
This situation can also be expressed using multiplication, but the unknown is a factor, rather than the product:

$$6 \cdot ? = 20$$

To find the unknown, we cannot simply multiply, but we can think of it as a division problem:

$$20 \div 6 = ?$$

- Now, suppose we have 40 ounces of water to pour into bottles, 12 ounces in each bottle, but the number of bottles is not given. Here we have an unknown number of groups, 12 in each group, and a total of 40.



Again, we can think of this in terms of multiplication, with a different factor being the unknown:

$$? \cdot 12 = 40$$

Likewise, we can use division to find the unknown:

$$40 \div 12 = ?$$

Whenever we have a multiplication situation, one factor tells us *how many groups* there are, and the other factor tells us *how much is in each group*.

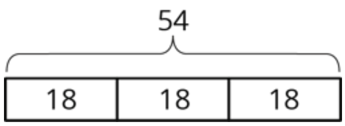
Sometimes we want to find the total. Sometimes we want to find how many groups there are. Sometimes we want to find how much is in each group. Anytime we want to find out how many groups there are or how much is in each group, we can represent the situation using division.

Lesson 3 Practice Problems

Problem 1

Statement

Write a multiplication equation and a division equation that this diagram could represent.



Solution

Multiplication: $3 \cdot 18 = 54$ (or $18 \cdot 3 = 54$), division: $54 \div 18 = 3$ (or $54 \div 3 = 18$)

Problem 2

Statement

Consider the problem: Mai has $36 to spend on movie tickets. Each movie ticket costs $4.50. How many tickets can she buy?

a. Write a multiplication equation and a division equation to represent this situation.

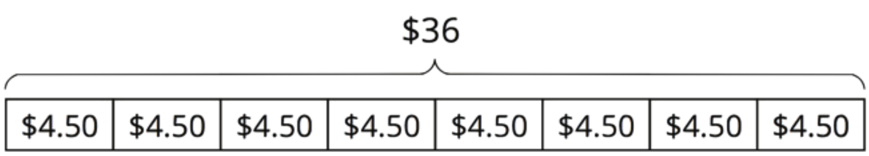
b. Find the answer. Draw a diagram, if needed.

c. Use the multiplication equation to check your answer.

Solution

a. Multiplication: $? \cdot (4.50) = 36$ (or equivalent), division: $36 \div 4.50 = ?$ (or equivalent)

b. Mai can buy 8 movie tickets.



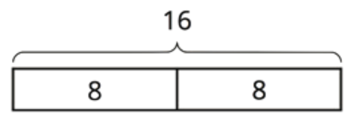
c. 8 is correct because $8 \cdot (4.50) = 36$.

Problem 3

Statement

Kiran said that this diagram can show the solution to $16 \div 8 = ?$ or $16 \div 2 = ?$, depending on how we think about the equations and the "?".

Explain or show how Kiran is correct.



Solution

The diagram can illustrate $16 \div 8 = ?$ if we interpret the equation and the "?" to mean: "How many groups of 8 are in 16?" The diagram can illustrate $16 \div 2 = ?$ if we interpret the equation and the "?" to mean: "What is in each group if 16 is divided into 2 equal groups?"

Problem 4

Statement

Write a sentence describing a situation that could be represented by the equation $4 \div 1\frac{1}{3} = ?$.

Solution

Answers vary. Sample responses:

- A group of friends met for lunch and got 4 small pizzas to share. Each person had $1\frac{1}{3}$ pizzas. How many friends went for lunch?
- A baker is filling equal-sized containers with sugar. Four pounds of sugar fill $1\frac{1}{3}$ containers. How many pounds fit in each container?

(From Unit 4, Lesson 2.)

Problem 5

Statement

Noah said, "When you divide a number by a second number, the result will always be smaller than the first number."

Jada said, "I think the result could be larger or smaller, depending on the numbers."

Do you agree with either of them? Explain or show your reasoning.

Solution

I agree with Jada. Explanations vary. Sample explanation: If number is divided by a number that is between 0 and 1, then the result is bigger than the first number. For example, $1 \div 0.1 = 10$, which is bigger than 1. But $1 \div 2 = 0.5$, which is smaller than 1.

(From Unit 4, Lesson 1.)

Problem 6

Statement

Mini muffins cost $3.00 per dozen.

- Andre says, “I have $2.00, so I can afford 8 muffins.”
- Elena says, “I want to get 16 muffins, so I’ll need to pay $4.00."

Do you agree with either of them? Explain your reasoning.

Solution

They are both correct. Each muffin costs 25 cents because $3 \div 12 = 0.25$. Andre can afford 8 muffins because $2 \div 0.25 = 8$, and Elena will need 4 dollars because $16 \boldsymbol{\cdot} 0.25 = 4$.

(From Unit 3, Lesson 7.)

Problem 7

Statement

A family has a monthly budget of $2,400. How much money is spent on each category?

a. 44% is spent on housing.

b. 23% is spent on food.

c. 6% is spent on clothing.

d. 17% is spent on transportation.

e. The rest is put into savings.

Solution

a. $1,056, because $(0.44) \boldsymbol{\cdot} 2,400 = 1{,}056$

b. $552, because $(0.23) \boldsymbol{\cdot} 2,400 = 552$.

c. $144, because $(0.06) \boldsymbol{\cdot} 2,400 = 144$.

d. $408, because $(0.17) \boldsymbol{\cdot} 2,400 = 408$.

e. $240, because there is 10% remaining for savings, and $(0.1) \boldsymbol{\cdot} 2,400 = 240$.

(From Unit 3, Lesson 15.)

Section: Meanings of Fraction Division

Lesson 4: How Many Groups? (Part 1)

Goals

- Coordinate multiplication equations and pattern block diagrams in which the yellow hexagon represents one whole.
- Create a diagram to represent and solve a problem asking “How many groups?” in which the divisor is a unit fraction, and explain (orally) the solution method.

Learning Targets

- I can find how many groups there are when the amount in each group is not a whole number.
- I can use diagrams and multiplication and division equations to represent “how many groups?” questions.

Lesson Narrative

This lesson and the next one extend the “how many groups?” interpretation of division to situations where the “group” can be fractional. This builds on the work in earlier grades on dividing whole numbers by unit fractions.

Students use pattern blocks to answer questions about how many times a fraction goes into another number (e.g., how many $\frac{2}{3}$s are in 2?), and to represent multiplication and division equations involving fractions. In this lesson, they focus on situations where the quotient (the number of groups) is a whole number.

This lesson is the first in a group of six lessons that trace out a gradual progression of learning—from reasoning with specific quantities, to using a symbolic formula for division of fractions (MP8).

Alignments

Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
- 5.NF.B.7: Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR3: Clarify, Critique, Correct

Required Materials

Pattern blocks

Required Preparation

Prepare enough pattern blocks so that each group of 3–4 students has at least 2 hexagons and 6 of each of the other shapes (triangle, rhombus, and trapezoid).

Student Learning Goals

Let’s play with blocks and diagrams to think about division with fractions.

4.1 Equal-sized Groups

Warm Up: 5 minutes

This warm-up reviews the idea of multiplication as representing equal-sized groups and the relationship between multiplication and division.

There are multiple equations students can write for each of the problems; the equations that connect multiplication and division to equal-sized groups are the important ones to highlight. As students work, identify students whose equations reflect these ideas.

Building On

- 5.NF.B.4
- 5.NF.B.7

Launch

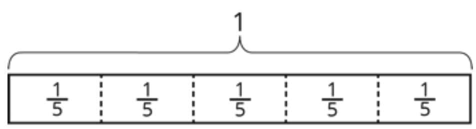
Give students 2 minutes of quiet think time, followed by a whole-class discussion.

Anticipated Misconceptions

Some students may struggle to frame repeated addition as multiplication. To help them see the connection, refer to one of their addition statements and ask questions such as, “How many same-sized groups are being added?” or “What is in each group?”.

Student Task Statement

Write a multiplication equation and a division equation for each sentence or diagram.

1. Eight \$5 bills are worth \$40.
2. There are 9 thirds in 3 ones.
3. 

Student Response

1. $8 \cdot 5 = 40$ (or $5 \cdot 8 = 40$) and $40 \div 5 = 8$ (or $40 \div 8 = 5$)
2. $9 \cdot \frac{1}{3} = 3$ (or $\frac{1}{3} \cdot 9 = 3$) and $3 \div 9 = \frac{1}{3}$ (or $3 \div \frac{1}{3} = 9$)
3. $5 \cdot \frac{1}{5} = 1$ (or $\frac{1}{5} \cdot 5 = 1$) and $1 \div 5 = \frac{1}{5}$ (or $1 \div \frac{1}{5} = 5$)

Activity Synthesis

Select 1–2 students to share their responses. Record the responses for all to see. Ask students to indicate whether they agree or disagree with each one.

As students present the equations for each problem, connect the pieces in each equation to the idea of equal-sized groups. Ask questions such as:

- “Which number in the multiplication equation refers to the number of groups?”
- “Which number in the multiplication equation refers to how much is in each group?”
- “In this case, what does the division $3 \div 9 = \frac{1}{3}$ mean?”
- “In this case, what does the division $1 \div \frac{1}{5} = 5$ mean?”

4.2 Reasoning with Pattern Blocks

25 minutes (there is a digital version of this activity)
In this activity, students use the relationships between the areas of geometric shapes to reason about division situations that involve fractions. The focus is on the “how many groups?” interpretation of division.

Students start by using pattern blocks to represent multiplication of a whole number and a fraction. For example, if a hexagon represents 1 and six triangles make a hexagon, then each triangle represents $\frac{1}{6}$. They can then use six triangles to represent $6 \cdot \frac{1}{6} = 1$.

Later, students use the blocks to reason in the opposite direction, answering questions such as, “How many $\frac{1}{2}$s are in 4?” These kinds of questions serve as a stepping stone to more abstract questions such as, “What is 4 divided by $\frac{1}{2}$?”

Addressing

- 6.NS.A.1

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

Arrange students in groups of 3–4. Provide access to pattern blocks. Give students 10–12 minutes to collaborate on the first three questions and 3–4 minutes of quiet think time for the last question.

Remind students of the following:

- We can think of a fraction such as $\frac{1}{2}$ or $\frac{1}{3}$ in relation to 1 whole. In this task, the hexagon is 1 whole.
- We worked with the same shapes earlier in the course. We saw that two triangles make a rhombus, because if we place two triangles (joined along one side with no gap) on top of a rhombus, the triangles would match the rhombus exactly. This means that a triangle is half of a rhombus.

Classrooms with no access to pattern blocks or those using the digital materials can use the provided applet. Physical pattern blocks are still preferred, however.

Support for Students with Disabilities

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. Once students have determined what fraction of a hexagon each of the shapes represents, pause the class. Invite students to demonstrate and share their strategies for finding each fraction using pattern blocks to justify their reasoning. Create a display that includes an image of each shape labeled with the name and the fraction it represents of a hexagon. Keep this display visible as students move on to the next problems.
Supports accessibility for: Conceptual processing; Memory

Support for English Language Learners

Representing, Writing: MLR3 Clarify, Critique, Correct. At the appropriate time, pause the class for a brief discussion of the first question. Display the following incorrect response that reflects a possible common misunderstanding: "The area of the rhombus is 3 because 3 fit inside the hexagon." Ask students, "Do you agree with the statement? Why or why not?" Invite students to identify the error, correct the statement, and draw a diagram to represent the situation. Improved statements should include fractional language and direct connections to the diagram. This will help students evaluate and improve on the written mathematical arguments of others.
Design Principle(s): Maximize meta-awareness; Optimize output (for justification)

Anticipated Misconceptions

Some students may not remember the names of the shapes for these blocks. Consider reviewing the names of these shapes before beginning the activity and having students write them next to the pictures for reference.

Some students may simply look at the blocks and incorrectly guess the size of each block relative to the hexagon. Encourage them to place the blocks on top of the hexagon, to use non-hexagons to compose a hexagon, or to otherwise manipulate the blocks in order to make comparisons.

Student Task Statement

Your teacher will give you pattern blocks as shown here. Use them to answer the questions.

1. If a hexagon represents 1 whole, what fraction does each of the following shapes represent? Be prepared to show or explain your reasoning.
    - 1 triangle
    - 1 rhombus
    - 1 trapezoid
    - 4 triangles
    - 3 rhombuses
    - 2 hexagons
    - 1 hexagon and 1 trapezoid

2. Here are Elena’s diagrams for $2 \cdot \frac{1}{2} = 1$ and $6 \cdot \frac{1}{3} = 2$. Do you think these diagrams represent the equations? Explain or show your reasoning.

3. Use pattern blocks to represent each multiplication equation. Remember that a hexagon represents 1 whole.

   a. $3 \cdot \frac{1}{6} = \frac{1}{2}$

   b. $2 \cdot \frac{3}{2} = 3$

4. Answer the questions. If you get stuck, consider using pattern blocks.

   a. How many $\frac{1}{2}$s are in 4?

   b. How many $\frac{2}{3}$s are in 2?

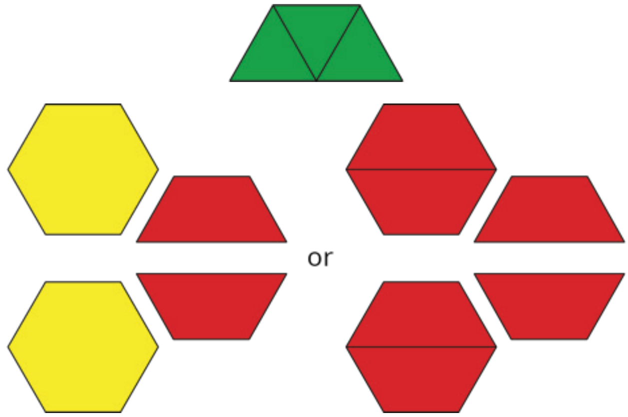
   c. How many $\frac{1}{6}$s are in $1\frac{1}{2}$?

Student Response

1.
   - $\frac{1}{6}$
   - $\frac{2}{6}$ (or $\frac{1}{3}$)
   - $\frac{1}{2}$
   - $\frac{4}{6}$ (or $\frac{2}{3}$)
   - 1
   - 2
   - $1\frac{1}{2}$

2. Agree. Sample reasoning: In the first representation, each trapezoid is $\frac{1}{2}$ of a hexagon, so 2 of them make 1 whole or
   1 hexagon. In the second representation, each rhombus is $\frac{1}{3}$ of a hexagon, so 3 rhombuses make 1 hexagon or 1 whole and 6 rhombuses make 2 wholes.

a.



3. b.

4. a. 8

   b. 3

   c. 9

Activity Synthesis

Select a few students to show their pattern-block arrangements or drawings for $3 \cdot \frac{1}{6} = \frac{1}{2}$ and $2 \cdot \frac{3}{2} = 3$. After each person shares, poll the class to see if others did it the same way or had alternative solutions.

Select other students to share their responses and reasoning for the last set of questions. If no one reasoned about the questions by using pattern blocks, show how the blocks could be used to answer the questions. For instance:

- For "how many $\frac{1}{2}$s are in 4?", we could use 8 trapezoids (each representing $\frac{1}{2}$) to make 4 hexagons.
- For "how many $\frac{2}{3}$s are in 2?", we could use 2 rhombuses (each representing $\frac{2}{3}$) to make 2 hexagons.
- For "how many $\frac{1}{6}$s are in $1\frac{1}{2}$?", we could use 9 triangles (each representing $\frac{1}{6}$) to make $1\frac{1}{2}$ hexagons.

Highlight that, in each case, we know the size of each group (or each block) and are trying to find out how many groups (or how many blocks) are needed to equal a particular area.

You may choose to use the applet at https://ggbm.at/VmEqZvke in the discussion.

Lesson Synthesis

In this lesson, we learned that we can reason about division with fractions as we have done in division with whole numbers—by thinking in terms of equal-sized groups. We can use pattern blocks, diagrams, and equations to think about questions such as "how many $\frac{3}{4}$s are in 6?"

- “How do we know which number represents the size of a group, and which represents a total?” (We can often tell by the context of the problem, or by interpreting the question carefully. For the question “how many $\frac{3}{4}$s are in 6?,” we are interested in groups of $\frac{3}{4}$s and we have a total amount of 6.)
- “How do diagrams or pattern blocks help us find the answers to these questions?” (Diagrams often allow us to count or see the number of groups.)
- “What equations can we write to represent the question ‘how many $\frac{3}{4}$s are in 6’?” (We can start with multiplication: “there are ? groups of $\frac{3}{4}$ in 6” can be written as $? \cdot \frac{3}{4} = 6$. The division equation $6 \div \frac{3}{4} = ?$ represents the same question.)

4.3 Halves, Thirds, and Sixths

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Launch

Give students continued access to pattern blocks, if needed.

Student Task Statement

1. The hexagon represents 1 whole.

Draw a pattern-block diagram that represents the equation $4 \cdot \frac{1}{3} = 1\frac{1}{3}$.

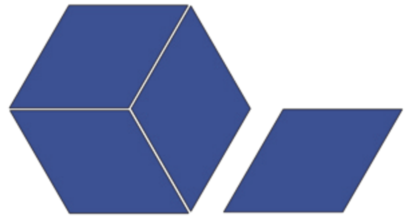
2. Answer the following questions. If you get stuck, consider using pattern blocks.

    a. How many $\frac{1}{2}$s are in $3\frac{1}{2}$?

    b. How many $\frac{1}{3}$s are in $2\frac{2}{3}$?

    c. How many $\frac{1}{6}$s are in $\frac{2}{3}$?

Student Response

1. 

2. a. There are seven $\frac{1}{2}$s in $3\frac{1}{2}$.

   b. There are eight $\frac{1}{3}$s in $2\frac{2}{3}$.

   c. There are four $\frac{1}{6}$s in $\frac{2}{3}$.

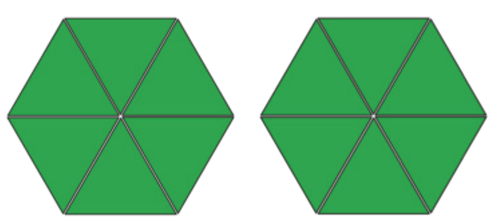
Student Lesson Summary

Some problems that involve equal-sized groups also involve fractions. Here is an example: "How many $\frac{1}{6}$ are in 2?" We can express this question with multiplication and division equations.

$$? \cdot \frac{1}{6} = 2$$
$$2 \div \frac{1}{6} = ?$$

Pattern-block diagrams can help us make sense of such problems. Here is a set of pattern blocks.

If the hexagon represents 1 whole, then a triangle must represent $\frac{1}{6}$, because 6 triangles make 1 hexagon. We can use the triangle to represent the $\frac{1}{6}$ in the problem.



Twelve triangles make 2 hexagons, which means there are 12 groups of $\frac{1}{6}$ in 2.

If we write the 12 in the place of the "?" in the original equations, we have:

$$12 \cdot \frac{1}{6} = 2$$

$$2 \div \frac{1}{6} = 12$$

Lesson 4 Practice Problems

Problem 1

Statement

Consider the problem: A shopper buys cat food in bags of 3 lbs. Her cat eats $\frac{3}{4}$ lb each week. How many weeks does one bag last?

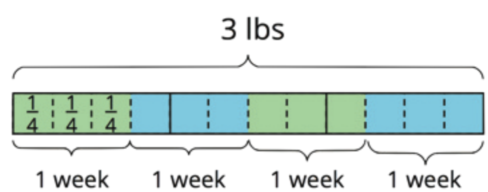
a. Draw a diagram to represent the situation and label your diagram so it can be followed by others. Answer the question.

b. Write a multiplication or division equation to represent the situation.

c. Multiply your answer in the first question (the number of weeks) by $\frac{3}{4}$. Did you get 3 as a result? If not, revise your previous work.

Solution

a.



There are 4 servings of $\frac{3}{4}$ lbs in the 3 lbs bag. The bag lasts 4 weeks.

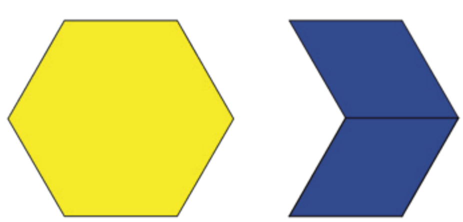
b. $? \cdot \frac{3}{4} = 3$ or $3 \div \frac{3}{4} = ?$

c. The answer is correct because $4 \cdot \frac{3}{4} = 3$.

Problem 2

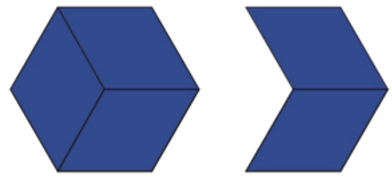
Statement

Use the diagram to answer the question: How many $\frac{1}{3}$s are in $1\frac{2}{3}$? The hexagon represents 1 whole. Explain or show your reasoning.



Solution

If the hexagon represents 1, then the rhombus represents $\frac{1}{3}$ because the hexagon is composed of three rhombuses. The diagram of one hexagon and two rhombuses matches up exactly with five rhombuses. So there are five $\frac{1}{3}$s in $1\frac{2}{3}$.



Problem 3

Statement

Which question can be represented by the equation $? \cdot \frac{1}{8} = 3$?

A. How many 3s are in $\frac{1}{8}$?

B. What is 3 groups of $\frac{1}{8}$?

C. How many $\frac{1}{8}$s are in 3?

D. What is $\frac{1}{8}$ of 3?

Solution

C

Problem 4

Statement

Write two division equations for each multiplication equation.

a. $15 \cdot \frac{2}{5} = 6$

b. $6 \cdot \frac{4}{3} = 8$

c. $16 \cdot \frac{7}{8} = 14$

Solution

a. $6 \div \frac{2}{5} = 15$ and $6 \div 15 = \frac{2}{5}$

b. $8 \div 6 = \frac{4}{3}$ and $8 \div \frac{4}{3} = 6$

c. $14 \div 16 = \frac{7}{8}$ and $14 \div \frac{7}{8} = 16$

Problem 5

Statement

Noah and his friends are going to an amusement park. The total cost of admission for 8 students is $100, and all students share the cost equally. Noah brought $13 for his ticket. Did he bring enough money to get into the park? Explain your reasoning.

Solution

Responses vary. Sample response: Yes, he did bring enough money, since $100 \div 8 = 12.5$. So if the friends share the cost equally, each pays $12.50. (Also $8 \cdot 13$ is bigger than 100, so if everybody brought $13, they would have more money than they need.)

(From Unit 4, Lesson 2.)

Problem 6

Statement

Write a division expression with a quotient that is:

a. greater than $8 \div 0.001$

b. less than $8 \div 0.001$

c. between $8 \div 0.001$ and $8 \div \frac{1}{10}$

Solution

Answers vary. Sample responses:

a. $9 \div 0.001$ or $8 \div 0.0001$

b. $7 \div 0.01$ or $8 \div 0.01$

c. $8 \div 0.01$ or $6 \div 0.001$

(From Unit 4, Lesson 1.)

Problem 7

Statement

Find each unknown number.

a. 12 is 150% of what number?

b. 5 is 50% of what number?

c. 10% of what number is 300?

d. 5% of what number is 72?

e. 20 is 80% of what number?

Solution

a. 8

b. 10

c. 3,000

d. 1,440

e. 25

Sample reasoning likely to include reasoning about benchmark percentages, or about percentages as rates per 100. For example, for "5% of what is 72."

- To reason about benchmark percentages, reason that if 5% is 72, then 10% is 144. 100% is ten times as much, so 100% is 1,440.
- To reason about rates per 100, create a double number line or a table of equivalent ratios, as shown. Since 5 is multiplied by 20 to reach 100, multiply 72 by 20 as well.

amount	percentage
72	5
1,440	100

(From Unit 3, Lesson 14.)

Lesson 5: How Many Groups? (Part 2)

Goals

- Coordinate multiplication and division equations and pattern block diagrams in which the red trapezoid represents one whole.
- Create a diagram to represent and solve a problem asking “How many groups?” in which the divisor is a non-unit fraction, and explain (orally) the solution method.
- Identify or generate a multiplication or division equation that represents a given situation involving a fractional divisor.

Learning Targets

- I can find how many groups there are when the number of groups and the amount in each group are not whole numbers.

Lesson Narrative

In this lesson, students continue to work with division situations involving questions like “how many groups?” or “how many of this in that?” Unlike in the previous lesson, they encounter situations where the quotient is not a whole number, and they must attend to the whole when representing the answer as a fraction (MP6). They represent the situations with multiplication equations (e.g., “? groups of $\frac{1}{2}$ make 8” can be expressed as $? \cdot \frac{1}{2} = 8$) and division equations ($8 \div \frac{1}{2} = ?$).

Alignments

Building On

- 5.NF.B.7: Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Group Presentations

- MLR8: Discussion Supports

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pattern blocks

Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Prepare enough pattern blocks such that each group of 3–4 students has at least 1 hexagon and 4 of each of the other shapes (triangle, rhombus, and trapezoid).

Student Learning Goals

Let's use blocks and diagrams to understand more about division with fractions.

5.1 Reasoning with Fraction Strips

Warm Up: 5 minutes
In this warm-up, students continue to think of division in terms of equal-sized groups, using fraction strips as an additional tool for reasoning.

Notice how students transition from concrete questions (the first three) to symbolic ones (the last three). Framing division expressions as "how many of this fraction in that number?" may not yet be intuitive to students. They will further explore that connection in this lesson. For now, support them using whole-number examples (e.g., ask: "how do you interpret $6 \div 2$?").

The divisors used here involve both unit fractions and non-unit fractions. The last question shows a fractional divisor that is not on the fraction strips. This encourages students to transfer the reasoning used with fraction strips to a new problem, or to use an additional strategy (e.g., by first writing an equivalent fraction).

As students work, notice those who are able to modify their reasoning effectively, even if the approach may not be efficient (e.g., adding a row of $\frac{1}{10}$s to the fraction strips). Ask them to share later.

Building On

- 5.NF.B.7

Addressing

- 6.NS.A.1

Launch

Give students 2–3 minutes of quiet work time.

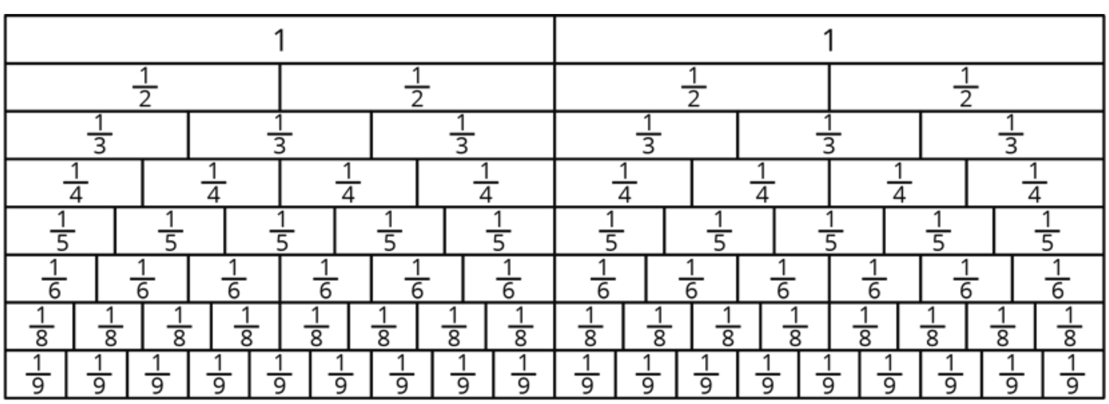
Anticipated Misconceptions

Since the fraction strips do not show tenths, students might think that it is impossible to answer the last question. Ask them if they can think of another fraction that is equivalent to $\frac{2}{10}$.

Student Task Statement

Write a fraction or whole number as an answer for each question. If you get stuck, use the fraction strips. Be prepared to share your reasoning.

1. How many $\frac{1}{2}$s are in 2?
2. How many $\frac{1}{5}$s are in 3?
3. How many $\frac{1}{8}$s are in $1\frac{1}{4}$?
4. $1 \div \frac{2}{6} = ?$
5. $2 \div \frac{2}{9} = ?$
6. $4 \div \frac{2}{10} = ?$



Student Response

1. 4
2. 15

3. 10

4. 3

5. 9

6. 20

Activity Synthesis

For each of the first five questions, select a student to share their response and ask the class to indicate whether they agree or disagree.

Focus the discussion on two things: how students interpreted expressions such as $1 \div \frac{2}{6}$, and on how they reasoned about $4 \div \frac{2}{10}$. Select a few students to share their reasoning.

For the last question, highlight strategies that are effective and efficient, such as using a unit fraction that is equivalent to $\frac{2}{10}$, finding out how many groups of $\frac{1}{5}$ are in 1 and then multiplying it by 4, etc.

5.2 More Reasoning with Pattern Blocks

25 minutes (there is a digital version of this activity)
This activity serves two purposes: to explicitly bridge “how many of this in that?” questions and division expressions, and to explore division situations in which the quotients are not whole numbers. (Students explored similar questions previously, but the quotients were whole numbers.)

Once again students move from reasoning concretely and visually to reasoning symbolically. They start by thinking about “how many rhombuses are in a trapezoid?” and then express that question as multiplication
($? \cdot \frac{2}{3} = 1$ or $\frac{2}{3} \cdot ? = 1$) and division ($1 \div \frac{2}{3}$). Students think about how to deal with a remainder in such problems.

As students discuss in groups, listen for their explanations for the question “How many rhombuses are in a trapezoid?” Select a couple of students to share later—one person to elaborate on Diego's argument, and another to support Jada's argument.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 3–4. Provide access to pattern blocks and geometry toolkits. Give students 10 minutes of quiet work time for the first three questions and a few minutes to discuss their responses and collaborate on the last question.

Classrooms with no access to pattern blocks or those using the digital materials can use the provided applet. Physical pattern blocks are still preferred, however.

Support for Students with Disabilities

*Representation: Develop Language and Symbols.*Display or provide charts with symbols and meanings. Emphasize the difference between this activity where students must find what fraction of a trapezoid each of the shapes represents, compared to the hexagon in the previous lesson. Create a display that includes an image of each shape labeled with the name and the fraction it represents of a trapezoid. Keep this display visible as students move on to the next problems.
Supports accessibility for: Conceptual processing; Memory

Anticipated Misconceptions

Some students may not notice that in this task, the trapezoid—not the hexagon—represents 1 whole. Encourage them to revisit the task statement to check.

Student Task Statement

Your teacher will give you pattern blocks. Use them to answer the questions.

1. If the trapezoid represents 1 whole, what do each of the other shapes represent? Be prepared to show or explain your reasoning.

2. Use pattern blocks to represent each multiplication equation. Use the trapezoid to represent 1 whole.

   a. $3 \cdot \frac{1}{3} = 1$

   b. $3 \cdot \frac{2}{3} = 2$

3. Diego and Jada were asked “How many rhombuses are in a trapezoid?”

- Diego says, "$1\frac{1}{3}$. If I put 1 rhombus on a trapezoid, the leftover shape is a triangle, which is $\frac{1}{3}$ of the trapezoid."
- Jada says, "I think it's $1\frac{1}{2}$. Since we want to find out 'how many rhombuses,' we should compare the leftover triangle to a rhombus. A triangle is $\frac{1}{2}$ of a rhombus."

Do you agree with either of them? Explain or show your reasoning.

4. Select **all** the equations that can be used to answer the question: "How many rhombuses are in a trapezoid?"

- $\frac{2}{3} \div ? = 1$
- $1 \div \frac{2}{3} = ?$
- $? \div \frac{2}{3} = 1$
- $? \cdot \frac{2}{3} = 1$
- $1 \cdot \frac{2}{3} = ?$

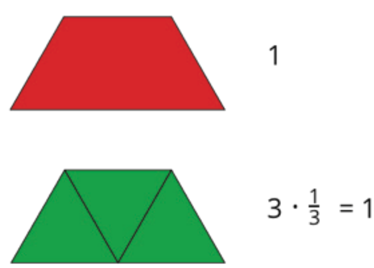
Student Response

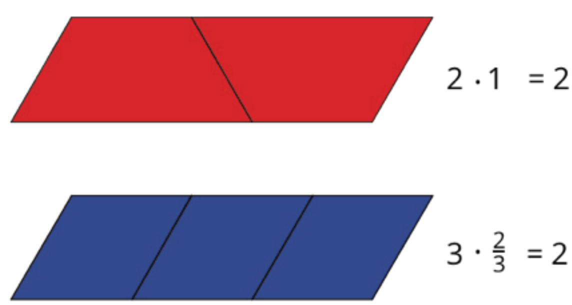
1. a. $\frac{1}{3}$ because three triangles make a trapezoid.

   b. $\frac{2}{3}$ because two triangles make a rhombus and each triangle represents $\frac{1}{3}$.

   c. 2 because two trapezoids make a hexagon.

a.





2. b.

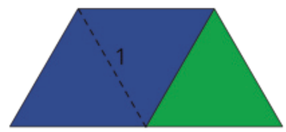
3. Jada's $1\frac{1}{2}$ is the right answer. Sample reasoning: Since the question is "how many rhombuses," the leftover space should be compared to a rhombus. A triangle is half of a rhombus, so we can fit $1\frac{1}{2}$ rhombuses in a trapezoid.

4. $? \cdot \frac{2}{3} = 1$ and $1 \div \frac{2}{3} = ?$

Activity Synthesis

Focus the whole-class discussion on the last two questions, especially on how the visual representation helps us reason about Jada and Diego's points of view, and on the connections between the verbal and numerical representations of the situation.

Select two previously identified students to explain why Diego or Jada are correct. Display a visual representation of “how many rhombus are in a trapezoid?” for all to see (as shown here), or use the applet at https://ggbm.at/VmEqZvke for illustration.



To highlight number of groups and size of one group in the problem, discuss questions such as:

- “This is a ‘how many groups of this in that?’ question. What makes 1 group, in this case?” (One rhombus.)
- “How do we know whether to compare the remainder to the rhombus or the trapezoid?” (Since a rhombus makes 1 group, we need to compare the remainder to the rhombus.)

If students struggle to compare the remainder to a rhombus, ask: “How many triangles are in a trapezoid?” and point out that the answer is “3 triangles.” Here, the answer to “how many rhombuses are in a trapezoid?” would be “(some number of) rhombuses.”

The fact that there are two 1 wholes to keep track of may be a source of confusion (the trapezoid represents the quantity 1 and the rhombus represents 1 group). Students will have opportunities to make clearer distinctions between the two 1 wholes in upcoming activities.

Support for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. Use this routine to support whole-class discussion. After a student shares their reasoning for whether they agree with Diego or Jada, ask students to restate what they heard using precise mathematical language. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

5.3 Drawing Diagrams to Show Equal-sized Groups

Optional: 20 minutes

This activity gives students additional practice in using diagrams and equations to represent division situations involving whole numbers and fractions.

For each problem, many kinds of visual representations are possible, but creating a meaningful representation may be challenging nonetheless. Urge students to use the contexts to generate ideas for useful diagrams, and to start with a draft and modify it as needed. Students may also use the fraction strips in the warm-up as a starting point for drawing diagrams.

As students work and discuss, monitor for effective diagrams or those that can be generalized to different situations (e.g., rectangles, tape diagrams, and number lines). Assign one problem for each group to record on a visual display.

Addressing

- 6.NS.A.1

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports

Launch

Arrange students in group of 3–4. Give students 8–10 minutes of quiet work time and a few minutes to share their responses with their group. Tell each group they will be asked to present their solution to one problem. Provide access to geometry toolkits and tools for creating a visual display.

During group discussion, ask students to exchange feedback on each other's diagrams and to notice any that might be particularly effective, efficient, or easy to understand. Then, they should record the diagram and equations for their assigned problem on a visual display and be prepared to explain them.

Support for Students with Disabilities

Representation: Internalize Comprehension. Demonstrate and encourage students to use color coding and annotations to illustrate connections between representations. For example, use the same color to represent the 3 miles in the diagram and the equation, $3 \div \frac{3}{2} = 2$, then label each as "dividend."
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Students may mistake the divisor and the dividend in the problems. Ask students to discuss (in their groups) the number or quantity being divided, and the reasonableness of the different ways of setting up each problem given the context. Representing the situations with objects may also help.

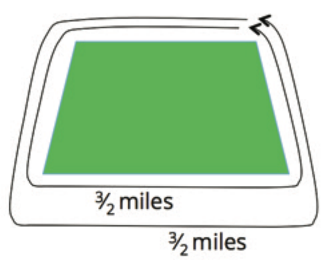
Student Task Statement

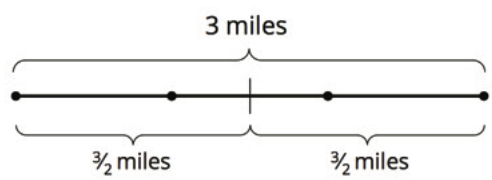
For each situation, draw a diagram for the relationship of the quantities to help you answer the question. Then write a multiplication equation or a division equation for the relationship. Be prepared to share your reasoning.

1. The distance around a park is $\frac{3}{2}$ miles. Noah rode his bicycle around the park for a total of 3 miles. How many times around the park did he ride?
2. You need $\frac{3}{4}$ yard of ribbon for one gift box. You have 3 yards of ribbon. How many gift boxes do you have ribbon for?
3. The water hose fills a bucket at $\frac{1}{3}$ gallon per minute. How many minutes does it take to fill a 2-gallon bucket?

Student Response

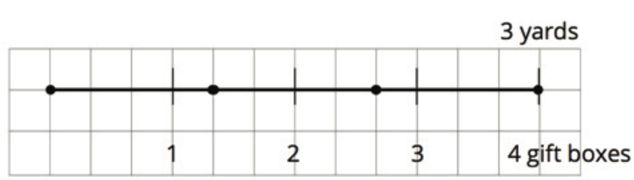
1. 2 (times around the park). Sample diagrams:





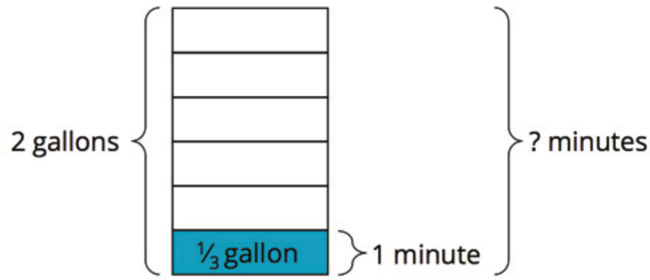
   Equation: $2 \cdot \frac{3}{2} = 3$, or $3 \div \frac{3}{2} = 2$

2. 4 (gift boxes). Sample diagram:



   Equation: $4 \cdot \frac{3}{4} = 3$, or $3 \div \frac{3}{4} = 4$

3. 6 (minutes). Sample diagram:



Equation: $6 \cdot \frac{1}{3} = 2$, or $2 \div \frac{1}{3} = 6$

Are You Ready for More?

How many heaping teaspoons are in a heaping tablespoon? How would the answer depend on the shape of the spoons?

Student Response

Answers vary. Sample response: There are around 3 heaping teaspoons in a heaping tablespoon, since there are 3 level teaspoons in 1 tablespoon. The size of the heaping spoonfuls depends on the shape of the spoon, since a wide and flat spoon will be able to have more material sit on top of the leveled spoonful.

Activity Synthesis

Invite each group to present the solution on their visual display. Ask the rest of the class to think about two things: whether the equations make sense, and how the presented diagram shows the number of groups, the size of each group, and a total amount. Doing so will help students see the structure of the problems in the equations and diagrams.

Make sure students understand how each situation can be expressed by multiplication and division equations. Students should recognize that a question such as "how many batches are in 4 cups if each batch requires $\frac{2}{3}$ cups?" can be written as both $? \cdot \frac{2}{3} = 4$ and $4 \div \frac{2}{3} = ?$. With repeated reasoning, they see that a division expression such as $5 \div \frac{3}{8}$ can be interpreted as "how many $\frac{3}{8}$s are in 5?"

Support for English Language Learners

Speaking, Representing: MLR8 Discussion Supports. Give students additional time to make sure that everyone in their group can explain their visual display and the relationship between the quantities represented. Prompt groups to rehearse what they will say when they share with the whole class. Rehearsing provides students with additional opportunities to speak and clarify their thinking. This will help students improve their explanations of their group's reasoning during the whole-class discussion.
Design Principle(s): Optimize output (for explanation)

Lesson Synthesis

In this lesson, we answered questions such as “how many of this fraction in that number?” or “how many groups of this in that?” We used pattern blocks, fraction strips, diagrams, and equations to help us make sense of those situations. The answers to those questions, we noticed, may not be whole numbers.

- “We can think of the question ‘how many $\frac{3}{4}$ are in 2?’ in terms of equal-size groups. What do the $\frac{3}{4}$ and 2 represent? What are we looking for?” ($\frac{3}{4}$ is the size of each group. The 2 is the total amount. We are looking for the number of groups.)
- “What multiplication equation can we write for this situation?” ($? \cdot \frac{3}{4} = 2$)
- “What division equation can we write?” ($2 \div \frac{3}{4} = ?$)
- “We can draw a diagram and count how many groups of $\frac{3}{4}$ there are in 2. How many whole groups of $\frac{3}{4}$ are there?” (2 whole groups.)
- “How do we deal with a remainder that is less than one whole group?” (We can compare the size of the remainder with the amount in one group. In $2 \div \frac{3}{4} = ?$, each group is $\frac{3}{4}$, and the remainder is $\frac{2}{4}$, which is $\frac{2}{3}$ of one group.)

5.4 Bags of Tangerines

Cool Down: 5 minutes

This cool-down assesses students’ ability to represent a division situation with diagrams and equations. The diagrams they create do not need to be of a specific type.

Addressing

- 6.NS.A.1

Student Task Statement

A grocery store sells tangerines in $\frac{2}{5}$ kg bags. A customer bought 4 kg of tangerines for a school party. How many bags did he buy?

1. Select **all** equations that represent the situation.

    a. $4 \cdot \frac{2}{5} = ?$

    b. $? \cdot \frac{2}{5} = 4$

    c. $\frac{2}{5} \div 4 = ?$

    d. $4 \div \frac{2}{5} = ?$

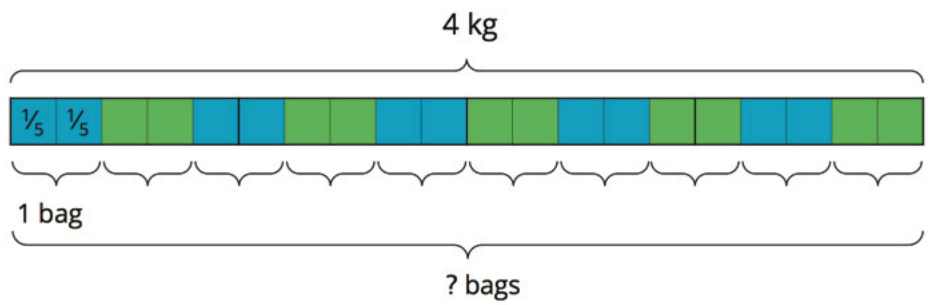
    e. $? \div \frac{2}{5} = 4$

2. Draw a diagram to represent the situation. Answer the question.

Student Response

1. B ($? \cdot \frac{2}{5} = 4$) and D ($4 \div \frac{2}{5} = ?$)

2. 10 bags. Diagrams vary. Sample diagram:



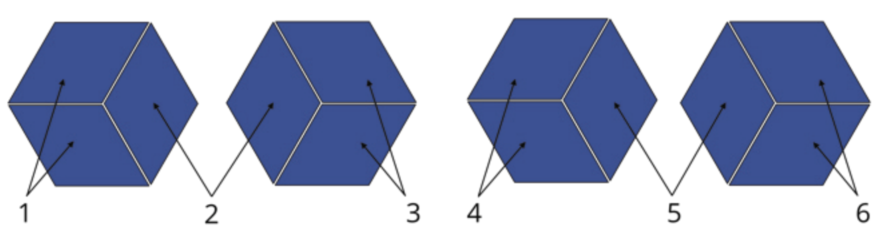
Student Lesson Summary

Suppose one batch of cookies requires $\frac{2}{3}$ cup flour. How many batches can be made with 4 cups of flour?

We can think of the question as being: "How many $\frac{2}{3}$ are in 4?" and represent it using multiplication and division equations.

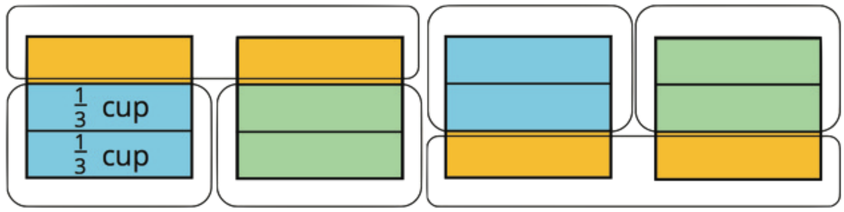
$$? \cdot \frac{2}{3} = 4$$
$$4 \div \frac{2}{3} = ?$$

Let's use pattern blocks to visualize the situation and say that a hexagon is 1 whole.



Since 3 rhombuses make a hexagon, 1 rhombus represents $\frac{1}{3}$ and 2 rhombuses represent $\frac{2}{3}$. We can see that 6 pairs of rhombuses make 4 hexagons, so there are 6 groups of $\frac{2}{3}$ in 4.

Other kinds of diagrams can also help us reason about equal-sized groups involving fractions. This example shows how we might reason about the same question from above: "How many $\frac{2}{3}$-cups are in 4 cups?"



We can see each "cup" partitioned into thirds, and that there are 6 groups of $\frac{2}{3}$-cup in 4 cups. In both diagrams, we see that the unknown value (or the "?" in the equations) is 6. So we can now write:

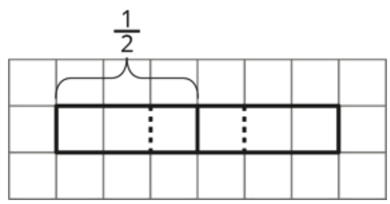
$$6 \cdot \frac{2}{3} = 4$$
$$4 \div \frac{2}{3} = 6$$

Lesson 5 Practice Problems

Problem 1

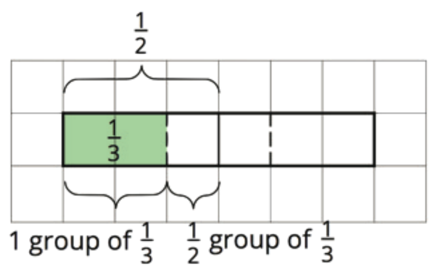
Statement

Use the tape diagram to find the value of $\frac{1}{2} \div \frac{1}{3}$. Show your reasoning.



Solution

$1\frac{1}{2}$



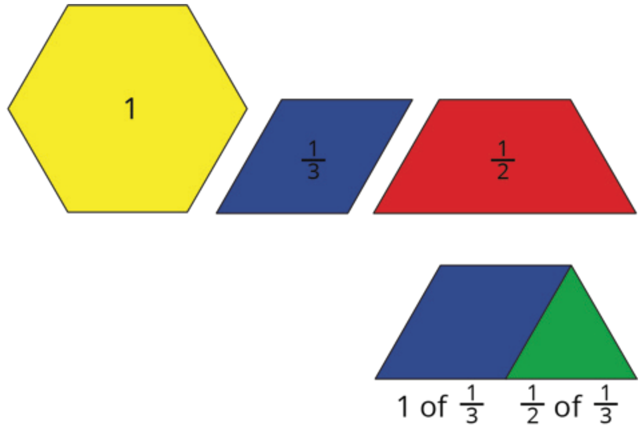
Problem 2

Statement

What is the value of $\frac{1}{2} \div \frac{1}{3}$? Use pattern blocks to represent and find this value. The yellow hexagon represents 1 whole. Explain or show your reasoning.

Solution

$1\frac{1}{2}$. Explanations vary. Sample explanations:



One rhombus and $\frac{1}{2}$ of a rhombus compose one trapezoid.

Problem 3

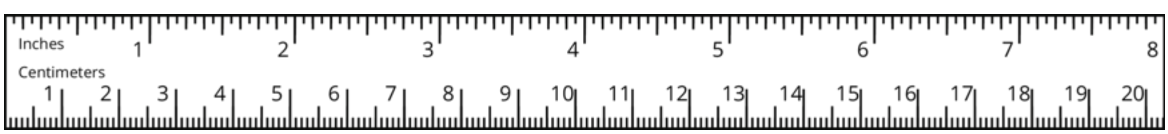
Statement

Use a standard inch ruler to answer each question. Then, write a multiplication equation and a division equation that answer the question.

a. How many $\frac{1}{2}$s are in 7?

b. How many $\frac{3}{8}$s are in 6?

c. How many $\frac{5}{16}$s are in $1\frac{7}{8}$?



Solution

a. Multiplication: $14 \cdot \frac{1}{2} = 7$ (or equivalent), division: $7 \div \frac{1}{2} = 14$

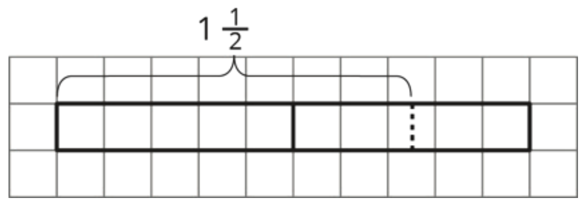
b. Multiplication: $16 \cdot \frac{3}{8} = 6$ (or equivalent), division: $6 \div \frac{3}{8} = 16$

c. Multiplication: $6 \cdot \frac{5}{16} = 1\frac{7}{8}$ (or equivalent), division: $1\frac{7}{8} \div \frac{5}{16} = 6$

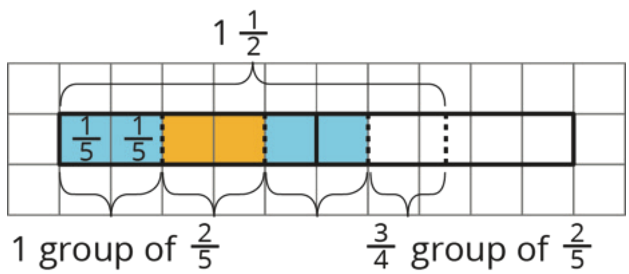
Problem 4

Statement

Use the tape diagram to answer the question: How many $\frac{2}{5}$s are in $1\frac{1}{2}$? Show your reasoning.



Solution



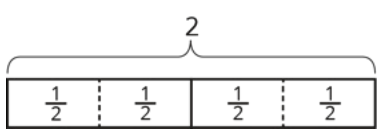
$3\frac{3}{4}$

Problem 5

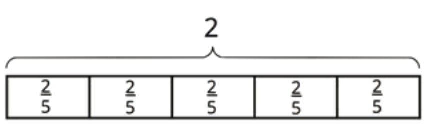
Statement

Write a multiplication equation and a division equation to represent each sentence or diagram.

a. There are 12 fourths in 3.

b. 

c. How many $\frac{2}{3}$s are in 6?

d. 

Solution

a. $12 \cdot \frac{1}{4} = 3$ (or equivalent, e.g., $3 \div 12 = \frac{1}{4}$ or $3 \div \frac{1}{4} = 12$)

b. $4 \cdot \frac{1}{2} = 2$ (or equivalent, e.g., $2 \div 4 = \frac{1}{2}$ or $2 \div \frac{1}{2} = 4$)

c. $? \cdot \frac{2}{3} = 6$ (or equivalent), $6 \div \frac{2}{3} = ?$ (or equivalent)

d. $5 \cdot \frac{2}{5} = 2$ (or equivalent), $2 \div 5 = \frac{2}{5}$ (or equivalent)

(From Unit 4, Lesson 4.)

Problem 6

Statement

At a farmer's market, two vendors sell fresh milk. One vendor sells 2 liters for \$3.80, and another vendor sells 1.5 liters for \$2.70. Which is the better deal? Explain your reasoning.

Solution

Answers vary. Sample response:

- 1.5 liters at \$2.70 is a better deal. The 1.5-liter-size costs \$1.80 per liter since $2.70 \div 1.5 = 1.80$. The 2-liter size costs \$1.90 per liter because $3.80 \div 2 = 1.90$. The 1.5-liter bottle is less expensive per liter.

(From Unit 3, Lesson 5.)

Problem 7

Statement

A recipe uses 5 cups of flour for every 2 cups of sugar.

a. How much sugar is used for 1 cup of flour?

b. How much flour is used for 1 cup of sugar?

c. How much flour is used with 7 cups of sugar?

d. How much sugar is used with 6 cups of flour?

Solution

a. $\frac{2}{5}$ or 0.4 cups of sugar are used for every cup of flour.

b. $\frac{5}{2}$ or 2.5 cups of flour are used for every cup of sugar.

c. 17.5. $(2.5) \cdot 7 = 17.5$ so with 7 cups of sugar, there will be 17.5 or $17\frac{1}{2}$ cups of flour.

d. $(0.4) \cdot 6 = 2.4$ so with 6 cups of flour, there will be 2.4 or $2\frac{2}{5}$ cups of sugar.

flour (cups)	sugar (cups)
5	2
1	$\frac{2}{5}$
$\frac{5}{2}$	1
$\frac{35}{2}$	7
6	$\frac{12}{5}$

(From Unit 3, Lesson 6.)

Lesson 6: Using Diagrams to Find the Number of Groups

Goals

- Explain (orally) how to create a tape diagram to represent and solve a problem asking “How many groups?”
- Justify (orally and using other representations) the answer to a problem asking “How many groups?” in which the divisor is a non-unit fraction and the quotient is a fraction greater than 1.

Learning Targets

- I can use a tape diagram to represent equal-sized groups and find the number of groups.

Lesson Narrative

This is the second lesson in a series of three lessons exploring the “how many groups?” interpretation of division in situations involving fractions.
In the preceding lesson and in this one, the number of groups in each given situation is 1 or greater. In the next lesson, students find the number of groups that is less than 1 (“what fraction of a group?”).

Students have used different diagrams to represent multiplication and division. In this lesson, tape diagrams are spotlighted and used more explicitly. They are more abstract and more flexible than other representations students may have chosen for thinking about division problems that involve fractions. Because they use measurement along the length of the tape, tape diagrams are closer to the number line representation of fractions, and ultimately help students visualize division problems on the number line. (Students are not required to do that in this lesson, however.)

Students continue to make the journey from reasoning with concrete quantities to reasoning with abstract representations of fraction
division (MP2).

Alignments

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's draw tape diagrams to think about division with fractions.

6.1 How Many of These in That?

Warm Up: 5 minutes
Students have previously seen tape diagrams used to represent equal-sized groups. This warm-up gives students a chance to create tape diagrams to represent division expressions in a scaffolded way. Each tape is started on a grid and pre-labeled with the known quantity. Each grid square represents 1.

Addressing

- 6.NS.A.1

Instructional Routines

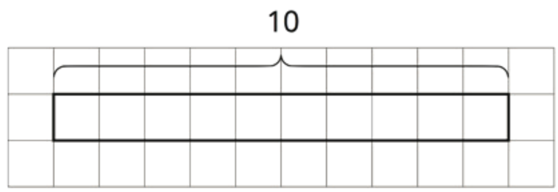
- Think Pair Share

Launch

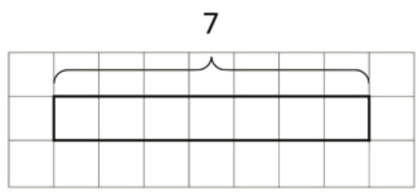
Arrange students in groups of 2. Give students 2 minutes of quiet work time and another minute to share their diagrams with their partner.

Student Task Statement

1. We can think of the division expression $10 \div 2\frac{1}{2}$ as the question: "How many groups of $2\frac{1}{2}$ are in 10?" Complete the tape diagram to represent this question. Then find the answer.

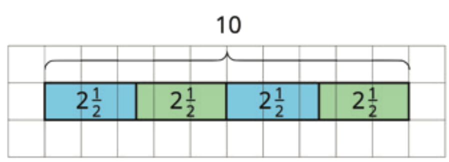


2. Complete the tape diagram to represent the question: "How many groups of 2 are in 7?" Then find the answer.

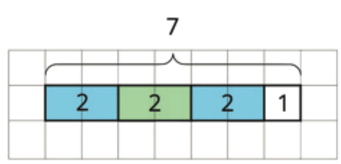


Student Response

1. There are 4 groups of $2\frac{1}{2}$ in 10.



2. There are $3\frac{1}{2}$ groups of 2 in 7.



Activity Synthesis

Select a few students to share their diagrams and answers. After each person shares, poll the class to see if others agree or disagree.

Discuss questions such as:

- "How did you know how large each part of the diagram should be?"
  (In the first question, the length of the tape represents 10, and there are 10 grid squares, so each grid represents 1. Since the size of each group is $2\frac{1}{2}$, each part needs to have $2\frac{1}{2}$ squares.)

- “In the second question, we see three groups of 2 and an extra square of 1. How did you know that the 1 is $\frac{1}{2}$ of a group and not $\frac{1}{7}$ of a group?” (The question asks “how many groups of 2 . . .,” so the size of each group is 2, not 7.)

Tell students they will use tape diagrams to help solve other division problems.

6.2 Representing Groups of Fractions with Tape Diagrams

20 minutes

In this lesson, students transition away from using concrete tools (pattern blocks and fraction strips) for reasoning about division to using a tool that is more abstract and more flexible. To represent “how many groups?” or “how many of this in that?” questions, they draw tape diagrams on a grid.

Students continue to write multiplication and division equations to make sense of given situations, but here, they also think about the reasonableness of their answers. They see that the solution to a division problem can be checked using the corresponding multiplication equation.

Some students may write answers as fractions and others as mixed numbers. Both are acceptable. Depending on the situation, one may be more useful than the other. For example, in a measurement situation, a mixed number is easier to interpret, but if we need to perform further calculations with an answer, a fraction is easier to work with. In the course of the unit, students should become comfortable with both forms and be flexible in using them.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 3–4. Tell students that they have solved “how many groups?” and “how many of this in that?” problems using pattern blocks and fraction strips. Now they will solve them by drawing tape diagrams.

Give students 2–3 minutes to discuss the first question and write their response. Follow with a brief whole-class discussion. Select 1–2 students to explain how a tape diagram shows us how many $\frac{2}{3}$s are in 1.

Then, give students 8–10 minutes of quiet work time to complete the rest of the task. Ask them to discuss their diagrams only after attempting at least 2 of the 3 remaining questions. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Support for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. For each equation, provide students with a blank template of a tape diagram for students to complete and find the unknown quantities.
Supports accessibility for: Visual-spatial processing; Organization

Support for English Language Learners

Representing, Conversing: MLR2 Collect and Display. As groups discuss the first question, circulate and record language students use to explain how Andre's tape diagram can be used to solve the equation. Listen for phrases such as "equal parts," "same size," and "group of $\frac{2}{3}$s." Consider asking "How are the number of groups represented in the tape diagram?", "How are the values in the equation represented in the diagram?" or "What do the blue and white parts represent?" Display the collected language for all to see and remind students to refer to it throughout the rest of the lesson. This will help students develop mathematical language to explain how a tape diagram can be used to solve a division problem.
Design Principle(s); Maximize meta-awareness

Anticipated Misconceptions

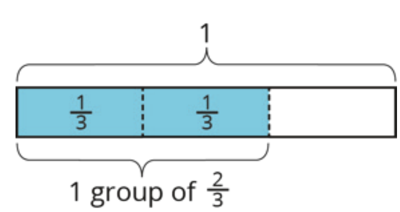
Students may misinterpret the given diagram and think that the answer can only be a whole number because in the diagram, "1 group" is shaded, but the fraction of a group is not. Remind students of their work with pattern blocks in which we saw that the answer to the question "how many of this in that?" or "how many rhombuses are in a trapezoid" could involve a whole number and a fraction.

Student Task Statement

To make sense of the question "How many $\frac{2}{3}$s are in 1?," Andre wrote equations and drew a tape diagram.

$$? \cdot \frac{2}{3} = 1$$

$$1 \div \frac{2}{3} = ?$$



1. In an earlier task, we used pattern blocks to help us solve the equation $1 \div \frac{2}{3} = ?$. Explain how Andre's tape diagram can also help us solve the equation.

2. Write a multiplication equation and a division equation for each question. Then, draw a tape diagram and find the answer.

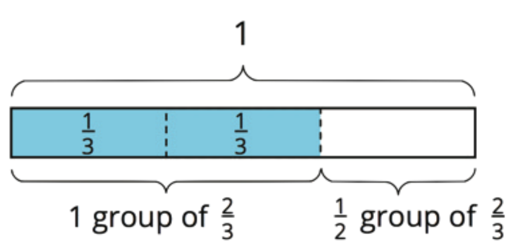
   a. How many $\frac{3}{4}$s are in 1?

   b. How many $\frac{2}{3}$s are in 3?

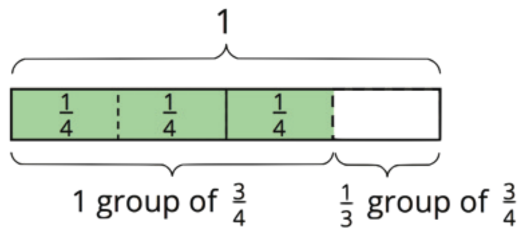
   c. How many $\frac{3}{2}$s are in 5?

Student Response

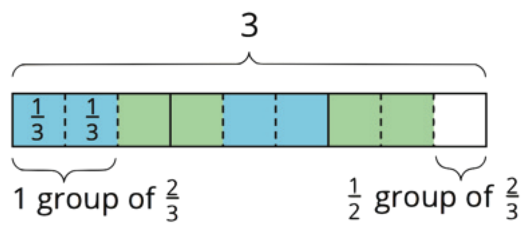
1. Answers vary. Sample reasoning: The tape diagram shows 1 whole broken into 3 equal parts. Each part is $\frac{1}{3}$, so two parts make $\frac{2}{3}$. There is one group of size $\frac{2}{3}$, plus an extra $\frac{1}{3}$, which is half of a group of the same size. This means there are $1\frac{1}{2}$ groups of size $\frac{2}{3}$ in 1, or $1 \div \frac{2}{3} = 1\frac{1}{2}$.



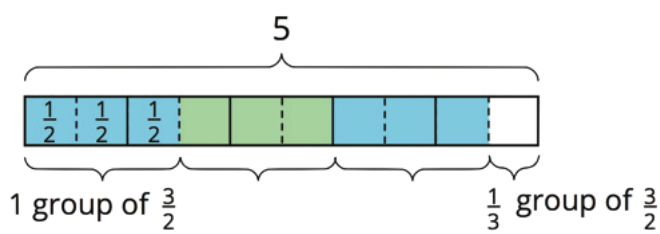
2. a. Multiplication equation: $? \cdot \frac{3}{4} = 1$ (or $\frac{3}{4} \cdot ? = 1$), division equation: $1 \div \frac{3}{4} = ?$ (or $1 \div ? = \frac{3}{4}$), solution: $\frac{4}{3}$ (or $1\frac{1}{3}$).



b. Multiplication equation: $? \cdot \frac{2}{3} = 3$ (or $\frac{2}{3} \cdot ? = 3$), division equation: $3 \div \frac{2}{3} = ?$ (or $3 \div ? = \frac{2}{3}$), solution: $\frac{9}{2}$ (or $4\frac{1}{2}$).



c. Multiplication equation: $? \cdot \frac{3}{2} = 5$ (or $\frac{3}{2} \cdot ? = 5$), division equation: $5 \div \frac{3}{2} = ?$ (or $5 \div ? = \frac{3}{2}$), solution: $\frac{10}{3}$ (or $3\frac{1}{3}$).



Activity Synthesis

Focus the whole-class discussion on two goals: reflecting on the process of creating and using the diagrams, and discussing how we can check our solutions.

For each problem in the second question, select a student to share their response and ask how many others had the same diagram.
If students' diagrams cannot be easily displayed for all to see, consider showing the ones in the Possible Responses. To help students reflect on their process, discuss:

- "How did you begin the diagram? How did you know how to partition the pieces in the diagram?"
- "Which of the two equations—multiplication or division—was helpful in setting up the diagram? How so?"
- "How did you determine how many groups there are?"

- “Can we use pattern blocks to represent these situations? Which might be preferable and why?”

Reinforce the idea that the size of 1 group is what we use as the unit for counting and to find out how many groups there are.

To prompt students to think about the reasonableness of their answers, ask: “How would you know if your answer is correct?” If not mentioned by students, point out how to use multiplication to check their solution to the division problem. For example:

- We wrote $? \cdot \frac{2}{3} = 1$ to represent “how many $\frac{2}{3}$ are in 1?”.
- We found the answer to be $1\frac{1}{2}$, so we can substitute $1\frac{1}{2}$ for the “?” and see if $1\frac{1}{2} \cdot \frac{2}{3}$ is indeed 1.
- $1\frac{1}{2} = \frac{3}{2}$, so we can rewrite that expression as $\frac{3}{2} \cdot \frac{2}{3}$.
- $\frac{3}{2} \cdot \frac{2}{3} = 1$, so the answer is correct.

For classes using the digital materials, consider demonstrating an applet to represent a division problem https://ggbm.at/atUteypU. This video shows how to use the tool to answer the question “How many $\frac{1}{2}$ are in 3?” https://vimeo.com/184879045.

6.3 Finding Number of Groups

15 minutes

In this activity, students apply the reasoning and strategies from the previous activity to solve division problems in context. Though the instructions do not prompt students to draw tape diagrams, students may find them to be a handy option.

Students may also choose to express each whole-number or mixed-number dividend as a fraction. For example, they may express the 6 inches as $\frac{48}{8}$ inches and then see how many $\frac{3}{8}$s are in $\frac{48}{8}$. As students work, notice the different strategies they use.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Keep students in the same groups. Give students 8–10 minutes of quiet work time followed by a few minutes to discuss their work with their group.

Some students may wish to use graph paper to draw tape diagrams and colored pencils to mark up parts of the diagrams. Provide access to geometry toolkits. Encourage students to think about the tools and strategies at their disposal and to check their solutions using multiplication.

Student Task Statement

1. Write a multiplication equation or a division equation for each question. Then, find the answer and explain or show your reasoning.

    a. How many $\frac{3}{8}$-inch thick books make a stack that is 6 inches tall?

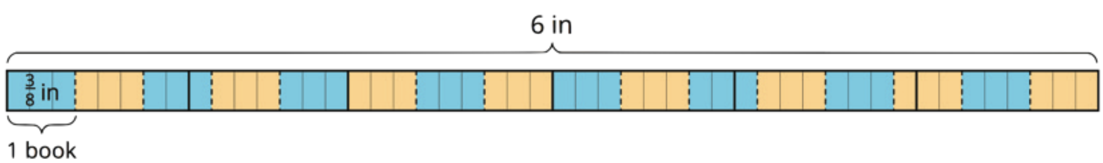
    b. How many groups of $\frac{1}{2}$ pound are in $2\frac{3}{4}$ pounds?

2. Write a question that can be represented by the division equation $5 \div 1\frac{1}{2} = ?$. Then, find the answer and explain or show your reasoning.

Student Response

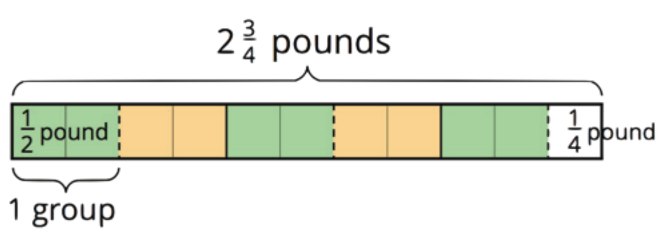
1. a. Multiplication equation: $? \cdot \frac{3}{8} = 6$ (or $\frac{3}{8} \cdot ? = 6$), Division equation: $6 \div \frac{3}{8} = ?$ (or $6 \div ? = \frac{3}{8}$).

    Answer: 16 books. Sample reasoning:



   b. Multiplication equation: $? \cdot \frac{1}{2} = 2\frac{3}{4}$ (or $\frac{1}{2} \cdot ? = 2\frac{3}{4}$), Division equation: $2\frac{3}{4} \div \frac{1}{2} = ?$ (or $2\frac{3}{4} \div ? = \frac{1}{2}$).

    Answer: $5\frac{1}{2}$ groups. Sample reasoning:



2. Questions vary. Sample response: How many $1\frac{1}{2}$ hours are in 5 hours?

    Answer: $3\frac{1}{3}$.

Activity Synthesis

Because the activity is intended to help students reason about division problems quantitatively and abstractly, consider wrapping it up by asking students to write a brief reflection about their reasoning. Display the following prompts for all to see and ask students to write their response on an index card or a sheet of paper so it can be collected.

"A friend is unsure what $2 \div \frac{4}{5}$ means and isn't sure how to find its value. How would you help your friend make sense of the expression? How do you think about it? Share two ways that you find helpful for reasoning about an expression like this."

Support for English Language Learners

Writing, Conversing: MLR1 Stronger and Clearer Each Time. Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. Give students time to meet with 2–3 partners to share and get feedback on their response to the synthesis prompt. Display feedback prompts that will help students strengthen their ideas and clarify their language. For example, "Can you describe that a different way?" and "Can you use both a tape diagram and multiplication equation to explain this?" Give students 1–2 minutes to revise their writing based on the feedback they receive before moving on to the whole-class discussion.
Design Principle(s): Optimize output; Cultivate conversation

Lesson Synthesis

In this lesson, we used tape diagrams to find the number of groups in division situations involving fractions. Consider asking some of the following questions:

- "In the question 'how many $\frac{3}{4}$s are in 6?' (or $6 \div \frac{3}{4} = ?$), what does each number represent?" ($\frac{3}{4}$ represents the size of each group. 6 is the total amount.)
- "What would a tape diagram for this situation show?" (A total tape length representing 6, broken into parts of $\frac{3}{4}$ each.)
- "How does the diagram help us answer the question?" (It allows us to see and count the number of groups.)
- "What if that length cannot be broken equally into $\frac{3}{4}$s? How do we deal with the remainder?" (We can think of it as a partial group. We can compare it to the size of 1 group and see what fraction of a group it is.)
- "We have used pattern blocks, fraction strips, and other diagrams to help us think about division with fractions. How are tape diagrams and these other tools alike?" (They all allow us to represent equal-sized groups so we can see the relationships between the numbers.)

- "How are tape diagrams different than those tools for reasoning about, say, $6 \div \frac{3}{4}$ or $18 \div \frac{2}{5}$?" (Tape diagrams may be more practical since we can specify what the length of the tape and what each part represent. If we use pattern blocks, we may need a lot of blocks to represent certain numbers. If we use fraction strips, we may need to draw much longer strips. Also, not all fractions are shown on the strips, so we may need to add them.)

6.4 How Many in 2?

Cool Down: 5 minutes
This cool-down assesses students' ability to use diagrams to reason about division situations. Two questions are given here, but if time is limited, consider asking students to answer only the second question.

Addressing

- 6.NS.A.1

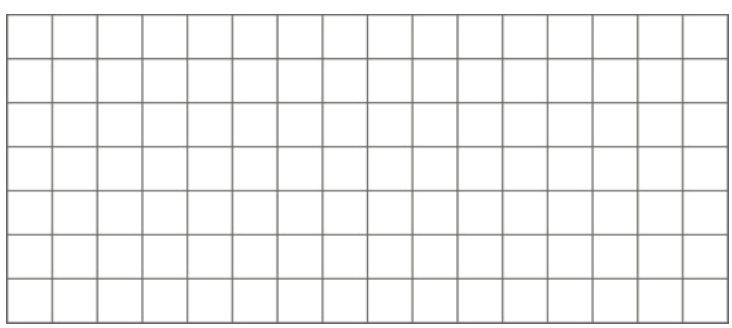
Launch

Provide continued access to colored pencils.

Student Task Statement

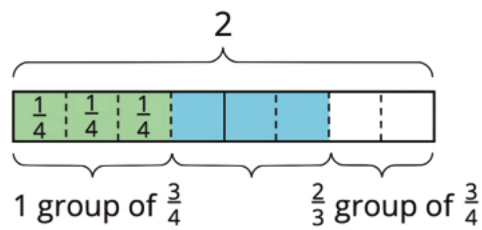
Consider the problem: How many $\frac{3}{4}$s are in 2?

1. Write a multiplication equation and a division equation that can be used to answer the question.
2. Draw a tape diagram and answer the question. Use the grid to help you draw, if needed.



Student Response

1. $? \cdot \frac{3}{4} = 2$. $2 \div \frac{3}{4} = ?$
2. There are two and two-thirds $\frac{3}{4}$s in 2.



Student Lesson Summary

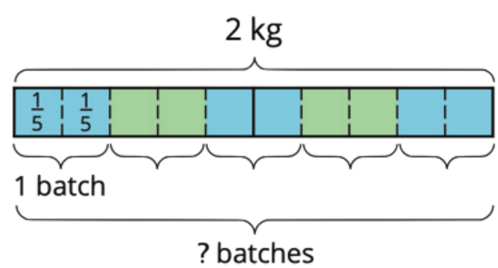
A baker used 2 kilograms of flour to make several batches of a pastry recipe. The recipe called for $\frac{2}{5}$ kilogram of flour per batch. How many batches did she make?

We can think of the question as: "How many groups of $\frac{2}{5}$ kilogram make 2 kilograms?" and represent that question with the equations:

$$? \cdot \frac{2}{5} = 2$$

$$2 \div \frac{2}{5} = ?$$

To help us make sense of the question, we can draw a tape diagram. This diagram shows 2 whole kilograms, with each kilogram partitioned into fifths.



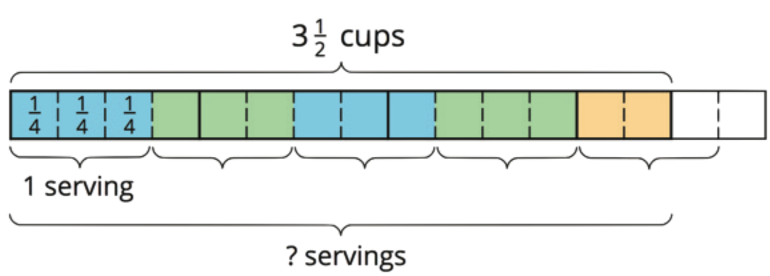
We can see there are 5 groups of $\frac{2}{5}$ in 2. Multiplying 5 and $\frac{2}{5}$ allows us to check this answer: $5 \cdot \frac{2}{5} = \frac{10}{5}$ and $\frac{10}{5} = 2$, so the answer is correct.

Notice the number of groups that result from $2 \div \frac{2}{5}$ is a whole number. Sometimes the number of groups we find from dividing may not be a whole number. Here is an example:

Suppose one serving of rice is $\frac{3}{4}$ cup. How many servings are there in $3\frac{1}{2}$ cups?

$$? \cdot \frac{3}{4} = 3\frac{1}{2}$$

$$3\frac{1}{2} \div \frac{3}{4} = ?$$



Looking at the diagram, we can see there are 4 full groups of $\frac{3}{4}$, plus 2 fourths. If 3 fourths make a whole group, then 2 fourths make $\frac{2}{3}$ of a group. So the number of servings (the "?" in each equation) is $4\frac{2}{3}$. We can check this by multiplying $4\frac{2}{3}$ and $\frac{3}{4}$.

$4\frac{2}{3} \cdot \frac{3}{4} = \frac{14}{3} \cdot \frac{3}{4}$, and $\frac{14}{3} \cdot \frac{3}{4} = \frac{14}{4}$, which is indeed equivalent to $3\frac{1}{2}$.

Lesson 6 Practice Problems

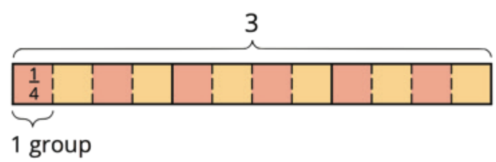
Problem 1

Statement

We can think of $3 \div \frac{1}{4}$ as the question “How many groups of $\frac{1}{4}$ are in 3?” Draw a tape diagram to represent this question. Then find the answer.

Solution

12. Sample diagram:



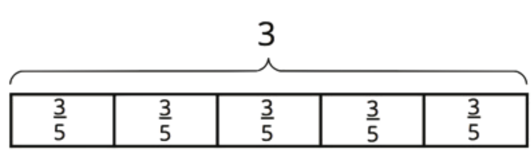
Problem 2

Statement

Describe how to draw a tape diagram to represent and answer $3 \div \frac{3}{5} = ?$ for a friend who was absent.

Solution

Answers vary. Sample explanation: Draw a rectangle whose length represents 3. Partition it into 3 equal parts to show 3 groups of 1. Partition each 1 whole into 5 fifths. There are 15 fifths in 3. Shade each group of 3 fifths, then count how many groups there are in 3.



Problem 3

Statement

How many groups of $\frac{1}{2}$ day are in 1 week?

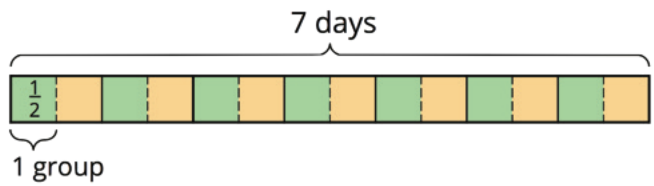
a. Write a multiplication equation or a division equation to represent the question.

b. Draw a tape diagram to show the relationship between the quantities and to answer the question. Use graph paper, if needed.

Solution

a. $? \cdot \frac{1}{2} = 7$ (or equivalent), $7 \div \frac{1}{2} = ?$

b. There are 14 groups of $\frac{1}{2}$-day in a week. Sample diagram:



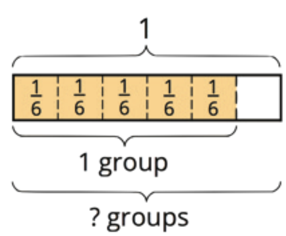
Problem 4

Statement

Diego said that the answer to the question "How many groups of $\frac{5}{6}$ are in 1?" is $\frac{6}{5}$ or $1\frac{1}{5}$. Do you agree with him? Explain or show your reasoning.

Solution

Agree. Sample reasoning:

- $\frac{6}{5} \cdot \frac{5}{6} = \frac{30}{30}$, which equals 1.
- There are 6 sixths in 1. We can make 1 group of $\frac{5}{6}$s and have $\frac{1}{6}$ remaining. $\frac{1}{6}$ is one fifth of $\frac{5}{6}$, so there are $1\frac{1}{5}$ groups of $\frac{5}{6}$ in 1.
- 

Problem 5

Statement

Select **all** the equations that can represent the question: "How many groups of $\frac{4}{5}$ are in 1?"

A. $? \cdot 1 = \frac{4}{5}$

B. $1 \cdot \frac{4}{5} = ?$

C. $\frac{4}{5} \div 1 = ?$

D. $? \cdot \frac{4}{5} = 1$

E. $1 \div \frac{4}{5} = ?$

Solution

["D", "E"]

(From Unit 4, Lesson 5.)

Problem 6

Statement

Calculate each percentage mentally.

a. What is 10% of 70?

b. What is 10% of 110?

c. What is 25% of 160?

d. What is 25% of 48?

e. What is 50% of 90?

f. What is 50% of 350?

g. What is 75% of 300?

h. What is 75% of 48?

Solution

a. 7

b. 11

c. 40

d. 12

e. 45

f. 175

g. 225

h. 36

(From Unit 3, Lesson 14.)

Lesson 7: What Fraction of a Group?

Goals

- Comprehend the phrase “What fraction of a group?” (in spoken and written language) as a variation of the question “How many groups?” that is used when the quotient is less than 1.
- Create a tape diagram to represent and solve a problem asking “How many groups?” in which the quotient is a fraction less than 1.
- Write multiplication and division equations to represent a problem asking “How many times as long?”

Learning Targets

- I can tell when a question is asking for the number of groups and that number is less than 1.
- I can use diagrams and multiplication and division equations to represent and answer “what fraction of a group?” questions.

Lesson Narrative

In the previous three lessons, students explored the “how many groups?” interpretation of division. Their explorations included situations where the number of groups was a whole number or a mixed number. In this lesson, they extend the work to include cases where the number of groups is a fraction less than 1, that is, situations in which the total amount is smaller than the size of 1 group. In such situations, the question becomes “what fraction of a group?”.

Students notice that they can use the same reasoning strategies as in situations with a whole number of groups, because the structure

$$(\text{number of groups}) \cdot (\text{size of a group}) = (\text{total amount})$$

is the same as before (MP7). They write multiplication equations of this form and for the corresponding division equations.

Throughout the lesson, students practice attending to details (in diagrams, descriptions, or equations) about how the given quantities relate to the size of 1 group.

Alignments

Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR6: Three Reads
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's think about dividing things into groups when we can't even make one whole group.

7.1 Estimating a Fraction of a Number

Warm Up: 5 minutes
In this warm-up, students estimate the value of a fraction of a number (e.g., $\frac{1}{3}$ of 7) using what they know about the size of the given fraction. Then, they write multiplication expressions to represent the verbal questions. The goal is to activate prior understandings that a fraction of a number can be found by multiplication, preparing students to explore division problems in which the quotient is less than 1 whole.

Building On

- 5.NF.B.4

Launch

Ask students to keep their materials closed. Display one estimation question at a time. (If all questions are displayed, ask students to work on one question at a time and to begin when cued.) Give students 30 seconds of quiet think time per question and ask them to give a signal when they have an answer and can explain their strategy.

Select 1–2 students to briefly share their estimates and how they made them. Record and display their estimates for all to see. After discussing the third estimation question, ask students to write a multiplication expression to represent each of the three questions.

Anticipated Misconceptions

Some students may try to find the exact answers to the questions instead of estimating. Encourage them to think about benchmark fractions that could help them estimate.

Student Task Statement

1. Estimate the quantities:
    a. What is $\frac{1}{3}$ of 7?
    b. What is $\frac{4}{5}$ of $9\frac{2}{3}$?
    c. What is $2\frac{4}{7}$ of $10\frac{1}{9}$?
2. Write a multiplication expression for each of the previous questions.

Student Response

1. Answers vary. Possible responses:
    a. Less than $3\frac{1}{2}$, because that would be $\frac{1}{2}$ of 7, but greater than 2, because that would be $\frac{1}{3}$ of 6.
    b. A little less than 8, because $9\frac{2}{3}$ is just under 10, and $\frac{4}{5}$ of 10 is 8.
    c. A little more than 25, because $2\frac{4}{7}$ is a bit more than $2\frac{1}{2}$, and $10\frac{1}{9}$ is just a little over 10. $2\frac{1}{2}$ of 10 is 25.
2. a. $\frac{1}{3} \cdot 7$ (or $7 \cdot \frac{1}{3}$)
    b. $\frac{4}{5} \cdot 9\frac{2}{3}$ (or $9\frac{2}{3} \cdot \frac{4}{5}$)
    c. $2\frac{4}{7} \cdot 10\frac{1}{9}$ (or $10\frac{1}{9} \cdot 2\frac{4}{7}$)

Activity Synthesis

Ask a few students to share the expressions they wrote for the questions. Record and display the expressions for all to see. Ask the class to indicate if they agree or disagree with each expression.

If not already brought up in students' explanations, highlight the idea that we can find the exact value of a fraction of a number (e.g., $\frac{4}{5}$ of $9\frac{2}{3}$) by multiplying the fraction and the number. It does not matter whether the number is a whole number, a mixed number, or another fraction.

To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"

- “Did anyone solve the problem the same way but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

7.2 Fractions of Ropes

10 minutes (there is a digital version of this activity)
This task helps to transition students from thinking about “how many groups?” to “what fraction of a group?”.

Students compare different lengths of ropes and express their relative lengths in multiplicative terms. Rope B and C are 5 and $2\frac{1}{2}$ times as long as rope A, respectively, but rope D is shorter than rope A, so then we say that it is $\frac{3}{4}$ times as long as rope A. They see that it is possible that the answer to a “how many groups?” question is a number less than 1 when the given amount is smaller than the size of a group.

As students work, notice how they go about making multiplicative comparisons. Select students who write clear and concise questions for the equations in the last problem so they can share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 4–5 minutes of quiet think time and then a couple of minutes to compare their responses with a partner and discuss any disagreements. Clarify that rope A is 4 units long.

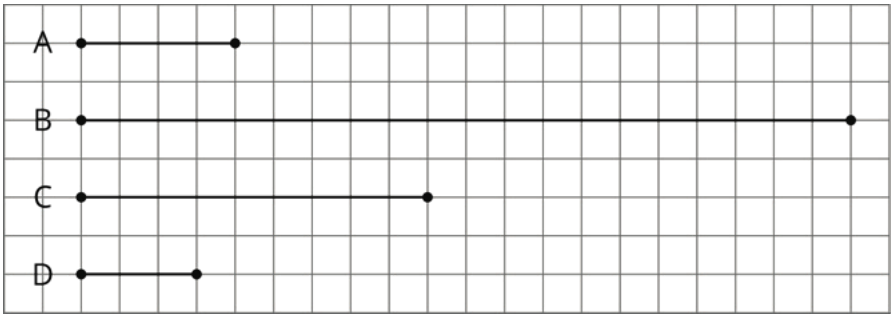
Students using the digital materials can use the applet at ggbm.at/kZUgANCC to compare the segments. The segments can be moved by dragging endpoints with open circles. The yellow “pins” can help students keep track of the groups.

Anticipated Misconceptions

Some students might associate the wrong lengths with the ropes or confuse the order of comparison (e.g., comparing A to C instead of C to A). Encourage them to put the length of each rope next to the diagram and attend more closely to the ropes being compared.

Student Task Statement

Here is a diagram that shows four ropes of different lengths.



1. Complete each sentence comparing the lengths of the ropes. Then, use the measurements shown on the grid to write a multiplication equation and a division equation for each comparison.

    a. Rope B is ______ times as long as Rope A.

    b. Rope C is ______ times as long as Rope A.

    c. Rope D is ______ times as long as Rope A.

2. Each equation can be used to answer a question about Ropes C and D. What could each question be?

    a. $? \cdot 3 = 9$ and $9 \div 3 = ?$

    b. $? \cdot 9 = 3$ and $3 \div 9 = ?$

Student Response

1. a. Rope B is 5 times as long as Rope A. $5 \cdot 4 = 20$ (or $4 \cdot 5 = 20$) and $20 \div 4 = 5$ (or $20 \div 5 = 4$)

    b. Rope C is $2\frac{1}{4}$ (or equivalent) times as long as Rope A. $2\frac{1}{4} \cdot 4 = 9$ (or $4 \cdot 2\frac{1}{4} = 9$) and $9 \div 4 = 2\frac{1}{4}$ (or $9 \div 2\frac{1}{4} = 4$)

    c. Rope D is $\frac{3}{4}$ (or equivalent) times as long as Rope A. $\frac{3}{4} \cdot 4 = 3$ (or $4 \cdot \frac{3}{4} = 3$) and $3 \div 4 = \frac{3}{4}$ (or $3 \div \frac{3}{4} = 4$)

2. a. Responses vary. Sample response: How many times as long as Rope D is Rope C? (or how many times does the length of Rope D go into that of Rope C?)

    b. Responses vary. Sample response: How many times as long as Rope C is Rope D? (or how many times does the length of Rope C go into that of Rope D?)

Activity Synthesis

Display the solutions to the first set of problems for all to see. Give students a minute to check their answers and ask questions. Then, focus class discussion on two key ideas:

- The connection between “how many groups?” questions and “how many times as long?” questions. Ask students how these two types of questions are similar and different. Make sure students see that both have the structure of $? \cdot a = b$, where a is the size of 1 group (or the unit we are using for comparison), and b is a given number.
- The language commonly used when referring to a situation in which the number of groups is less than 1 whole. Explain that we have seen equal-sized groups where the number of groups is greater than 1, but some situations involve a part of 1 group. So instead of saying “the number of groups” or asking “how many groups?,” we would ask “what fraction of a group?” or “what part of a group?”. For example, in the case of rope D, where the answer is less than 1, we can ask, “What fraction of rope A is rope D?”

Ask 1–2 previously identified students to share the question they wrote for the last pair of equations ($? \cdot 9 = 3$ and $3 \div 9 = ?$). Make sure students see that this pair of equations represent a situation with a fractional group
(i.e., rope D is shorter than rope C, so the length of rope D is a fraction of that of rope C).

Support for Students with Disabilities

Representation: Internalize Comprehension. Use color and annotations to illustrate connections between representations. As students describe their reasoning about “how many groups?” and “how many times as long?”, use color and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing

Support for English Language Learners

Speaking: MLR8 Discussion Supports. As students compare and contrast these two types of division questions, provide a sentence frame such as: “Something these two types of questions have in common is . . .” and "A difference between these two types of questions is . . . ." This will help students produce and make sense of the language needed to communicate their ideas about the relationship between multiplication and division equations.
Design Principle(s): Support sense-making; Optimize output (for comparison)

7.3 Fractional Batches of Ice Cream

20 minutes
In this activity, students make sense of quotients that are less than 1 and greater than 1 in the same context. Later in the task, students generalize their reasoning to solve division problems (where the quotient is less than 1) without contexts.

Given the amount of milk required for 1 batch of ice cream (i.e., the size of 1 group), students find out how many batches (i.e., the number of groups or what fraction of a group) can be made with different amounts of milk. They continue to use tape diagrams and write equations to reason about the situations, but this time, they are not prompted to write multiplication equations.

As students work, identify students students who drew clear and effective diagrams for the ice cream problems. Select them to share later.

Addressing

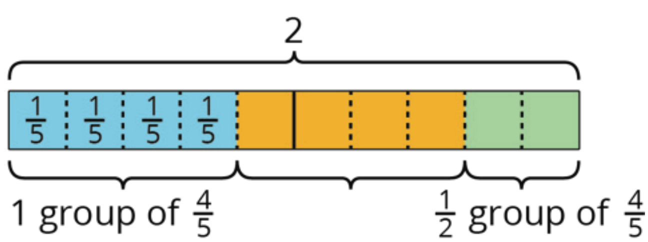
- 6.NS.A.1

Instructional Routines

- MLR6: Three Reads
- Think Pair Share

Launch

Keep students in groups of 2. Display an example of a tape diagram that students have used in a previous lesson. The diagram for the question "how many $\frac{4}{5}$s are in 2?" from a previous cool-down is shown here.



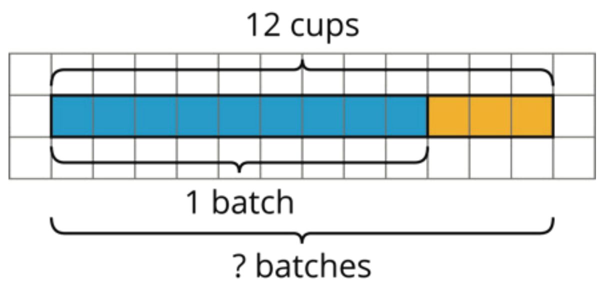
Point out how the diagram shows both full groups of $\frac{4}{5}$ and a partial group. Tell students that they will see more situations involving partial groups in this activity.

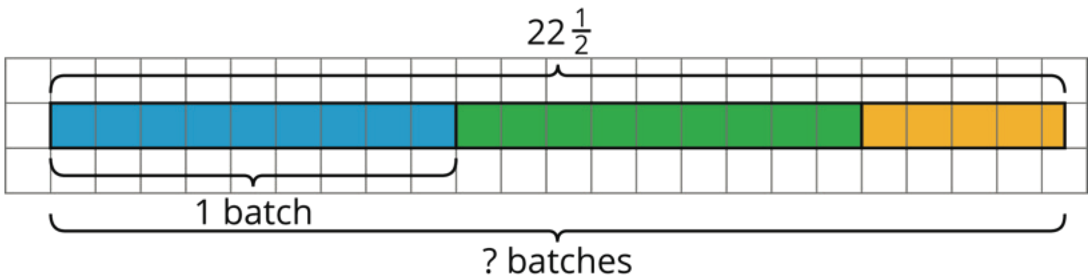
Give students 6–8 minutes of quiet work time for the first two sets of questions about ice cream. Ask students to make a quick estimate on whether each answer will be greater than or less than 1 before solving the problem. Provide access to colored pencils, as some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Give students 2–3 minutes to discuss their responses with their partner. Follow with a whole-class discussion before students return to the last set of questions.

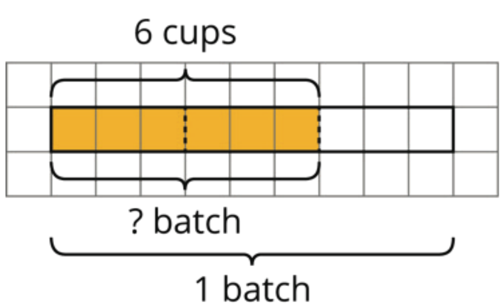
Ask previously identified students to share their diagrams for the ice cream problems, or display the following diagrams.

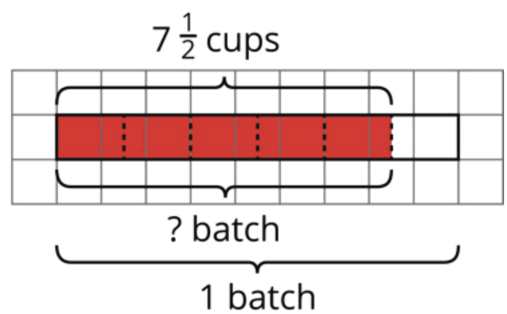
Monday and Tuesday:





Thursday and Friday:





To help students see the structure in the diagrams, ask: “How are the diagrams for Monday and Tuesday like and unlike those for Thursday and Friday?” If not brought up in their responses, point out that:

- The size of 1 group (or the amount of milk in 1 batch) is the same in all diagrams, but the amounts we are comparing to 1 group vary. Those amounts are greater than 1 batch (9 cups) on Monday and Tuesday, and less than 1 batch on Thursday and Friday.
- This comparison to the size of 1 group is also reflected in the questions. We ask “how many batches?” for the first two, and “what fraction of a batch?” for the other two.

To help students notice the structure in the equations, ask: “How are the division equations for Monday and Tuesday different than those for Thursday and Friday? How are they the same?”

$$12 \div 9 = ?$$

$$22\frac{1}{2} \div 9 = ?$$

$$6 \div 9 = ?$$

$$7\frac{1}{2} \div 9 = ?$$

Highlight that, regardless of whether the answer is greater than 1 or less than 1, the equations show that the questions "how many batches (of 9 cups)?" and "what fraction of a batch (of 9 cups)?" can be expressed with a division by 9, because the multiplication counterparts of these situations all have the structure of "what number times 9 equals a given amount of milk?" or

$$? \cdot 9 = b$$

where b is a given amount of milk.

Give students quiet time to complete the last set of questions.

Support for English Language Learners

Reading: MLR6 Three Reads. Use this routine to support reading comprehension of this word problem, without solving it for students. Use the first read to orient students to the situation. After a shared reading, ask students "what is this situation about?" (A chef makes makes different amounts of ice cream on different days). After the second read, students list any quantities that can be counted or measured, without focusing on specific values (number of cups of milk needed for every batch of ice cream, number of cups of milk used each day). Listen for, and amplify, the two important quantities that vary in relation to each other in this situation: number of cups of milk, and number of (or part of) batches of ice cream. After the third read, ask students to brainstorm possible strategies to answer the question.
Design Principle(s): Support sense-making

Anticipated Misconceptions

If students are not sure how to begin representing a situation with a tape diagram, ask them to represent one quantity or number at a time. For example, they could begin by showing the amount of milk used as a tape with a particular length, and then mark the second quantity (the amount of milk in 1 batch) on the same tape and with the same starting point. Or they could represent the amounts in the opposite order.

Student Task Statement

One batch of an ice cream recipe uses 9 cups of milk. A chef makes different amounts of ice cream on different days. Here are the amounts of milk she used:

- Monday: 12 cups
- Tuesday: $22\frac{1}{2}$ cups
- Thursday: 6 cups
- Friday: $7\frac{1}{2}$ cups

1. How many batches of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.

    a. Monday

b. Tuesday

2. What fraction of a batch of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.

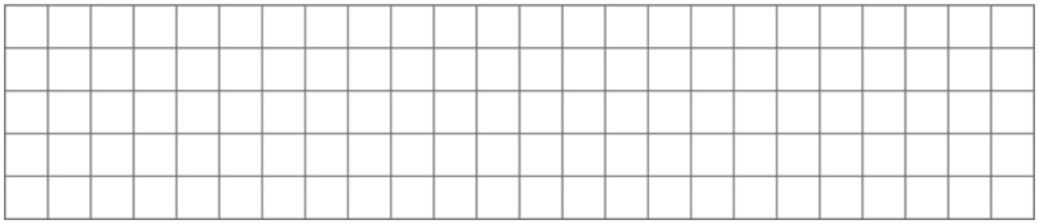
   a. Thursday

   b. Friday

3. For each question, write a division equation, draw a tape diagram, and find the answer.

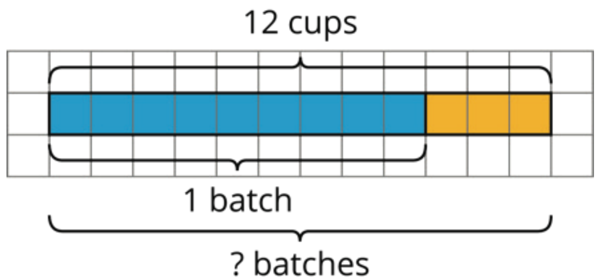
   a. What fraction of 9 is 3?

   b. What fraction of 5 is $\frac{1}{2}$?

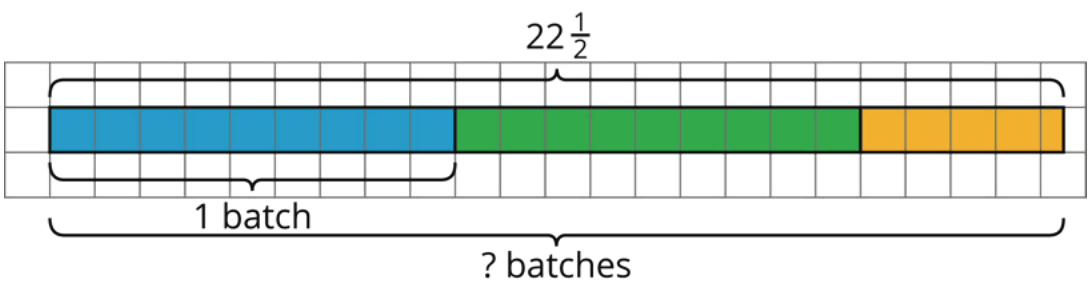


Student Response

1. a. Equation: $12 \div 9 = ?$ (or $12 \div ? = 9$). Solution: $1\frac{1}{3}$ batches (or equivalent).

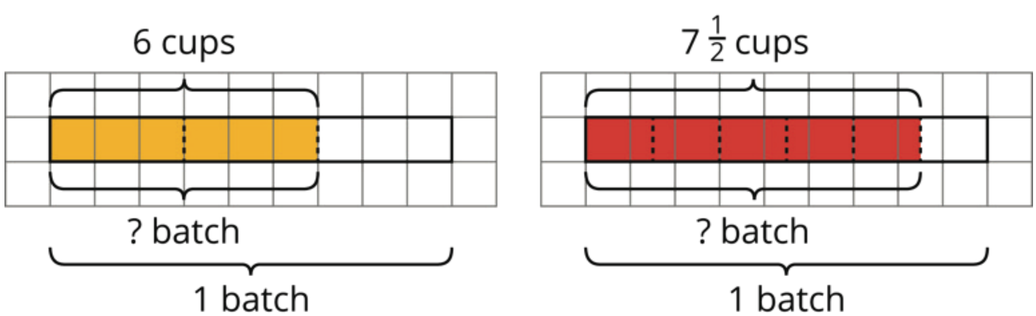


   b. Equation: $22\frac{1}{2} \div 9 = ?$ (or $22\frac{1}{2} \div ? = 9$). Solution: $2\frac{1}{2}$ batches (or equivalent).



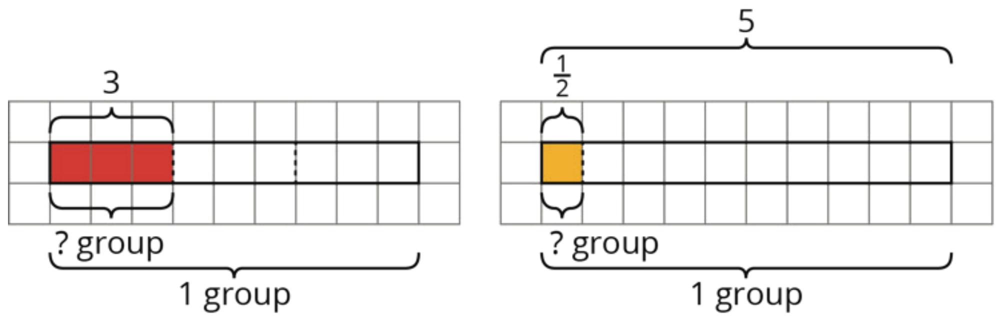
2. a. Equation: $6 \div 9 = ?$ (or $6 \div ? = 9$). Solution: $\frac{2}{3}$ of a batch (or equivalent).

   b. Equation: $7\frac{1}{2} \div 9 = ?$ (or $7\frac{1}{2} \div ? = 9$). Solution: $\frac{5}{6}$ of a batch (or equivalent).



3. a. Equation: $3 \div 9 = ?$ (or $3 \div ? = 9$). Solution: $\frac{1}{3}$.

   b. Equation: $\frac{1}{2} \div 5 = ?$ (or $\frac{1}{2} \div ? = 5$). Solution: $\frac{1}{10}$ (or 0.1).

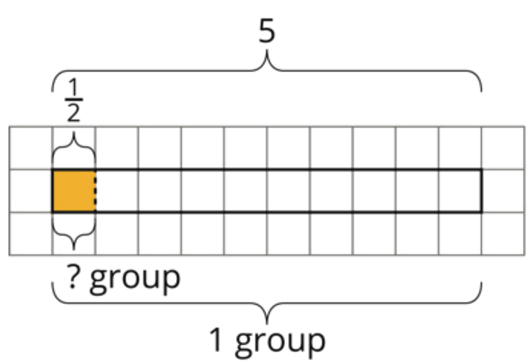


Activity Synthesis

After students worked on the last set of problems, discuss the question “what fraction of 5 is $\frac{1}{2}$?” To help students connect this question to previous ones, consider asking:

- “How can we tell if the answer is greater than 1 or less than 1 before calculating?” (The phrase “what fraction of” offers a clue that it is less than 1. Or, we are comparing $\frac{1}{2}$ to 5 and can see that $\frac{1}{2}$ is less than 5.)
- “What is the size of 1 group here? How do we know?” (We can tell that 5 is the size of 1 group because that is the value to which another number is being compared.)
- “How do we write a multiplication equation for this question? A division equation?” ($? \cdot 5 = \frac{1}{2}$, and $\frac{1}{2} \div 5 = ?$)

Select a student to display a correct diagram for the problem, or display this diagram for all to see. Discuss how the two given values and the solution are represented in the diagram.



Lesson Synthesis

In this lesson, we saw that a division problem can represent the idea of equal-sized groups but may have a total amount that is less than the size of one full group. Instead of “how many of this is in that?”, the question is now “what fraction of this is that?”.

- “How can we tell if a division situation involves less than one whole group?” (The total amount is less than the size of the a group, or the question asks “what fraction of. . .?”)

- “How do we find quotients that are less than 1?” (We can write a multiplication equation that corresponds to the situation and draw a tape diagram to help us reason about what fraction of 1 group the given amount is.)

We also explored division problems as representing the answers to comparison questions about measurement. Instead of “how many groups?”, we can ask “how many times as long (or as heavy)?” For example: $16 \div 4 = ?$ corresponds to $? \cdot 4 = 16$, which can represent the question “how many times as long as 4 cm is 16 cm?” We can reason that 16 cm is 4 times as long as 4 cm.

In the same context, $3 \div 4$ corresponds to $? \cdot 4 = 3$, and would mean “how many times as long as 4 cm is 3 cm?” Here, we can see that the answer will be a fraction less than 1. Because 3 is $\frac{3}{4}$ of 4, we can say “3 cm is $\frac{3}{4}$ as long as 4 cm.”

7.4 A Partially Filled Container

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

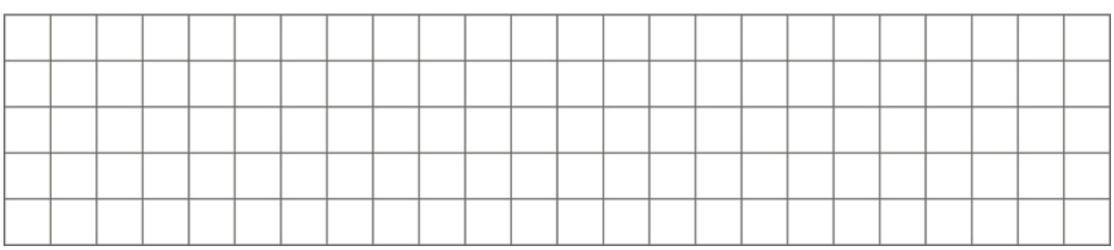
Launch

Provide continued access to colored pencils.

Student Task Statement

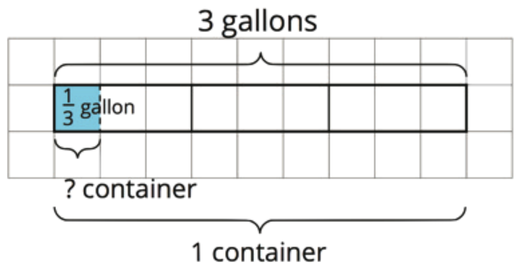
There is $\frac{1}{3}$ gallon of water in a 3-gallon container. What fraction of the container is filled?

1. Write a multiplication equation and a division equation to represent the situation.
2. Draw a tape diagram to represent the situation. Then, answer the question.



Student Response

1. $? \cdot 3 = \frac{1}{3}$ (or $3 \cdot ? = \frac{1}{3}$) and $\frac{1}{3} \div 3 = ?$ (or $\frac{1}{3} \div ? = 3$)
2. Diagrams vary. Sample diagram:

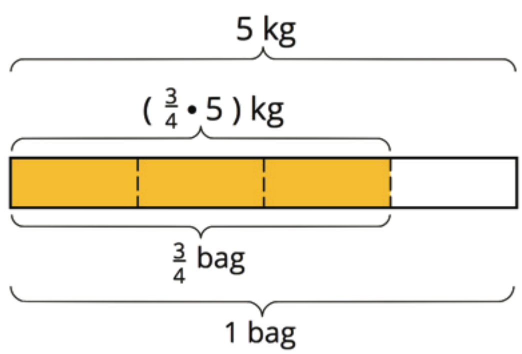


$\frac{1}{9}$ of the container is filled.

Student Lesson Summary

It is natural to think about groups when we have more than one group, but we can also have a *fraction of a group*.

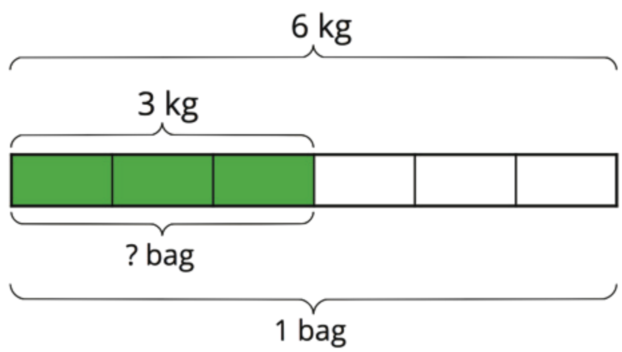
To find the amount in a fraction of a group, we can multiply the fraction by the amount in the whole group. If a bag of rice weighs 5 kg, $\frac{3}{4}$ of a bag would weigh $(\frac{3}{4} \cdot 5)$ kg.



Sometimes we need to find what fraction of a group an amount is. Suppose a full bag of flour weighs 6 kg. A chef used 3 kg of flour. What fraction of a full bag was used? In other words, what fraction of 6 kg is 3 kg?

This question can be represented by a multiplication equation and a division equation, as well as by a diagram.

$$? \cdot 6 = 3$$
$$3 \div 6 = ?$$

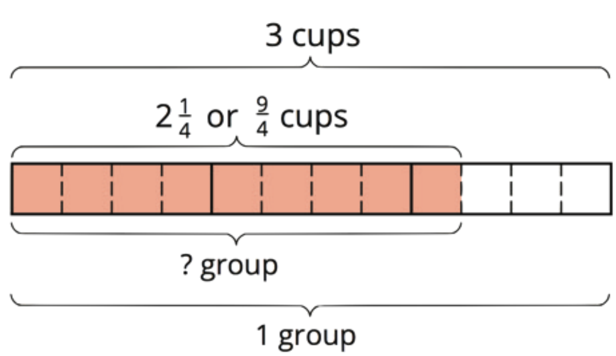


We can see from the diagram that 3 is $\frac{1}{2}$ of 6, and we can check this answer by multiplying: $\frac{1}{2} \cdot 6 = 3$.

In *any* situation where we want to know what fraction one number is of another number, we can write a division equation to help us find the answer.

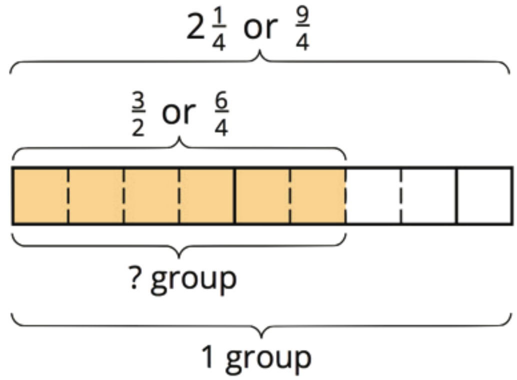
For example, "What fraction of 3 is $2\frac{1}{4}$?" can be expressed as $? \cdot 3 = 2\frac{1}{4}$, which can also be written as $2\frac{1}{4} \div 3 = ?$.

The answer to "What is $2\frac{1}{4} \div 3$?" is also the answer to the original question.



The diagram shows that 3 wholes contain 12 fourths, and $2\frac{1}{4}$ contains 9 fourths, so the answer to this question is $\frac{9}{12}$, which is equivalent to $\frac{3}{4}$.

We can use diagrams to help us solve other division problems that require finding a fraction of a group. For example, here is a diagram to help us answer the question: "What fraction of $\frac{9}{4}$ is $\frac{3}{2}$?," which can be written as $\frac{3}{2} \div \frac{9}{4} = ?$.



We can see that the quotient is $\frac{6}{9}$, which is equivalent to $\frac{2}{3}$. To check this, let's multiply. $\frac{2}{3} \boldsymbol{\cdot} \frac{9}{4} = \frac{18}{12}$, and $\frac{18}{12}$ is, indeed, equal to $\frac{3}{2}$.

Lesson 7 Practice Problems

Problem 1

Statement

A recipe calls for $\frac{1}{2}$ lb of flour for 1 batch. How many batches can be made with each of these amounts?

a. 1 lb

b. $\frac{3}{4}$ lb

c. $\frac{1}{4}$ lb

Solution

a. 2

b. $1\frac{1}{2}$

c. $\frac{1}{2}$

Problem 2

Statement

Whiskers the cat weighs $2\frac{2}{3}$ kg. Piglio weighs 4 kg. For each question, write a multiplication equation and a division equation, decide whether the answer is greater than 1 or less than 1, and then find the answer.

a. How many times as heavy as Piglio is Whiskers?

b. How many times as heavy as Whiskers is Piglio?

Solution

a. Multiplication: $? \cdot 4 = 2\frac{2}{3}$ (or $4 \cdot ? = 2\frac{2}{3}$), division: $\left(2\frac{2}{3}\right) \div 4 = ?$. Less than 1. Cat A is $\frac{8}{12}$ (or $\frac{2}{3}$) as heavy as Cat B.

b. Multiplication: $? \cdot \left(2\frac{2}{3}\right) = 4$ (or $\left(2\frac{2}{3}\right) \cdot ? = 4$), division: $4 \div \left(2\frac{2}{3}\right) = ?$. Bigger than 1. Cat B is $1\frac{4}{8}$ or $1\frac{1}{2}$ times as heavy Cat A.

Problem 3

Statement

Andre is walking from his home to a festival that is $1\frac{5}{8}$ kilometers away. He walks $\frac{1}{3}$ kilometer and then takes a quick rest. Which question can be represented by the equation $? \cdot 1\frac{5}{8} = \frac{1}{3}$ in this situation?

A. What fraction of the trip has Andre completed?

B. What fraction of the trip is left?

C. How many more kilometers does Andre have to walk to get to the festival?

D. How many kilometers is it from home to the festival and back home?

Solution

A

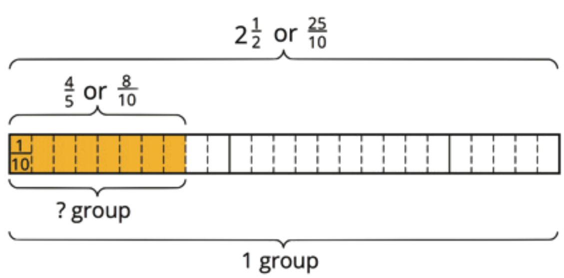
Problem 4

Statement

Draw a tape diagram to represent the question: What fraction of $2\frac{1}{2}$ is $\frac{4}{5}$?
Then find the answer.

Solution

$\frac{8}{25}$. Sample diagram:



Problem 5

Statement

How many groups of $\frac{3}{4}$ are in each of these quantities?

a. $\frac{11}{4}$

b. $6\frac{1}{2}$

Solution

a. $3\frac{2}{3}$ Sample reasoning: create a tape diagram showing $\frac{11}{4}$ divided into groups of $\frac{3}{4}$ each.

b. $8\frac{2}{3}$ Sample reasoning: create a tape diagram showing $6\frac{1}{2}$ divided into groups of $\frac{3}{4}$ each.

(From Unit 4, Lesson 6.)

Problem 6

Statement

Which question can be represented by the equation $4 \div \frac{2}{7} = ?$

A. What is 4 groups of $\frac{2}{7}$?

B. How many $\frac{2}{7}$s are in 4?

C. What is $\frac{2}{7}$ of 4?

D. How many 4s are in $\frac{2}{7}$?

Solution

B

(From Unit 4, Lesson 4.)

Lesson 8: How Much in Each Group? (Part 1)

Goals

- Compare and contrast (orally) strategies for solving problems about "how many groups?" and "how much in 1 group?"
- Create a tape diagram to represent and solve a problem asking "How much in 1 group?" where the dividend, divisor, and quotient may be fractions, and explain (orally) the solution method.
- Write multiplication and division equations to represent a problem asking "How much in 1 group?"

Learning Targets

- I can tell when a question is asking for the amount in one group.
- I can use diagrams and multiplication and division equations to represent and answer "how much in each group?" questions.

Lesson Narrative

Previously, students looked at division situations in which the number of groups (or the fraction of a group) was unknown. They interpreted division expressions as a way to find out that number (or fraction) of groups. In this lesson, students encounter situations where the number of groups is known but the size of each group is not. They interpret division expressions as a way to answer "how much in a group?" questions.

Students use the same tools—multiplication and division equations and tape diagrams—and the same structure of equal-sized groups to reason about "how much in a group?" questions (MP7). They also continue to relate their reasoning in quantitative contexts to their reasoning on abstract representations (MP2). Students find both whole-number and non-whole-number quotients, recognizing that, like the number of groups, the amount in one group can also be a whole number or a fraction.

Alignments

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR8: Discussion Supports
- Notice and Wonder

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's look at division problems that help us find the size of one group.

8.1 Inventing a Situation

Warm Up: 5 minutes
By now students have written many division equations based on verbal descriptions of situations. This warm-up prompts them to go in the other direction: to interpret a division expression and write a fitting question the expression could help answer. Then, they trade descriptions with a partner and reason about the value of the same expression in someone else's story.

As students write their descriptions, look out for scenarios that are unclear and ask their authors clarifying questions. Select 2–3 students whose descriptions clearly match the expression to share later.

Students may struggle to answer their partner's question because the descriptions are confusing or do not match the given expression. Encourage them to ask the questions they have about the description, but still try to find the answer to the given expression.

Addressing

- 6.NS.A.1

Launch

Arrange students in groups of 2. Explain to students the two halves of this activity: to write a situation and a question that can be represented by the equation $12 \div \frac{2}{3} = ?$, and to trade papers with their partner and find the answer to their partner's question.

Give 2 minutes of quiet think time for the first question, another minute for the second question, and follow with a whole-class discussion.

Anticipated Misconceptions

Some students may struggle writing a context. Ask them to reflect on the work they have done and what the division expression means for them, or suggest that they write related multiplication expressions to help them understand the division expression.

Student Task Statement

1. Think of a situation with a question that can be represented by the equation $12 \div \frac{2}{3} = ?$ Describe the situation and the question.

2. Trade descriptions with your partner, and answer your partner's question.

Student Response

1. Answers vary. Possible response: I have 12 feet of ribbon. The bows I am making each require $\frac{2}{3}$ ft of ribbon. How many bows can I make with the ribbon I have?

2. 18

Activity Synthesis

Ask selected students to share their stories with the whole group. After each student shares, have another student explain how the story matches the division equation. Afterward, ask these and other students for the value of the given expression, and write and display the completed division equation for all to see.

If not already mentioned by students, highlight that the answer is the same for all scenarios because they are all based on the same division expression.

8.2 How Much in One Batch?

15 minutes

In this task, students explore division situations (in the context of baking cookies) where the number of groups and a total amount are given, but the size of 1 group is unknown. They write multiplication equations in which the missing factor answers the question "how much in each group?" instead of "how many groups?"

Students continue to use equations, diagrams, and the connection between multiplication and division in their reasoning. No grid is provided here, but allow students to use graph paper to support their diagram drawing if desired or needed.

As students work, monitor how they start their diagrams and which quantity they represent first. If they are not quite sure how to show a particular quantity, ask them to refer to earlier diagrams and notice how the number of groups, the size of each group, and the total amount were represented on a single diagram. Select students whose diagrams would be instructive to others to share later.

Addressing

- 6.NS.A.1

Launch

Arrange students in groups of 3–4. Give groups 5–6 minutes of quiet work time and 2 minutes to share their responses in their group. Provide access to geometry toolkits (especially graph paper and colored pencils).

Anticipated Misconceptions

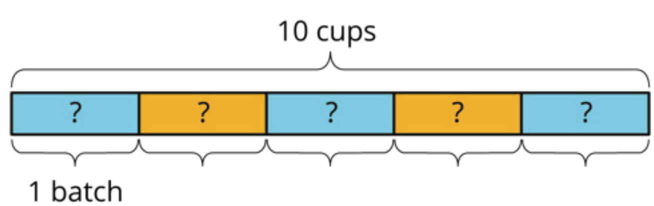
When writing multiplication equations, some students might simply use the smaller number as a factor and the larger number as the product. Or, when writing division equations, they might simply divide the larger number by the smaller number without attending to what the numbers mean. Encourage them to think about what the numbers represent. To help them see the connection between verbal statements and mathematical equations, support their reasoning with simple examples, e.g., "To make 2 batches of cookies, we need 6 cups of flour. How many cups are needed for 1 batch?" This can be interpreted as "2 groups of what equals 6?" and then $2 \cdot ? = 6$.

Student Task Statement

To make 5 batches of cookies, 10 cups of flour are required. Consider the question: How many cups of flour does each batch require?

We can write equations and draw a diagram to represent this situation.

$$5 \cdot ? = 10$$
$$10 \div 5 = ?$$



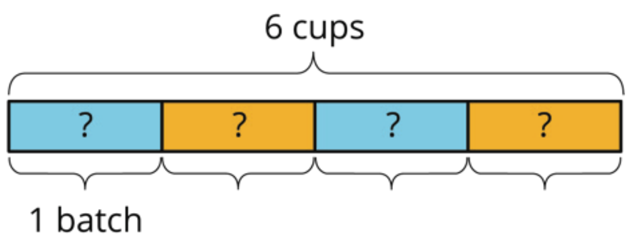
This helps us see that each batch requires 2 cups of flour.

For each question, write a multiplication equation and a division equation, draw a diagram, and find the answer.

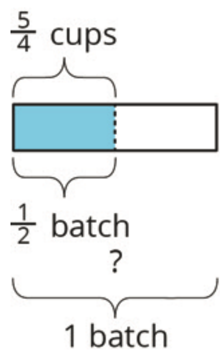
1. To make 4 batches of cupcakes, it takes 6 cups of flour. How many cups of flour are needed for 1 batch?
2. To make $\frac{1}{2}$ batch of rolls, it takes $\frac{5}{4}$ cups of flour. How many cups of flour are needed for 1 batch?
3. Two cups of flour make $\frac{2}{3}$ batch of bread. How many cups of flour make 1 batch?

Student Response

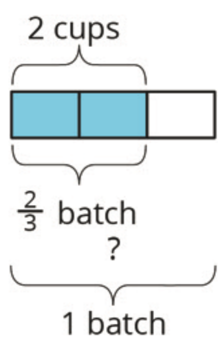
1. Multiplication equation: $4 \cdot ? = 6$ (or $? \cdot 4 = 6$), division equation: $6 \div 4 = ?$ (or $6 \div ? = 4$). Each batch takes $\frac{6}{4}$ (or $1\frac{1}{2}$) cups of flour.



2. Multiplication equation: $\frac{1}{2} \cdot ? = \frac{5}{4}$ (or $? \cdot \frac{1}{2} = \frac{5}{4}$), division equation: $\frac{5}{4} \div \frac{1}{2} = ?$ (or $\frac{5}{4} \div ? = \frac{1}{2}$). Each batch takes $\frac{10}{4}$ (or $2\frac{1}{2}$) cups of flour.

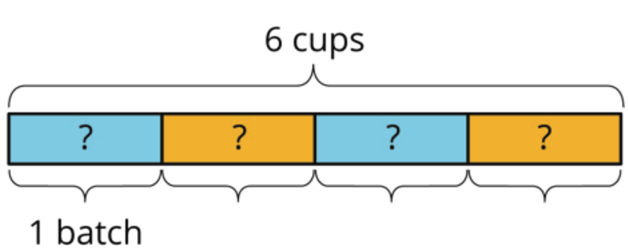


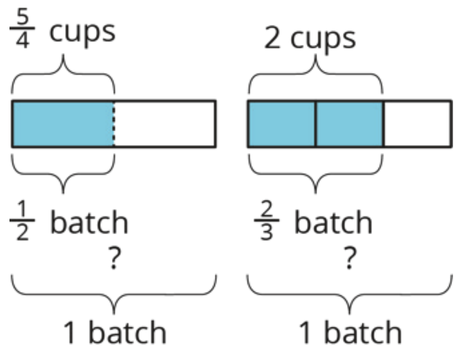
3. Multiplication equation: $\frac{2}{3} \cdot ? = 2$ (or $? \cdot \frac{2}{3} = 2$), division equation: $2 \div \frac{2}{3} = ?$ (or $2 \div ? = \frac{2}{3}$). Each batch takes 3 cups of flour.



Activity Synthesis

Select previously identified students to share their diagrams and reasoning, or display the ones here for all to see. If other students used alternative approaches, invite a couple of them to share.





To highlight some key ideas in this lesson and in the unit, discuss:

- "How are these diagrams like those in previous lessons? How are they different?" (Alike: We can use them to show a total amount, the number of groups, the size of 1 group. Different: We don't know the value of 1 group here. In past diagrams, we were given the size of 1 group, but needed to find the number of groups.)

- "How are these equations and the ones in previous lessons alike and different?" (Alike: The multiplication equations have one unknown factor. The division equations have the same setup. Different: The unknown factor here represents a different quantity than in past lessons.)

- "How are the diagrams for the second and third questions different from the one in the first question?" (In the first one, the given amount is greater than the size of 1 batch. In the other two, the given amount is less than the size of 1 batch or is a fraction of a batch.)

Support for Students with Disabilities

Representation: Internalize Comprehension. Use color and annotations to illustrate connections between representations. As students share their diagrams and reasoning, use color and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing

8.3 One Container and One Section of Highway

15 minutes
In this activity, students continue to reason about the size of 1 group in situations involving equal-sized groups. In the first part, the given total amount is a whole number. In the second, the given amount is a fraction. The visual representations for both parts are very similar, allowing students to notice the structure of the relationships and generalize their observations (MP7).

Addressing

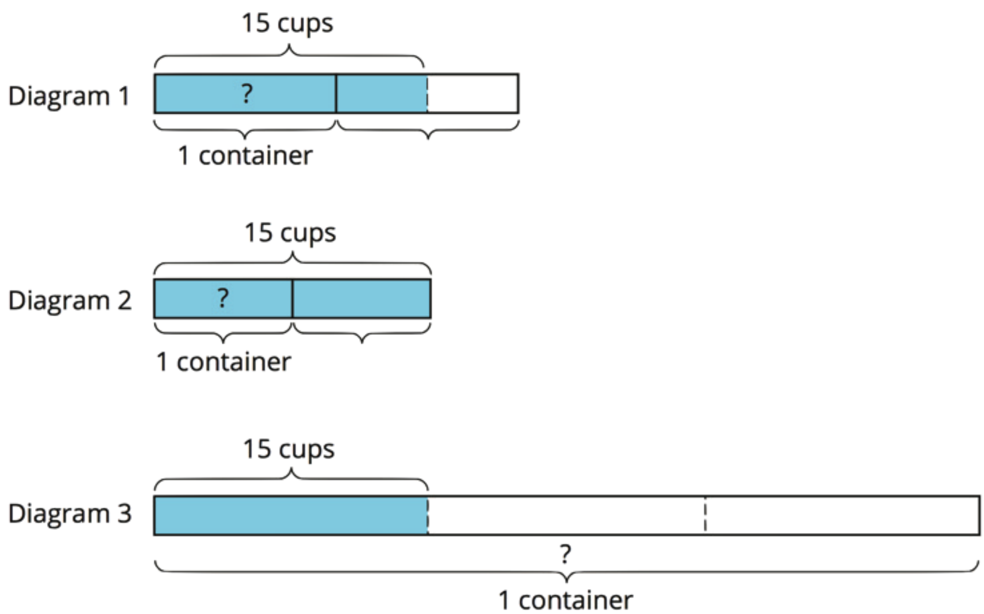
- 6.NS.A.1

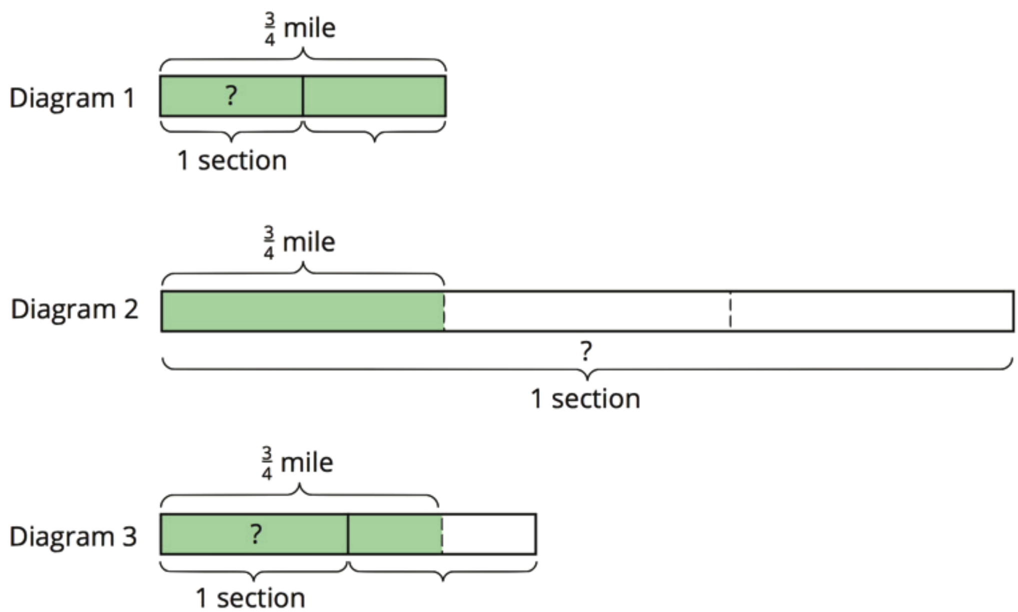
Instructional Routines

- MLR8: Discussion Supports
- Notice and Wonder

Launch

Display the two sets of images in the activity. Give students a minute to observe them. Ask them to be prepared to share at least one thing they notice and one thing they wonder. Then, solicit a few observations and questions from students.





Keep students in groups of 3–4. Give students 3–4 minutes of quiet think time for the first set of questions (about water in containers) and ask them to pause for a brief group discussion before continuing on to the second set (about highways). Tell students to compare their responses and discuss any questions or discrepancies until they reach an agreement. Afterward, give them another 4–5 minutes of quiet think time to complete the remaining questions and, if possible, time to discuss their responses in groups.

Support for English Language Learners

Speaking, Conversing: MLR8 Discussion Supports. Display sentence frames that will help students produce verbal justifications. For example, "Diagram ___ represents situation ___ because . . .", "A multiplication equation that represents this situation is _____ because . . ." and "A division equation that represents this situation is _____ because . . . ."
Design Principle(s): Support sense-making, Optimize output (for justification)

Student Task Statement

Here are three tape diagrams that represent situations about filling containers of water.

Diagram 1

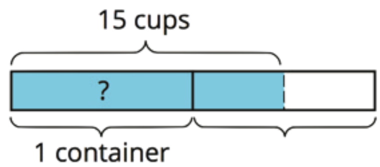


Diagram 2

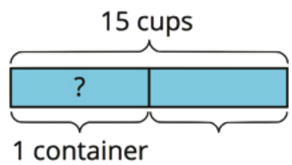
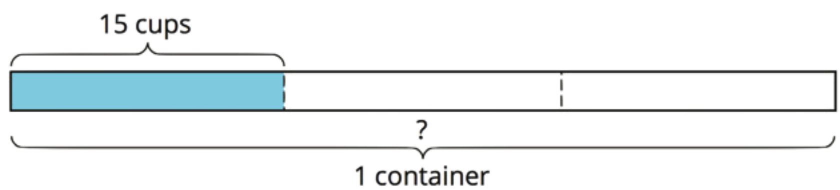


Diagram 3



Match each situation to a diagram and use the diagram to help you answer the question. Then, write a multiplication equation and a division equation to represent the situation.

1. Tyler poured a total of 15 cups of water into 2 equal-sized bottles and filled each bottle. How much water was in each bottle?
2. Kiran poured a total of 15 cups of water into equal-sized pitchers and filled $1\frac{1}{2}$ pitchers. How much water was in the full pitcher?
3. It takes 15 cups of water to fill $\frac{1}{3}$ pail. How much water is needed to fill 1 pail?

Here are tape diagrams that represent situations about cleaning sections of highway.

Diagram 1

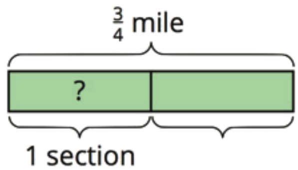


Diagram 2

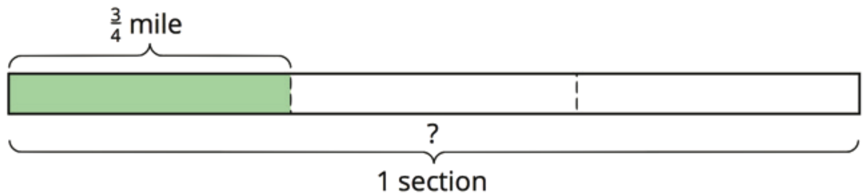
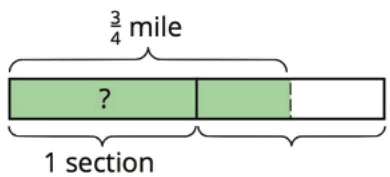


Diagram 3



Match each situation to a diagram and use the diagram to help you answer the question. Then, write a multiplication equation and a division equation to represent the situation.

4. Priya's class has adopted two equal sections of a highway to keep clean. The combined length is $\frac{3}{4}$ of a mile. How long is each section?

5. Lin's class has also adopted some sections of highway to keep clean. If $1\frac{1}{2}$ sections are $\frac{3}{4}$ mile long, how long is each section?

6. A school has adopted a section of highway to keep clean. If $\frac{1}{3}$ of the section is $\frac{3}{4}$ mile long, how long is the section?

Student Response

1. Diagram 2, multiplication equation: $2 \cdot ? = 15$ (or $? \cdot 2 = 15$), division equation: $15 \div 2 = ?$ (or $15 \div ? = 2$), answer: $7\frac{1}{2}$ cups.

2. Diagram 1, multiplication equation: $(1\frac{1}{2}) \cdot ? = 15$ (or $? \cdot (1\frac{1}{2}) = 15$), division equation: $15 \div 1\frac{1}{2} = ?$ (or $15 \div ? = 1\frac{1}{2}$), answer: 10 cups.

3. Diagram 3, multiplication equation: $\frac{1}{3} \cdot ? = 15$ (or $? \cdot \frac{1}{3} = 15$), division equation: $15 \div \frac{1}{3} = ?$ (or $15 \div ? = \frac{1}{3}$), answer: 45 cups.

4. Diagram 1, multiplication equation: $2 \cdot ? = \frac{3}{4}$ (or $? \cdot 2 = \frac{3}{4}$), division equation: $\frac{3}{4} \div 2 = ?$ (or $\frac{3}{4} \div ? = 2$), answer: $\frac{3}{8}$ mile.

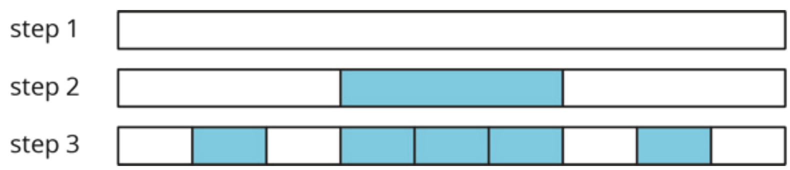
5. Diagram 3, multiplication equation: $(1\frac{1}{2}) \cdot ? = \frac{3}{4}$ (or $? \cdot (1\frac{1}{2}) = \frac{3}{4}$), division equation: $\frac{3}{4} \div 1\frac{1}{2} = ?$ (or $\frac{3}{4} \div ? = 1\frac{1}{2}$), answer: $\frac{1}{2}$ mile.

6. Diagram 2, multiplication equation: $\frac{1}{3} \cdot ? = \frac{3}{4}$ (or $? \cdot \frac{1}{3} = \frac{3}{4}$), division equation: $\frac{3}{4} \div \frac{1}{3} = ?$ (or $\frac{3}{4} \div ? = \frac{1}{3}$), answer: $\frac{9}{4}$ miles.

Are You Ready for More?

To make a Cantor ternary set:

- Start with a tape diagram of length 1 unit. This is step 1.
- Color in the middle third of the tape diagram. This is step 2.
- Do the same to each remaining segment that is not colored in. This is step 3.
- Keep repeating this process.



1. How much of the diagram is colored in after step 2? Step 3? Step 10?

2. If you continue this process, how much of the tape diagram will you color?

3. Can you think of a different process that will give you a similar result? For example, color the first fifth instead of the middle third of each strip.

Student Response

1. Step 2: $\frac{1}{3}$, Step 3: $\frac{5}{9}$, Step 10: $\frac{19,171}{19,683}$

2. The whole thing.

3. Answers vary.

Activity Synthesis

Most of the discussions are to take place in small groups. Reconvene as a class to discuss any unresolved disagreements or common misconceptions, and to highlight the following points:

- “In both sets of questions, what information was missing?” (The size of 1 group.)
- “How were the division equations in the two contexts different?” (In the water context, the dividend, which is the given amount of water, is a whole number and the quotients are greater than 1. In the highway context, the dividend is a fraction and the quotient could be greater or less than 1.)

Lesson Synthesis

Remind students that there are two multiplication equations that correspond to $7 \div \frac{1}{2} = ?$. We can write:

- $? \cdot \frac{1}{2} = 7$, which can be interpreted as: "how many groups of $\frac{1}{2}$ are in 7?"
- $\frac{1}{2} \cdot ? = 7$, which can be interpreted as: "$\frac{1}{2}$ of what number is 7?"

Because both multiplication equations are related to $7 \div \frac{1}{2} = ?$, we can use division to answer both questions. In this lesson, we looked at the second interpretation, where we know the number of groups but not the amount in 1 group.

To help students synthesize these meanings of division, consider asking them to draw a tape diagram for each interpretation of $7 \div \frac{1}{2} = ?$ and show that in both cases, the quotient is 14.

8.4 Funding a Camping Trip

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

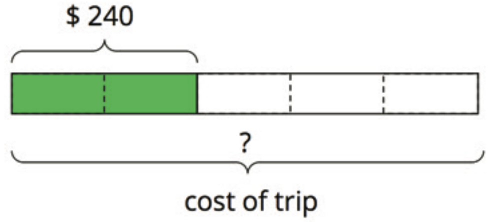
Student Task Statement

Consider the problem: Students in a sixth-grade class are raising money for an end-of-year camping trip. So far, they have raised \$240. This is $\frac{2}{5}$ of the cost of the trip. How much does the trip cost?

Write a multiplication equation and a division equation and draw a diagram to represent the situation. Then, find the answer and show your reasoning.

Student Response

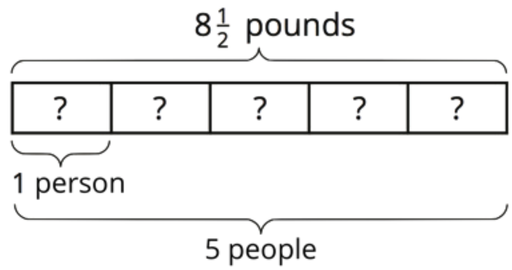
$\frac{2}{5} \cdot ? = 240$ (or $? \cdot \frac{2}{5} = 240$) and $240 \div \frac{2}{5} = ?$ (or $240 \div ? = \frac{2}{5}$). The trip costs \$600. If $\frac{2}{5}$ of the cost is \$240, then each $\frac{1}{5}$ of the cost is \$120, so the cost would be 5 times \$120, which is \$600.



Student Lesson Summary

Sometimes we know the amount for *multiple* groups, but we don't know how much is in one group. We can use division to find out.

For example, if 5 people share $8\frac{1}{2}$ pounds of cherries equally, how many pounds of cherries does each person get?



We can represent this situation with a multiplication equation and a division equation:

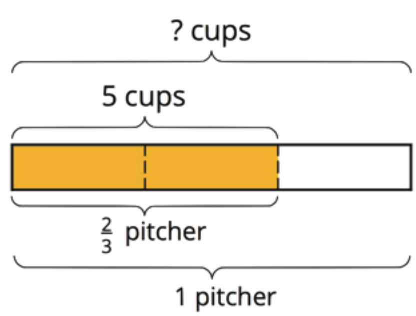
$$5 \cdot ? = 8\frac{1}{2}$$

$$8\frac{1}{2} \div 5 = ?$$

$8\frac{1}{2} \div 5$ can be written as $\frac{17}{2} \div 5$. Dividing by 5 is equivalent to multiplying by $\frac{1}{5}$, and $\frac{17}{2} \cdot \frac{1}{5} = \frac{17}{10}$. This means each person gets $1\frac{7}{10}$ pounds.

Other times, we know the amount for *a fraction* of a group, but we don't know the size of one whole group. We can also use division to find out.

For example, Jada poured 5 cups of iced tea in a pitcher and filled $\frac{2}{3}$ of the pitcher. How many cups of iced tea fill the entire pitcher?



We can represent this situation with a multiplication equation and a division equation:

$$\frac{2}{3} \cdot ? = 5$$

$$5 \div \frac{2}{3} = ?$$

The diagram can help us reason about the answer. If $\frac{2}{3}$ of a pitcher is 5 cups, then $\frac{1}{3}$ of a pitcher is half of 5, which is $\frac{5}{2}$. Because there are 3 thirds in 1 whole, there would be $(3 \cdot \frac{5}{2})$ or $\frac{15}{2}$ cups in one whole pitcher. We can check our answer by multiplying: $\frac{2}{3} \cdot \frac{15}{2} = \frac{30}{6}$, and $\frac{30}{6} = 5$.

Notice that in the first example, the number of groups is greater than 1 (5 people) and in the second, the number of groups is less than 1 ($\frac{2}{3}$ of a pitcher), but the division and multiplication equations for both situations have the same structures.

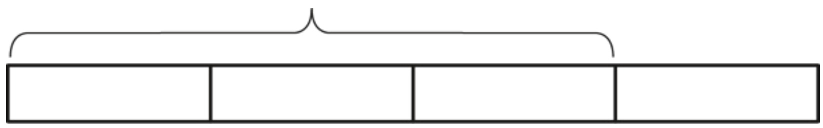
Lesson 8 Practice Problems

Problem 1

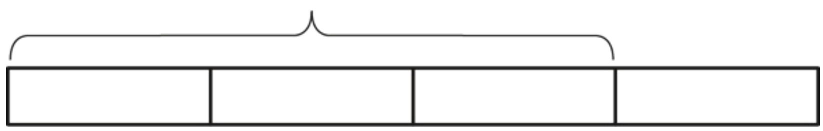
Statement

For each situation, complete the tape diagram to represent and answer the question.

a. Mai has picked 1 cup of strawberries for a cake, which is enough for $\frac{3}{4}$ of the cake. How many cups does she need for the whole cake?

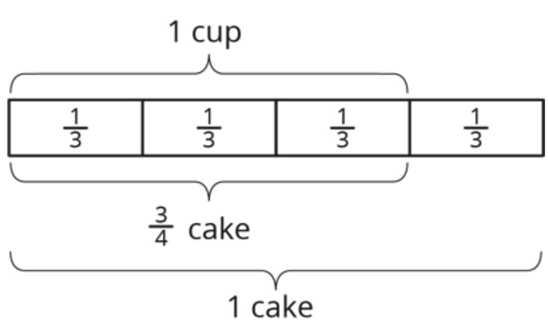


b. Priya has picked $1\frac{1}{2}$ cups of raspberries, which is enough for $\frac{3}{4}$ of a cake. How many cups does she need for the whole cake?

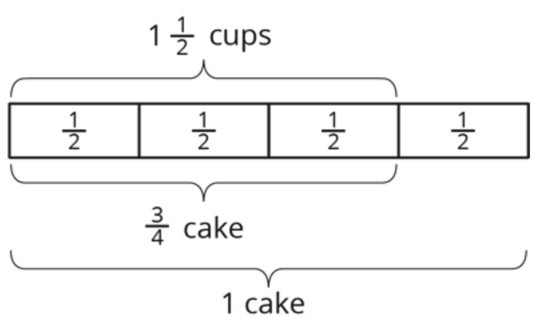


Solution

a. $1\frac{1}{3}$ cups of strawberries



b. 2 cups of raspberries



Problem 2

Statement

Consider the problem: Tyler painted $\frac{9}{2}$ square yards of wall area with 3 gallons of paint. How many gallons of paint does it take to paint each square yard of wall?

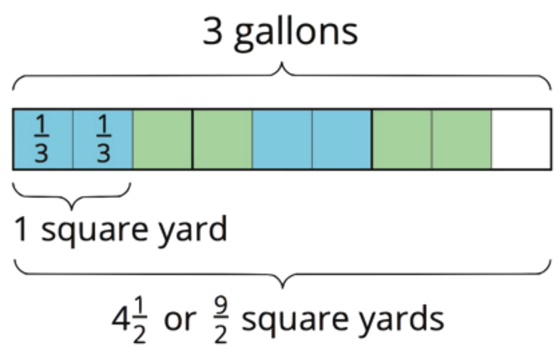
a. Write multiplication and division equations to represent the situation.

b. Draw a diagram to represent and answer the question.

Solution

a. Multiplication: $\frac{9}{2} \cdot ? = 3$, division: $3 \div \frac{9}{2} = ?$

b. Diagrams vary. Sample diagram:



It takes $\frac{2}{3}$ gallons of paint for each square yard of wall. (The answer is correct because $\frac{9}{2} \cdot \frac{2}{3} = 3$.)

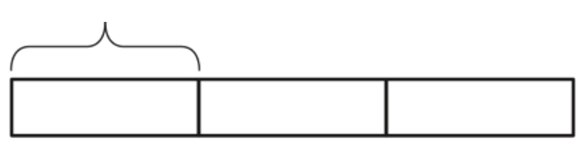
Problem 3

Statement

Consider the problem: After walking $\frac{1}{4}$ mile from home, Han is $\frac{1}{3}$ of his way to school. What is the distance between his home and school?

a. Write multiplication and division equations to represent this situation.

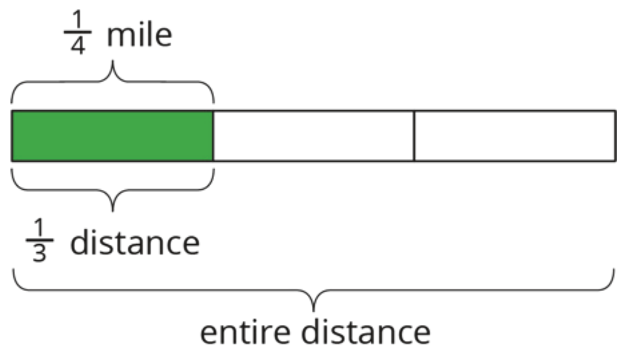
b. Complete the diagram to represent and answer the question.



Solution

a. Multiplication: $\frac{1}{3} \cdot ? = \frac{1}{4}$, division: $\frac{1}{4} \div \frac{1}{3} = ?$

b. Diagrams vary. Sample diagram:



$\frac{3}{4}$ mile ($\frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$)

Problem 4

Statement

Here is a division equation: $\frac{4}{5} \div \frac{2}{3} = ?$

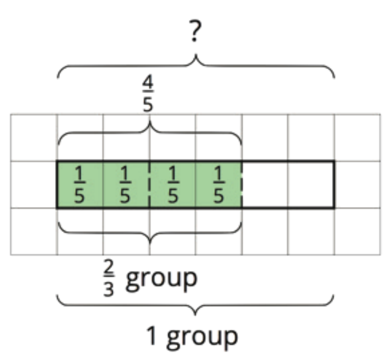
a. Write a multiplication equation that corresponds to the division equation.

b. Draw a diagram to represent and answer the question.

Solution

a. $\frac{2}{3} \cdot ? = \frac{4}{5}$ (or equivalent)

b. $\frac{6}{5}$. Sample diagram:



(From Unit 4, Lesson 7.)

Problem 5

Statement

Consider the problem: A set of books that are each 1.5 inches wide are being organized on a bookshelf that is 36 inches wide. How many books can fit on the shelf?

a. Write multiplication and division equations to represent the situation.

b. Find the answer. Draw a diagram, if needed.

c. Use the multiplication equation to check your answer.

Solution

a. $? \cdot (1.5) = 36$ (or equivalent), $36 \div 1.5 = ?$ (or equivalent)

b. 24 books can fit on the shelf.

c. $24 \cdot (1.5) = 36$

(From Unit 4, Lesson 3.)

Problem 6

Statement

a. Without calculating, order the quotients from smallest to largest.

$56 \div 8$ $\quad$ $56 \div 8{,}000{,}000$ $\quad$ $56 \div 0.000008$

b. Explain how you decided the order of the three expressions.

c. Find a number n so that $56 \div n$ is greater than 1 but less than 7.

Solution

a. $56 \div 8{,}000{,}000$, $56 \div 8$, $56 \div 0.000008$

b. Since the dividend is the same for all three expressions, the larger the divisor, the smaller the quotient.

c. Answers vary. Possible response: $n = 10$ would work since $56 \div 8 = 7$ and $56 \div 56 = 1$. (Any number between $n = 8$ and $n = 56$ would work.)

(From Unit 4, Lesson 1.)

Lesson 9: How Much in Each Group? (Part 2)

Goals

- Interpret a situation (presented in written language or using other representations) involving equal-sized groups, and generate mathematical questions that could be asked about it.
- Solve a problem involving division of fractions, and present the solution method (orally, in writing, and using other representations).

Learning Targets

- I can find the amount in one group in different real-world situations.

Lesson Narrative

This lesson allows students to practice finding the amount in one group, and to interpret, represent, and solve different kinds of division problems with less scaffolding. In one activity, students are not explicitly told whether the division situations involve finding the number of groups or finding the amount in each group. They decide on an interpretation that would enable them to solve a division problem. Students are also required to identify relevant information (from a video, a picture, or written statements) that would help them answer questions.

Because the tasks in this lesson are not scaffolded, students will need to make sense of the problems and persevere to solve them (MP1). As students move back and forth between the contexts and the abstract equations and diagrams that represent them, they reason abstractly and quantitatively (MP2).

Alignments

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Building Towards

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Group Presentations
- MLR8: Discussion Supports
- Number Talk
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Student Learning Goals

Let’s practice dividing fractions in different situations.

9.1 Number Talk: Greater Than 1 or Less Than 1?

Warm Up: 5 minutes
This warm-up prompts students to interpret division of fractions in terms of the number of groups of one fraction in the other (i.e., “how many groups of this in that?” question). Students do not calculate the exact value of each expression. Instead, they decide if at least one group the size of the divisor is in the dividend. This requires students to use what they know about benchmark fractions and equivalent fractions to reason about the size of the numbers.

Building Towards

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports

- Number Talk

Launch

Display one problem at a time. Give students 30 seconds of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Follow with a whole-group discussion.

Support for Students with Disabilities

Representation: Internalize Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organization

Anticipated Misconceptions

If students have trouble getting started, remind them that, in a previous lesson, one way they interpreted $4 \div \frac{1}{2} = ?$ was as "how many $\frac{1}{2}$s are in 4?" Ask them if that interpretation could apply here. Also encourage students to recall how the size of the divisor affects the quotient.

Some students may struggle with the last expression because they don't easily see the relationship between fifths and eighths. Ask if they can think of a fraction, in eighths, that is close to $\frac{3}{5}$, or a fraction in fifths that is close to $\frac{7}{8}$. If not, ask them how they would reason about the expression if it was $2\frac{6}{8}$ instead of $2\frac{3}{5}$.

Student Task Statement

Decide whether each quotient is greater than 1 or less than 1.

$\frac{1}{2} \div \frac{1}{4}$

$1 \div \frac{3}{4}$

$\frac{2}{3} \div \frac{7}{8}$

$2\frac{7}{8} \div 2\frac{3}{5}$

Student Response

- Greater than 1 (because $2 \cdot \frac{1}{4} = \frac{1}{2}$)
- Greater than 1 (because $\frac{4}{3} \cdot \frac{3}{4} = 1$)
- Less than 1 (because $\frac{7}{8}$ is greater than $\frac{2}{3}$)
- Greater than 1 (because $2\frac{3}{5}$ is less than $2\frac{6}{8}$ and $2\frac{6}{8}$ would go into $2\frac{7}{8}$ at least one time)

Activity Synthesis

Ask students to share their reasoning after completing each problem and before they think about the next problem. Record and display their explanations for all to see.

Students may reason about the answers by thinking about the relative sizes of the two fractions, which is a valid approach. This shows an understanding of how the size of the dividend and that of the divisor affect the quotient.

To encourage students to connect the division expressions to multiplication, however, ask students to support their response to each of the first two problems with a related multiplication expression or equation, or in terms of equal-sized groups. For example, $\frac{1}{2} \div \frac{1}{4}$ could be connected to $? \boldsymbol{\cdot} \frac{1}{4} = 2$, or the question "how many $\frac{1}{4}$s are in $\frac{1}{2}$?".

To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone think about the expression the same way but would explain it differently?"
- "Did anyone find the value in a different way?"
- "Does anyone want to add on to _____'s reasoning?"
- "Do you agree or disagree? Why?"

Support for English Language Learners

Speaking: MLR8 Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimize output (for explanation)

9.2 Two Water Containers

15 minutes

This activity prompts students to solve a problem involving division of fractions in a less-scaffolded way. Students can see two relevant numbers to work with, but they need to interpret the context, the visual information, and the written question to decide whether the missing value is the size of one group, the number of groups, or the given amount (MP1). By carefully making sense of the context, students see that part of a group is known and they have to find the size of the whole group.

As students work, monitor the strategies they use and the way they reason about the situation. Students may draw directly on the photo to make sense of the quantities. Some may draw a tape diagram or another type of diagram to represent the partially-filled water container. Others may bypass diagramming and instead reason verbally or by writing equations. Select a few students who use different but equally effective strategies to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

Video 'Water in Containers' available here: https://player.vimeo.com/video/304137827.

Arrange students in groups of 3–4. Show the short video. Ask students what questions we could ask about the amount of water in this situation that would require working with fractions to determine the answers. Give groups a moment to think about their questions. If needed, show the video again, or refer to the photos to identify the fractions.

Possible questions:

- How much water fits into the whole water dispenser?
- How much water is indicated by each tick mark of the water dispenser?
- How far would 1 liter of water fill the water dispenser?

After hearing students' ideas, give students 4–5 minutes of quiet work time and then another 2 minutes to share their responses with their group. Ask students to discuss any disagreements they might have about their interpretation of the problem, the solving process, or the answer. Remind students they can check their solution using the multiplication equation.

Support for Students with Disabilities

Representation: Internalize Comprehension. Guide information processing and visualization. To support working memory, play the video multiple times. Students may also benefit from explicit guidance on which aspects of the video to focus on each time.
Supports accessibility for: Memory; Organization

Anticipated Misconceptions

For the second question, students may not immediately see that, to answer the question "how many liters of water fit in the dispenser?" they need to relate the amount in liters (as shown in the measuring cup) to the fraction of the dispenser that is filled with water. Suggest that they think about the problem in terms of equal-sized groups and try to identify the groups, number of groups, etc.

Student Task Statement

1. After looking at these pictures, Lin says, “I see the fraction $\frac{2}{5}$.” Jada says, “I see the fraction $\frac{3}{4}$.” What quantities are Lin and Jada referring to?

2. Consider the problem: How many liters of water fit in the water dispenser?

    a. Write a multiplication equation and a division equation for the question.

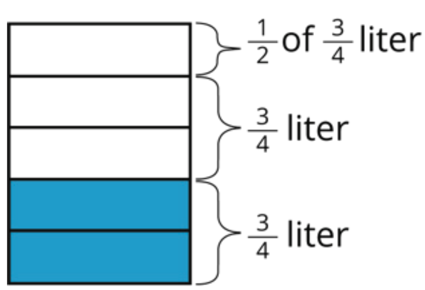
    b. Find the answer and explain your reasoning. If you get stuck, consider drawing a diagram.

    c. Check your answer using the multiplication equation.

Student Response

1. Lin is seeing that $\frac{2}{5}$ of the water dispenser is filled. Jada is seeing that the amount of water is $\frac{3}{4}$ liters.

2. Multiplication: $\frac{2}{5} \boldsymbol{\cdot} ? = \frac{3}{4}$ (or $? \boldsymbol{\cdot} \frac{2}{5} = \frac{3}{4}$), division: $\frac{3}{4} \div \frac{2}{5} = ?$ (or $\frac{3}{4} \div ? = \frac{2}{5}$). Diagrams vary. Sample diagram:



$\frac{3}{4} + \frac{3}{4} + \frac{1}{2} \boldsymbol{\cdot} \frac{3}{4} = 5 \boldsymbol{\cdot} \frac{1}{2} \boldsymbol{\cdot} \frac{3}{4}$, so there are $\frac{15}{8}$ (or $1\frac{7}{8}$) liters. Check: $\frac{2}{5} \boldsymbol{\cdot} \frac{15}{8} = \frac{3}{4}$

Activity Synthesis

Invite previously identified students to share their solutions and reasoning. Sequence their presentations so that the more concrete strategies are presented before the more abstract (e.g., a student who used only equations should present last). Display their diagrams and record their reasoning for all to see.

Connect the reasoning done using the diagram to the mathematical operations. Regardless of the path students took, finding the amount of water that fills the entire dispenser requires dividing $\frac{3}{4}$ liters by 2 (to find the amount in $\frac{1}{5}$ of a dispenser), and then multiplying that amount by 5. Consider revisiting this activity later, after students have learned an algorithm for dividing a fraction, and using this problem to reinforce that multiplying by the denominator and dividing by the numerator makes sense as a way to divide by a fraction.

9.3 Amount in One Group

15 minutes

In this activity, students practice reasoning about the amount in one group in division situations. They continue to write equations and draw diagrams to support their reasoning. In two problems (odd-numbered), the given number of groups is greater than 1. In the other two problems (even-numbered), a fraction of a group is given. Though this does not affect the structure of the equations students write, students need to take care to reflect this information correctly in their diagrams.

Students choose two problems to solve and work in groups of 2. Time permitting, students create a visual display of their work and conclude the activity with a gallery walk.

Addressing

- 6.NS.A.1

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Ask each group to choose two problems: an even-numbered problem and an odd-numbered problem. Give students 7–8 minutes of quiet time to work on their two chosen problems, and a few minutes to share their work with their partner.

Provide access to geometry toolkits and tools for creating a visual display. If time permits, ask each group to create a visual display of their solution and reasoning for one problem. Emphasize that they should organize their reasoning so it can be followed by others.

Support for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to blank tape diagrams to support information processing.
Supports accessibility for: Visual-spatial processing; Organization

Student Task Statement

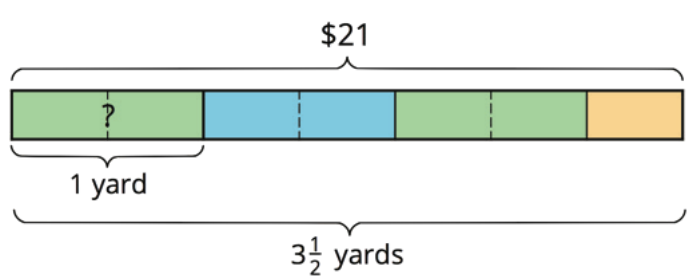
Write a multiplication equation and a division equation and draw a diagram to represent each situation. Then, find the answer and explain your reasoning.

1. Jada bought $3\frac{1}{2}$ yards of fabric for $21. How much did each yard cost?
2. $\frac{4}{9}$ kilogram of baking soda costs $2. How much does 1 kilogram of baking soda cost?
3. Diego can fill $1\frac{1}{5}$ bottles with 3 liters of water. How many liters of water fill 1 bottle?
4. $\frac{5}{4}$ gallons of water fill $\frac{5}{6}$ of a bucket. How many gallons of water fill the entire bucket?

Student Response

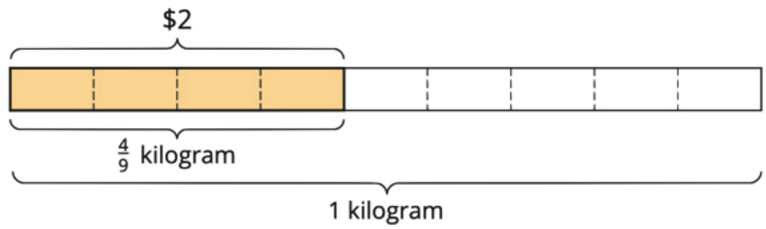
1. Multiplication equation: $3\frac{1}{2} \cdot ? = 21$ (or $? \cdot 3\frac{1}{2} = 21$), division equation: $21 \div 3\frac{1}{2} = ?$ (or $21 \div ? = 3\frac{1}{2}$).

   Each yard costs $6. Sample reasoning: There are seven $\frac{1}{2}$ yards in $3\frac{1}{2}$. If seven $\frac{1}{2}$ yards cost $21, then each $\frac{1}{2}$ yard is $\$21 \div 7$ (or $3). This means each yard is $6.

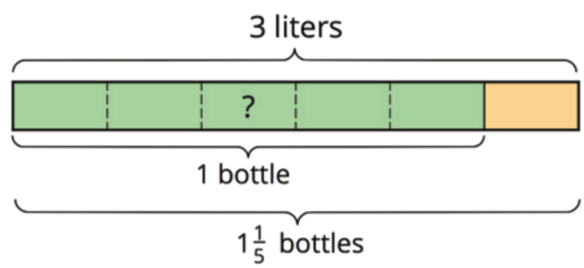
2. Multiplication equation: $\frac{4}{9} \cdot ? = 2$ (or $? \cdot \frac{4}{9} = 2$), division equation: $2 \div \frac{4}{9} = ?$ (or $2 \div ? = \frac{4}{9}$).

   Each kilogram of baking soda costs $4.50. Sample reasoning: Since $\frac{4}{9}$ kilogram costs $2, $\frac{1}{9}$ kilogram costs one fourth of $2 (or $0.50). One kilogram is nine $\frac{9}{9}$ kilogram, so it would cost 9 times $0.50, which is $4.50.



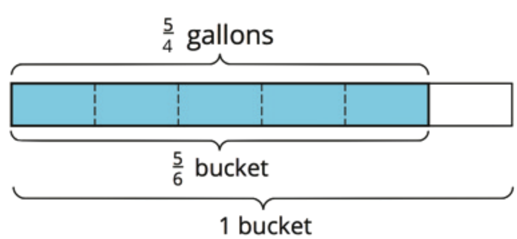
3. Multiplication equation: $1\frac{1}{5} \cdot ? = 3$ (or $? \cdot 1\frac{1}{5} = 3$), division equation: $3 \div 1\frac{1}{5} = ?$ (or $3 \div ? = 1\frac{1}{5}$).

   $\frac{5}{2}$ (or $2\frac{1}{2}$) liters fill 1 bottle. Sample reasoning: $1\frac{1}{5}$ is $\frac{6}{5}$. If $\frac{6}{5}$ bottles contain 3 liters, then each $\frac{1}{5}$ bottle contains of $(3 \div 6$ liters, which is $\frac{1}{2}$ liter. This means 1 bottle, or $\frac{5}{5}$ bottle, contains 5 times $\frac{1}{2}$, which is $\frac{5}{2}$ (or $2\frac{1}{2}$ liters).



4. Multiplication equation: $\frac{5}{6} \cdot ? = \frac{5}{4}$ (or $? \cdot \frac{5}{6} = \frac{5}{4}$), division equation: $\frac{5}{4} \div \frac{5}{6} = ?$ (or $\frac{5}{4} \div ? = \frac{5}{6}$).

   $1\frac{1}{2}$ gallons fill the entire bucket. Sample reasoning: $\frac{5}{4}$ gallons fill $\frac{5}{6}$ bucket, so each $\frac{1}{4}$ gallon fills $\frac{1}{6}$ bucket. One whole bucket is $\frac{4}{4}$ bucket; it would take $(6 \cdot \frac{1}{4})$ or $\frac{6}{4}$ or $1\frac{1}{2}$ gallons to fill it.



Are You Ready for More?

The largest sandwich ever made weighed 5,440 pounds. If everyone on Earth shares the sandwich equally, how much would you get? What fraction of a regular sandwich does this represent?

Student Response

Answers vary. Sample response: Each person would get $\frac{17}{22{,}959{,}375}$ pound. This amount is $\frac{34}{22{,}959{,}375}$ of a regular sandwich. There are approximately 7,347,000,000 people on Earth, so one serving of the sandwich would be the total weight of 5,440 pounds divided by the number of servings:

$\frac{5{,}440}{7{,}347{,}000{,}000} = \frac{17}{22{,}959{,}375}$. If a regular sandwich weighs approximately a half pound, then one serving of the largest sandwich would be $\frac{17}{22{,}959{,}375} \div \frac{1}{2}$ (or $\frac{34}{22{,}959{,}375}$) of a regular sandwich.

Activity Synthesis

Arrange for groups that are assigned the same problem to present their visual displays near one another. Give students a few minutes to visit the displays and to see how others reasoned about the same two problems they chose. Before students begin a gallery walk, ask them to be prepared to share a couple of observations about how their strategies and diagrams are the same as or different from others'.

After the gallery walk, invite a couple of students to share their observations. Tell them that, in upcoming lessons, they will investigate other ways to reason efficiently about division problems.

Support for English Language Learners

Speaking, Representing: MLR8 Discussion Supports. Give students additional time to make sure that everyone in their group can explain the equations and diagrams they created. Prompt groups to rehearse what they will say when they share with another group. Rehearsing provides students with additional opportunities to speak and clarify their thinking. This will also help students improve the quality of their explanations during the whole-class discussion. *Design Principle(s): Optimize output (for explanation)*

9.4 Inventing Another Situation

Optional: 15 minutes

This open-ended activity gives students a chance to formulate their own scenarios for a division expression, find the value of that expression, and make sense of the value in the context of their scenario (MP2).

By now students will have seen a variety of situations in which a division means finding "how many groups of this in that?" or finding "how much in each group?" and can refer to these two interpretations of division to get started. If they are stuck, encourage them to consult the examples they have seen so far (e.g., the problems about the water container, the lengths of ropes, cleaning the highway, etc.)

As students work, notice the different types of problems students are writing, the range of attributes (length, volume, weight, etc.) involved, the different types of diagrams used, and the interpretation of division chosen (number of groups vs. size of a group). Select a couple of students who interpret the expression in different ways to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports

Launch

Arrange students in groups of 4. If desired, arrange students in groups of 4 in two dimensions. (Assign each student into a group, and then to a label within it, so that new groups—consisting of one student from each of the original groups—can be formed later.) Provide continued access to tools for creating a visual display.

Give students 4–5 minutes of quiet think time to invent a scenario and a question. Then, before they find the answer to the question, ask them to trade their scenarios with a person in their group, give each other feedback about how well the question fits the expression, and revise the question. When they are ready, ask them to write their question on the visual display.

Support for English Language Learners

Speaking, Representing: MLR8 Discussion Supports. Use this routine to support peer feedback. Display the following sentence frames: "This situation and equation do/ do not match the equation because . . .", and "To improve your situation (or equation), you could . . . ." Encourage students to consider what details are important include and how they will explain their reasoning using mathematical language.
Design Principle(s): Support sense-making

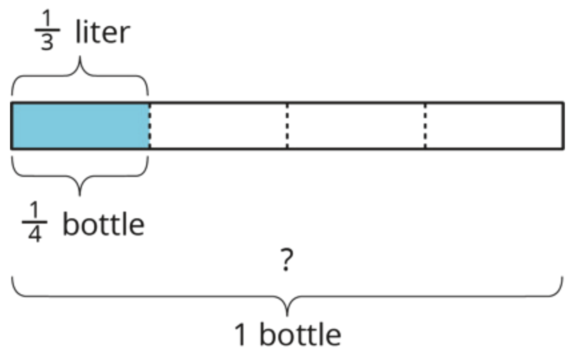
Student Task Statement

1. Think of a situation with a question that can be represented by $\frac{1}{3} \div \frac{1}{4} = ?$. Describe the situation and the question.
2. Trade descriptions with a partner.
    - Review each other's description and discuss whether each question matches the equation.
    - Revise your description based on the feedback from your partner.
3. Find the answer to your question. Explain or show your reasoning. If you get stuck, consider drawing a diagram.

Student Response

1. Answers vary. Possible answer: Lin poured $\frac{1}{3}$ liters of water into a bottle and it filled $\frac{1}{4}$ of the bottle. How much water fits into the bottle?
2. No answer is required.

3. $4 \cdot \frac{1}{3} = \frac{4}{3}$, so $\frac{4}{3}$ liters.



Activity Synthesis

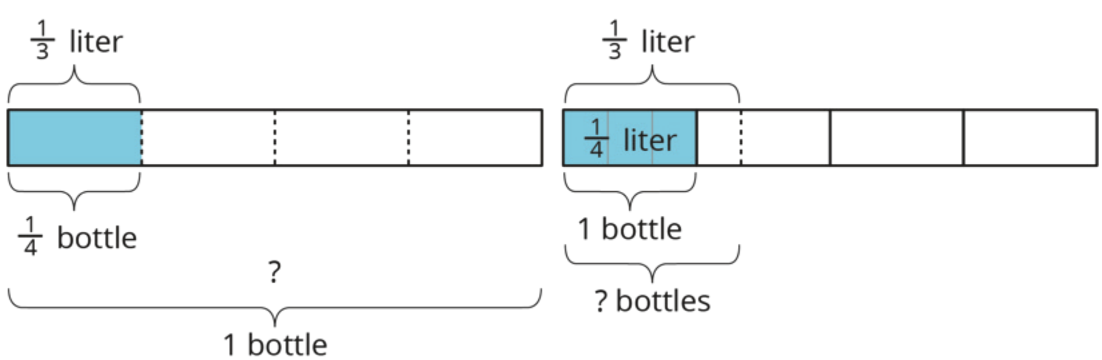
Consider rearranging students into new groups of 4 so they can share their questions and solutions in small groups.

If possible, select one person who came up with a "$\frac{1}{4}$ of what number is $\frac{1}{3}$?" type of question, and another who wrote a "how many $\frac{1}{4}$s in $\frac{1}{3}$?" type of question to share their questions and explain their solution paths to the class.

If needed, use these examples to illustrate two different interpretations of the division statement.

- First interpretation: If $\frac{1}{3}$ liter fills $\frac{1}{4}$ of a bottle, how many liters fill 1 bottle?
- Second interpretation: If 1 bottle contains $\frac{1}{4}$ liter of water, how many bottles can be filled with $\frac{1}{3}$ liter of water?

In this case, students will likely find the first interpretation of division easier to represent and to solve using a diagram.



The diagram on the left shows the content of $\frac{1}{4}$ of a bottle, which is $\frac{1}{3}$ liter, being multiplied by 4 to get the content of 1 bottle, which is $\frac{4}{3}$ liters. The diagram on the right shows that 1 bottle contains $\frac{1}{4}$ liter, or 3 parts of $\frac{1}{12}$ liter. In $\frac{1}{3}$ liter there are 4 parts of $\frac{1}{12}$, so $\frac{1}{3}$ liter fills $1\frac{1}{3}$ bottles.

Lesson Synthesis

In this lesson, we solved problems that involved finding the amount in one group. Present this problem and give students a minute of quiet think time: How long is a whole trip if $\frac{2}{3}$ of a trip is $\frac{4}{5}$ mile? Then, discuss some of the following questions.

- "What is the 'one group' we are interested in here?" (One full trip.)
- "Do we know the number of groups or the fraction of a group?" (Yes, $\frac{2}{3}$ of a group.)
- "What multiplication equation and division equations can we write to represent this situation?" ($\frac{2}{3} \boldsymbol{\cdot} ? = \frac{4}{5}$ and $\frac{4}{5} \div \frac{2}{3} = ?$)
- "How can we interpret $\frac{2}{3} \boldsymbol{\cdot} ? = \frac{4}{5}$? in this context?" (We can think of it as asking "$\frac{2}{3}$ of what distance is $\frac{4}{5}$ mile?")
- "How might we set up a tape diagram to help us answer the question?" (The tape should show two rectangles to represent $\frac{2}{3}$ of a trip and $\frac{4}{5}$ mile, which allows us to see $\frac{1}{3}$ of the trip as $\frac{2}{5}$ mile and the whole trip as $3 \boldsymbol{\cdot} \frac{2}{5}$ or $\frac{6}{5}$ miles.)

Explain to students that sometimes it is not always obvious whether a division situation involves finding the number of groups or the size of 1 group. There may be two 1 wholes to keep track of and two possible questions that could be asked. We need to analyze the situation carefully to reason correctly.

9.5 Refilling a Soap Dispenser

Cool Down: 5 minutes

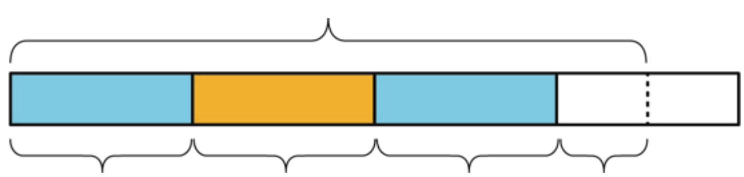
Addressing

- 6.NS.A.1

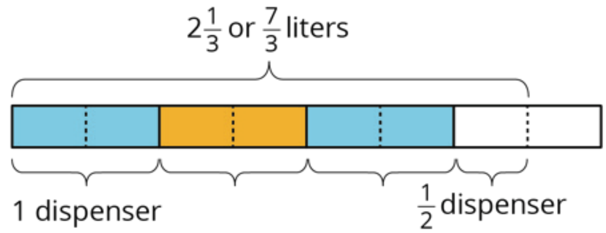
Student Task Statement

Noah fills a soap dispenser from a big bottle that contains $2\frac{1}{3}$ liters of liquid soap. That amount of soap will fill $3\frac{1}{2}$ dispensers. How many liters of soap fit into one dispenser?

Use the diagram below to answer the question. Label all relevant parts of the diagram.



Student Response



He can refill the dispenser $3\frac{1}{2}$ times, which is $\frac{7}{2}$ times. So each time, he uses $\frac{2}{3}$ liters of soap.

Student Lesson Summary

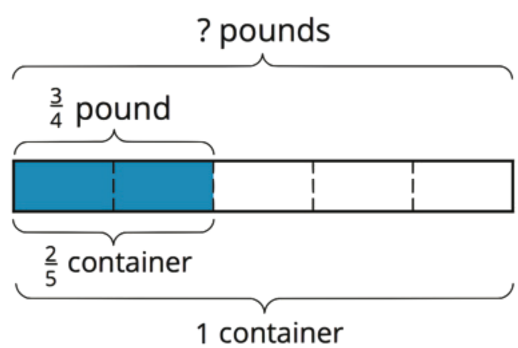
Sometimes we have to think carefully about how to solve a problem that involves multiplication and division. Diagrams and equations can help us.

For example, $\frac{3}{4}$ of a pound of rice fills $\frac{2}{5}$ of a container. There are two whole amounts to keep track of here: 1 whole pound and 1 whole container. The equations we write and the diagram we draw depend on what question we are trying to answer.

- How many pounds fill 1 container?

$$\frac{2}{5} \cdot ? = \frac{3}{4}$$

$$\frac{3}{4} \div \frac{2}{5} = ?$$

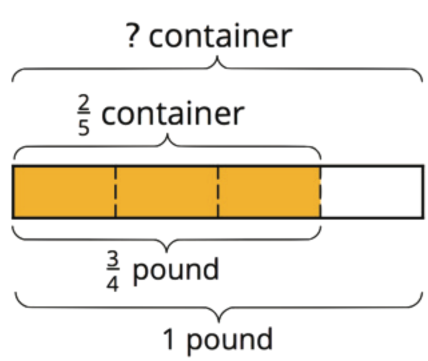


If $\frac{2}{5}$ of a container is filled with $\frac{3}{4}$ pound, then $\frac{1}{5}$ of a container is filled with half of $\frac{3}{4}$, or $\frac{3}{8}$, pound. One whole container then has $5 \cdot \frac{3}{8}$ (or $\frac{15}{8}$) pounds.

- What fraction of a container does 1 pound fill?

$$\frac{3}{4} \cdot ? = \frac{2}{5}$$

$$\frac{2}{5} \div \frac{3}{4} = ?$$



If $\frac{3}{4}$ pound fills $\frac{2}{5}$ of a container, then $\frac{1}{4}$ pound fills a third of $\frac{2}{5}$, or $\frac{2}{15}$, of a container. One whole pound then fills $4 \cdot \frac{2}{15}$ (or $\frac{8}{15}$) of a container.

Lesson 9 Practice Problems

Problem 1

Statement

A group of friends is sharing $2\frac{1}{2}$ pounds of berries.

a. If each friend received $\frac{5}{4}$ of a pound of berries, how many friends are sharing the berries?

b. If 5 friends are sharing the berries, how many pounds of berries does each friend receive?

Solution

a. 2 friends ($2\frac{1}{2} \div \frac{5}{4} = 2$)

b. $\frac{1}{2}$ pound ($2\frac{1}{2} \div 5 = \frac{1}{2}$)

Problem 2

Statement

$\frac{2}{5}$ kilogram of soil fills $\frac{1}{3}$ of a container. Can 1 kilogram of soil fit in the container? Explain or show your reasoning.

Solution

Yes. Reasonings vary. Sample reasonings:

- The container can fit $3 \cdot \frac{2}{5}$ or $\frac{6}{5}$ kilograms of soil, which is greater than 1 kilogram.
- $\frac{2}{5} \div \frac{1}{3}$ gives a quotient greater than 1, which means that the container can fit more than 1 kilogram.

Problem 3

Statement

After raining for $\frac{3}{4}$ of an hour, a rain gauge is $\frac{2}{5}$ filled. If it continues to rain at that rate for 15 more minutes, what fraction of the rain gauge will be filled?

a. To help answer this question, Diego wrote the equation $\frac{3}{4} \div \frac{2}{5} = ?$. Explain why this equation does *not* represent the situation.

b. Write a multiplication equation and a division equation that do represent the situation.

Solution

a. Explanations vary. Sample response:

- If Diego were correct, then the gauge would be overflowing after an hour of rain, because $\frac{3}{4} \div \frac{2}{5}$ is greater than 1.
- Less than half of the gauge has been filled after more than half an hour has gone by.

b. $\frac{3}{4} \cdot ? = \frac{2}{5}$, $\frac{2}{5} \div \frac{3}{4} = ?$

Problem 4

Statement

3 tickets to the museum cost $12.75. At this rate, what is the cost of:

a. 1 ticket?

b. 5 tickets?

Solution

a. $4.25 (because 12.75 divided by 3 is 4.25).

b. $21.25 (because the unit rate is 4.25, and $(4.25) \cdot 5 = 21.25$).

(From Unit 2, Lesson 8.)

Problem 5

Statement

Elena went 60 meters in 15 seconds. Noah went 50 meters in 10 seconds. Elena and Noah both moved at a constant speed.

a. How far did Elena go in 1 second?

b. How far did Noah go in 1 second?

c. Who went faster? Explain or show your reasoning.

Solution

a. Elena went 4 meters in 1 second because $60 \div 15 = 4$.

b. Noah went 5 meters in 1 second because $50 \div 10 = 5$.

c. Noah went faster; he ran more distance in 1 second. Once the distances traveled in 1 second are computed, they can be compared directly.

(From Unit 2, Lesson 9.)

Problem 6

Statement

The first row in the table shows a recipe for 1 batch of trail mix. Complete the table to show recipes for 2, 3, and 4 batches of the same type of trail mix.

number of batches	cups of cereal	cups of almonds	cups of raisins
1	2	$\frac{1}{3}$	$\frac{1}{4}$
2			
3			
4			

Solution

number of batches	cups of cereal	cups of almonds	cups of raisins
1	2	$\frac{1}{3}$	$\frac{1}{4}$
2	4	$\frac{2}{3}$	$\frac{1}{2}$
3	6	1 or equivalent	$\frac{3}{4}$
4	8	$\frac{4}{3}$	1 or equivalent

(From Unit 2, Lesson 11.)

Section: Algorithm for Fraction Division

Lesson 10: Dividing by Unit and Non-Unit Fractions

Goals

- Interpret and critique explanations (in spoken and written language, as well as in other representations) of how to divide by a fraction.
- Use a tape diagram to represent dividing by a non-unit fraction $\frac{a}{b}$ and explain (orally) why this produces the same result as multiplying the number by b and dividing by a.
- Use a tape diagram to represent dividing by a unit fraction $\frac{1}{b}$ and explain (orally and in writing) why this is the same as multiplying by b.

Learning Targets

- I can divide a number by a non-unit fraction $\frac{a}{b}$ by reasoning with the numerator and denominator, which are whole numbers.
- I can divide a number by a unit fraction $\frac{1}{b}$ by reasoning with the denominator, which is a whole number.

Lesson Narrative

This is the first of two lessons in which students pull together the threads of reasoning from the previous six lessons to develop a general algorithm for dividing fractions. Students start by recalling the idea from grade 5 that dividing by a unit fraction has the same outcome as multiplying by the reciprocal of that unit fraction. They use tape diagrams to verify this.

Next, they use the same diagrams to look at the effects of dividing by non-unit fractions. Through repetition, they notice a pattern in the steps of their reasoning (MP8) and structure in the visual representation of these steps (MP7). Students see that division by a non-unit fraction can be thought of as having two steps: dividing by the unit fraction, and then dividing the result by the numerator of the fraction. In other words, to divide by $\frac{2}{5}$ is equivalent to dividing by $\frac{1}{5}$, and then again by 2. Because dividing by a unit fraction $\frac{1}{5}$ is equivalent to multiplying by 5, we can evaluate division by $\frac{2}{5}$ by multiplying by 5 and dividing by 2.

Alignments

Building On

- 5.NF.B: Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let’s look for patterns when we divide by a fraction.

10.1 Dividing by a Whole Number

Warm Up: 10 minutes
In this warm-up, students use tape diagrams to revisit the idea that dividing by a whole number is equivalent to multiplying by a unit fraction. Though this is a review of a grade 5 expectation, connecting the division problems to diagrams allows students to see the equivalence in the related division and multiplication problems. It also prepares students to extend the reasoning and representations used here to division by non-unit fractions later.

Building On

- 5.NF.B

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Ask one person in each group to draw diagrams and answer the questions for Partner A, and the other to take on the questions for Partner B. Give students a few minutes of quiet time to complete the first two questions, and then ask them to collaborate on the last two.

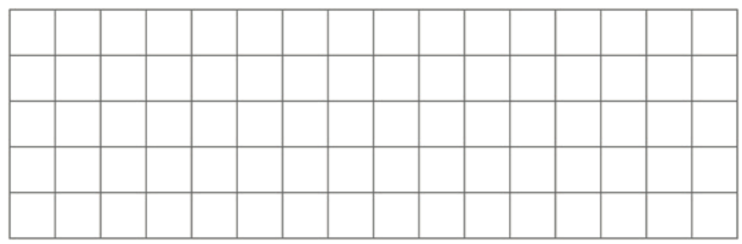
Student Task Statement

Work with a partner. One person solves the problems labeled “Partner A” and the other person solves those labeled “Partner B.” Write an equation for each question. If you get stuck, consider drawing a diagram.

1. Partner A:

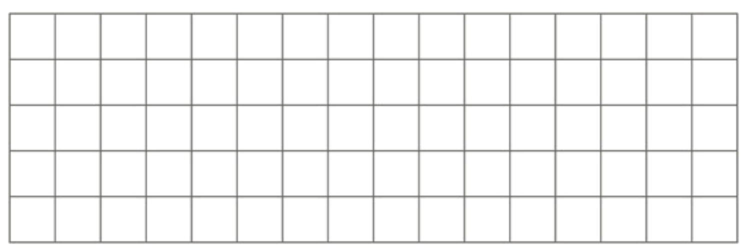
   How many 3s are in 12?

   Division equation:

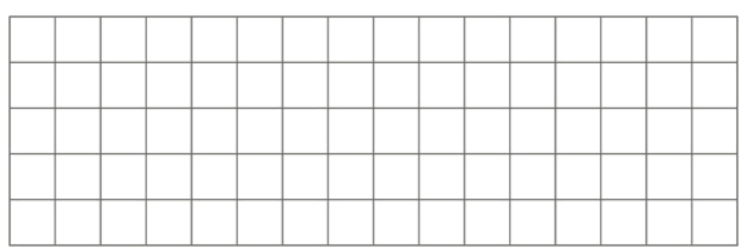
   How many 4s are in 12?

   Division equation:

   How many 6s are in 12?

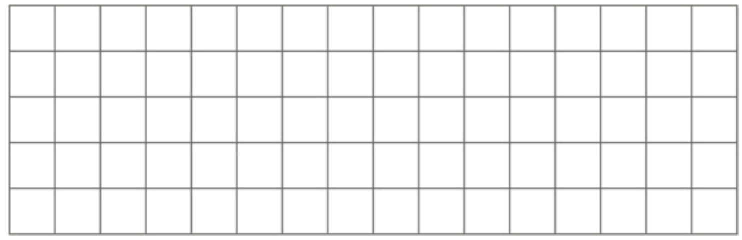
   Division equation:

   Partner B:

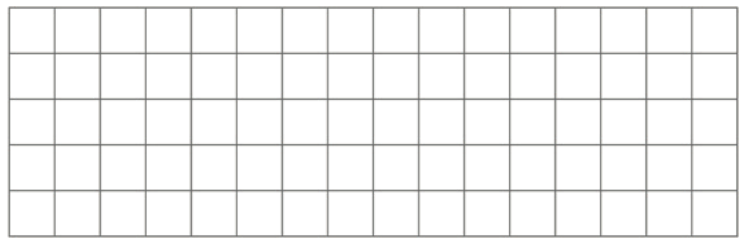
What is 12 groups of $\frac{1}{3}$?

Multiplication equation:



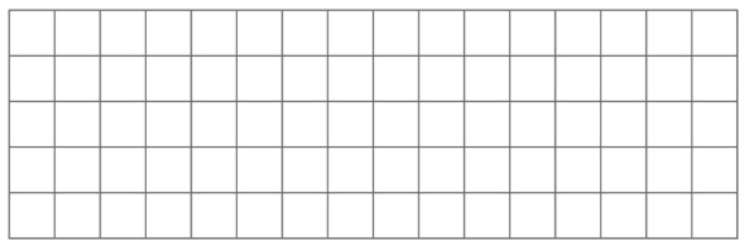
What is 12 groups of $\frac{1}{4}$?

Multiplication equation:



What is 12 groups of $\frac{1}{6}$?

Multiplication equation:

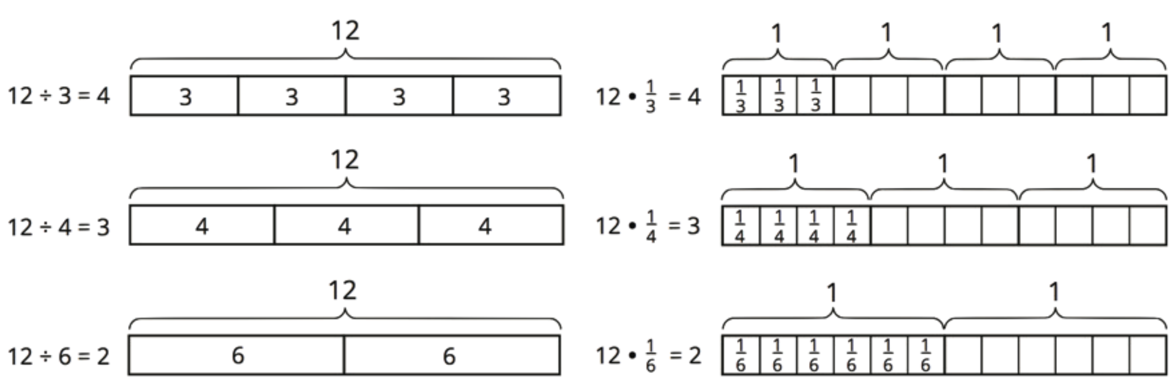


2. What do you notice about the diagrams and equations? Discuss with your partner.
3. Complete this sentence based on what you noticed:

   Dividing by a whole number a produces the same result as multiplying by ________.

Student Response

1.

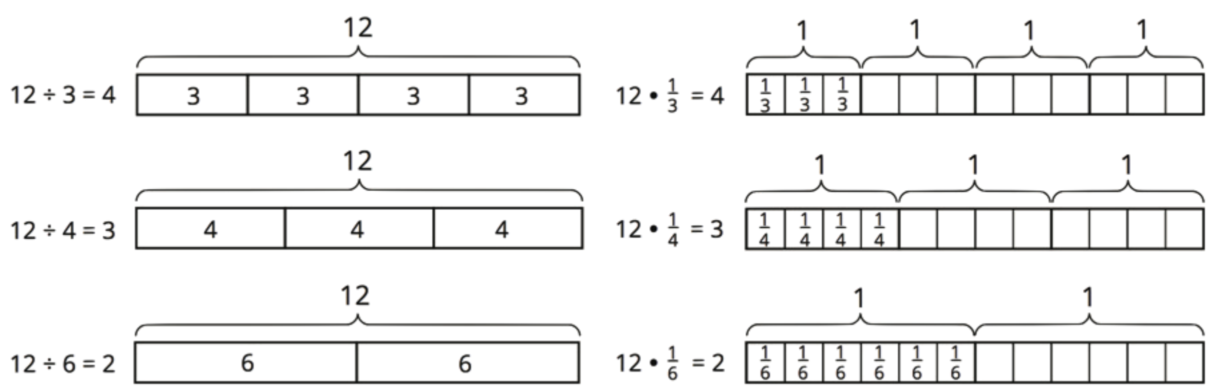


2. Answers vary. Sample responses:
    - The quotients in the division equations have the same value as the products in the multiplication equations.
    - Both sets of problems use the number 12. The division problems have divisors 3, 4, and 6, and the multiplication problems have a factor that is the reciprocal of those numbers ($\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$).
    - Each pair of diagrams are divided into the same number of major parts but they show different information.
    - Dividing by a whole number gives the same result as multiplying by its reciprocal.
3. Dividing by a whole number a produces the same result as multiplying by $\frac{1}{a}$.

Activity Synthesis

Invite a couple of students to share their observations about their group's diagrams and answers. Students should notice that the answers for the three division problems match those for the multiplication ones, even though the questions were not the same and their diagrams show groups of different sizes. Ask a few students to share their response to the last question.

Consider displaying the following image to reinforce the idea that dividing by a whole number has the same effect as multiplying by the reciprocal of that number.



10.2 Dividing by Unit Fractions

15 minutes

In this activity, students use tape diagrams and the meanings of division to divide a number by unit fractions. They do this as a first step toward generalizing the reasoning for dividing any two fractions. By reasoning repeatedly and noticing a pattern (MP8), students arrive at the conclusion that $a \div \frac{1}{b}$ is equivalent to $a \cdot b$.

Later, students put together those results to see that dividing by a fraction $\frac{c}{d}$ is equivalent to dividing by c and multiplying by d.

As students work, notice those who are able to transfer the reasoning in the first two problems to subsequent problems without the help of diagrams. Select several students to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 5–6 minutes of quiet work time and then time to share their work with a partner. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

For classes using the digital materials, an applet from an earlier lesson can also be used to represent the division problems here,

Video 'Fraction Division Tool Demo' available here: https://player.vimeo.com/video/304136674.

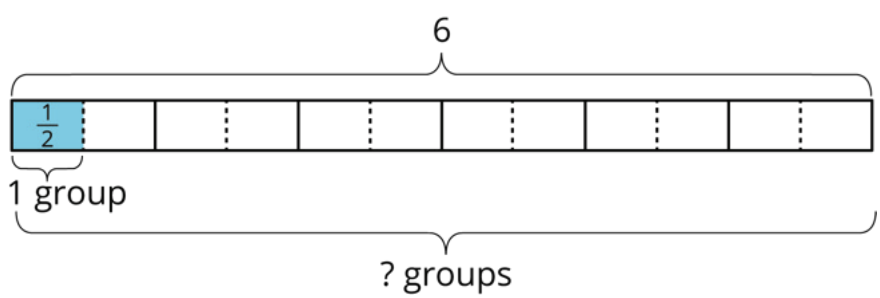
Anticipated Misconceptions

For some students, the structure in the division may still not be apparent by the time they get to problems such as $6 \div \frac{1}{25}$, and they may try to partition a section in a tape diagram into 25 parts. Ask them to study the diagrams as well as the answers to the previous questions carefully and to look for a pattern.

Student Task Statement

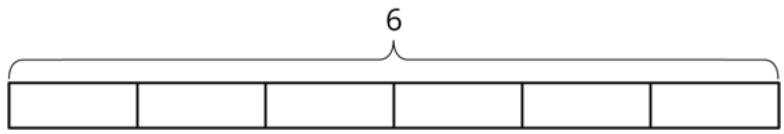
To find the value of $6 \div \frac{1}{2}$, Elena thought, "How many $\frac{1}{2}$s are in 6?" and then she drew this tape diagram. It shows 6 ones, with each one partitioned into 2 equal pieces.

$6 \div \frac{1}{2}$



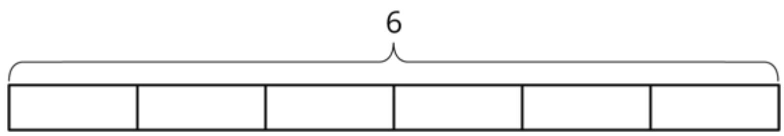
1. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression.

a. $6 \div \frac{1}{3}$



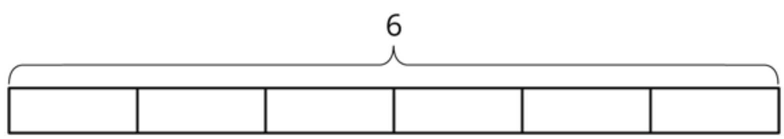
Value of the expression: ____________

b. $6 \div \frac{1}{4}$



Value of the expression: ____________

c. $6 \div \frac{1}{6}$



Value of the expression: ____________

2. Examine the expressions and answers more closely. Look for a pattern. How could you find how many halves, thirds, fourths, or sixths were in 6 without counting all of them? Explain your reasoning.

3. Use the pattern you noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.

   a. $6 \div \frac{1}{8}$

   b. $6 \div \frac{1}{10}$

   c. $6 \div \frac{1}{25}$

   d. $6 \div \frac{1}{b}$

4. Find the value of each expression.

   a. $8 \div \frac{1}{4}$

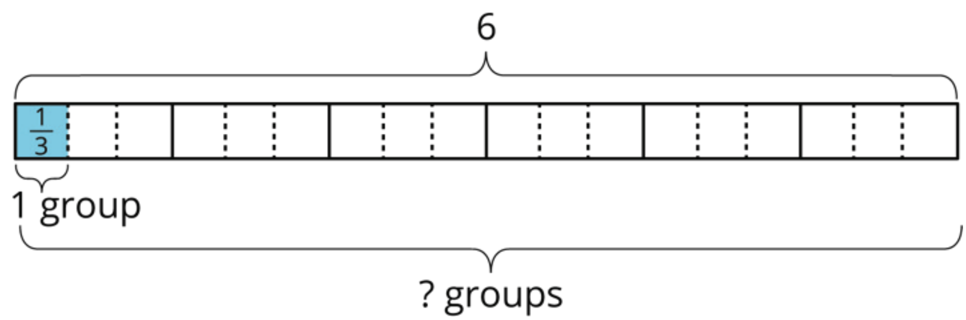
   b. $12 \div \frac{1}{5}$

   c. $a \div \frac{1}{2}$

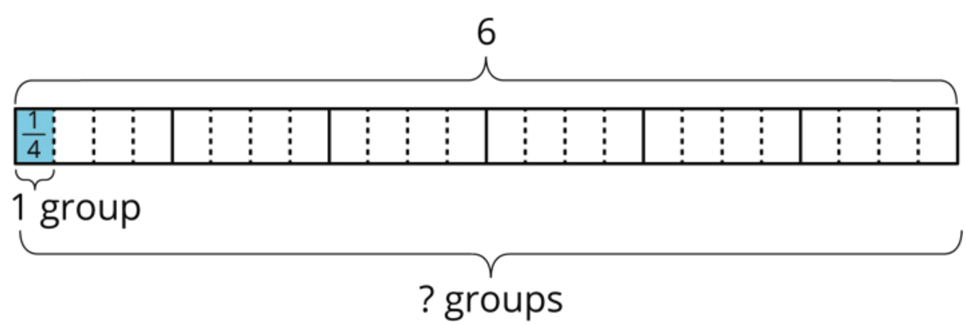
   d. $a \div \frac{1}{b}$

Student Response

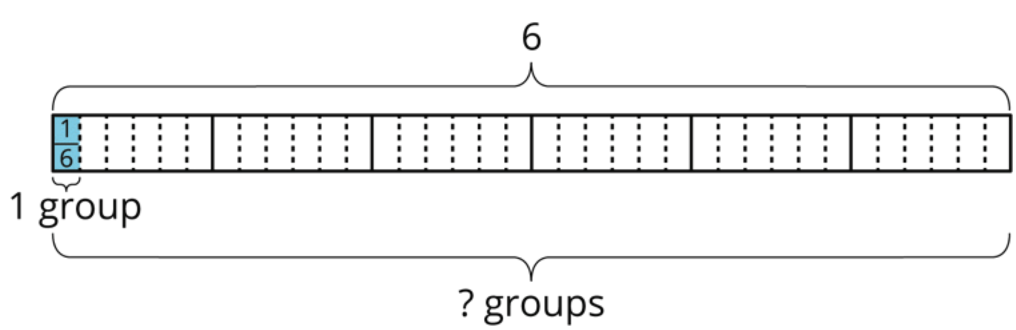
1. a. Value: 18



b. Value:24



c. Value: 36



2. Answers vary. Sample response: I divided each 1 whole in the tape diagram into the same number of pieces as in the number in the denominator. If the fraction was $\frac{1}{4}$, I broke each one into 4 pieces. To find how many pieces are in 6, I multiplied 6 by the denominator. If the fraction is $\frac{1}{4}$, there are 4 times as many pieces on my tape diagram as in the original tape diagram.

3. a. 48

   b. 60

   c. 150

   d. $6 \cdot b$

4. a. 32

b. 60

c. $a \cdot 2$

d. $a \cdot b$

Activity Synthesis

Select previously identified students to share their responses to the questions $6 \div \frac{1}{b}$ and $a \div \frac{1}{b}$. To highlight the connections between the diagram, division by a unit fraction, and multiplication by the reciprocal of the fraction ask students:

- "How is the division by a unit fraction depicted in the diagrams?" (The tape diagram is broken into equal parts. The unit fraction is the size of each part.)
- "Where in the diagrams do we see the multiplication?" (The multiplication shows the number of parts in each 1 whole.)
- "How are the two—the division by a unit fraction and the multiplication—related?" (When we divide a number by a unit fraction $\frac{1}{b}$, we end up with b times as many parts, so dividing by $\frac{1}{b}$ is the same as multiplying by b.)

If not already articulated by students, clarify that dividing a number by a unit fraction has the same result as multiplying by its reciprocal.

Discuss the usefulness and limits of diagrams. Ask:

- "How do we find the value of $1{,}000 \div \frac{1}{9}$ or $2 \div \frac{1}{250}$ using a diagram?" (By dividing each 1 whole into 9 parts or 250 parts)
- "Would you use diagrams to find these quotients? Why or why not?" (No, it is not practical.)
- "When working on the task, did you stop partitioning the tape diagrams at some point? If so, why?"
- "Why do we use diagrams? When can they be helpful?" (Diagrams can show us the structure or relationships between numbers and help us see the general process.)

If not mentioned by students, point out that for larger numbers, or smaller fractions, drawing a full diagram becomes increasingly cumbersome. Noticing the structure that is visible in the diagrams for easier cases allows us to use it for more difficult ones.

Support for English Language Learners

Conversing, Representing: MLR8 Discussion Supports. Use this routine to amplify mathematical language. After students share a response, invite them to repeat their reasoning using mathematical language relevant to the lesson (denominator, repeated, unit fraction, partitioned, reciprocal). For example, say, "Can you say that again, using the phrase 'reciprocal of the fraction'?" Consider inviting remaining students to chorally repeat these phrases to provide additional opportunities for all students to produce language verbally.
Design Principle(s): Support sense-making

10.3 Dividing by Non-unit Fractions

15 minutes
Here students continue to use tape diagrams to reason about division and extend their observations about unit fractions to non-unit fractions. Specifically, they explore how to represent the numerator of the fraction in the tape diagram and study its effect on the quotient. Students generalize their observations as operational steps and then as expressions, which they then use to solve other division problems.

As students work, notice those who effectively show the divisor on their diagrams, i.e., the multiplication by the denominator and division by the numerator, as well as those who could explain why the steps make sense.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

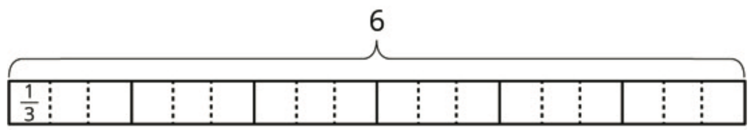
Keep students in groups of 2. Give students 5–7 minutes of quiet work time and then time to share their responses with their partner. Provide continued access to colored pencils.

Anticipated Misconceptions

Some students may read the phrase “partition 1 section into 4 parts” in Elena's reasoning and focus on the value of each small part $\left(\frac{1}{4}\right)$ instead on how many parts are now shown. Similarly, they may take “making of 3 of these parts into one piece” to imply multiplying the $\frac{1}{4}$ by 3, instead of looking at how it changes the number of pieces. Explain that Elena's diagram& suggests that she interpreted $6 \div \frac{3}{4}$ as “how many $\frac{3}{4}$s are in 6?” which tells us that we are looking for the number of groups, rather than the value of each part in the diagram.

Student Task Statement

1. To find the value of $6 \div \frac{2}{3}$, Elena started by drawing a diagram the same way she did for $6 \div \frac{1}{3}$.

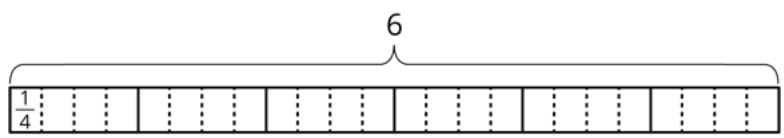


   a. Complete the diagram to show how many $\frac{2}{3}$s are in 6.

   b. Elena says, “To find $6 \div \frac{2}{3}$, I can just take the value of $6 \div \frac{1}{3}$ and then either multiply it by $\frac{1}{2}$ or divide it by 2.” Do you agree with her? Explain your reasoning.

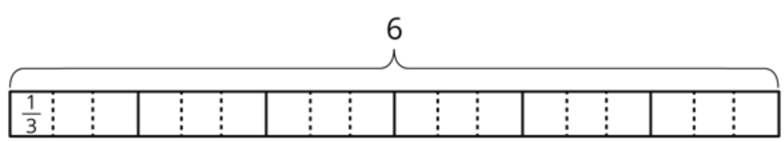
2. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression. Think about how you could find that value without counting all the pieces in your diagram.

   a. $6 \div \frac{3}{4}$



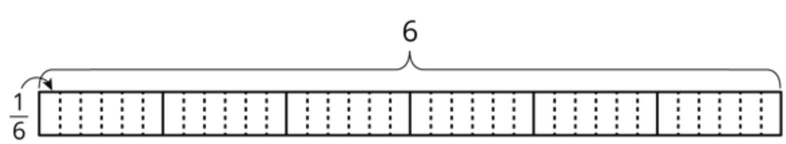
   Value of the expression:__________

   b. $6 \div \frac{4}{3}$



   Value of the expression:__________

   c. $6 \div \frac{4}{6}$



   Value of the expression:__________

3. Elena examined her diagrams and noticed that she always took the same two steps to show division by a fraction on a tape diagram. She said:

   “My first step was to divide each 1 whole into as many parts as the number in the denominator. So if the expression is $6 \div \frac{3}{4}$, I would break each 1 whole into 4 parts. Now I have 4 times as many parts.

My second step was to put a certain number of those parts into one group, and that number is the numerator of the divisor. So if the fraction is $\frac{3}{4}$, I would put 3 of the $\frac{1}{4}$s into one group. Then I could tell how many $\frac{3}{4}$s are in 6."

Which expression represents how many $\frac{3}{4}$s Elena would have after these two steps? Be prepared to explain your reasoning.

- $6 \div 4 \cdot 3$
- $6 \cdot 4 \div 3$
- $6 \div 4 \div 3$
- $6 \cdot 4 \cdot 3$

4. Use the pattern Elena noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.

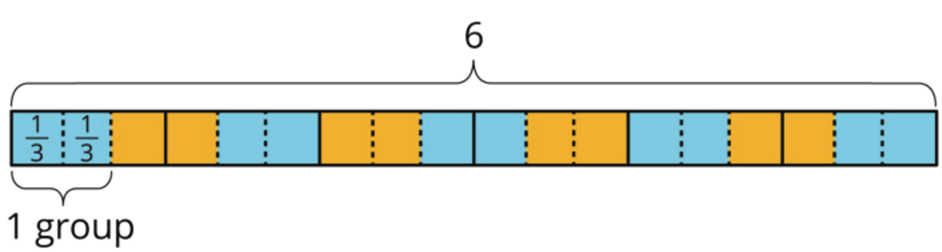
   a. $6 \div \frac{2}{7}$

   b. $6 \div \frac{3}{10}$

   c. $6 \div \frac{6}{25}$

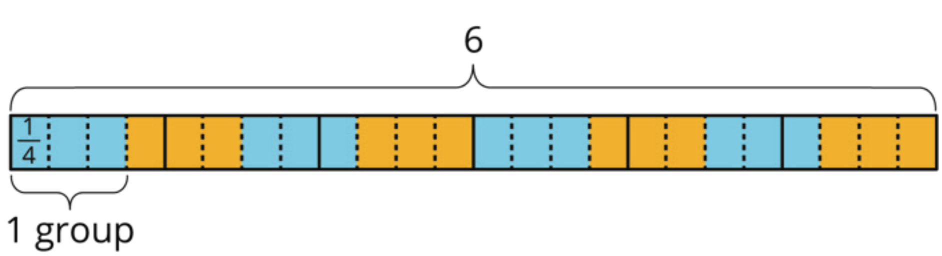
Student Response

1. a.

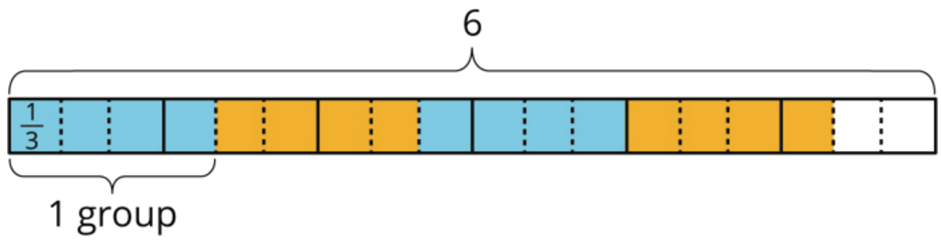


b. I agree. Sample reasoning: $\frac{2}{3}$ is two $\frac{1}{3}$s. If we put two $\frac{1}{3}$s in a group, we would have half as many pieces as we did for $6 \div \frac{1}{3}$.

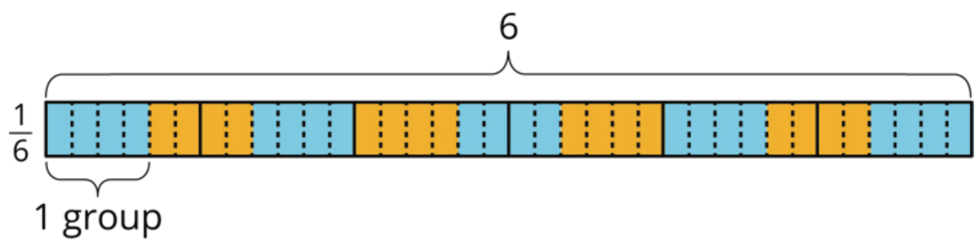
2. a. Value: 8



b. Value: $\frac{9}{2}$ (or $4\frac{1}{2}$)



c. Value: 9



3. $6 \cdot 4 \div 3$. Sample reasoning: Dividing each 1 whole into 4 parts makes 4 times as many pieces in the diagram. Making every 3 of those pieces into a group makes $\frac{1}{3}$ as many groups as there were pieces in the diagram. To get the number of groups, multiply the number of wholes—which is 6—by $\frac{1}{3}$ or divide it by 3.

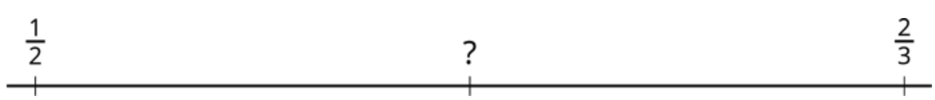
4. a. 21, because $6 \cdot 7 \div 2 = 21$

   b. 20, because $6 \cdot 10 \div 3 = 20$

   c. 25, because $6 \cdot 25 \div 6 = 25$

Are You Ready for More?

Find the missing value.

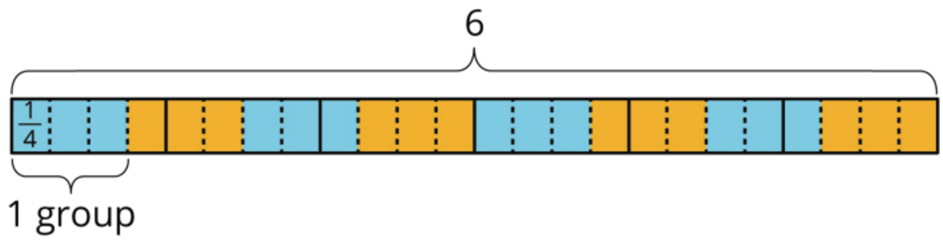


Student Response

$\frac{7}{12}$

Activity Synthesis

Select a few students to share their diagrams and explanations about why Elena's reasoning and method work. Display the following tape diagram for $6 \div \frac{3}{4}$, if needed. Ask students to point out where in the diagram the two steps are visible.



To involve more students in the conversation, consider asking questions such as:

- "Who can restate ___'s reasoning in different words?"
- "Did anyone think about the division the same way but would explain it differently?"
- "Does anyone want to add an observation to the way ____ reasoned about the division?"
- "Do you agree or disagree? Why?"

Highlight that dividing a number c by a fraction $\frac{a}{b}$ has the same result as multiplying by b, then dividing by a (or multiplying by $\frac{1}{a}$).

Support for English Language Learners

Writing, Conversing: MLR3 Clarify, Critique, Correct. Display the following statement: "The value of $6 \div \frac{2}{7}$ is 84 because $6 \div \frac{1}{7}$ is 42. Then I multiplied that by 2 to get 84." Keep students in groups of 2. Invite groups to consider the reasoning behind the statement, identify any mathematical or language errors they can see, and then write a correct explanation. As groups discuss, listen for the language students use to describe the reasoning they think contributed to this error. Call students' attention to the words and phrases they use that help them clarify their revised statements. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Support sense-making, Cultivate conversation

Lesson Synthesis

In this lesson, we reasoned about fraction division using tape diagrams. We traced the steps in our reasoning and analyzed the outcomes.

- "What did we notice about the result of dividing a number by a unit fraction? Can you explain with an example?" (It has the same outcome as multiplying by the denominator of the fraction.)
- "What observations did we make when dividing a number by a non-unit fraction? Can you explain with an example?" (It has the same outcome as multiplying by the denominator of the fraction and dividing by the numerator, and dividing by the numerator is the same as multiplying by its reciprocal.)

- “Suppose we are finding $5 \div \frac{7}{25}$. How might these observations help us find this quotient?” (We can multiply 5 by 25 and then divide by 7, instead of drawing a diagram and breaking each 1 into 25 parts, and so on.)

Emphasize that recognizing the patterns in how we reason about division of simpler fractions can help us divide other fractions more efficiently. Instead of drawing and reasoning with diagrams, which could be time consuming, we can follow the same series of steps to find quotients.

10.4 Dividing by $\frac{1}{3}$ and $\frac{3}{5}$

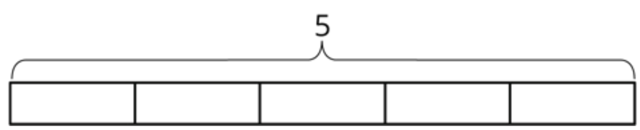
Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

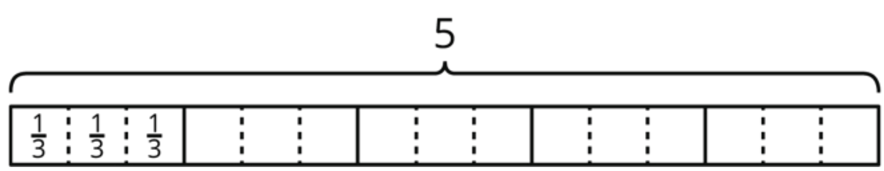
1. Explain or show how you could find $5 \div \frac{1}{3}$. You can use this diagram if it is helpful.



2. Find $12 \div \frac{3}{5}$. Try not to use a diagram, if possible. Show your reasoning.

Student Response

1. Answers vary. Sample reasoning: $5 \div \frac{1}{3}$ can mean “How many $\frac{1}{3}$s (thirds) are in 5?” There are 3 thirds in 1, so in 5, there are 5 times as many thirds. Five times as many is $5 \cdot 3$, so there are 15 thirds in 5.

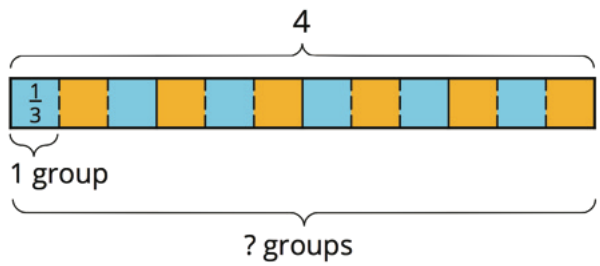


2. 20. Sample reasoning: $12 \div \frac{3}{5} = 12 \cdot 5 \cdot \frac{1}{3} = 20$

Student Lesson Summary

To answer the question “How many $\frac{1}{3}$s are in 4?” or “What is $4 \div \frac{1}{3}$?”, we can reason that there are 3 thirds in 1, so there are $(4 \cdot 3)$ thirds in 4.

In other words, dividing 4 by $\frac{1}{3}$ has the same result as multiplying 4 by 3.

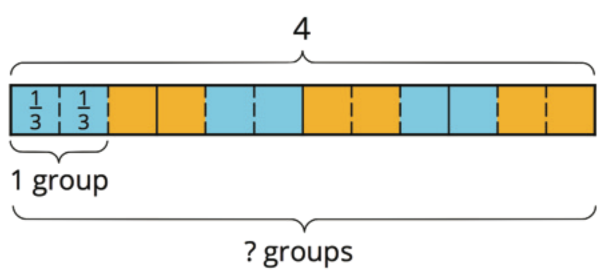


$$4 \div \frac{1}{3} = 4 \cdot 3$$

In general, dividing a number by a unit fraction $\frac{1}{b}$ is the same as multiplying the number by b, which is the **reciprocal** of $\frac{1}{b}$.

How can we reason about $4 \div \frac{2}{3}$?

We already know that there are $(4 \cdot 3)$ or 12 groups of $\frac{1}{3}$s in 4. To find how many $\frac{2}{3}$s are in 4, we need to put together every 2 of the $\frac{1}{3}$s into a group. Doing this results in half as many groups, which is 6 groups. In other words:



$$4 \div \frac{2}{3} = (4 \cdot 3) \div 2$$

or

$$4 \div \frac{2}{3} = (4 \cdot 3) \cdot \frac{1}{2}$$

In general, dividing a number by $\frac{a}{b}$, is the same as multiplying the number by b and then dividing by a, or multiplying the number by b and then by $\frac{1}{a}$.

Glossary

- reciprocal

Lesson 10 Practice Problems

Problem 1

Statement

Priya is sharing 24 apples equally with some friends. She uses division to determine how many people can have a share if each person gets a particular number of apples. For example, $24 \div 4 = 6$ means that if each person gets 4 apples, then 6 people can have apples. Here are some other calculations:

$24 \div 4 = 6$ $\quad$ $24 \div 2 = 12$ $\quad$ $24 \div 1 = 24$ $\quad$ $24 \div \frac{1}{2} = ?$

a. Priya thinks the "?" represents a number less than 24. Do you agree? Explain or show your reasoning.

b. In the case of $24 \div \frac{1}{2} = ?$, how many people can have apples?

Solution

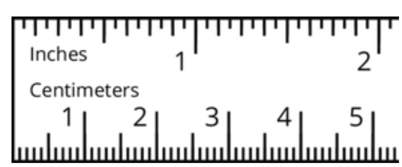
a. Disagree. Sample reasoning:
- As the amount for each person gets smaller, more people can have apples.
- There is a pattern in the numbers: when the number of apples per person is halved, the number of people doubles. Since 1 apple per person means 24 people can enjoy an apple, then $\frac{1}{2}$ apple per person means 48 (twice as many) people can enjoy some apple.

b. 48 apples

Problem 2

Statement

Here is a centimeter ruler.



a. Use the ruler to find $1 \div \frac{1}{10}$ and $4 \div \frac{1}{10}$.

b. What calculation did you do each time?

c. Use this pattern to find $18 \div \frac{1}{10}$.

d. Explain how you could find $4 \div \frac{2}{10}$ and $4 \div \frac{8}{10}$.

Solution

a. 10 and 40

b. Each time the dividend was multiplied by 10.

c. 180

d. Take the answer from $4 \div \frac{1}{10}$ and divide it by 2 or 8, getting 20 and 5, respectively.

Problem 3

Statement

Find each quotient.

a. $5 \div \frac{1}{10}$

b. $5 \div \frac{3}{10}$

c. $5 \div \frac{9}{10}$

Solution

a. 50

b. $\frac{50}{3}$ or $16\frac{2}{3}$

c. $\frac{50}{9}$ or $5\frac{5}{9}$

Problem 4

Statement

Use the fact that $2\frac{1}{2} \div \frac{1}{8} = 20$ to find $2\frac{1}{2} \div \frac{5}{8}$. Explain or show your reasoning.

Solution

Explanations vary. Sample response: There are 20 groups of $\frac{1}{8}$ in $2\frac{1}{2}$. If the size of each group is quintupled (from $\frac{1}{8}$ to $\frac{5}{8}$), then the number of groups will decrease by a factor of 5.

Problem 5

Statement

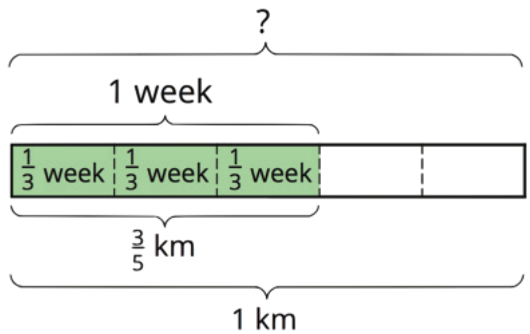
Consider the problem: It takes one week for a crew of workers to pave $\frac{3}{5}$ kilometer of a road. At that rate, how long will it take to pave 1 kilometer?

Write a multiplication equation and a division equation to represent the question. Then find the answer and show your reasoning.

Solution

a. $\frac{3}{5} \cdot ? = 1$ (or equivalent), $1 \div \frac{3}{5} = ?$

b. $1\frac{2}{3}$ weeks. Sample reasoning:



(From Unit 4, Lesson 9.)

Problem 6

Statement

A box contains $1\frac{3}{4}$ pounds of pancake mix. Jada used $\frac{7}{8}$ pound for a recipe. What fraction of the pancake mix in the box did she use? Explain or show your reasoning. Draw a diagram, if needed.

Solution

$\frac{1}{2}$. Sample explanations:

- $1\frac{3}{4}$ is $\frac{7}{4}$. $\frac{7}{8}$ is half of $\frac{7}{4}$.
- The question can be represented with: $? \cdot \frac{7}{4} = \frac{7}{8}$. The "?" has to be $\frac{1}{2}$ so that the product is $\frac{7}{8}$.

(From Unit 4, Lesson 7.)

Problem 7

Statement

Calculate each percentage mentally.

a. 25% of 400
b. 50% of 90
a. 75% of 200
b. 10% of 8,000
a. 5% of 20

Solution

a. 100

b. 45

c. 150

d. 800

e. 1

(From Unit 3, Lesson 14.)

Lesson 11: Using an Algorithm to Divide Fractions

Goals

- Coordinate (orally) different strategies for dividing by a fraction.
- Find the quotient of two fractions, and explain (orally, in writing, and using other representations) the solution method.
- Generalize a process for dividing a number by a fraction, and justify (orally) why this can be abstracted as $n \cdot \frac{b}{a}$.

Learning Targets

- I can describe and apply a rule to divide numbers by any fraction.

Lesson Narrative

In the previous lesson, students began to develop a general algorithm for dividing a fraction by a fraction. They complete that process in this lesson. Students calculate quotients using the steps they observed previously (i.e., to divide by $\frac{a}{b}$, we can multiply by b and divide by a), and compare them to quotients found by reasoning with a tape diagram. Through repeated reasoning, they notice that the two methods produce the same quotient and that the steps can be summed up as an algorithm: to divide by $\frac{a}{b}$, we multiply by $\frac{b}{a}$ (MP8). As students use the algorithm to divide different numbers (whole numbers and fractions), they begin to see its flexibility and efficiency.

Alignments

Building On

- 5.NF.B.4: Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

- MLR8: Discussion Supports
- Think Pair Share

Student Learning Goals

Let’s divide fractions using the rule we learned.

11.1 Multiplying Fractions

Warm Up: 5 minutes

This warm-up revisits multiplication of fractions from grade 5. Students will use this skill as they divide fractions throughout the lesson and the rest of the unit.

Building On

- 5.NF.B.4

Launch

Give students 2–3 minutes of quiet work time to complete the questions. Ask them to be prepared to explain their reasoning.

Student Task Statement

Evaluate each expression.

1. $\frac{2}{3} \cdot 27$
2. $\frac{1}{2} \cdot \frac{2}{3}$
3. $\frac{2}{9} \cdot \frac{3}{5}$
4. $\frac{27}{100} \cdot \frac{200}{9}$
5. $\left(1\frac{3}{4}\right) \cdot \frac{5}{7}$

Student Response

1. 18
2. $\frac{1}{3}$
3. $\frac{2}{15}$
4. 6
5. $\frac{5}{4}$ (or $1\frac{1}{4}$)

Activity Synthesis

Ask a student to share their answer and reasoning to each question, then ask if anyone disagrees. Invite students who disagree to share their explanations. If not mentioned in students' explanations, discuss strategies for multiplying fractions efficiently, how to multiply fractions in which a numerator and a denominator share at least one common factor, and how to multiply mixed numbers.

If students mention "canceling" a numerator and a denominator that share a common factor, demonstrate using the term "dividing" instead. For example, if a student suggests that in the second question ($\frac{1}{2} \cdot \frac{2}{3}$) the 2 in $\frac{1}{2}$ and the 2 in the $\frac{2}{3}$ "cancel out", rephrase the statement by saying that dividing the 2 in the numerator by the 2 in the denominator gives us 1, and multiplying by 1 does not change the other numerator or denominator.

11.2 Dividing a Fraction by a Fraction

15 minutes (there is a digital version of this activity)

This is the final task in a series that leads students toward a general procedure for dividing fractions. Students verify previous observations about the steps for dividing non-unit fractions (namely, multiplying by the denominator and dividing by the numerator) and contrast the results with those found using diagrams. They then generalize these steps as an algorithm and apply it to answer other division questions.

As students discuss in their groups, listen to their observations and explanations. Select students with clear explanations to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 5–7 minutes of quiet think time and 2–3 minutes to share their response with their partner. Provide access to colored pencils. Some students may find it helpful to identify whole groups and partial groups on a tape diagram by coloring.

Students using the digital materials can use an applet to investigate division of fractions.

Support for Students with Disabilities

Representation: Internalize Comprehension. Provide additional examples and counterexamples to illustrate concepts. Students may benefit from additional examples of division problems that they can "test" to reinforce understanding of their conclusion about how to divide a number by any fraction.
Supports accessibility for: Visual-spatial processing; Organization

Support for English Language Learners

Conversing, Representing: MLR8 Discussion Supports. Provide the following sentence frames for students to use when they share their response with their partner: "I know there are ___ $\frac{1}{8}$s in $\frac{3}{4}$ because....", "I drew the diagram like this ___ because....", and "First, I ___ because....". This will help students produce statements that describe how to divide a number by any fraction.
Design Principle(s): Cultivate conversation; Support sense-making

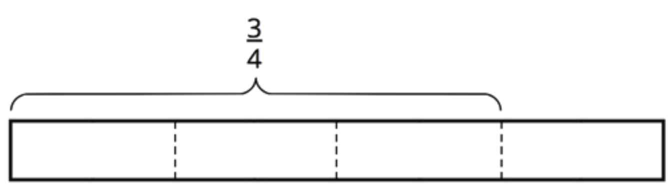
Student Task Statement

Work with a partner. One person works on the questions labeled "Partner A" and the other person works on those labeled "Partner B."

1. Partner A: Find the value of each expression by completing the diagram.

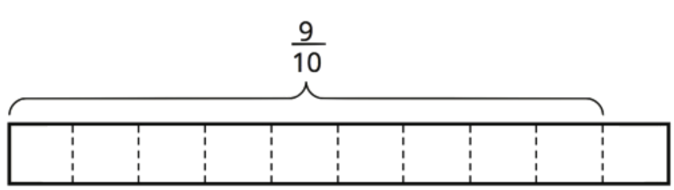
   a. $\frac{3}{4} \div \frac{1}{8}$

   How many $\frac{1}{8}$s in $\frac{3}{4}$?



   b. $\frac{9}{10} \div \frac{3}{5}$

   How many $\frac{3}{5}$s in $\frac{9}{10}$?



   Partner B:

   Elena said, "If I want to divide 4 by $\frac{2}{5}$, I can multiply 4 by 5 and then divide it by 2 or multiply it by $\frac{1}{2}$."

Find the value of each expression using the strategy Elena described.

a. $\frac{3}{4} \div \frac{1}{8}$

b. $\frac{9}{10} \div \frac{3}{5}$

2. What do you notice about the diagrams and expressions? Discuss with your partner.

3. Complete this sentence based on what you noticed:

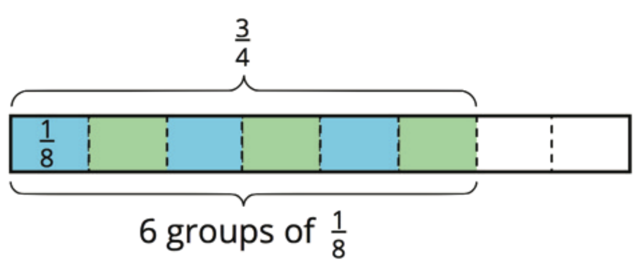
To divide a number n by a fraction $\frac{a}{b}$, we can multiply n by _______ and then divide the product by _______.

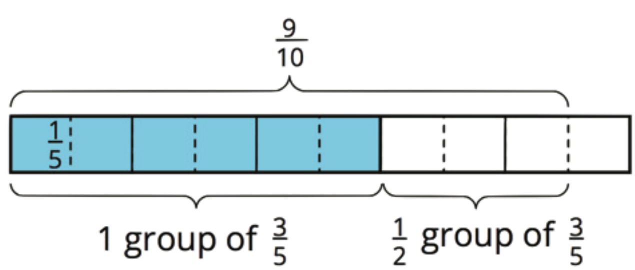
4. Select **all** the equations that represent the sentence you completed.

- $n \div \frac{a}{b} = n \cdot b \div a$
- $n \div \frac{a}{b} = n \cdot a \div b$
- $n \div \frac{a}{b} = n \cdot \frac{a}{b}$
- $n \div \frac{a}{b} = n \cdot \frac{b}{a}$

Student Response

1. Partner A

a. 

b. 

Partner B

a. $\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \cdot 8 = 6$

b. $\frac{9}{10} \div \frac{3}{5} = \frac{9}{10} \cdot 5 \div 3 = \frac{3}{2}$ or $1\frac{1}{2}$

2. Answers vary. Sample response: Multiplying by the denominator finds the total number of pieces in the diagram. Dividing by the numerator finds the number of groups. Sometimes you end up with a fraction of a group.

3. Multiply n by b and then divide the product by a.

4. $n \div \frac{a}{b} = n \cdot b \div a$ and $n \div \frac{a}{b} = n \cdot \frac{b}{a}$

Activity Synthesis

Invite a couple of students to share their conclusion about how to divide a number by any fraction. Then, review the sequence of reasoning that led us to this conclusion using both numerical examples and algebraic statements throughout. Remind students that in the past few activities, we learned that:

- Dividing by a whole number n is the same as multiplying by a unit fraction $\frac{1}{n}$ (e.g., dividing by 5 is the same as multiplying by $\frac{1}{5}$).
- Dividing by a unit fraction $\frac{1}{n}$ is the same as multiplying by a whole number n (e.g., dividing by $\frac{1}{7}$ is the same as multiplying by 7).
- Dividing by a fraction $\frac{a}{b}$ is the same as multiplying by a unit fraction $\frac{1}{a}$ and multiplying by a whole number b, which is the same as multiplying by $\frac{b}{a}$ (e.g., dividing by $\frac{5}{7}$ is the same as multiplying by $\frac{1}{5}$, and then by 7. Performing these two steps gives the same result as multiplying by $\frac{7}{5}$).

Finish the discussion by trying out the generalized method with other fractions such a $18 \div \frac{9}{7}$, or $\frac{15}{14} \div \frac{5}{2}$. Explain that although we now have a reliable and efficient method to divide any number by any fraction, sometimes it is still easier and more natural to think of the quotient in terms of a multiplication problem with a missing factor and to use diagrams to find the missing factor.

11.3 Using an Algorithm to Divide Fractions

15 minutes

This activity allows students to practice using the algorithm from earlier to solve division problems that involve a wider variety of fractions. Students can use any method of reasoning and are not expected to use the algorithm. As they encounter problems with less-friendly numbers, however, they notice that it becomes more challenging to use diagrams or other concrete strategies, and more efficient to use the algorithm. As they work through the activity, students choose their method.

Monitor the strategies students use and identify those with different strategies— including those who may not have used the algorithm—so they can share later.

Addressing

- 6.NS.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect
- Think Pair Share

Launch

Keep students in groups of 2. Give students 5–7 minutes of quiet work time, followed by 2–3 minutes to discuss their responses with a partner.

Support for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge. Provide students with access to blank tape diagrams. Encourage students to attempt more than one strategy for at least one of the problems.
Supports accessibility for: Visual-spatial processing; Organization

Student Task Statement

Calculate each quotient. Show your thinking and be prepared to explain your reasoning.

1. $\frac{8}{9} \div 4$
2. $\frac{3}{4} \div \frac{1}{2}$
3. $3\frac{1}{3} \div \frac{2}{9}$
4. $\frac{9}{2} \div \frac{3}{8}$
5. $6\frac{2}{5} \div 3$
6. After biking $5\frac{1}{2}$ miles, Jada has traveled $\frac{2}{3}$ of the length of her trip. How long (in miles) is the entire length of her trip? Write an equation to represent the situation, and then find the answer.

Student Response

1. $\frac{8}{9} \cdot \frac{1}{4} = \frac{2}{9}$
2. $\frac{3}{4} \cdot 2 = \frac{3}{2}$ or $1\frac{1}{2}$

3. $\frac{10}{3} \cdot 9 \div 2 = 15$

4. $\frac{9}{2} \cdot \frac{8}{3} = 12$

5. $\frac{32}{5} \cdot \frac{1}{3} = \frac{32}{15}$ or $2\frac{2}{15}$

6. Equation: $\frac{2}{3} \cdot ? = 5\frac{1}{2}$ (or $? \cdot \frac{2}{3} = 5\frac{1}{2}$, $5\frac{1}{2} \div \frac{2}{3} = ?$, $5\frac{1}{2} \div ? = \frac{2}{3}$). Answer: $8\frac{1}{4}$ miles. Sample reasoning: $5\frac{1}{2} \div \frac{2}{3} = \frac{11}{2} \cdot \frac{3}{2} = \frac{33}{4} = 8\frac{1}{4}$

Are You Ready for More?

Suppose you have a pint of grape juice and a pint of milk. You pour 1 tablespoon of the grape juice into the milk and mix it up. Then you pour 1 tablespoon of this mixture back into the grape juice. Which liquid is more contaminated?

Student Response

1 tablespoon is $\frac{1}{32}$ of a pint. This means that the pint of milk with the tablespoon of grape juice is $1\frac{1}{32}$ pints of mixed liquid and is $\frac{32}{33}$ of milk. When a tablespoon of the mixture is added back into the grape juice, there's less than a tablespoon of milk being added. This means that the pint of milk and tablespoon of grape juice is more contaminated by grape juice.

Activity Synthesis

Select previously identified students to share their responses. Sequence their presentations so that students with the more concrete strategies (e.g., drawing pictures) share before those with more abstract strategies. Students using the algorithm should share last. Find opportunities to connect the different methods. For example, point out where the multiplication by a denominator and division by a numerator are visible in a tape diagram.

Support for English Language Learners

Conversing, Representing: MLR7 Compare and Connect. As students consider the different strategies, invite them to make connections between the various representations and approaches. Ask, "What do each of the strategies have in common?", "How are the strategies different?" and "Which strategy is more efficient? Why?" Listen for and amplify observations that include mathematical language and reasoning.
Design Principle(s): Maximize meta-awareness; Optimize output (for comparison)

Lesson Synthesis

In this lesson, we noticed a more-efficient way to divide fractions. We found that to divide $\frac{3}{2}$ by $\frac{2}{5}$, for example, we can multiply $\frac{3}{2}$ by 5 and then by $\frac{1}{2}$, or simply multiply $\frac{3}{2}$ by $\frac{5}{2}$.

Let's see how this is the same or different than finding the quotient using tape diagrams. (If time permits, consider illustrating each diagram for all to see.)

- “Suppose we interpret $\frac{3}{2} \div \frac{2}{5}$ to mean ‘how many $\frac{2}{5}$ are in $\frac{3}{2}$?’ and use a tape diagram to find the answer. Where do we see the multiplication by 5 and by $\frac{1}{2}$ in the diagramming process?” (We draw a diagram to represent $\frac{3}{2}$ and draw equal parts, each with a value of $\frac{1}{5}$. We count how many groups of $\frac{2}{5}$ there are. Partitioning into fifths gives as 5 times as many parts. This is the multiplication by 5. Counting by two-fifths leads to half as many parts. This is the multiplication by $\frac{1}{2}$.)

- “Suppose we interpret $\frac{3}{2} \div \frac{2}{5}$ to mean ‘$\frac{2}{5}$ of what number is $\frac{3}{2}$?’ and use a tape diagram to find the answer. Where do we see the multiplication by 5 and by $\frac{1}{2}$ in the diagramming process?” (We draw a tape diagram to represent a whole group. We mark two-fifths of it as having a value of $\frac{3}{2}$. We divide that value by 2 (or multiply by $\frac{1}{2}$) to find one fifth of a group. To find out how much is in the whole group, we multiply by 5.)

Note that in both cases, there is a multiplication by $\frac{1}{2}$ and another multiplication by 5, which is the same as multiplication by $\frac{5}{2}$. Highlight that dividing by $\frac{a}{b}$ is equivalent to multiplying by b and then by $\frac{1}{a}$, or simply multiplying by $\frac{b}{a}$ (the reciprocal of $\frac{a}{b}$). This is true whether we interpreted the division problem in terms of finding the number of groups or finding the size of a group.

11.4 Watering A Fraction of House Plants

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

1. Find the value of $\frac{24}{25} \div \frac{4}{5}$. Show your reasoning.

2. If $\frac{4}{3}$ liters of water are enough to water $\frac{2}{5}$ of the plants in the house, how much water is necessary to water all the plants in the house? Write an equation to represent the situation, and then find the answer.

Student Response

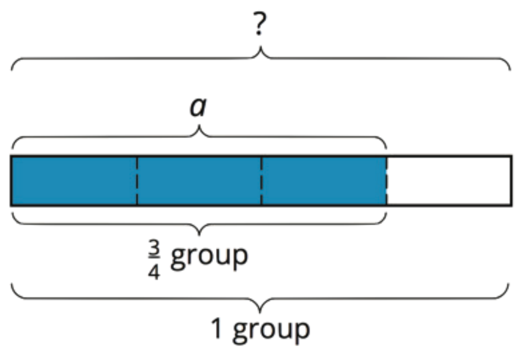
1. $\frac{6}{5}$, because $\frac{24}{25} \cdot \frac{5}{4} = \frac{6}{5}$.

2. Equations: $\frac{4}{3} \div \frac{2}{5} = ?$ or $\frac{4}{3} \div ? = \frac{2}{5}$ (or $\frac{2}{5} \cdot ? = \frac{4}{3}$ or $? \cdot \frac{2}{5} = \frac{4}{3}$).

   Solution: $\frac{4}{3} \cdot \frac{5}{2} = \frac{10}{3}$ (or $3\frac{1}{3}$)

Student Lesson Summary

The division $a \div \frac{3}{4} = ?$ is equivalent to $\frac{3}{4} \cdot ? = a$, so we can think of it as meaning "$\frac{3}{4}$ of what number is a?" and represent it with a diagram as shown. The length of the entire diagram represents the unknown number.



If $\frac{3}{4}$ of a number is a, then to find the number, we can first divide a by 3 to find $\frac{1}{4}$ of the number. Then we multiply the result by 4 to find the number.

The steps above can be written as: $a \div 3 \cdot 4$. Dividing by 3 is the same as multiplying by $\frac{1}{3}$, so we can also write the steps as: $a \cdot \frac{1}{3} \cdot 4$.

In other words: $a \div 3 \cdot 4 = a \cdot \frac{1}{3} \cdot 4$. And $a \cdot \frac{1}{3} \cdot 4 = a \cdot \frac{4}{3}$, so we can say that:

$$a \div \frac{3}{4} = a \cdot \frac{4}{3}$$

In general, dividing a number by a fraction $\frac{c}{d}$ is the same as multiplying the number by $\frac{d}{c}$, which is the reciprocal of the fraction.

Lesson 11 Practice Problems

Problem 1

Statement

Select **all** the statements that show correct reasoning for finding $\frac{14}{15} \div \frac{7}{5}$.

A. Multiplying $\frac{14}{15}$ by 5 and then by $\frac{1}{7}$.

B. Dividing $\frac{14}{15}$ by 5, and then multiplying by $\frac{1}{7}$.

C. Multiplying $\frac{14}{15}$ by 7, and then multiplying by $\frac{1}{5}$.

D. Multiplying $\frac{14}{15}$ by 5 and then dividing by 7.

E. Multiplying $\frac{15}{14}$ by 7 and then dividing by 5.

Solution

["A", "D"]

Problem 2

Statement

Clare said that $\frac{4}{3} \div \frac{5}{2}$ is $\frac{10}{3}$. She reasoned: $\frac{4}{3} \cdot 5 = \frac{20}{3}$ and $\frac{20}{3} \div 2 = \frac{10}{3}$.

Explain why Clare's answer and reasoning are incorrect. Find the correct quotient.

Solution

The correct quotient is $\frac{8}{15}$. Explanations vary. Sample response:

- Clare should have multiplied $\frac{4}{3}$ by 2 to find how many groups of $\frac{1}{2}$ are in $\frac{4}{3}$ and then divide the result by 5.
- Clare divided the fraction $\frac{4}{3}$ by the fraction $\frac{2}{5}$ instead of $\frac{5}{2}$.

Problem 3

Statement

Find the value of $\frac{15}{4} \div \frac{5}{8}$. Show your reasoning.

Solution

6. Reasoning varies. Sample reasoning: There are $\frac{15}{4} \cdot 8$ or 30 groups of $\frac{1}{8}$ in $\frac{15}{4}$. If five $\frac{1}{8}$s make a group, then the number of groups is $\frac{1}{5}$ of 30, which is 6.

Problem 4

Statement

Consider the problem: Kiran has $2\frac{3}{4}$ pounds of flour. When he divides the flour into equal-sized bags, he fills $4\frac{1}{8}$ bags. How many pounds fit in each bag?

Write a multiplication equation and a division equation to represent the question. Then, find the answer and show your reasoning.

Solution

$\frac{2}{3}$ pound per bag. Reasoning varies. Sample reasoning: $4\frac{1}{8} \cdot ? = 2\frac{3}{4}$ can be written as $2\frac{3}{4} \div 4\frac{1}{8} = ?$. Using the algorithm to divide: $2\frac{3}{4} \div 4\frac{1}{8} = \frac{11}{4} \div \frac{33}{8} = \frac{11}{4} \cdot \frac{8}{33} = \frac{2}{3}$.

Problem 5

Statement

Divide $4\frac{1}{2}$ by each of these unit fractions.

a. $\frac{1}{8}$

b. $\frac{1}{4}$

c. $\frac{1}{6}$

Solution

a. 36

b. 18

c. 27

(From Unit 4, Lesson 10.)

Problem 6

Statement

Consider the problem: After charging for $\frac{1}{3}$ of an hour, a phone is at $\frac{2}{5}$ of its full power. How long will it take the phone to charge completely?

Decide whether each equation can represent the situation.

a. $\frac{1}{3} \cdot ? = \frac{2}{5}$

b. $\frac{1}{3} \div \frac{2}{5} = ?$

c. $\frac{2}{5} \div \frac{1}{3} = ?$

d. $\frac{2}{5} \cdot ? = \frac{1}{3}$

Solution

a. No

b. Yes

c. No

d. Yes

(From Unit 4, Lesson 9.)

Problem 7

Statement

Elena and Noah are each filling a bucket with water. Noah's bucket is $\frac{2}{5}$ full and the water weighs $2\frac{1}{2}$ pounds. How much does Elena's water weigh if her bucket is full and her bucket is identical to Noah's?

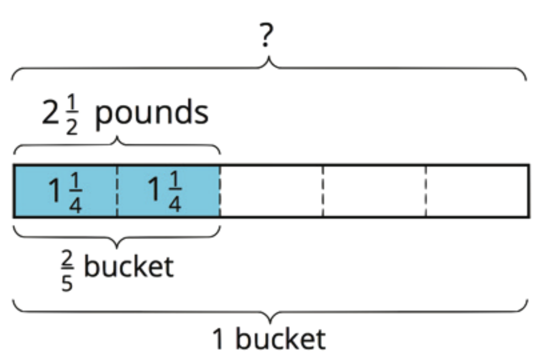
a. Write multiplication and division equations to represent the question.

b. Draw a diagram to show the relationship between the quantities and to find the answer.

Solution

a. $\frac{2}{5} \cdot ? = 2\frac{1}{2}$ (or equivalent), $2\frac{1}{2} \div \frac{2}{5} = ?$

b. $6\frac{1}{4}$ pounds. Sample diagram:



(From Unit 4, Lesson 8.)

Section: Fractions in Lengths, Areas, and Volumes

Lesson 12: Fractional Lengths

Goals

- Apply dividing by fractions to solve a problem about comparing lengths or measuring with non-standard units, and explain (orally and in writing) the solution method.
- Interpret a question (in written language) about multiplicative comparison, e.g., “How many times as long?” and write a division equation to represent it.

Learning Targets

- I can use division and multiplication to solve problems involving fractional lengths.

Lesson Narrative

This is the first of four lessons in which students use multiplication and division of fractions to solve geometric problems. In this lesson, they encounter problems involving fractional lengths. They use their understanding of the two interpretations of division—“how many groups?” and “how much in each group?”—to solve problems that involve multiplicative comparison (MP7).

In these geometry-themed lessons, students work with a wider range of fractions and mixed numbers, which gives them opportunities to choose their methods and tools for problem solving.

Alignments

Building On

- 3.OA.B.5: Apply properties of operations as strategies to multiply and divide.Students need not use formal terms for these properties. Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because 3/4 of 8/9 is 2/3. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR3: Clarify, Critique, Correct
- MLR4: Information Gap Cards
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports
- Notice and Wonder
- Number Talk

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the blackline master

Required Preparation

You will need the Info Gap: How Many Would It Take? blackline master for this lesson. Make 1 copy for every 4 students, and cut them up ahead of time.

Consider preparing the objects mentioned in the Info Gap: How Many Would It Take? activity for students to verify their answers. These objects are: $\frac{3}{4}$-inch square stickers, $1\frac{1}{4}$-inch binder clips, and $1\frac{3}{4}$-inch paper clips.

If the optional Comparing Paper Rolls activity is chosen, consider preparing the paper towel and toilet paper rolls as displayed in the image.

Student Learning Goals

Let’s solve problems about fractional lengths.

12.1 Number Talk: Multiplication Strategies

Warm Up: 5 minutes
This number talk encourages students to think carefully about the numbers in a computation problem and rely on what they know about structure, patterns, and properties of operations to mentally solve it. The reasoning helps students develop fluency and will support students in calculating products and quotients in upcoming work.

Building On

- 3.OA.B.5

Instructional Routines

- MLR8: Discussion Supports
- Number Talk

Launch

Give students 2 minutes of quiet think time and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

Support for Students with Disabilities

Representation: Internalize Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organization

Anticipated Misconceptions

When multiplying $19 \cdot 14$, students may only multiply the tens digits and multiply the ones digits and add them to get 136. Ask these students to estimate an answer for the problem and consider whether their answer makes sense.

Student Task Statement

Find the product mentally.

$19 \cdot 14$

Student Response

$19 \cdot 14 = 266$. Possible strategies:

- Think of 14 as $10 + 4$, multiply 19 by 10 and 19 by 4 separately, and add the two products: $19 \cdot 10 + 19 \cdot 4 = 266$
- Think of 19 as $20 - 1$, multiply 14 by 20 and 14 by 1 separately, and subtract the two products: $20 \cdot 14 - 1 \cdot 14 = 266$
- Picture a rectangle (an area diagram) with $10 + 9$ for one side length and $10 + 4$ for the other, partition the rectangle into 4 sub-rectangles, and find the sum of their areas: $(10 \cdot 10) + (10 \cdot 4) + (9 \cdot 10) + (9 \cdot 4) = 266$

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students if or how the factors in the problem impacted their strategy choice. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone solve the problem the same way but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

Support for English Language Learners

Speaking: MLR8 Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimize output (for explanation)

12.2 Info Gap: How Many Would It Take?

15 minutes
In this activity, students use division to solve problems involving lengths. No methods are specified for any of the questions, so students need to choose an appropriate strategy.

The info gap structure requires students to make sense of problems by determining what information is necessary, and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need (MP1). It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need (MP6).

Here is the text of the cards for reference and planning:

Info Gap: How Many Would It Take?

Problem Card 1

Jada is using square stickers to decorate the spine of a photo album. If she places the stickers in a line, side by side without gaps or overlaps, how many stickers will it take?

Info Gap: How Many Would It Take?

Data Card 1

- The photo album is $8\frac{1}{4}$ inches wide by $10\frac{1}{2}$ inches tall by $1\frac{1}{2}$ inches thick.
- The photo album's spine is $10\frac{1}{2}$ inches long.
- The side length of the stickers is $\frac{3}{4}$ inch.
- Jada places the stickers in one straight line along the length of the spine.

Info Gap: How Many Would It Take?

Problem Card 2

Tyler is using binder clips to decorate the edges of a poster. If he places the binder clips in a line, side by side without gaps or overlaps, how many binder clips will it take?

Info Gap: How Many Would It Take?

Data Card 2

- The poster is 16 inches wide by 20 inches tall.
- The binder clips are $1\frac{1}{4}$ inches wide.
- Tyler places the binder clips in two straight lines, one along the left side of the poster and one along the right side.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR4: Information Gap Cards

Launch

Arrange students in groups of 2. In each group, distribute the first problem card to one student and a data card to the other student. After debriefing on the first problem, distribute the cards for the second problem, in which students switch roles.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Display or provide students with a physical copy of the written directions. Check for understanding by inviting students to rephrase directions in their own words. Keep the display of directions visible throughout the activity.
Supports accessibility for: Memory; Organization

Support for English Language Learners

Conversing: This activity uses *MLR4 Information Gap* to give students a purpose for discussing information needed to solve problems involving lengths using division. Display questions or question starters for students who need a starting point such as: "Can you tell me . . . (specific piece of information)", and "Why do you need to know . . . (that piece of information)?"
Design Principle(s): Cultivate Conversation

Anticipated Misconceptions

Some students may not know what is meant by the "spine" of a book. Consider holding up a book and pointing out where its spine is.

If students struggle to represent the situations mathematically, suggest that they draw diagrams to represent the situations. They could start with sketches of the objects and then move toward other simpler or more abstract representations as they make better sense of the problems.

Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Student Response

1. It takes 14 stickers to make a line that is $10\frac{1}{2}$ inches long. $10\frac{1}{2} \div \frac{3}{4} = \frac{21}{2} \cdot \frac{4}{3} = 14$.
2. It takes 32 binder clips to make two lines that are each 20 inches long. $20 \div \frac{5}{4} = 20 \cdot \frac{4}{5} = 16$ and $16 \cdot 2 = 32$.

Are You Ready for More?

Lin has a work of art that is 14 inches by 20 inches. She wants to frame it with large paper clips laid end to end.

1. If each paper clip is $1\frac{3}{4}$ inch long, how many paper clips would she need? Show your reasoning and be sure to think about potential gaps and overlaps. Consider making a sketch that shows how the paper clips could be arranged.
2. How many paper clips are needed if the paper clips are spaced $\frac{1}{4}$ inch apart? Describe the arrangement of the paper clips at the corners of the frame.

Student Response

Answers vary. Sample response:

1. 38 paper clips.
    - One side of the paper is 20 inches long. $20 \div 1\frac{3}{4} = 11\frac{3}{7}$, so Lin can fit 11 paper clips along the side with a gap of $\frac{3}{4}$ inch since $11 \cdot 1\frac{3}{4} = 19\frac{1}{4}$. If the paper clips are centered along the 20-inch length, there will be $\frac{3}{8}$ inch of gap on either side.
    - The other side is 14 inches long. $14 \div 1\frac{3}{4} = 8$, so Lin can fit 8 paper clips along the side with no gap at all.
    - At each corner of the paper, two paper clips will meet. If the paper clip has a width that is about $\frac{3}{8}$ inch (to fit in the $\frac{3}{8}$ gap left by the 11 paper clips along the longer side), then there will be no gap or overlap.
    - Altogether, Lin will need 38 paper clips. $11 + 11 + 8 + 8 = 38$
2. 34 paper clips. If space is put between the paper clips, then fewer paper clips will be needed. If a gap of $\frac{1}{4}$ inch is between the paper clips, then each paper clip could have $\frac{1}{8}$ inch of space on either end so that the paper clip and its space takes up 2 inches. Then there are 7 paper clips along the sides of length 14 inches of the frame, and there are 10 paper clips along the sides of length 20 inches. There is a gap of $\frac{1}{8}$ inch between the end of the paper clip and the end of the frame.

Activity Synthesis

Select previously identified students to share their solutions and reasoning for each question. Start with students who used the more involved methods and move toward the more efficient ones. Record the approaches for all to see.

Highlight the connections between the different methods (e.g., between diagrams and equations, between a multiplication equation one student wrote and a division equation another person wrote for the same situation, etc.).

12.3 How Many Times as Tall or as Far?

15 minutes

In this activity, students practice performing division of fractions and using it to solve multiplicative comparison problems. The activity extends the work students have done earlier in the unit. In the Fractions of Ropes activity (in Lesson 7), students use diagrams to reason about how many times as long one rope is compared to another. Having had more experience in interpreting division situations and having learned a division algorithm, students can solve a wider range of problems that involve a greater variety of fractions. Minimal scaffolding is given here, so students need to decide what representations or strategies would be fruitful.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR5: Co-Craft Questions

Launch

Arrange students in groups of 4. Give students 1–2 minutes of quiet time to think about and draw a diagram for each question in the first problem. Ask them also to think about and write two equations that can represent the two questions. Afterward, give each group 2 minutes to compare their diagrams and equations.

Invite a couple of students to share their diagrams and equations. Ask the class whether they agree that the diagrams and equations represent the questions. Once students agree that the representations are appropriate, give them 8–10 minutes to complete the activity, either independently or collaboratively with their group. Encourage students to estimate the answer before calculating and to check their quotients using multiplication.

Support for English Language Learners

Reading, Conversing, Writing: MLR5 Co-Craft Questions. To help students make sense of the language of mathematical comparisons, start by displaying only the context for the first question ("A second-grade student is 4 feet tall. Her teacher is $5\frac{2}{3}$ feet tall."). Give students 1–2 minutes to write their own mathematical questions about the situation. Invite students to share their questions with the class, then reveal the activity's questions. Help students notice similarities and differences in how they phrased comparison questions and how the activity phrases these questions. For example, "Who's taller?" versus "How many times as tall . . . ." This will build student understanding of the language of mathematical comparisons and help ensure students interpret the task correctly.
Design Principle(s): Maximize meta-awareness; Cultivate conversation

Anticipated Misconceptions

If students have trouble drawing and using a diagram to compare lengths, ask them to revisit the Fractions of Ropes activity (in Lesson 7) and use the diagrams there as examples. Suggest that they try drawing a diagram on graph paper, as the grid could support them in drawing and making sense of the fractional lengths.

Student Task Statement

1. A second-grade student is 4 feet tall. Her teacher is $5\frac{2}{3}$ feet tall.

   a. How many times as tall as the student is the teacher?

   b. What fraction of the teacher's height is the student's height?

2. Find each quotient. Show your reasoning and check your answer.

   a. $9 \div \frac{3}{5}$

   b. $1\frac{7}{8} \div \frac{3}{4}$

3. Write a division equation that can help answer each of these questions. Then find the answer. If you get stuck, consider drawing a diagram.

   a. A runner ran $1\frac{4}{5}$ miles on Monday and $6\frac{3}{10}$ miles on Tuesday. How many times her Monday's distance was her Tuesday's distance?

   b. A cyclist planned to ride $9\frac{1}{2}$ miles but only managed to travel $3\frac{7}{8}$ miles. What fraction of his planned trip did he travel?

Student Response

1. a. The teacher is $\frac{17}{12}$ or $1\frac{5}{12}$ times as tall as the student. $5\frac{2}{3} \div 4 = \frac{17}{3} \cdot \frac{1}{4} = \frac{17}{12}$

   b. The student is $\frac{12}{17}$ as tall as the teacher. $4 \div 5\frac{2}{3} = 4 \cdot \frac{3}{17} = \frac{12}{17}$

2. a. $9 \cdot \frac{5}{3} = 15$

   b. $\frac{15}{8} \cdot \frac{4}{3} = \frac{5}{2} = 2\frac{1}{2}$

3. a. $6\frac{3}{10} \div 1\frac{4}{5}$. On Tuesday, she ran $3\frac{1}{2}$ times Monday's distance.
   $6\frac{3}{10} \div 1\frac{4}{5} = \frac{63}{10} \cdot \frac{5}{9} = \frac{7}{2} = 3\frac{1}{2}$

   b. $3\frac{7}{8} \div 9\frac{1}{2}$. He traveled $\frac{31}{76}$ of his planned trip. $3\frac{7}{8} \div 9\frac{1}{2} = \frac{31}{8} \cdot \frac{2}{19} = \frac{31}{76}$

Activity Synthesis

Consider giving students access to the answers so they can check their work. Much of the discussion will have happened in small groups. If time permits, reconvene as a class to discuss the last set of questions and the different ways they were represented and solved.

12.4 Comparing Paper Rolls

Optional: 15 minutes

This optional activity gives students another opportunity to solve a contextual problem using what they know about fractions, relationships between multiplication and division, and diagrams. Students observe a photograph of two paper rolls of differing lengths and estimate the relationship between the lengths. The photograph shows that the longer roll is about $2\frac{1}{2}$ or $\frac{5}{2}$ times as long as the shorter roll. Students use this observation to find out the length of the shorter roll.

The two paper rolls are from paper towels and toilet paper. If possible, consider providing one of each roll to each group of students so they can physically compare their lengths in addition to observing the picture.

As students work, notice the different starting equations or diagrams they use to begin solving the problems. Ask students using different entry points to share later.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR3: Clarify, Critique, Correct
- Notice and Wonder

Launch

Keep students in groups of 4. Ask students to keep their materials closed. Display the image of the paper rolls for all to see. Give students 1–2 minutes to observe the picture. Ask them to be prepared to share at least one thing they notice and one thing they wonder. Then, invite students to share their observations and questions. If no students mention the relationship between the lengths of the rolls, ask them questions such as:

- "What do you notice about the lengths of the paper rolls or the relationships between those lengths?"
- "What questions can you ask about the lengths of the rolls?"
- "What information would you need to answer these questions?"

Then, give students 7–8 minutes of quiet time to complete the questions.

Support for Students with Disabilities

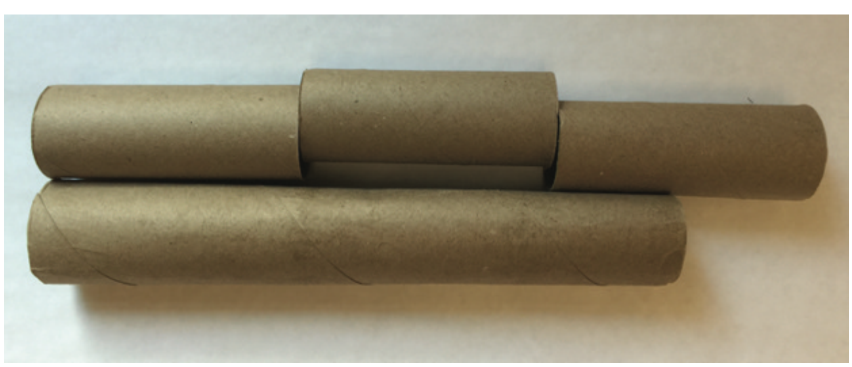
Action and Expression: Internalize Executive Functions. Provide students with a graphic organizer to include a table with the following headings: What do you notice?, What do you wonder?, What information do you need to answer these questions?
Supports accessibility for: Language; Organization

Anticipated Misconceptions

Students might estimate the relationships between the lengths of rolls by rounding too much. For example, they might say that the length of the shorter roll is $\frac{1}{3}$ the length of the longer roll, or that the longer roll is twice as long as the shorter roll. If this happens, ask students to take a closer look and make a more precise estimate. Suggest that they divide the larger roll into smaller segments, each of which matches the length of the shorter rolls.

Student Task Statement

The photo shows a situation that involves fractions.



1. Complete the sentences. Be prepared to explain your reasoning.

a. The length of the long tube is about _____ times the length of a short tube.

b. The length of a short tube is about _____ times the length of the long tube.

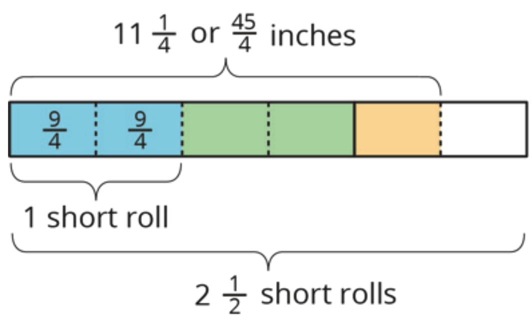
2. If the length of the long paper roll is $11\frac{1}{4}$ inches, what is the length of each short paper roll?

Student Response

1. a. About $\frac{5}{2}$ (or $2\frac{1}{2}$ or 2.5) times.

   b. About $\frac{2}{5}$ (or 0.4) times.

2. $4\frac{1}{2}$ (or equivalent) inches. Sample reasoning:
   - $11\frac{1}{4} \div 2\frac{1}{2} = \frac{45}{4} \div \frac{5}{2} = \frac{45}{4} \cdot \frac{2}{5} = 4\frac{1}{2}$
   - $\frac{2}{5} \cdot 11\frac{1}{4} = \frac{2}{5} \cdot \frac{45}{4} = \frac{9}{2} = 4\frac{1}{2}$



Activity Synthesis

Invite previously identified students to share their strategies for finding the length of the short roll. Display their diagram and record their reasoning for all to see.

To find the length of the short roll, some students may use $11\frac{1}{4} \div \frac{5}{2} =?$ and others $\frac{2}{5} \cdot \frac{45}{4} =?$, depending on how they view the relationship between the rolls. Highlight the idea that to find the length of the short roll, one way is to partition the length of the large roll into 5 equal pieces and find that length and multiply it by 2, because the length of the shorter roll is about $\frac{2}{5}$ of that of the longer roll. This is an opportunity to reinforce the structure behind the division algorithm.

If appropriate, discuss the merits of writing the numbers as fractions versus as mixed numbers. Mixed numbers such as $2\frac{1}{2}$ and $11\frac{1}{4}$ are easier to visualize but $\frac{5}{2}$ and $\frac{45}{4}$ are easier to work with. In fact, we have to use the fractions (rather than mixed numbers) to easily multiply. Explain that two forms serve different purposes and that it is helpful to be able to change from one to the other depending on what we aim to do. When writing them as solutions to problems, both forms are mathematically correct.

Support for English Language Learners

Writing, Conversing: MLR3 Clarify, Critique, Correct. Use this routine to help students evaluate the reasonableness of their answers. Display the following incorrect response: "The length of each short paper roll is $28\frac{1}{8}$ inches because I multiplied $\frac{45}{4} \cdot \frac{5}{2}$." Share with students that $\frac{45}{4} \cdot \frac{5}{2}$ is indeed $28\frac{1}{8}$. Give students 1 minute of quiet think time to consider this information. Ask students to discuss with their partner why the answer $28\frac{1}{8}$ doesn't make sense (the length of the short paper roll should be less than the length of the long paper roll). Next, ask students to identify why the equation $\frac{45}{4} \cdot \frac{5}{2} =?$ does not correctly represent this situation. Listen for and amplify corrections that include mathematical language and reasoning.
Design Principle(s): Maximize meta-awareness; Cultivate conversation

Lesson Synthesis

In this lesson, we used division to solve problems that involve fractional lengths. For example: How many $\frac{5}{8}$-inch paper clips, laid end to end, are in a length of $12\frac{1}{2}$ inches? Review how we can interpret such problems.

- "How is this question like those we have seen? How can division help us answer it?" (It is a 'how many groups?' question. We can think of it as 'how many $\frac{5}{8}$s in $12\frac{1}{2}$?' and solve it by finding $12\frac{1}{2} \div \frac{5}{8}$.)

- "Here is another question: 'What is the length of one stick if 9 sticks, laid end to end, make $12\frac{3}{8}$-inch?' How does division help us answer it?" (It is a 'how much in one group?' question. We can answer it by finding $12\frac{3}{8} \div 9$.)

We also saw that division can help us compare two lengths and find out how many times one is as long as the other. For example, suppose one hiking trail, Trail A, is $1\frac{1}{8}$ miles and another, Trail B, is $\frac{3}{4}$ miles.

- "How do we find out how many times as long as Trail A is Trail B?" (We can interpret the question as "___ times the length of A equals the length of B" or $? \cdot 1\frac{1}{8} = \frac{3}{4}$, and then find $\frac{3}{4} \div 1\frac{1}{8}$.)

- "What is another comparison question we could ask?" (How many times as long as Trail B is Trail A?)

- “How do we represent and answer that question?” (We can interpret it as “___ times the length of B is the length of A” or $? \cdot \frac{3}{4} = 1\frac{1}{8}$, and then find $1\frac{1}{8} \div \frac{3}{4}$.)

12.5 Building A Fence

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

A builder was building a fence. In the morning, he worked for $\frac{2}{5}$ of an hour. In the afternoon, he worked for $\frac{9}{10}$ of an hour. How many times as long as in the morning did he work in the afternoon?

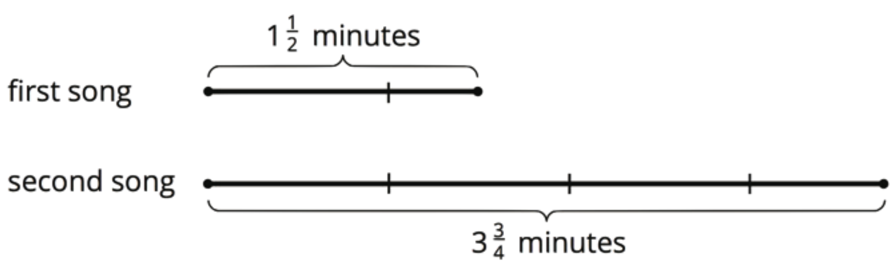
Write a division equation to represent this situation, then answer the question. Show your reasoning. If you get stuck, consider drawing a diagram.

Student Response

Division equation: $\frac{9}{10} \div \frac{2}{5} = ?$ (or $\frac{9}{10} \div ? = \frac{2}{5}$). In the afternoon, he worked $2\frac{1}{4}$ times as long as he did in the morning. Sample reasoning: $\frac{9}{10} \div \frac{2}{5} = \frac{9}{10} \cdot \frac{5}{2} = \frac{45}{20} = \frac{9}{4}$.

Student Lesson Summary

Division can help us solve comparison problems in which we find out how many times as large or as small one number is compared to another. For example, a student is playing two songs for a music recital. The first song is $1\frac{1}{2}$ minutes long. The second song is $3\frac{3}{4}$ minutes long.



We can ask two different comparison questions and write different multiplication and division equations to represent each question.

- How many times as long as the first song is the second song?

$$? \cdot 1\frac{1}{2} = 3\frac{3}{4}$$

$$3\frac{3}{4} \div 1\frac{1}{2} = ?$$

- What fraction of the second song is the first song?

$$? \cdot 3\frac{3}{4} = 1\frac{1}{2}$$

$$1\frac{1}{2} \div 3\frac{3}{4} = ?$$

We can use the algorithm we learned to calculate the quotients.

$$= \frac{15}{4} \div \frac{3}{2}$$

$$= \frac{15}{4} \cdot \frac{2}{3}$$

$$= \frac{30}{12}$$

$$= \frac{5}{2}$$

This means the second song is $2\frac{1}{2}$ times as long as the first song.

$$= \frac{3}{2} \div \frac{15}{4}$$

$$= \frac{3}{2} \cdot \frac{4}{15}$$

$$= \frac{12}{30}$$

$$= \frac{2}{5}$$

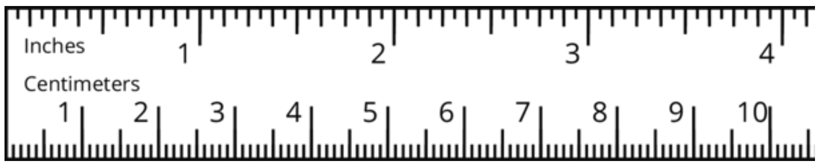
This means the first song is $\frac{2}{5}$ as long as the second song.

Lesson 12 Practice Problems

Problem 1

Statement

One inch is around $2\frac{11}{20}$ centimeters.



a. How many centimeters long is 3 inches? Show your reasoning.

b. What fraction of an inch is 1 centimeter? Show your reasoning.

c. What question can be answered by finding $10 \div 2\frac{11}{20}$ in this situation?

Solution

a. $7\frac{13}{20}$ centimeters. $3 \cdot 2\frac{11}{20} = \frac{3}{1} \cdot \frac{51}{20} = \frac{153}{20}$, which is $7\frac{13}{20}$.

b. $\frac{20}{51}$. $1 \div 2\frac{11}{20} = 1 \cdot \frac{20}{51}$, which is $\frac{20}{51}$.

c. How many inches are in 10 centimeters?

Problem 2

Statement

A zookeeper is $6\frac{1}{4}$ feet tall. A young giraffe in his care is $9\frac{3}{8}$ feet tall.

a. How many times as tall as the zookeeper is the giraffe?

b. What fraction of the giraffe's height is the zookeeper's height?

Solution

a. $9\frac{3}{8} \div 6\frac{1}{4} = \frac{75}{8} \div \frac{25}{4}$, and $\frac{75}{8} \div \frac{25}{4} = \frac{75}{8} \cdot \frac{4}{25}$, which equals $\frac{3}{2}$. The giraffe is $\frac{3}{2}$ or $1\frac{1}{2}$ times as tall as the zookeeper.

b. $6\frac{1}{4} \div 9\frac{3}{8} = \frac{25}{4} \div \frac{75}{8}$, and $\frac{25}{4} \div \frac{75}{8} = \frac{25}{4} \cdot \frac{8}{75}$, which equals $\frac{2}{3}$. The zookeeper's height is $\frac{2}{3}$ of the giraffe's height.

Problem 3

Statement

A rectangular bathroom floor is covered with square tiles that are $1\frac{1}{2}$ feet by $1\frac{1}{2}$ feet. The length of the bathroom floor is $10\frac{1}{2}$ feet and the width is $6\frac{1}{2}$ feet.

a. How many tiles does it take to cover the length of the floor?

b. How many tiles does it take to cover the width of the floor?

Solution

a. 7 tiles ($10\frac{1}{2} \div 1\frac{1}{2} = \frac{21}{2} \div \frac{3}{2}$, and $\frac{21}{2} \div \frac{3}{2} = \frac{21}{2} \boldsymbol{\cdot} \frac{2}{3}$, which equals 7.)

b. $4\frac{1}{3}$ tiles ($6\frac{1}{2} \div 1\frac{1}{2} = \frac{13}{2} \div \frac{3}{2}$, and $\frac{13}{2} \div \frac{3}{2} = \frac{13}{2} \boldsymbol{\cdot} \frac{2}{3}$, which equals $\frac{13}{3}$ or $4\frac{1}{3}$)

Problem 4

Statement

The Food and Drug Administration (FDA) recommends a certain amount of nutrient intake per day called the "daily value." Food labels usually show percentages of the daily values for several different nutrients—calcium, iron, vitamins, etc.

Consider the problem: In $\frac{3}{4}$ cup of oatmeal, there is $\frac{1}{10}$ of the recommended daily value of iron. What fraction of the daily recommended value of iron is in 1 cup of oatmeal?

Write a multiplication equation and a division equation to represent the question. Then find the answer and show your reasoning.

Solution

$\frac{3}{4} \boldsymbol{\cdot} ? = \frac{1}{10}$ (or equivalent), $\frac{1}{10} \div \frac{3}{4} = ?$.

$\frac{2}{15}$ of the daily value of iron. Reasoning varies. Sample reasoning: $\frac{1}{10} \div \frac{3}{4} = \frac{1}{10} \boldsymbol{\cdot} \frac{4}{3} = \frac{4}{30}$ or $\frac{2}{15}$.

(From Unit 4, Lesson 11.)

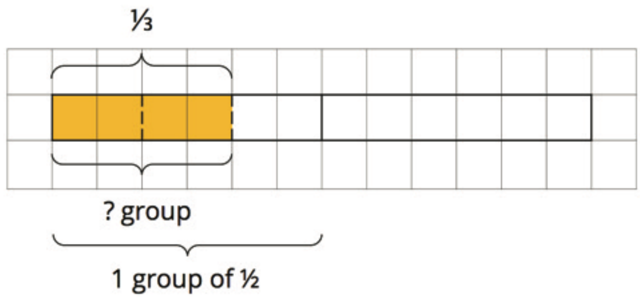
Problem 5

Statement

What fraction of $\frac{1}{2}$ is $\frac{1}{3}$? Draw a tape diagram to represent and answer the question. Use graph paper if needed.

Solution

$\frac{2}{3}$



(From Unit 4, Lesson 7.)

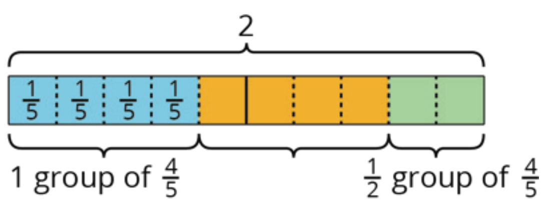
Problem 6

Statement

Noah says, “There are $2\frac{1}{2}$ groups of $\frac{4}{5}$ in 2.” Do you agree with him? Draw a tape diagram to show your reasoning. Use graph paper, if needed.

Solution

Agree. Sample diagram:



(From Unit 4, Lesson 6.)

Lesson 13: Rectangles with Fractional Side Lengths

Goals

- Apply dividing by fractions to calculate the side length of a rectangle, given its area and the other side length.
- Coordinate (orally) diagrams and equations that represent the area of a rectangle with fractional side lengths.
- Draw and label a diagram to justify the area of a rectangle with fractional side lengths.

Learning Targets

- I can use division and multiplication to solve problems involving areas of rectangles with fractional side lengths.

Lesson Narrative

This lesson builds on students' work on area and fractions in grade 5. Students solve problems involving the relationship between area and side lengths of rectangles, in cases where these measurements can be fractions. Knowing that the area of a rectangle can be found by multiplying its side lengths, and knowing the relationship between multiplication and division, they use division to find an unknown side length when the other side length and the area are given.

Alignments

Building On

- 5.NF.B.4.b: Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Building Towards

- 6.G.A.2: Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas

$V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR6: Three Reads
- MLR7: Compare and Connect
- Think Pair Share

Required Materials

$\frac{1}{4}$-inch graph paper

Straightedges

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Consider having the objects mentioned in the How Many Would it Take? activity available for students to verify their answers. These objects are: $\frac{3}{4}$-inch square stickers, $\frac{5}{8}$-binder clips, and $1\frac{3}{4}$-inch paper clips.

Student Learning Goals

Let’s explore rectangles that have fractional measurements.

13.1 Areas of Squares

Warm Up: 5 minutes

In this warm-up, students review how to find and record the area of a square with whole-number and fractional side lengths. The first question is open-ended to encourage students to notice many things about the area of each square, the relationships between them, and other geometric ideas they might remember from earlier grades. The second question prepares students for the work in this lesson. Focus class discussion on this question.

As students discuss the second question, note those who think of the area of a $\frac{1}{3}$-inch square in terms of:

- Tiling, i.e., determining how many squares with $\frac{1}{3}$-inch side length cover a square with 1-inch side length and dividing the area of 1 square inch by that number
- Multiplying $\frac{1}{3} \cdot \frac{1}{3}$

Building On

- 5.NF.B.4.b

Launch

Arrange students in groups of 2. Display the image and the first question for all to see. Give students 1 minute of quiet time to make observations about the squares. Follow with a brief whole-class discussion.

If not mentioned by students, ask students what they notice about the following:

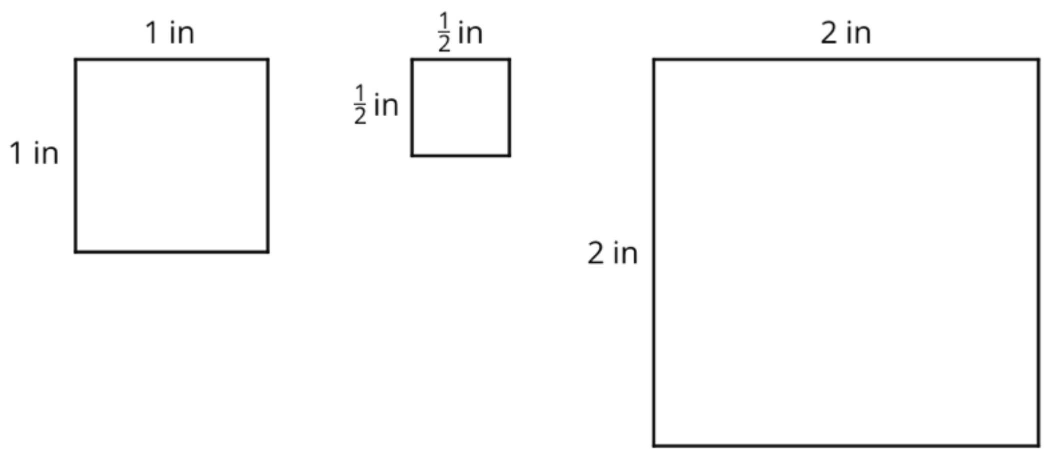
- The area of each square
- How to record the area in square inches
- Whether one square could tile another square completely

Then, give students 1–2 minutes to discuss the second question with their partner.

Anticipated Misconceptions

Some students may struggle in getting started with the second question. Suggest that they try marking up the given 1-inch square to show $\frac{1}{3}$-inch squares.

Student Task Statement



1. What do you notice about the areas of the squares?
2. Kiran says "A square with side lengths of $\frac{1}{3}$ inch has an area of $\frac{1}{3}$ square inches." Do you agree? Explain or show your reasoning.

Student Response

1. Answers vary.
2. No, there are 9 squares with side lengths of $\frac{1}{3}$ inch in a square with side lengths of 1 inch, so the area of a square with side length of $\frac{1}{3}$ inch is $\frac{1}{9}$ in^2.

Activity Synthesis

Consider telling students that we can call a square with 1-inch side length "a 1-inch square."

Ask previously identified students to share their response to the second question. Illustrate their reasoning for all to see. After each person shares, poll the class on whether they agree with the answer and the explanation. If not mentioned in students' explanations, highlight the following ideas:

- A square with a side length of 1 inch (a 1-inch square) has an area of 1 in^2.
- A 2-inch square has an area of 4 in^2, because 4 squares with 1-inch side length are needed to cover it.
- A $\frac{1}{2}$-inch square has an area of $\frac{1}{4}$ in^2 because 4 of them are needed to completely cover a 1-inch square.
- A $\frac{1}{3}$-inch square has a side length of $\frac{1}{3}$ inch, so it would take 9 squares to cover a 1-inch square. Its area is therefore $\frac{1}{9}$ square inch.

13.2 Areas of Squares and Rectangles

20 minutes

This activity serves two purposes:

- To review and illustrate the idea from grade 5 that the area of a rectangle with fractional side lengths can be found by multiplying the two fractions, just as in the case of whole numbers.
- To prepare students to reason about a prism with fractional edge lengths. Students connect the area of a square with fractional side length with that of a unit square. Later, they transfer this idea to find the volume of prisms with fractional edge lengths. They will then compare whole cubic units and fractional cubic units.

As students work, monitor the ways students represent and reason about the area of the rectangle with fractional side lengths in the last question. A few possibilities are shown in the Possible Responses. Select students who use different strategies to share later.

Building On

- 5.NF.B.4.b

Building Towards

- 6.G.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- MLR7: Compare and Connect
- Think Pair Share

Launch

Keep students in groups of 2. Give students 7–8 minutes of quiet work time and 2–3 minutes to share their responses and drawings with their partner. Provide each student with $\frac{1}{4}$-inch graph paper and a straightedge.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, pause to check for understanding after 3–5 minutes of work time. *Supports accessibility for: Organization; Attention*

Anticipated Misconceptions

Some students may have trouble counting grid squares or using a ruler on graph paper and struggle to measure the lengths of the rectangle. Consider preparing pre-drawn copies of the rectangle for students who may benefit from them.

Student Task Statement

Your teacher will give you graph paper and a ruler.

1. On the graph paper, draw a square with side lengths of 1 inch. Inside this square, draw another square with side lengths of $\frac{1}{4}$ inch.

   Use your drawing to answer the questions.

   a. How many squares with side lengths of $\frac{1}{4}$ inch can fit in a square with side lengths of 1 inch?

   b. What is the area of a square with side lengths of $\frac{1}{4}$ inch? Explain or show your reasoning.

2. On the graph paper, draw a rectangle that is $3\frac{1}{2}$ inches by $2\frac{1}{4}$ inches.

   For each question, write a division expression and then find the answer.

   a. How many $\frac{1}{4}$-inch segments are in a length of $3\frac{1}{2}$ inches?

   b. How many $\frac{1}{4}$-inch segments are in a length of $2\frac{1}{4}$ inches?

3. Use your drawing to show that a rectangle that is $3\frac{1}{2}$ inches by $2\frac{1}{4}$ inches has an area of $7\frac{7}{8}$ square inches.

Student Response

1. Drawing on graph paper should show a square that is 4 units by 4 units. (Each unit is $\frac{1}{4}$ inch.)

    a. 16 squares

    b. $\frac{1}{16}$ square inches

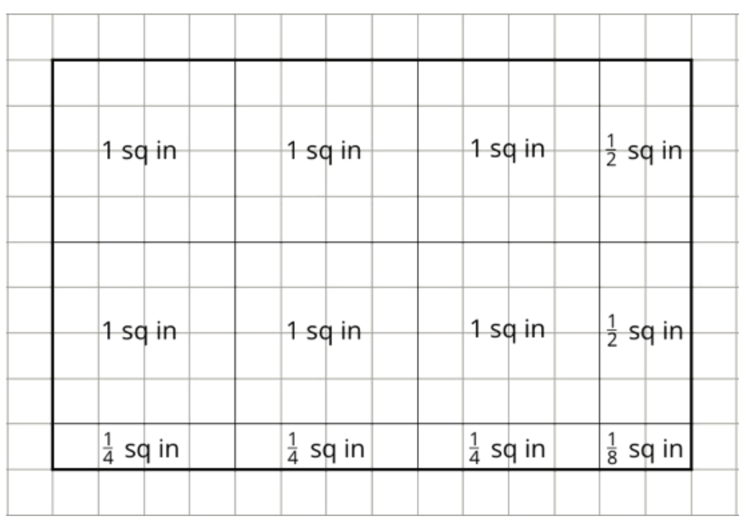
2. Drawing on graph paper should show a rectangle that is 14 units by 9 units. (Each unit is $\frac{1}{4}$ inch.)

    a. $3\frac{1}{2} \div \frac{1}{4} = ?$. Fourteen $\frac{1}{4}$-inch segments.

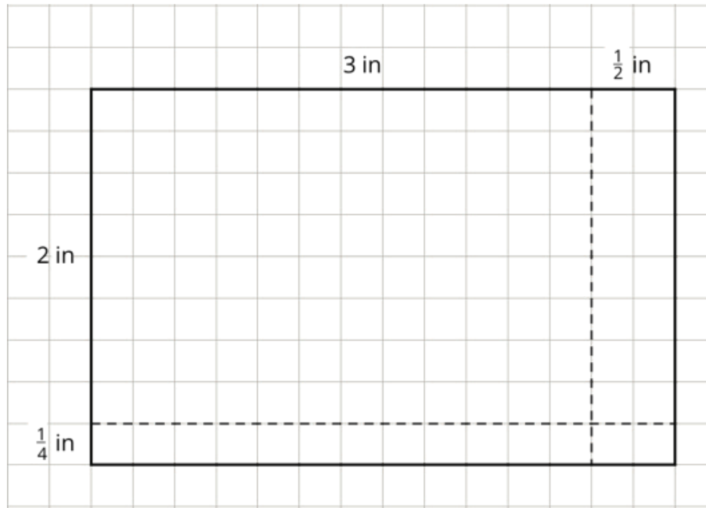
    b. $2\frac{1}{4} \div \frac{1}{4} = ?$. Nine $\frac{1}{4}$-inch segments.

3. Reasoning varies. Sample reasoning:

    - Multiplying the number of $\frac{1}{4}$-inch segments in the length and width (from the second question) to find the number of $\frac{1}{4}$-inch squares, then multiplying by $\frac{1}{16}$ to find the area in square inches. $14 \cdot 9 = 126$ and $126 \cdot \frac{1}{16} = 7\frac{7}{8}$
    - Filling the rectangle with squares with side lengths of 1 inch and with side lengths of $\frac{1}{4}$ inch and adding their areas. $(6 \cdot 1) + (2 \cdot \frac{1}{2}) + (3 \cdot \frac{1}{4}) + (1 \cdot \frac{1}{8}) = 6 + 1 + \frac{3}{4} + \frac{1}{8} = 7\frac{7}{8}$



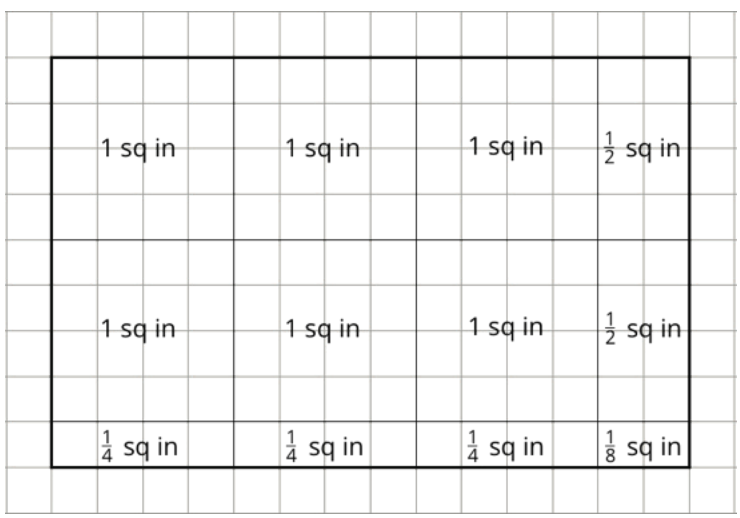
    - Using partial products and the distributive property. $(3 \cdot 2) + (3 \cdot \frac{1}{4}) + (\frac{1}{2} \cdot 2) + (\frac{1}{2} \cdot \frac{1}{4}) = 6 + \frac{3}{4} + 1 + \frac{1}{8} = 7\frac{7}{8}$

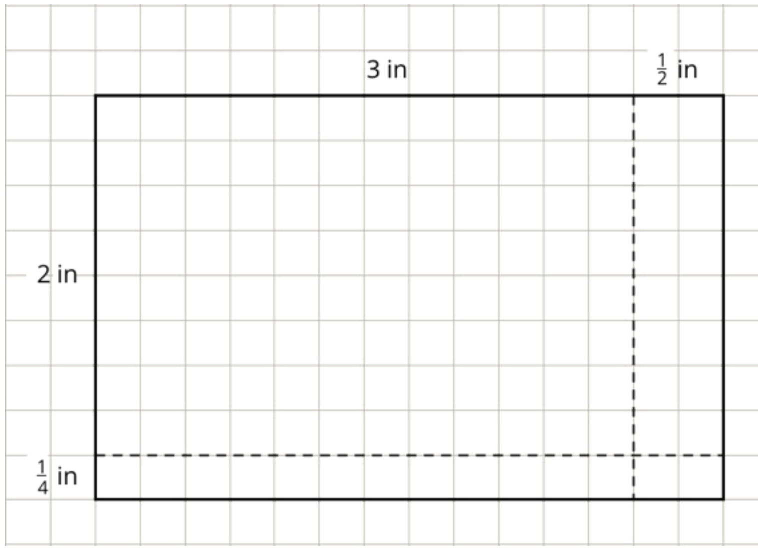


Activity Synthesis

Focus the whole-class discussion on the last question. Invite previously selected students to share their answers and diagrams in the sequence shown in the Possible Responses. Ask students to explain how they found that $3\frac{1}{2} \cdot 2\frac{1}{4}$ equals $7\frac{7}{8}\text{ in}^2$. Record their reasoning for all to see.

Consider displaying the following images and highlighting the areas of the sub-rectangles with fractional side lengths.





Compare and contrast the different strategies. Then, ask students how the area they found would compare to the product $3\frac{1}{2} \cdot 2\frac{1}{4}$. Ask them to calculate the product. Make sure students see that the product of the two numbers is equal to the area of the rectangle.

$$3\frac{1}{2} \cdot 2\frac{1}{4} = \frac{7}{2} \cdot \frac{9}{4} = \frac{63}{8} = 7\frac{7}{8}$$

Support for English Language Learners

Representing: MLR7 Compare and Connect. Use this routine after share their answers and diagrams for the last question. Ask students, “What is the same and what is different?” about the different strategies. Help students make connections by asking, “How was multiplication used in each strategy?”. This will help strengthen students’ mathematical language use and reasoning about the connection between the area of rectangles and multiplication of fractions. *Design Principle(s): Support sense-making; Maximize meta-awareness*

13.3 Areas of Rectangles

Optional: 10 minutes

This activity also revisits grade 5 work on finding the area of a rectangle with fractional side lengths. Students interpret and match numerical expressions and diagrams. Because the diagrams are unlabeled, students need to use the structure in the expressions and in the diagrams to make a match (MP7). This work reinforces their understanding of the area of rectangles and of multiplication. Specifically, it helps them see how the product of two mixed numbers (or two fractions that are greater than 1) can be found using partial products.

Building On

- 5.NF.B.4.b

Building Towards

- 6.G.A.2

Launch

Give students 2–3 minutes of quiet work time. Emphasize the direction that states, "All regions shaded in light blue have the same area" before students begin working.

Anticipated Misconceptions

In answering the second question (showing that $2\frac{1}{2} \cdot 4\frac{3}{4} = 11\frac{7}{8}$), some students may neglect to use the diagram and simply multiply the whole numbers in the side lengths (the 2 and 4), multiply the fractions (the $\frac{1}{2}$ and $\frac{3}{4}$), and then add them. Allow them to pursue this path of reasoning, but later, when they recognize their answer is less than $11\frac{7}{8}$, refer them to the diagram. Ask them to identify the rectangles whose areas they have calculated and those they have not accounted for, and to think about how they could find the area of the entire rectangle.

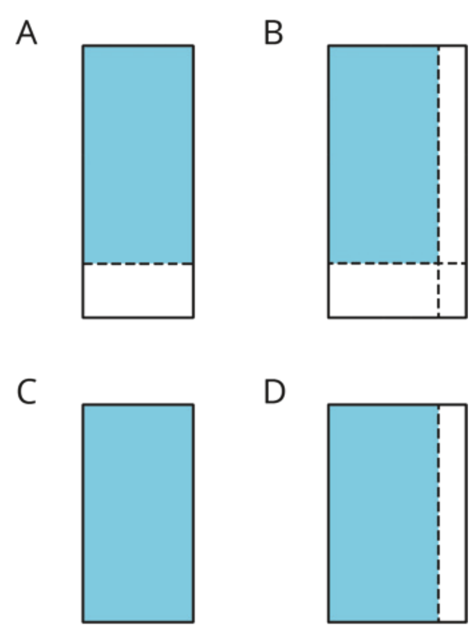
When adding partial products with fractions in different denominators, some students may simply add the numerators and denominators. Remind them to attend to the size of the fractional parts when adding or subtracting fractions.

Student Task Statement

Each of these multiplication expressions represents the area of a rectangle.

$2 \cdot 4$ $\quad$ $2\frac{1}{2} \cdot 4$ $\quad$ $2 \cdot 4\frac{3}{4}$ $\quad$ $2\frac{1}{2} \cdot 4\frac{3}{4}$

1. All regions shaded in light blue have the same area. Match each diagram to the expression that you think represents its area. Be prepared to explain your reasoning.



2. Use the diagram that matches $2\frac{1}{2} \cdot 4\frac{3}{4}$ to show that the value of $2\frac{1}{2} \cdot 4\frac{3}{4}$ is $11\frac{7}{8}$.

Student Response

1. $2 \cdot 4$ is Figure C, $2\frac{1}{2} \cdot 4$ is Figure D, $2 \cdot 4\frac{3}{4}$ is Figure A, $2\frac{1}{2} \cdot 4\frac{3}{4}$ is Figure B.
2. $(2 \cdot 4) + \left(\frac{1}{2} \cdot 4\right) + \left(\frac{3}{4} \cdot 2\right) + \left(\frac{3}{4} \cdot \frac{1}{2}\right) = 8 + 2 + \frac{3}{2} + \frac{3}{8} = 11\frac{7}{8}$

Are You Ready for More?

The following rectangles are composed of squares, and each rectangle is constructed using the previous rectangle. The side length of the first square is 1 unit.

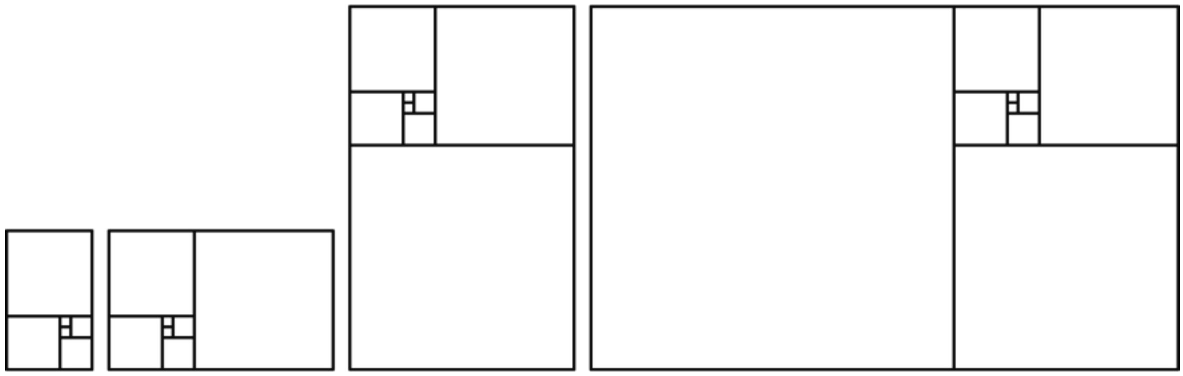
1. Draw the next four rectangles that are constructed in the same way. Then complete the table with the side lengths of the rectangle and the fraction of the longer side over the shorter side.

short side	long side	$\frac{\text{long side}}{\text{short side}}$
1		
1		
2		
3		

2. Describe the values of the fraction of the longer side over the shorter side. What happens to the fraction as the pattern continues?

Student Response

1.



short side	long side	$\frac{\text{long side}}{\text{short side}}$
1	1	1
1	2	2
2	3	$1\frac{1}{2}$
3	5	$1\frac{2}{3}$
5	8	$1\frac{3}{5}$
8	13	$1\frac{5}{8}$
13	21	$1\frac{8}{13}$
21	34	$1\frac{13}{21}$
34	55	$1\frac{21}{34}$

2. Answers vary. The fractions go up and down around a value that is near $1\frac{2}{3}$.

Activity Synthesis

For each diagram, ask one or more students to share which expression they think matches. Ask students to share their reasoning for how they matched the figure to the expression. Consider displaying the four figures for all to see and recording students' reasoning or explanations on the figures. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s reasoning?"
- "Do you agree or disagree? Why?"

For the second question, ask students for the area of each section in Figure B. Label each section with its side lengths and its area and display for all to see. If not already articulated by students, highlight that combining all the partial areas gives us a sum of $11\frac{7}{8}$, which is the area of the entire rectangle.

13.4 How Many Would it Take? (Part 2)

15 minutes

This activity consolidates prior work on the area of rectangles and the current work on division of fractions. Students determine how many tiles with fractional side lengths are needed to completely cover another rectangular region that also has fractional side lengths. Besides dividing fractions, students also need to plan their approach, think about how the orientation of the tiles affects their calculation and solution, and attend carefully to the different measurements and steps in their calculation. The experience here prepares students to work with lengths and volumes in the culminating lesson (in which students determine how many small boxes with fractional edge lengths will fit into larger boxes that also have fractional edge lengths).

As students work, identify those whose diagrams or solutions show different tile orientations. Also notice students who consider both ways of laying the tiles before finding the solutions.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR6: Three Reads

Launch

Keep students in groups of 2. Give students 7–8 minutes of quiet work time and 2–3 minutes to share their responses with their partner, or give 10 minutes for them to complete the activity in groups.

Support for English Language Learners

Reading: MLR6 Three Reads. Use this routine to support reading comprehension. Ask students to keep their books or devices closed and display only the task statement without revealing the questions that follow. Use the first read to orient students to the situation. After a shared reading, ask students "what is this situation about?" (Noah is covering a tray with tiles). Consider using photos of these items to ensure that all students understand the words tray, tile, gaps, and overlaps. After the second read, students list any quantities that can be counted or measured, without focusing on specific values (length of tray, in inches; width of tray, in inches; area of tray, in square inches; length of each tile, in inches; width of each tile, in inches; area of each tile, in square inches). During the third read, the question or prompt is revealed. Invite students to discuss possible strategies, referencing the relevant quantities named after the second read.
Design Principle(s): Support sense-making

Anticipated Misconceptions

Students might only determine the amount of tiles needed to line the four sides of the tray. If this happens, suggest that they look at their drawing of the tray and check whether their tiles cover the entire area of the tray.

Student Task Statement

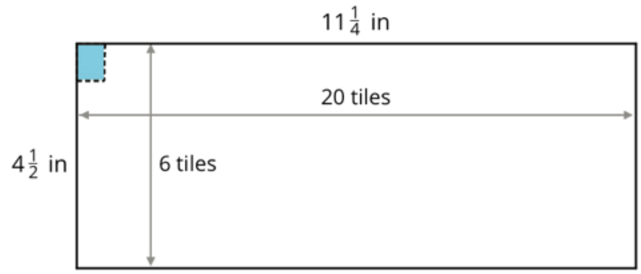
Noah would like to cover a rectangular tray with rectangular tiles. The tray has a width of $11\frac{1}{4}$ inches and an area of $50\frac{5}{8}$ square inches.

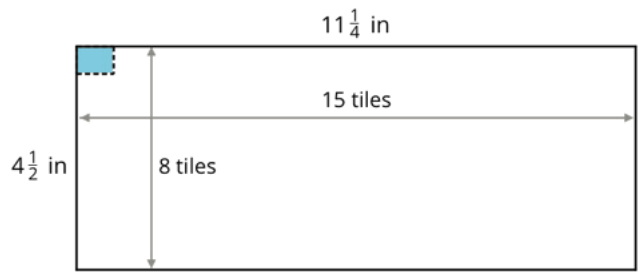
1. Find the length of the tray in inches.
2. If the tiles are $\frac{3}{4}$ inch by $\frac{9}{16}$ inch, how many would Noah need to cover the tray completely, without gaps or overlaps? Explain or show your reasoning.
3. Draw a diagram to show how Noah could lay the tiles. Your diagram should show how many tiles would be needed to cover the length and width of the tray, but does not need to show every tile.

Student Response

1. $50\frac{5}{8} \div 11\frac{1}{4} = \frac{405}{8} \div \frac{45}{4}$, so the length of the tray is $4\frac{1}{2}$ in.
2. If we lay the $\frac{3}{4}$-inch side of the tiles along the $4\frac{1}{2}$-inch side of the tray, we need 6 tiles, because $\left(4\frac{1}{2}\right) \div \frac{3}{4} = 6$, and we'll need 20 tiles along the other side, because $\left(11\frac{1}{4}\right) \div \frac{9}{16} = 20$. So the total number of tiles needed is $6 \cdot 20 = 120$. If we orient the tiles in the other direction, we will need the same number of tiles, but we'll have 8 along the short side and 15 along the long side.

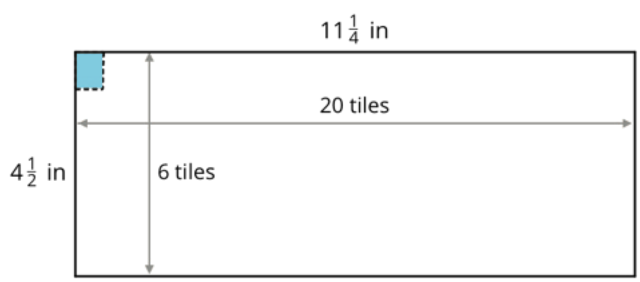
3.

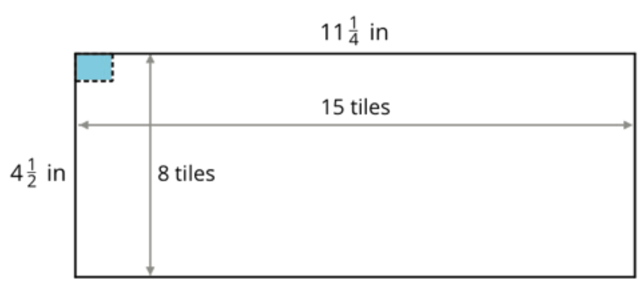




Activity Synthesis

Invite students who chose different tile orientations to show their diagrams and explain their reasoning. Display these two diagrams, if needed.





Point out how in this problem, the two different tile orientations do not matter, as the length and the width of the tiles are factors of both the length and the width of the tray. This means we can fit a whole number of tiles in either direction, and can fit the same number of tiles to cover the tray regardless of orientation.

But if the side lengths of the tiles do not both fit into $11\frac{1}{4}$ and $4\frac{1}{2}$ evenly, then the orientation of the tiles does matter (i.e., we may need more or fewer tiles, or we may not be able to tile the entire tray without gaps if the tiles are oriented a certain way).

Use the opportunity to point out that a diagram does not have to show all the details (i.e., every single tile) to be useful.

Lesson Synthesis

Review the different ways of reasoning about the area of a rectangle with fractional side lengths. If time permits, consider asking students to illustrate each reasoning strategy mentioned.

- "What are some ways that we can find the area of a rectangle that is $5\frac{1}{2}$ cm by $3\frac{1}{2}$ cm?" We can:
    - See how many $\frac{1}{2}$-cm squares cover the rectangle completely and multiply it by the area of each square, which is $\frac{1}{4}$ sq cm
    - Decompose the rectangle into whole centimeter squares (with 1-cm side length) and other rectangles with fractional side lengths, find their areas, and add them
    - Decompose the rectangle into sub-rectangles with whole-number side lengths and fractional side lengths, find their areas, and add them by multiplying the side lengths of the rectangle.

Emphasize that because we can multiply the side lengths of a rectangle (even if they are not whole numbers) to find its area, if we know the area of a rectangle and one side length, we can find the length of the other side by dividing.

- "Suppose we know that the width of a rectangle is $4\frac{3}{5}$ cm and the area is $16\frac{1}{10}$ sq cm. How can we find its length?" (We can find $16\frac{1}{10} \div 4\frac{3}{5} = ?$)
- "How do we check our quotient?" (We can multiply it by the width $4\frac{3}{5}$ and see if we get the given area.)

13.5 Two Frames

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

Two rectangular picture frames have the same area of 45 square inches but have different side lengths. Frame A has a length of $6\frac{3}{4}$ inches, and Frame B has a length of $7\frac{1}{2}$ inches.

1. Without calculating, predict which frame has the shorter width. Explain your reasoning.

2. Find the width that you predicted to be shorter. Show your reasoning.

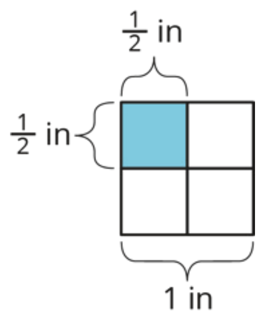
Student Response

1. Frame B has a longer length, so its width is shorter if the two pairs of side lengths produce the same product of 45.

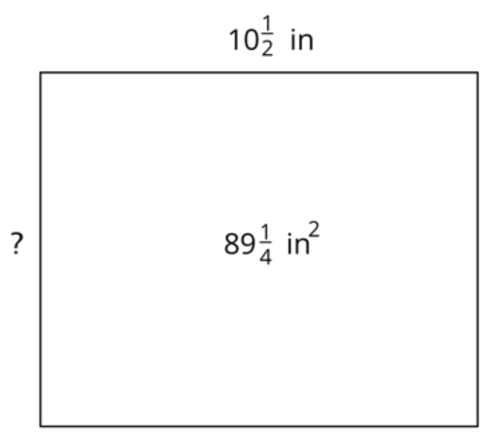
2. 6 inches. Sample reasoning: $45 \div 7\frac{1}{2} = 45 \div \frac{15}{2} = 45 \cdot \frac{2}{15} = 6$

Student Lesson Summary

If a rectangle has side lengths a units and b units, the area is $a \cdot b$ square units. For example, if we have a rectangle with $\frac{1}{2}$-inch side lengths, its area is $\frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{4}$ square inches.



This means that if we know the *area* and *one side length* of a rectangle, we can divide to find the *other* side length.



If one side length of a rectangle is $10\frac{1}{2}$ in and its area is $89\frac{1}{4}$ in^2, we can write this equation to show their relationship:

$$10\frac{1}{2} \cdot ? = 89\frac{1}{4}$$

Then, we can find the other side length, in inches, using division:

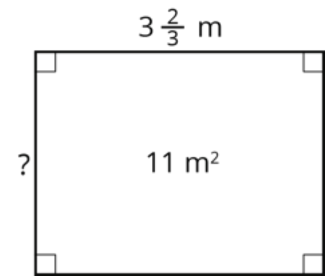
$$89\frac{1}{4} \div 10\frac{1}{2} = ?$$

Lesson 13 Practice Problems

Problem 1

Statement

a. Find the unknown side length of the rectangle if its area is 11 m^2. Show your reasoning.



b. Check your answer by multiplying it by the given side length ($3\frac{2}{3}$). Is the resulting product 11? If not, revise your previous work.

Solution

a. 3 m, because $11 \div \left(3\frac{2}{3}\right) = 3$

b. $3\frac{2}{3} \cdot 3 = 11$

Problem 2

Statement

A worker is tiling the floor of a rectangular room that is 12 feet by 15 feet. The tiles are square with side lengths $1\frac{1}{3}$ feet. How many tiles are needed to cover the entire floor? Show your reasoning.

Solution

$101\frac{1}{4}$ or 102 tiles. Reasoning varies. Sample reasoning: $12 \div \frac{4}{3} = 9$, so 9 tiles are needed to cover the 12 feet of length. $15 \div \frac{4}{3} = \frac{45}{4}$, so $11\frac{1}{4}$ tiles are needed to cover the 15 feet of length. To find the number of tiles, we multiply: $9 \cdot \frac{45}{4} = \frac{405}{4}$ or $101\frac{1}{4}$ tiles, which can be rounded to 102 tiles.

Problem 3

Statement

A television screen has length $16\frac{1}{2}$ inches, width w inches, and area 462 square inches. Select **all** the equations that represent the relationship of the side lengths and area of the television.

A. $w \cdot 462 = 16\frac{1}{2}$

B. $16\frac{1}{2} \cdot w = 462$

C. $462 \div 16\frac{1}{2} = w$

D. $462 \div w = 16\frac{1}{2}$

E. $16\frac{1}{2} \cdot 462 = w$

Solution

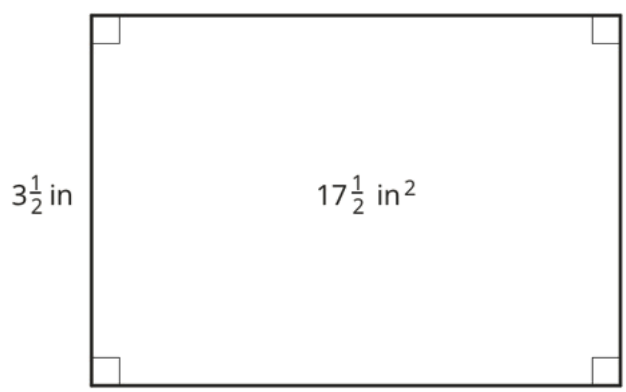
["B", "C", "D"]

Problem 4

Statement

The area of a rectangle is $17\frac{1}{2}$ in^2 and its shorter side is $3\frac{1}{2}$ in. Draw a diagram that shows this information. What is the length of the longer side?

Solution



5 in. (The sides perpendicular to the $3\frac{1}{2}$-inch side each have length in inches of $\left(17\frac{1}{2}\right) \div \left(3\frac{1}{2}\right) = \frac{35}{2} \cdot \frac{2}{7} = 5$.)

Problem 5

Statement

A bookshelf is 42 inches long.

a. How many books of length $1\frac{1}{2}$ inches will fit on the bookshelf? Explain your reasoning.

b. A bookcase has 5 of these bookshelves. How many feet of shelf space is there? Explain your reasoning.

Solution

a. 28 books. $42 \div 1\frac{1}{2} = 42 \div \frac{3}{2} = \frac{84}{3} = 28$

b. $17\frac{1}{2}$ feet. $5 \cdot 42 = 210$. 210 inches is $17\frac{1}{2}$ feet, since $210 \div 12 = 17\frac{1}{2}$.

(From Unit 4, Lesson 12.)

Problem 6

Statement

Find the value of $\frac{5}{32} \div \frac{25}{4}$. Show your reasoning.

Solution

$\frac{1}{40}$ ($\frac{5}{32} \div \frac{25}{4} = \frac{5}{32} \cdot \frac{4}{25}$, which is equal to $\frac{1}{40}$)

(From Unit 4, Lesson 11.)

Problem 7

Statement

How many groups of $1\frac{2}{3}$ are in each of these quantities?

a. $1\frac{5}{6}$

b. $4\frac{1}{3}$

c. $\frac{5}{6}$

Solution

a. $1\frac{1}{10}$

b. $2\frac{3}{5}$

c. $\frac{1}{2}$

(From Unit 4, Lesson 6.)

Problem 8

Statement

It takes $1\frac{1}{4}$ minutes to fill a 3-gallon bucket of water with a hose. At this rate, how long does it take to fill a 50-gallon tub? If you get stuck, consider using a table.

Solution

$\frac{125}{6}$ minutes (or equivalent). Possible strategy:

gallons of water	time in minutes
3	$\frac{5}{4}$
300	125 (or equivalent)
50	$\frac{125}{6}$ (or equivalent)

(From Unit 2, Lesson 14.)

Lesson 14: Fractional Lengths in Triangles and Prisms

Goals

- Apply dividing by fractions to calculate the base or height of a triangle, given its area and the other measurement.
- Determine the volume of a rectangular prism by counting how many 12-inch or 13-inch cubes it takes to build, and explain (orally and in writing) the solution method.
- Generalize that the volume of a rectangular prism with fractional edge lengths can be found by multiplying the edge lengths.

Learning Targets

- I can explain how to find the volume of a rectangular prism using cubes that have a unit fraction as their edge length.
- I can use division and multiplication to solve problems involving areas of triangles with fractional bases and heights.
- I know how to find the volume of a rectangular prism even when the edge lengths are not whole numbers.

Lesson Narrative

In this transitional lesson, students conclude their work with area and begin to explore volume of rectangular prisms. First, they extend their work on area to include triangles, using division to find the length of a base or a height in a triangle when the area is known. Second, they undertake a key activity for extending their understanding of how to find the volume of a prism.

In previous grades, students learned that the volume of a prism with whole-number edge lengths is the product of the edge lengths. Now they consider the volume of a prism with dimensions $1\frac{1}{2}$ inch by 2 inches by $2\frac{1}{2}$ inches. They picture it as being packed with cubes whose edge length is $\frac{1}{2}$ inch, making it a prism that is 3 cubes by 4 cubes by 5 cubes, for a total of 60 cubes, because $3 \cdot 4 \cdot 5 = 60$. At the same time, they see that each of these $\frac{1}{2}$-inch cubes has a volume of $\frac{1}{8}$ cubic inches, because we can fit 8 of them into a unit cube. They conclude that the volume of the prism is $60 \cdot \frac{1}{8} = 7\frac{1}{2}$ cubic inches.

In the next lesson, by repeating this reasoning and generalizing (MP8), students see that the volume of a rectangular prism with fractional edge lengths can also be found by multiplying its edge lengths directly (e.g., $\left(1\frac{1}{2}\right) \cdot 2 \cdot \left(2\frac{1}{2}\right) = 7\frac{1}{2}$).

Alignments

Addressing

- 6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
- 6.G.A.2: Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.
- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

$\frac{1}{2}$-inch cubes

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

For the Volumes of Cubes and Prisms activity, prepare 20 half-inch cubes for every group of 3–4 students. Wooden ones are available inexpensively at craft stores. If you have access to centimeter cubes, you could use those instead. Tell students that we will consider them half-inch cubes for the purposes of that activity.

Student Learning Goals

Let’s explore area and volume when fractions are involved.

14.1 Area of Triangle

Warm Up: 5 minutes

This warm-up allows students to review calculation of triangular area and prepares them to use division of fractions to solve area problems involving triangles later. Students calculate the area of a triangle given fractional base and height measurements.

Addressing

- 6.G.A.1
- 6.NS.A.1

Instructional Routines

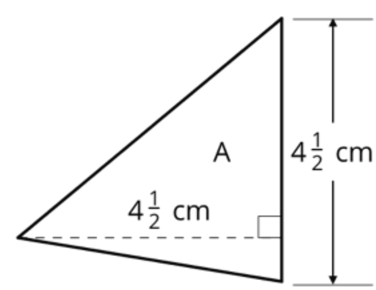
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time, followed by 1 minute of partner discussion. Before students begin, review the formula for the area of a triangle. Consider displaying a drawing of a triangle with one side labeled as a base and a corresponding height shown and labeled as such.

Student Task Statement

Find the area of Triangle A in square centimeters. Show your reasoning.



Student Response

The area of any triangle is $A = \frac{1}{2} \boldsymbol{\cdot} \text{base} \boldsymbol{\cdot} \text{height}$. The area of this triangle is $10\frac{1}{8}$ cm^2, because $\frac{1}{2} \boldsymbol{\cdot} \left(4\frac{1}{2}\right) \boldsymbol{\cdot} \left(4\frac{1}{2}\right) = 10\frac{1}{8}$.

Activity Synthesis

Ask a student to share their solution and reasoning. Record it for all to see. Poll the class to see whether students agree or disagree. Ask if others had alternative ways of reasoning or calculating.

Tell students that they will solve more problems involving the area of triangles in this lesson.

14.2 Bases and Heights of Triangles

10 minutes
In this activity, students apply their knowledge of division of fractions to answer questions about bases and heights of triangles, which they studied earlier in the year. In the warm-up, students recalled how to find the area of a triangle given a pair of base and height. Here, they find a missing length given the area of a triangle and a fractional base or height.

The formula for the area of a triangle $A = \frac{1}{2} \cdot b \cdot h$ presents a different multiplication situation than students have seen in this unit—there are three factors at play. Students are likely to approach in a number of ways. They may:

- Draw a duplicate of the triangle and compose a parallelogram with the same base and height, double the given area (to represent the area of the parallelogram), and then divide by the known length to find the unknown length
- Without drawing, multiply the given area by 2 to find the value of $b \cdot h$, and then divide it by the known length
- Perform division twice, i.e., dividing the area by $\frac{1}{2}$ and then by the known base or length, or vice versa
- Multiply the known length and $\frac{1}{2}$ first, so there are only two factors to work with (e.g., for $\frac{1}{2} \cdot b \cdot \frac{8}{3} = 8$, they may apply the commutative and associative properties of operations and write $\frac{1}{2} \cdot \frac{8}{3} \cdot b = 8$ and then $\frac{4}{3} \cdot b = 8$)

Monitor for these or other approaches as students work. Select students who use different strategies to share later.

Addressing

- 6.G.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR8: Discussion Supports

Launch

Keep students in groups of 2. Give students 4–5 minutes of quiet work time and 2 minutes to discuss their responses and complete the activity with their partner. Keep the formula for the area of a triangle and a labeled drawing of a triangle displayed.

Support for Students with Disabilities

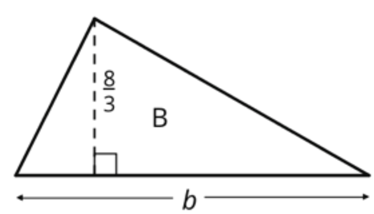
Action and Expression: Develop Expression and Communication. Activate or supply background knowledge. During the launch, take time to review the following terms from previous lessons that students will need to access for this activity: base and height of a triangle, formula for area of a triangle.
Supports accessibility for: Memory; Language

Anticipated Misconceptions

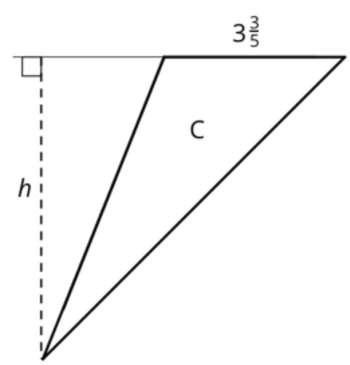
Some students may not know how to find a missing factor when three—instead of two—factors are involved. Ask if there is any step to take or any calculation to make first so they are only working with two factors.

Student Task Statement

1. The area of Triangle B is 8 square units. Find the length of b. Show your reasoning.



2. The area of Triangle C is $\frac{54}{5}$ square units. What is the length of h? Show your reasoning.



Student Response

1. 6 units. $\frac{1}{2} \cdot \frac{8}{3} \cdot b = 8$, so $b = 6$.
2. 6 units. $\frac{1}{2} \cdot \left(3\frac{3}{5}\right) \cdot h = \frac{54}{5}$, so $h = 6$.

Activity Synthesis

Ask previously selected students to share their solutions and reasoning. Sequence their presentations so that students who reasoned concretely (e.g., by duplicating the triangle to compose a parallelogram) present first and those who reasoned symbolically (e.g., only by manipulating expressions or equations) present last. After each student shares, ask all students to indicate whether they reasoned the same way.

Clarify that there is no single way to solve problems such as these. Point out specific points in the solving process where division of fractions enabled them to complete their reasoning. Emphasize that what we learned about fractions and operations in this unit can help us reason more effectively about problems in other areas of mathematics.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this to routine amplify students' use of mathematical language to communicate reasoning about finding the area of triangles. When students share their solutions and reasoning, remind them to use words such as compose, quadrilateral, parallelogram, base, height, area, and rearrange. Invite students to chorally repeat the phrases that include these words in context. Students used these words in a previous unit, so this will help students be more precise with their language.
Design Principle(s): Optimize output (for explanation)

14.3 Volumes of Cubes and Prisms

20 minutes (there is a digital version of this activity)
This activity extends students' understanding about the volume of rectangular prisms from earlier grades. Previously, students learned that the volume of a rectangular prism with whole-number edge lengths can be found by computing the number of unit cubes that can be packed into the prism. Here, they draw on the same idea to find the volume of a prism with fractional edge lengths. The edge length of the cubes used as units of measurement are not 1 unit long, however. Instead, they have a unit fraction ($\frac{1}{2}$, $\frac{1}{4}$, etc.) for their edge length. Students calculate the number of these smaller cubes in a prism and use it to find the volume in a standard unit of volume measurement (cubic inches, in this case).

By reasoning repeatedly with small cubes (with $\frac{1}{2}$-inch edge lengths), students notice that the volume of a rectangular prism with fractional edge lengths can also be found by directly multiplying the edge lengths in inches.

Addressing

- 6.G.A.2

Instructional Routines

- MLR2: Collect and Display

Launch

Tell students to look at the image of the 1-inch cube in the task (or display it for all to see). Ask students:

- "This cube has an edge length of 1 inch. What is its volume in cubic inches?" (1 cubic inch.) "How do you know?" ($1 \cdot 1 \cdot 1 = 1$)
- "How do we find the volume of a cube with an edge length of 2 inches?" ($2 \cdot 2 \cdot 2 = 8$. Or, since we can pack it with eight 1-inch cubes, we can tell its area is $8 \cdot 1$ or 8 cubic inches.)

If no students mentioned using the 1-inch cube to find the volume of the 2-inch cube, bring it up. Consider telling students that we can call a cube with edge length of 1 inch a "1-inch cube."

Arrange students in groups of 3–4. Give each group 20 cubes and 5 minutes to complete the first set of questions. Ask them to pause for a brief class discussion afterwards.

Invite students to share how they found the volume of a cube with $\frac{1}{2}$-inch edge length and the prism composed of 4 stacked cubes. For the $\frac{1}{2}$-inch cube, if students do not mention one of the two ways shown in the Possible Responses, bring it up. For the tower, if they don't mention multiplying the volume of a $\frac{1}{2}$-inch cube, which is $\frac{1}{8}$ cubic inch, ask if that is a possible way to find the volume of the prism.

Next, give students 7–8 minutes to complete the rest of the activity.

For classes using the digital materials, an applet is provided, but using physical cubes is preferred and recommended. Adapted from an applet made in GeoGebra by Susan Addington.

Support for English Language Learners

Conversing, Representing: MLR2 Collect and Display. As students work on the first set of questions, listen for the words and phrases students use as they discuss how to find the volume of each prism (length, width, height, and volume). Display collected language for all to see, and include any diagrams that students use to represent their thinking. Continue to update the display throughout the lesson. Remind students to borrow language from the display as needed, as this will help students produce language related to volume.
Design Principle(s): Support sense-making

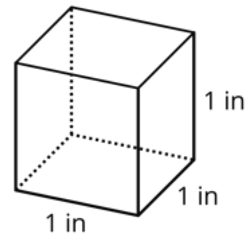
Anticipated Misconceptions

Students may need a reminder to label volume in cubic units.

Student Task Statement

Your teacher will give you cubes that have edge lengths of $\frac{1}{2}$ inch.

1. Here is a drawing of a cube with edge lengths of 1 inch.



a. How many cubes with edge lengths of $\frac{1}{2}$ inch are needed to fill this cube?

b. What is the volume, in cubic inches, of a cube with edge lengths of $\frac{1}{2}$ inch? Explain or show your reasoning.

2. Four cubes are piled in a single stack to make a prism. Each cube has an edge length of $\frac{1}{2}$ inch. Sketch the prism, and find its volume in cubic inches.

3. Use cubes with an edge length of $\frac{1}{2}$ inch to build prisms with the lengths, widths, and heights shown in the table.

   a. For each prism, record in the table how many $\frac{1}{2}$-inch cubes can be packed into the prism and the volume of the prism.

prism length (in)	prism width (in)	prism height (in)	number of $\frac{1}{2}$-inch cubes in prism	volume of prism (in^3)
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		
1	1	$\frac{1}{2}$		
2	1	$\frac{1}{2}$		
2	2	1		
4	2	$\frac{3}{2}$		
5	4	2		
5	4	$2\frac{1}{2}$		

   b. Examine the values in the table. What do you notice about the relationship between the edge lengths of each prism and its volume?

4. What is the volume of a rectangular prism that is $1\frac{1}{2}$ inches by $2\frac{1}{4}$ inches by 4 inches? Show your reasoning.

Student Response

1. a. 8 cubes

   b. The volume of a $\frac{1}{2}$-inch cube is $\frac{1}{8}$ in^3 because it is $\frac{1}{8}$ of the volume of a 1-inch cube.

2. $\frac{4}{8}$ in^3 or $\frac{1}{2}$ in^3

3. a.

prism length (in)	prism width (in)	prism height (in)	number of $\frac{1}{2}$-inch cubes in prism	volume of prism (cu in)
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{8}$
1	1	$\frac{1}{2}$	4	$\frac{1}{2}$
2	1	$\frac{1}{2}$	8	1
2	2	1	32	4
4	2	$\frac{3}{2}$	96	12
5	4	2	320	40
5	4	$2\frac{1}{2}$	400	50

   b. Answers vary. Sample response: I noticed that the volume is equal to $l \cdot w \cdot h$.

4. $13\frac{1}{2}$ in^3. Sample reasoning: $\left(1\frac{1}{2}\right) \cdot \left(2\frac{1}{4}\right) = \frac{27}{8}$, and $\frac{27}{8} \cdot 4 = 13\frac{1}{2}$.

Are You Ready for More?

A unit fraction has a 1 in the numerator.

- These are unit fractions: $\frac{1}{3}, \frac{1}{100}, \frac{1}{1}$.
- These are *not* unit fractions: $\frac{2}{9}, \frac{8}{1}, 2\frac{1}{5}$.

1. Find three unit fractions whose sum is $\frac{1}{2}$. An example is: $\frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$ How many examples like this can you find?
2. Find a box whose surface area in square units equals its volume in cubic units. How many like this can you find?

Student Response

Answers vary. Sample response:

1. $\frac{1}{3} + \frac{1}{12} + \frac{1}{12}$
   $\frac{1}{5} + \frac{1}{5} + \frac{1}{10}$
   $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
2. The denominators of the fractions that work for the first question can be used as the length, width, and height of the box.

Activity Synthesis

Display a completed table for all to see. Give students a minute to check their responses. Invite a few students to share how they determined the volume of the prisms, and their observations about the relationships between the number of cubes (with $\frac{1}{2}$-inch edge lengths) and the volume of prism in cubic inches. Highlight that the volume can be found by calculating the number of cubes and multiplying it by $\frac{1}{8}$, because each small cube has a volume of $\frac{1}{8}$ cubic inch.

If students do not notice a pattern in their table, ask them what they notice about the edge lengths of the prisms in the second, fourth, and sixth rows of the table and their corresponding volumes. Make sure they see that the volume of each prism can also be found by multiplying its side lengths. In other words, we can find the volume of a rectangular prism with fractional edge lengths the same way we find that of a prism with whole-number edge lengths.

Lesson Synthesis

In this lesson, we used division to solve for a missing base or height in a triangle. Review with students that we used a given area to find an unknown base or height.

- "How is finding an unknown base or height in a triangle different than finding an unknown side length in a rectangle?" (In a triangle, there are three factors at play.)
- "What multiplication equation can we write to help us find the height of a triangle that has a base of $\frac{5}{4}$ cm and an area of 10 sq cm?" ($\frac{1}{2} \boldsymbol{\cdot} \frac{5}{4} \boldsymbol{\cdot} h = 10$.)
- "How do we go about finding the unknown length?" (We can multiply the two known factors first. In this case, we can multiply $\frac{1}{2} \boldsymbol{\cdot} \frac{5}{4}$ to get $\frac{5}{8}$, and then divide 10 by $\frac{5}{8}$. Or we can divide by the factors one at a time—divide by $\frac{1}{2}$ first, and then by $\frac{5}{4}$.)

What we know about fractions and operations can also help us find the volume of rectangular prisms when the edge lengths are not whole numbers.

- How can we use cubes with $\frac{1}{2}$-inch edge lengths to find the volume of a prism that is $\frac{1}{2}$ inch by 2 inch by $3\frac{1}{2}$ inches? (We can think of the prism as being built of these cubes; the prism will have whole numbers of cubes for its edge lengths. In this case, they are 1, 4, and 7, so there are 28 cubes with $\frac{1}{2}$-inch edge length. Each of these cubes is $\frac{1}{8}$ cubic inch because 8 of them fit in 1 cubic inch. The volume of the prism, in cubic inches, is then $28 \boldsymbol{\cdot} \frac{1}{8}$, which is $3\frac{1}{2}$.)

- How else might we find the volume of a rectangular prism with fractional edge lengths? (We can multiply the three edge lengths in inches directly. $\frac{1}{2} \cdot 2 \cdot \frac{7}{2} = \frac{7}{2} = 3\frac{1}{2}$.)

14.4 Triangles and Cubes

Cool Down: 5 minutes
If time is limited, ask students to complete only the second problem.

Addressing

- 6.G.A.1
- 6.G.A.2
- 6.NS.A.1

Student Task Statement

1. A triangle has a base of $3\frac{2}{5}$ inches and an area of $5\frac{1}{10}$ square inches. Find the height of the triangle. Show your reasoning.
2. Answer each question and show your reasoning.
    a. How many cubes with edge lengths of $\frac{1}{3}$ inch are needed to build a cube with an edge length of 1 inch?
    b. What is the volume, in cubic inches, of one cube with an edge length of $\frac{1}{3}$ inch?

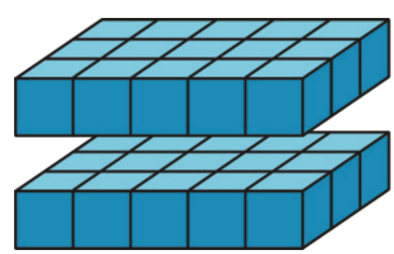
Student Response

1. $\frac{1}{2} \cdot \left(3\frac{2}{5}\right) \cdot h = 5\frac{1}{10}$, so h is $5\frac{1}{10} \div \left(\frac{1}{2} \cdot 3\frac{2}{5}\right)$ and the height is 3 inches.
2. a. 27
    b. $\frac{1}{27}$ in^3

Student Lesson Summary

If a rectangular prism has edge lengths of 2 units, 3 units, and 5 units, we can think of it as 2 layers of unit cubes, with each layer having $(3 \cdot 5)$ unit cubes in it. So the volume, in cubic units, is:

$$2 \cdot 3 \cdot 5$$



To find the volume of a rectangular prism with fractional edge lengths, we can think of it as being built of cubes that have a unit fraction for their edge length. For instance, if we build a prism that is $\frac{1}{2}$-inch tall, $\frac{3}{2}$-inch wide, and 4 inches long using cubes with a $\frac{1}{2}$-inch edge length, we would have:

- A height of 1 cube, because $1 \cdot \frac{1}{2} = \frac{1}{2}$.
- A width of 3 cubes, because $3 \cdot \frac{1}{2} = \frac{3}{2}$.
- A length of 8 cubes, because $8 \cdot \frac{1}{2} = 4$.

The volume of the prism would be $1 \cdot 3 \cdot 8$, or 24 cubic units. How do we find its volume in cubic inches? We know that each cube with a $\frac{1}{2}$-inch edge length has a volume of $\frac{1}{8}$ cubic inch, because $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Since the prism is built using 24 of these cubes, its volume, in cubic inches, would then be $24 \cdot \frac{1}{8}$, or 3 cubic inches.

The volume of the prism, in cubic inches, can also be found by multiplying the fractional edge lengths in inches: $\frac{1}{2} \cdot \frac{3}{2} \cdot 4 = 3$

Lesson 14 Practice Problems

Problem 1

Statement

Clare is using little wooden cubes with edge length $\frac{1}{2}$ inch to build a larger cube that has edge length 4 inches. How many little cubes does she need? Explain your reasoning.

Solution

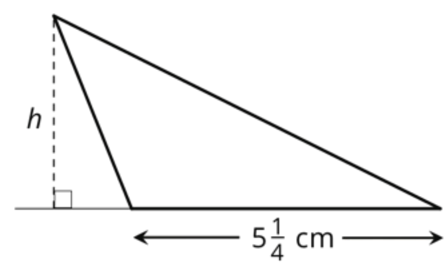
512. Since there are 8 half inches in 4 inches, Clare needs $8 \cdot 8 \cdot 8$ little cubes. $8 \cdot 8 \cdot 8 = 512$.

Problem 2

Statement

The triangle has an area of $7\frac{7}{8}$ cm^2 and a base of $5\frac{1}{4}$ cm.

What is the length of h? Explain your reasoning.



Solution

3 cm. One half of the base ($2\frac{5}{8}$ cm) times the height is $7\frac{7}{8}\text{ cm}^2$. So the height in cm is $\left(7\frac{7}{8}\right) \div \left(2\frac{5}{8}\right) = 3$.

Problem 3

Statement

a. Which expression can be used to find how many cubes with edge length of $\frac{1}{3}$ unit fit in a prism that is 5 units by 5 units by 8 units? Explain or show your reasoning.

- $(5 \cdot \frac{1}{3}) \cdot (5 \cdot \frac{1}{3}) \cdot (8 \cdot \frac{1}{3})$
- $5 \cdot 5 \cdot 8$
- $(5 \cdot 3) \cdot (5 \cdot 3) \cdot (8 \cdot 3)$
- $(5 \cdot 5 \cdot 8) \cdot (\frac{1}{3})$

b. Mai says that we can also find the answer by multiplying the edge lengths of the prism and then multiplying the result by 27. Do you agree with her? Explain your reasoning.

Solution

a. $(5 \cdot 3) \cdot (5 \cdot 3) \cdot (8 \cdot 3)$. Sample reasoning: It takes three $\frac{1}{3}$ units to make 1 unit. In terms of the edge length of the small cube, the prism is 15 by 15 by 24.

b. Mai is correct. Sample reasoning: Because it takes three $\frac{1}{3}$ units to make 1 unit, it takes $3 \cdot 3 \cdot 3$ cubes with edge length of $\frac{1}{3}$ unit to make one cube with edge length 1 unit.

Problem 4

Statement

A builder is building a fence with $6\frac{1}{4}$-inch-wide wooden boards, arranged side-by-side with no gaps or overlaps. How many boards are needed to build a fence that is 150 inches long? Show your reasoning.

Solution

24 boards. $(150 \div 6\frac{1}{4} = 150 \cdot \frac{4}{25} = 24)$

(From Unit 4, Lesson 12.)

Problem 5

Statement

Find the value of each expression. Show your reasoning and check your answer.

a. $2\frac{1}{7} \div \frac{2}{7}$

b. $\frac{17}{20} \div \frac{1}{4}$

Solution

a. $\frac{15}{2}$ or $7\frac{1}{2}$

b. $\frac{17}{5}$ or $3\frac{2}{5}$

(From Unit 4, Lesson 12.)

Problem 6

Statement

Consider the problem: A bucket contains $11\frac{2}{3}$ gallons of water and is $\frac{5}{6}$ full. How many gallons of water would be in a full bucket?

Write a multiplication and a division equation to represent the situation. Then, find the answer and show your reasoning.

Solution

14 gallons. Equations: $\frac{5}{6} \cdot ? = 11\frac{2}{3}$ and $11\frac{2}{3} \div \frac{5}{6} = ?$. $11\frac{2}{3} \div \frac{5}{6} = \frac{35}{3} \cdot \frac{6}{5}$, which equals 14.

(From Unit 4, Lesson 11.)

Problem 7

Statement

There are 80 kids in a gym. 75% are wearing socks. How many are *not* wearing socks? If you get stuck, consider using a tape diagram.

Solution

20. Sample reasoning: if 75% are wearing socks, then 25% are not wearing socks. 25% of a number is the same as $\frac{1}{4}$ of the number, and $\frac{1}{4}$ of 80 is 20.

(From Unit 3, Lesson 12.)

Problem 8

Statement

a. Lin wants to save \$75 for a trip to the city. If she has saved \$37.50 so far, what percentage of her goal has she saved? What percentage remains?

b. Noah wants to save \$60 so that he can purchase a concert ticket. If he has saved \$45 so far, what percentage of his goal has he saved? What percentage remains?

Solution

a. 50% has been saved, and 50% remains, (37.50 is half of 75).

b. 75% has been saved, and 25% remains, ($\frac{1}{4}$ of 60 is 15, so 15 is 25%.)

(From Unit 3, Lesson 11.)

Lesson 15: Volume of Prisms

Goals

- Apply dividing by fractions to calculate one edge length of a rectangular prism, given its volume and the other two edge lengths.
- Explain (orally, in writing, and using other representations) how to solve a problem involving the volume of a rectangular prism with fractional edge lengths.
- Generalize that it takes more smaller cubes or fewer larger cubes to fill the same volume.

Learning Targets

- I can solve volume problems that involve fractions.

Lesson Narrative

In this lesson, students complete their understanding of why the method of multiplying the edge lengths works for finding the volume of a prism with fractional edge lengths, just as it did for prisms with whole-number edge lengths. They use this understanding to find the volume of rectangular prisms given the edge lengths, and to find unknown edge lengths given the volume and other edge lengths.

Problems about rectangles and triangles in the previous two lessons involved three quantities: length, width, and area; or base, height, and area. Problems in this lesson involve four quantities: length, width, height, and volume. So finding an unknown quantity might involve an extra step, for example, multiplying two known lengths first and then dividing the volume by this product, or dividing the volume twice, once by each known length.

In tackling problems with increasing complexity and less scaffolding, students must make sense of problems and persevere in solving them (MP1).

Alignments

Addressing

- 6.G.A.2: Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

Instructional Routines

- MLR8: Discussion Supports

Student Learning Goals

Let’s look at the volume of prisms that have fractional measurements.

15.1 A Box of Cubes

Warm Up: 5 minutes

This warm-up reviews the volume work students had done previously to prepare for the work in this lesson. It reinforces the idea of using unit cubes and fractional-unit cubes as a way to measure the volume of a rectangular prism.

Addressing

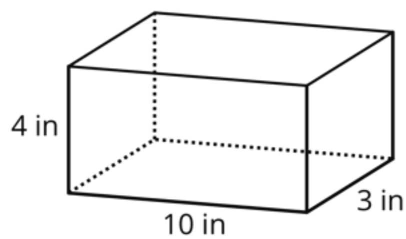
- 6.G.A.2

Launch

Give students 2–3 minutes of quiet work time. Follow with a class discussion.

Student Task Statement

1. How many cubes with an edge length of 1 inch fill this box?



2. If the cubes had an edge length of 2 inches, would you need more or fewer cubes to fill the box? Explain your reasoning.

3. If the cubes had an edge length of $\frac{1}{2}$ inch, would you need more or fewer cubes to fill the box? Explain your reasoning.

Student Response

1. 120 cubes with edge length of 1 inch fill the box since $10 \cdot 3 \cdot 4 = 120$.
2. Fewer. Sample explanation: 8 cubes with edge length of 1 inch fit into a cube with edge length of 2 inches, so there will be fewer cubes.
3. More. Sample explanation: Each cube with edge length of 1 inch can be packed with 8 cubes with edge length of $\frac{1}{2}$ inch, so there will be more cubes.

Activity Synthesis

Select several students to share their responses and reasoning. After each person explains, ask students to indicate whether they agree. To involve more students in the discussion, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s reasoning?"

- “Do you agree or disagree with the reasoning? Why?”

Tell students that they will use their understanding of the volume of rectangular prisms to solve other geometric problems.

15.2 Cubes with Fractional Edge Lengths

20 minutes

In this activity, students continue the work on finding the volume of a right rectangular prism with fractional edge lengths. This time, they do so by packing it with unit cubes of different unit fractions for their edge lengths—$\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{4}$ of an inch. They use these cubes to find the volume of the prism in cubic inches, decide which unit fraction works better to accomplish this goal and why, and explain whether cubes of different fractional edge lengths would lead to the same volume in cubic inches (MP3).

As students work, notice those who are able to clearly explain why cubes with a particular fractional edge length are preferable as a unit of measurement and why the volume in cubic inches will be the same regardless of the cubes used. Invite them to share later.

Addressing

- 6.G.A.2

Launch

Arrange students in groups of 3–4. Give them 8–10 minutes of quiet work time, and then 5 minutes to discuss their responses with their group. Ask groups to be sure to discuss the third question. Encourage students to draw a sketch to help with reasoning, if needed.

Support for Students with Disabilities

Representation: Internalize Comprehension. Activate or supply background knowledge about measurement. Students may benefit from watching a quick demonstration or video of packing cubes into a box. Review terms such as dimensions, volume, and cubic units.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

If students have trouble getting started, ask them to revisit their work with $\frac{1}{2}$-inch cubes from a previous lesson. Remind them that a cube with $\frac{1}{2}$-inch edge length has a volume of $\frac{1}{8}$ in^3 (because we can fit 8 of such cubes in a 1-inch cube). Ask them to think about how many $\frac{1}{3}$-inch cubes can fit into a 1-inch cube, or think about what the volume of a $\frac{1}{3}$-inch cube is in cubic inches.

Some students may not be able to visualize and keep track of the measurements of the boxes in this task. Encourage students to draw and label the measurements of the boxes described in these questions.

Student Task Statement

1. Diego says that 108 cubes with an edge length of $\frac{1}{3}$ inch are needed to fill a rectangular prism that is 3 inches by 1 inch by $1\frac{1}{3}$ inch.
    a. Explain or show how this is true. If you get stuck, consider drawing a diagram.
    b. What is the volume, in cubic inches, of the rectangular prism? Explain or show your reasoning.
2. Lin and Noah are packing small cubes into a larger cube with an edge length of $1\frac{1}{2}$ inches. Lin is using cubes with an edge length of $\frac{1}{2}$ inch, and Noah is using cubes with an edge length of $\frac{1}{4}$ inch.
    a. Who would need more cubes to fill the $1\frac{1}{2}$-inch cube? Be prepared to explain your reasoning.
    b. If Lin and Noah each use their small cubes to find the volume of the larger $1\frac{1}{2}$-inch cube, will they get the same answer? Explain or show your reasoning.

Student Response

1. a. Sample reasoning: There are 9 groups of $\frac{1}{3}$ inch in 3 inches, 3 groups in 1 inch, and 4 groups in $\frac{4}{3}$ inches. So it would take $9 \cdot 3 \cdot 4$ (or 108 cubes) to pack the prism.

    b. 4 in^3. Sample reasoning:
    - Each cube with edge length of $\frac{1}{3}$ inch has a volume of $\frac{1}{27}$ in^3. $108 \cdot \frac{1}{27} = 4$
    - $3 \cdot 1 \cdot \frac{4}{3} = 4$
2. a. Noah would need more cubes, because his small cubes are smaller.

    b. Yes, they should get the same volume. Sample reasoning:
    - There are three $\frac{1}{2}$-inches in $1\frac{1}{2}$ inches, so Lin would need $3 \cdot 3 \cdot 3$ or 27 cubes with edge length of $\frac{1}{2}$ inch. The volume of each $\frac{1}{2}$-inch cube is $\frac{1}{8}$ in^3, so the volume of the $1\frac{1}{2}$-inch cube is $27 \cdot \frac{1}{8}$ (or $\frac{27}{8}$ in^3).
    - There are six $\frac{1}{4}$-inches in $1\frac{1}{2}$ inches, so Noah would need $6 \cdot 6 \cdot 6$ or 216 cubes with edge length of $\frac{1}{4}$ inch. The volume of each cube with edge length of $\frac{1}{4}$ inch is $\frac{1}{64}$ in^3, so the volume of the $1\frac{1}{2}$-inch cube is $216 \cdot \frac{1}{64}$ (or $\frac{27}{8}$ in^3).

- Both sets of cubes can be packed into the $1\frac{1}{2}$-inch cube. In both cases, the volume can be calculated using $\left(1\frac{1}{2}\right) \cdot \left(1\frac{1}{2}\right) \cdot \left(1\frac{1}{2}\right)$, which equals $\frac{27}{8}$ (or $3\frac{3}{8}$ in^3).

Activity Synthesis

Select several students to share their responses and articulate their reasoning. Compare the different strategies students used for finding the volume of the rectangular prism. Ask students:

- “Does it matter which fractional-unit cubes we use to find the volume? Why or why not?” (As long as the unit fraction can fit evenly into all three edge lengths of the prism, it doesn’t matter what unit fraction we use.)
- “Do certain unit fractions work better as edge lengths of the small cubes than others?” (It helps to use as large a unit fraction as possible, since it means using fewer cubes and working with fractions that are closer to 1.)
- “Is there another way of finding the volume of a rectangular prism with fractional edge length besides using these small cubes?” (Multiply the fractional edge lengths.)

Point out that it is helpful to use a unit fraction that is a common factor of the fractional edge lengths of the prism. Make sure students also recognize that multiplying the edge lengths of the prism is a practical way to find the volume of such a rectangular prism.

15.3 Fish Tank and Baking Pan

Optional: 20 minutes
In this activity, students solve word problems that involve finding the volume of rectangular prisms that have fractional edge lengths, and calculate unknown edge lengths given other measurements. The last question in the activity requires students to interpret how the same volume of liquid would fit in two different containers in the shape of rectangular prisms.

As they work, monitor for different representations students use to solve the problems.

Addressing

- 6.G.A.2

Instructional Routines

- MLR8: Discussion Supports

Launch

Keep students in groups of 3–4. Give students 5 minutes of quiet work time for the first question and 2–3 minutes to discuss their responses with their group. Then, give students time to complete the second question either individually or with their group. Encourage students to draw a sketch to help with reasoning, if needed.

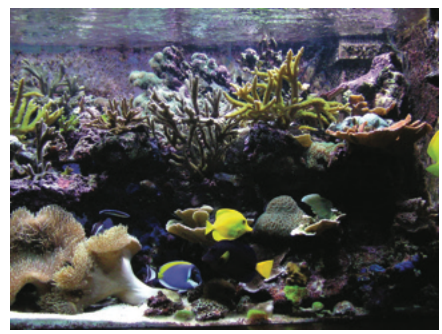
Support for Students with Disabilities

Representation: Access for Perception. Read all problems aloud. Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems, and other text-based content.
Supports accessibility for: Language; Conceptual processing

Student Task Statement

1. A nature center has a fish tank in the shape of a rectangular prism. The tank is 10 feet long, $8\frac{1}{4}$ feet wide, and 6 feet tall.

    a. What is the volume of the tank in cubic feet? Explain or show your reasoning.

    b. The nature center's caretaker filled $\frac{4}{5}$ of the tank with water. What was the volume of the water in the tank, in cubic feet? What was the height of the water in the tank? Explain or show your reasoning.

    c. Another day, the tank was filled with 330 cubic feet of water. The height of the water was what fraction of the height of the tank? Show your reasoning.

2. Clare's recipe for banana bread won't fit in her favorite pan. The pan is $8\frac{1}{2}$ inches by 11 inches by 2 inches. The batter fills the pan to the very top, and when baking, the batter spills over the sides. To avoid spills, there should be about an inch between the top of the batter and the rim of the pan.

    Clare has another pan that is 9 inches by 9 inches by $2\frac{1}{2}$ inches. If she uses this pan, will the batter spill over during baking?

Student Response

1. a. 495 ft^3, because $10 \cdot \left(8\frac{1}{4}\right) \cdot (6) = 495$.

    b. 396 ft^3, because $(10) \cdot \left(8\frac{1}{4}\right) \cdot \left(\frac{4}{5} \cdot 6\right) = 396$. The water is $4\frac{4}{5}$ feet deep.

    c. $\frac{2}{3}$ of the height of the tank. $330 \div \left(10 \cdot 8\frac{1}{4}\right) = 4$ and $4 = \frac{2}{3} \cdot 6$.

2. Yes, it would spill. Sample reasoning: The volume of the batter is 187 inch, since $\left(8\frac{1}{2}\right) \cdot 11 \cdot 2 = 187$. In the second pan, the batter will have a height of a little over 2 inches, because $187 \div (9 \cdot 9) \approx 2\frac{1}{3}$. Since the pan is only $2\frac{1}{2}$ inches deep, there would not be at least 1 inch between the top of batter and the rim of the pan.

Are You Ready for More?

1. Find the area of a rectangle with side lengths $\frac{1}{2}$ and $\frac{2}{3}$.
2. Find the volume of a rectangular prism with side lengths $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.
3. What do you think happens if we keep multiplying fractions $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \ldots$?
4. Find the area of a rectangle with side lengths $\frac{1}{1}$ and $\frac{2}{1}$.
5. Find the volume of a rectangular prism with side lengths $\frac{1}{1}$, $\frac{2}{1}$, and $\frac{1}{3}$.
6. What do you think happens if we keep multiplying fractions $\frac{1}{1} \cdot \frac{2}{1} \cdot \frac{1}{3} \cdot \frac{4}{1} \cdot \frac{1}{5} \ldots$?

Student Response

1. $\frac{1}{3}$
2. $\frac{1}{4}$
3. Approach the value of 0
4. 2
5. $\frac{2}{3}$
6. Approach the value of 1

Activity Synthesis

Invite a few students to share their solutions, explanations, and drawings (if any). Record and display their solutions for all to see. To involve more students in the discussion, ask students to indicate whether they agree or disagree with their classmate’s reasoning, if they approached it the same way but could explain it differently, or if they have an alternative path.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. After a student explains their solution to the class, call on students to use mathematical language to restate and/or revoice the strategy presented (volume, cubic feet, cubic units, rectangular prism, edge lengths). Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. This will provide additional opportunities for all students to produce language that describes strategies for finding the volume of rectangular prisms.
Design Principle(s): Support sense-making; Maximize meta-awareness

Lesson Synthesis

In this lesson, we used fraction multiplication and division to solve several kinds of problems about the volume of rectangular prisms. Consider using this time to help students reflect on their problem-solving process and asking questions such as:

- "How was finding the volume of a prism with fractional edge lengths like finding the volume of a prism with whole-number edge lengths? How is it different?"
- "When calculating volume, did you find it harder to work with mixed numbers than with fractions less than 1? Why or why not?" (Working with mixed numbers is a little harder since it often involves an extra step of converting them into fractions. If an error is made then, the work that follows is affected. It is easier, however, to make sense of the size of a quantity when it is written as a mixed number.)
- "How was the process of finding an unknown length of a rectangle the same or different than finding an unknown length of a prism?" (In both cases, there is one missing factor. When working with area, there are 3 quantities to keep track of: area, base, and height. When working with volume, there are 4 quantities to consider: volume, length, width, and height.)
- "Were there certain parts of calculating a volume or an unknown length that you found challenging or were prone to making mistakes? If so, which parts?"

15.4 Storage Box

Cool Down: 5 minutes
Addressing

- 6.G.A.2

Launch

Encourage students to draw a sketch to help with reasoning, if needed.

Student Task Statement

A storage box has a volume of 56 cubic inches,. The base of the box is 4 inches by 4 inches.

1. What is the height of the box?
2. Lin's teacher uses the box to store her set of cubes with an edge length of $\frac{1}{2}$ inch. If the box is completely full, how many cubes are in the set?

Student Response

1. $3\frac{1}{2}$ (or equivalent) inches. $56 \div (4 \cdot 4) = \frac{56}{16} = 3\frac{8}{16}$
2. 448 cubes. Each cubic inch fits 8 cubes with edge length of $\frac{1}{2}$ inch. $56 \cdot 8 = 448$

Student Lesson Summary

If a rectangular prism has edge lengths a units, b units, and c units, the volume is the product of a, b, and c.

$$V = a \cdot b \cdot c$$

This means that if we know the *volume* and *two edge lengths*, we can divide to find the *third* edge length.

Suppose the volume of a rectangular prism is $400\frac{1}{2}$ cm^3, one edge length is $\frac{11}{2}$ cm, another is 6 cm, and the third edge length is unknown. We can write a multiplication equation to represent the situation:

$$\frac{11}{2} \cdot 6 \cdot ? = 400\frac{1}{2}$$

We can find the third edge length by dividing:

$$400\frac{1}{2} \div \left(\frac{11}{2} \cdot 6\right) = ?$$

Lesson 15 Practice Problems

Problem 1

Statement

A pool in the shape of a rectangular prism is being filled with water. The length and width of the pool is 24 feet and 15 feet. If the height of the water in the pool is $1\frac{1}{3}$ feet, what is the volume of the water in cubic feet?

Solution

480 cubic feet. ($24 \cdot 15 = 360$, and $360 \cdot \frac{4}{3} = 480$.)

Problem 2

Statement

A rectangular prism measures $2\frac{2}{5}$ inches by $3\frac{1}{5}$ inches by 2 inch.

a. Priya said, "It takes more cubes with edge length $\frac{2}{5}$ inch than cubes with edge length $\frac{1}{5}$ inch to pack the prism." Do you agree with Priya? Explain or show your reasoning.

b. How many cubes with edge length $\frac{1}{5}$ inch fit in the prism? Show your reasoning.

c. Explain how you can use your answer in the previous question to find the volume of the prism in cubic inches.

Solution

a. Disagree. Sample reasoning: Cubes with side lengths $\frac{2}{5}$ inch are larger than cubes with side lengths $\frac{1}{5}$ inch, so it would take fewer of the former to pack the same prism.

b. 1,920 cubes. Reasoning varies. Sample reasoning: $2\frac{2}{5} \div \frac{1}{5} = 12$, $3\frac{1}{5} \div \frac{1}{5} = 16$, and $2 \div \frac{1}{5} = 10$. We can fit 12 cubes along the length of the prism, 16 cubes along the width, and 10 cubes along the height, so the number of cubes is: $12 \cdot 16 \cdot 10 = 1{,}920$.

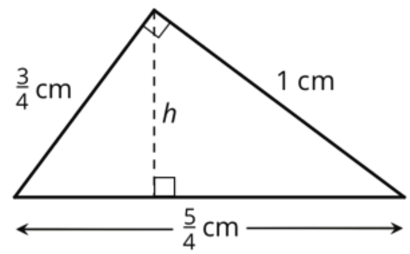
c. Each unit cube (edge length $\frac{1}{5}$ inch) has a volume of $\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5}$ or $\frac{1}{125}$ cubic inch. There are 1,920 of these unit cubes, so the volume is $1{,}920 \cdot \frac{1}{125}$ or 15.36 cubic inches.

Problem 3

Statement

a. Here is a right triangle. What is its area?

b. What is the height h for the base that is $\frac{5}{4}$ units long? Show your reasoning.



Solution

a. $\frac{3}{8}$ cm^2. Sample reasoning: The area of a triangle is found with $\frac{1}{2} \cdot b \cdot h$. We can use the two perpendicular sides as the base and the height. $\frac{1}{2} \cdot \frac{3}{4} \cdot 1 = \frac{3}{8}$

b. $\frac{3}{5}$ cm. Sample reasoning: The area of the triangle is $\frac{3}{8}$ cm^2 and we can also write the area using the $\frac{5}{4}$ side and h. $\frac{1}{2} \cdot \frac{5}{4}h = \frac{3}{8}$, so $\frac{5}{8}h = \frac{3}{8}$. To find what we could multiply by $\frac{5}{8}$ to get $\frac{3}{8}$, we can divide $\frac{3}{8} \div \frac{5}{8}$, which is $\frac{3}{8} \cdot \frac{8}{5}$, which is $\frac{3}{5}$.

(From Unit 4, Lesson 14.)

Problem 4

Statement

To give their animals essential minerals and nutrients, farmers and ranchers often have a block of salt—called "salt lick"—available for their animals to lick.

a. A rancher is ordering a box of cube-shaped salt licks. The edge lengths of each salt lick are $\frac{5}{12}$ foot. Is the volume of one salt lick greater or less than 1 cubic foot? Explain your reasoning.

b. The box that contains the salt lick is $1\frac{1}{4}$ feet by $1\frac{2}{3}$ feet by $\frac{5}{6}$ feet. How many cubes of salt lick fit in the box? Explain or show your reasoning.

Solution

a. Less than 1 cubic foot. Reasoning varies. Sample reasoning: A cube with edge length 1 foot has a volume of 1 cubic foot. A salt-lick cube has edge length $\frac{5}{12}$ foot, which is less than 1 foot, so its volume ($\frac{5}{12} \cdot \frac{5}{12} \cdot \frac{5}{12}$) is less than 1 cubic foot.

b. 24 cubes. Reasoning varies. Sample reasoning: The length of the box can fit $\frac{5}{4} \div \frac{5}{12}$ or 3 cubes. The width of the box can fit $\frac{5}{3} \div \frac{5}{12}$ or 4 cubes. The height of the box can fit $\frac{5}{6} \div \frac{5}{12}$ or 2 cubes. The box can fit $(3 \cdot 4 \cdot 2)$ or 24 cubes.

Problem 5

Statement

a. How many groups of $\frac{1}{3}$ inch are in $\frac{3}{4}$ inch?

b. How many inches are in $1\frac{2}{5}$ groups of $1\frac{2}{3}$ inches?

Solution

a. $2\frac{1}{4}$. Sample reasoning: To find "how many groups," compute $\frac{3}{4} \div \frac{1}{3}$, which is $\frac{3}{4} \cdot \frac{3}{1}$, which is $\frac{9}{4}$ or $2\frac{1}{4}$

b. $2\frac{1}{3}$. Sample reasoning: To find "how many inches," compute $1\frac{2}{5} \cdot 1\frac{2}{3}$, which is $\frac{7}{5} \cdot \frac{5}{3}$, which is $\frac{7}{3}$ or $2\frac{1}{3}$.

(From Unit 4, Lesson 12.)

Problem 6

Statement

Here is a table that shows the ratio of flour to water in an art paste. Complete the table with values in equivalent ratios.

cups of flour	cups of water
1	$\frac{1}{2}$
4	
	3
$\frac{1}{2}$	

Solution

cups of flour	cups of water
1	$\frac{1}{2}$
4	2
6	3
$\frac{1}{2}$	$\frac{1}{4}$

(From Unit 2, Lesson 12.)

Section: Let's Put It to Work

Lesson 16: Solving Problems Involving Fractions

Goals

- Apply operations with fractions to solve problems in a variety of situations, and explain (orally and in writing) the reasoning.
- Generate an equation to represent a situation involving fractions, and justify (orally) the operation chosen.

Learning Targets

- I can use mathematical expressions to represent and solve word problems that involve fractions.

Lesson Narrative

In this lesson, students use their understanding of and their facility with all four operations to represent and solve problems involving fractions. The last activity requires students to make sense of the problem and persevere in solving it (MP1).

Alignments

Addressing

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- Notice and Wonder
- Think Pair Share

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

For the second activity (Pairs of Problems), plan an efficient way to assign at least 1 division problem and 1 problem involving another operation to each student (or group).

Student Learning Goals

Let's add, subtract, multiply, and divide fractions.

16.1 Operations with Fractions

Warm Up: 5 minutes
This warm-up reinforces students' understanding of what each of the four operations (addition, subtraction, multiplication, and division) does when performed on fractions. The same pair of fractions are used in each problem so that students can focus on the meaning of the operation rather than on the values.

Addressing

- 6.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Display problems for all to see. Give students 1 minute of quiet think time.

Tell students not to calculate exact values of the expressions. Ask them to estimate the value of each expression by reasoning about the operation and the fractions, and then put the expressions in order based on their values, from least to greatest. Ask students to give a signal as soon as they have determined an order and can support it with an explanation.

Give students 1 minute to discuss their reasoning with a partner and agree on a correct order.

Anticipated Misconceptions

Some students may think the division expression would have the lowest value because they still assume that division always produces a number that is less than the dividend. This is not true for division by a positive number less than 1, which is the case here. If this misconception arises, consider addressing it during whole-class discussion.

Student Task Statement

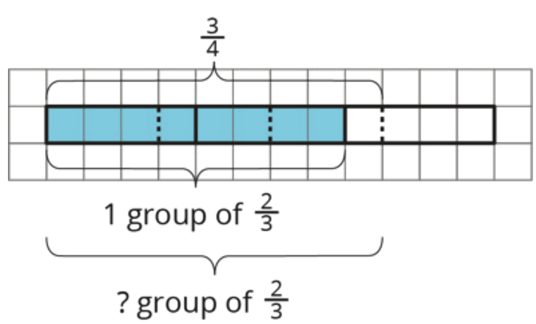
Without calculating, order the expressions according to their values from least to greatest. Be prepared to explain your reasoning.

$\frac{3}{4}+\frac{2}{3}$ $\frac{3}{4}-\frac{2}{3}$ $\frac{3}{4}\cdot\frac{2}{3}$ $\frac{3}{4}\div\frac{2}{3}$

Student Response

The order from smallest to largest is $\frac{3}{4}-\frac{2}{3}$, $\frac{3}{4}\cdot\frac{2}{3}$, $\frac{3}{4}\div\frac{2}{3}$, $\frac{3}{4}+\frac{2}{3}$. Sample reasoning:

- $\frac{3}{4}+\frac{2}{3}$ and $\frac{3}{4}\div\frac{2}{3}$ are both greater than 1.
- $\frac{3}{4}-\frac{2}{3}$ and $\frac{3}{4}\cdot\frac{2}{3}$ are both less than 1.
- $\frac{3}{4}$ is greater than $\frac{1}{2}$ and $\frac{2}{3}$ is also greater than $\frac{1}{2}$, so $\frac{3}{4}+\frac{2}{3}$ is greater than 1.
- $\frac{3}{4}$ is 0.75 and $\frac{2}{3}$ is about 0.67, so their sum is a little less than 1.5.
- $\frac{3}{4}\div\frac{2}{3}$ can be viewed as "how many $\frac{2}{3}$s are in $\frac{3}{4}$?". Since $\frac{3}{4}$ is just a little over $\frac{2}{3}$, the quotient is a little more than 1. If we were to draw a tape diagram, we can see it is just a little bit more than 1.



- $\frac{3}{4}\cdot\frac{2}{3}$ can be viewed as $\frac{3}{4}$ of $\frac{2}{3}$, so the product is less than $\frac{2}{3}$ but more than $\frac{1}{3}$.
- $\frac{3}{4}$ is $\frac{1}{4}$ more than $\frac{1}{2}$, and $\frac{2}{3}$ is greater than $\frac{1}{2}$ by an even smaller amount, so $\frac{3}{4}-\frac{2}{3}$ is less than $\frac{1}{4}$.

Activity Synthesis

Ask 1–2 groups to share how they ordered their expressions from least to greatest. If everyone agrees on one answer, ask a few students to share their reasoning. Record it for all to see. If there are disagreements, ask students with opposing views to explain their reasoning and discuss it to reach an agreement on a correct order.

16.2 Situations with $\frac{3}{4}$ and $\frac{1}{2}$

Optional: 15 minutes

This activity offers an additional opportunity for students to make sense of word problems, set up an appropriate representation, use that representation for reasoning, and estimate before solving. Students are presented with four situations that involve only fractions. Two of them require multiplication to solve, and the other two require division. Students decide which operation is needed to answer each question, and before solving, estimate the answer based on the given context.

As students work, monitor how they determine appropriate operations to use. Note any common challenges so they can be discussed later.

Addressing

- 6.NS.A.1

Instructional Routines

- Think Pair Share

Launch

Keep students in groups of 2. Explain to students that the situations presented in this activity all involve the same two fractions, but they do not all require the same operation to solve. Encourage them to make sense of each situation carefully before calculating or reasoning about the answer. Provide access to geometry toolkits (especially graph paper and colored pencils).

Give students 8–10 minutes to work on the activity either individually or with their partner, and then some time to discuss or check their responses. If time is limited, consider asking students to answer either the first two or the last two questions.

Student Task Statement

Here are four situations that involve $\frac{3}{4}$ and $\frac{1}{2}$.

- Before calculating, decide if each answer is greater than 1 or less than 1.
- Write a multiplication equation or division equation for the situation.
- Answer the question. Show your reasoning. Draw a tape diagram, if needed.

1. There was $\frac{3}{4}$ liter of water in Andre's water bottle. Andre drank $\frac{1}{2}$ of the water. How many liters of water did he drink?

2. The distance from Han’s house to his school is $\frac{3}{4}$ kilometers. Han walked $\frac{1}{2}$ kilometers. What fraction of the distance from his house to the school did Han walk?
3. Priya’s goal was to collect $\frac{1}{2}$ kilograms of trash. She collected $\frac{3}{4}$ kilograms of trash. How many times her goal was the amount of trash she collected?
4. Mai’s class volunteered to clean a park with an area of $\frac{1}{2}$ square mile. Before they took a lunch break, the class had cleaned $\frac{3}{4}$ of the park. How many square miles had they cleaned before lunch?

Student Response

Reasoning varies.

1. Estimate: less than 1; equation: $\frac{1}{2} \cdot \frac{3}{4} = ?$; answer: $\frac{3}{8}$
2. Estimate: less than 1; equation: $\frac{1}{2} \div \frac{3}{4} = ?$ or $? \cdot \frac{3}{4} = \frac{1}{2}$; answer: $\frac{2}{3}$
3. Estimate: greater than 1; equation: $\frac{3}{4} \div \frac{1}{2} = ?$ or $? \cdot \frac{1}{2} = \frac{3}{4}$; answer: $1\frac{1}{2}$
4. Estimate: less than 1; equation: $\frac{3}{4} \cdot \frac{1}{2} = ?$; answer: $\frac{3}{8}$

Activity Synthesis

Display the solutions for all to see and give students time to check their work. If time permits, discuss students’ reasoning. Ask:

- “How did you estimate the answers?”
- “How did you know what operation you needed to perform to find the answer?”
- “For which problems was it difficult to tell what operation to use?”
- “Did you draw diagrams or write equations? What diagrams or equations were helpful?”

Some students may notice that the second and third questions involve the phrases “how many times?” and “what fraction of,” which suggest that division might be involved. Ask them to identify the size of 1 group in those cases.

16.3 Pairs of Problems

20 minutes

This activity prompts students to make sense of and write equations for a variety of situations involving fractions and all four operations. After writing equations, students are assigned two problems to solve, at least one of which is a division problem. Before calculating, students first estimate their answer. Doing so helps them to attend to the meaning of the operation and to the reasonableness of their calculated answer in the context of the situation.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR7: Compare and Connect
- Notice and Wonder
- Think Pair Share

Launch

Give students a minute to skim the two sets of problems. Ask them to be prepared to share at least one thing they notice and one thing they wonder. Then, invite a few students to share their observations and questions.

Keep students in groups of 2. Tell students they will practice writing equations to represent situations in context. Ask one person to write an equation for each question labeled with a letter and the number 1, and the other person to do the same for each question labeled with a letter and the number 2. Give groups 4–5 minutes to write their equations, and another 4–5 minutes to check each other's equations and discuss any questions or issues.

Afterward, briefly discuss and compare the equations as a class. Point out equations that correctly represent the same problem (and are thus equivalent) but are expressed differently. For example, a student may write a multiplication equation with a missing factor, while another writes a division equation with an unknown quotient.

Next, assign at least 1 division problem and 1 problem involving another operation for each student (or group) to solve. Consider preparing the assignments, or an efficient way to assign the problems, in advance. Give students 4–5 minutes of quiet work time or collaboration time.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, pause to check for understanding after the first 2–3 minutes of work time.
Supports accessibility for: Organization; Attention

Student Task Statement

1. Work with a partner to write equations for the following questions. One person works on the questions labeled A1, B1, . . . , E1 and the other person works on those labeled A2, B2, . . . , E2.

A1. Lin's bottle holds $3\frac{1}{4}$ cups of water. She drank 1 cup of water. What fraction of the water in the bottle did she drink?

A2. Lin's bottle holds $3\frac{1}{4}$ cups of water. After she drank some, there were $1\frac{1}{2}$ cups of water in the bottle. How many cups did she drink?

B1. Plant A is $\frac{16}{3}$ feet tall. This is $\frac{4}{5}$ as tall as Plant B. How tall is Plant B?

B2. Plant A is $\frac{16}{3}$ feet tall. Plant C is $\frac{4}{5}$ as tall as Plant A. How tall is Plant C?

C1. $\frac{8}{9}$ kilogram of berries is put into a container that already has $\frac{7}{3}$ kilogram of berries. How many kilograms are in the container?

C2. A container with $\frac{8}{9}$ kilogram of berries is $\frac{2}{3}$ full. How many kilograms can the container hold?

D1. The area of a rectangle is $14\frac{1}{2}$ sq cm and one side is $4\frac{1}{2}$ cm. How long is the other side?

D2. The side lengths of a rectangle are $4\frac{1}{2}$ cm and $2\frac{2}{5}$ cm. What is the area of the rectangle?

E1. A stack of magazines is $4\frac{2}{5}$ inches high. The stack needs to fit into a box that is $2\frac{1}{8}$ inches high. How many inches too high is the stack?

E2. A stack of magazines is $4\frac{2}{5}$ inches high. Each magazine is $\frac{2}{5}$-inch thick. How many magazines are in the stack?

2. Trade papers with your partner, and check your partner's equations. If you disagree, work to reach an agreement.
3. Your teacher will assign 2 or 3 questions for you to answer. For each question:
    a. Estimate the answer before calculating it.
    b. Find the answer, and show your reasoning.

Student Response

Estimates and reasoning vary.

A1. Equation: $1 \div \left(3\frac{1}{4}\right) = ?$

1. Estimate: Less than $\frac{1}{3}$ of the water in the bottle.
2. $\frac{4}{13}$ of the bottle. $1 \div \left(3\frac{1}{4}\right) = 1 \div \frac{13}{4} = 1 \cdot \frac{4}{13} = \frac{4}{13}$

A2. Equation: $3\frac{1}{4} - ? = 1\frac{1}{2}$

1. Estimate: A little less than 2 cups.
2. $1\frac{3}{4}$ cups. $3\frac{1}{4} - 1\frac{1}{2} = 1\frac{3}{4}$

B1. Equation: $\frac{16}{3} = \frac{4}{5} \cdot ?$

1. Estimate: Around 7 feet.
2. $6\frac{2}{3}$ feet. $\frac{16}{3} \div \frac{4}{5} = \frac{16}{3} \cdot \frac{5}{4} = \frac{20}{3} = 6\frac{2}{3}$

B2. Equation: $\frac{4}{5} \cdot \left(5\frac{1}{3}\right) = ?$

1. Estimate: A little bit less than 5 feet.
2. $4\frac{4}{15}$ feet. $\frac{4}{5} \cdot \frac{16}{3} = \frac{64}{15} = 4\frac{4}{15}$

C1. Equation: $\frac{8}{9} + \frac{7}{3} = ?$

1. Estimate: Around 3 kg.
2. $3\frac{2}{9}$ kg. $\frac{8}{9} + \frac{7}{3} = \frac{29}{9} = 3\frac{2}{9}$

C2. Equation: $\frac{8}{9} = \frac{2}{3} \cdot ?$

1. Estimate: Between 1 and 2 kg.
2. $\frac{4}{3}$ kg. $\frac{8}{9} \div \frac{2}{3} = \frac{4}{3}$

D1. Equation: $\left(14\frac{1}{2}\right) \div \left(4\frac{1}{2}\right) = ?$

1. Estimate: Between 3 and 4 cm.
2. $3\frac{2}{9}$ cm. $\left(14\frac{1}{2}\right) \div \left(4\frac{1}{2}\right) = \frac{29}{2} \div \frac{9}{2} = \frac{29}{9}$

D2. Equation: $4\frac{1}{2} \cdot 2\frac{2}{5} = ?$

1. Estimate: Around 10 square centimeters.
2. $10\frac{4}{5}$ square centimeters. $\left(4\frac{1}{2}\right) \cdot \left(2\frac{2}{5}\right) = \frac{9}{2} \cdot \frac{12}{5} = \frac{54}{5} = 10\frac{4}{5}$

E1. Equation: $4\frac{2}{5} - 2\frac{1}{8} = ?$

1. Estimate: A little more than 2 inches.
2. $2\frac{11}{40}$ inches. $4\frac{2}{5} - 2\frac{1}{8} = 4\frac{16}{40} - 2\frac{5}{40} = 2\frac{11}{40}$.

E2. Equation: $\left(4\frac{2}{5}\right) \div \frac{2}{5} = ?$

1. Estimate: A little bit more than 10 magazines in the stack.
2. 11 magazines. $\left(4\frac{2}{5}\right) \div \frac{2}{5} = 11$.

Activity Synthesis

Much of the discussion will have occurred in small groups, so a whole-class discussion is not essential unless there are common issues or misconceptions to be addressed. Consider having the solutions accessible for students to check their answers.

Support for English Language Learners

Representing: MLR7 Compare and Connect. Use this routine to prepare students for the whole-class discussion. Invite groups to discuss, "What is the same and what is different?" about the equations they used to solve each problem. Circulate and listen for the words and phrases students use as they make connections between the operations they selected to represent the same situations. Ask students to restate or revoice their explanations using mathematical language as needed. For example, say "Can you say that again using . . . ?" This will help students use mathematical language to describe their reasoning about their choice of operation in relation to the context of a situation.
Design Principle(s): Maximize meta-awareness; Support sense-making

16.4 Baking Cookies

Optional: 15 minutes
This optional activity gives students another opportunity to use what they have learned about all operations to model and solve a problem in a baking context. Students need to make sense of the problem and persevere in solving it (MP1).

Students may approach the problem in different ways (by drawing diagrams, making computations, reasoning verbally, etc.). Students may also choose different operations to obtain the information they need. For instance, instead of dividing by a fraction, they may perform repeated subtraction. Notice the different methods students use and identify strategies or explanations that should be shared with the class.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR5: Co-Craft Questions
- Think Pair Share

Launch

Consider arranging students in new groups of 2. Give students 7–8 minutes of quiet work time and then 1–2 minutes to discuss their response with their partner. Ask students to be prepared to explain their reasoning.

Support for English Language Learners

Conversing, Writing: MLR5 Co-craft Questions. Use this routine to increase students' awareness of the language of mathematical questions. Display the problem about baking cookies, but withhold the final section that includes the question. Give students 2–3 minutes to write down any mathematical questions they have about the situation presented, then to share their questions with a partner. Circulate and listen for questions that involve combining fractional amounts of ingredients. Reveal the actual question and ask students to compare their questions with the original. Listen for and amplify any questions involving the relationships between the two quantities in this task (number of cups of flour and number of cups of butter).
Design Principle(s): Cultivate conversation; Support sense-making

Student Task Statement

Mai, Kiran, and Clare are baking cookies together. They need $\frac{3}{4}$ cup of flour and $\frac{1}{3}$ cup of butter to make a batch of cookies. They each brought the ingredients they had at home.

- Mai brought 2 cups of flour and $\frac{1}{4}$ cup of butter.
- Kiran brought 1 cup of flour and $\frac{1}{2}$ cup of butter.
- Clare brought $1\frac{1}{4}$ cups of flour and $\frac{3}{4}$ cup of butter.

If the students have plenty of the other ingredients they need (sugar, salt, baking soda, etc.), how many whole batches of cookies can they make? Explain your reasoning.

Student Response

The students brought $4\frac{1}{4}$ cups of flour and $1\frac{1}{2}$ cups of butter. Flour: $2+1+1\frac{1}{4}=4\frac{1}{4}$. Butter: $\frac{1}{4}+\frac{1}{2}+\frac{3}{4}=1\frac{1}{2}$.

They have enough flour for $5\frac{2}{3}$ batches, because $4\frac{1}{4}\div\frac{3}{4}=\frac{17}{4}\cdot\frac{4}{3}=\frac{17}{3}=5\frac{2}{3}$.

They have enough butter for $4\frac{1}{2}$ batches, because $1\frac{1}{2}\div\frac{1}{3}=\frac{3}{2}\cdot\frac{3}{1}=\frac{9}{2}=4\frac{1}{2}$. Given the amount of butter, they can make 4 whole batches of cookies.

Activity Synthesis

Consider combining every 2–3 groups of 2 students and having students discuss their responses and methods in larger groups of 4–6.

If time permits, reconvene as a class to highlight a couple of strategies and reflect on the effectiveness and efficiency of students' strategies. For example, if some students performed repeated addition instead of multiplying (or repeated subtraction instead of dividing), ask if repeated addition (or subtraction) is always as efficient as multiplication (or division), or under what circumstances one method might be preferred over the other.

Lesson Synthesis

This lesson gave students opportunities to use operations to solve a variety of contextual problems that involve fractions. Review the operations with students and help them reflect on their problem-solving process. Ask questions such as:

- "How did you add or subtract fractions with different denominators?"
- "How did you multiply fractions?"
- "What method(s) did you use to divide a number by a fraction?"
- "How did you know which operations to use for each situation? How did you know if you chose the right operation?"

16.5 A Box of Pencils

Cool Down: 5 minutes

Addressing

- 6.NS.A.1

Student Task Statement

A box of pencils is $5\frac{1}{4}$ inches wide. Seven pencils, laid side by side, take up $2\frac{5}{8}$ inches of the width.

1. How many inches of the width of the box is *not* taken up by the pencils? Explain or show your reasoning.
2. All 7 pencils have the same width. How wide is each pencil? Explain or show your reasoning.

Student Response

1. $2\frac{5}{8}$ inches, because $5\frac{1}{4} - 2\frac{5}{8} = 2\frac{5}{8}$
2. $\frac{3}{8}$ inch, because $2\frac{5}{8} \div 7 = \frac{21}{8} \cdot \frac{1}{7} = \frac{3}{8}$

Student Lesson Summary

We can add, subtract, multiply, and divide both whole numbers and fractions. Here is a summary of how we add, subtract, multiply, and divide fractions.

- To add or subtract fractions, we often look for a common denominator so the pieces involved are the same size. This makes it easy to add or subtract the pieces.

$$\frac{3}{2} - \frac{4}{5} = \frac{15}{10} - \frac{8}{10}$$

- To multiply fractions, we often multiply the numerators and the denominators.

$$\frac{3}{8} \cdot \frac{5}{9} = \frac{3 \cdot 5}{8 \cdot 9}$$

- To divide a number by a fraction $\frac{a}{b}$, we can multiply the number by $\frac{b}{a}$, which is the reciprocal of $\frac{a}{b}$.

$$\frac{4}{7} \div \frac{5}{3} = \frac{4}{7} \cdot \frac{3}{5}$$

Lesson 16 Practice Problems

Problem 1

Statement

An orange has about $\frac{1}{4}$ cup of juice. How many oranges are needed to make $2\frac{1}{2}$ cups of juice? Select **all** the equations that represent this question.

A. $? \cdot \frac{1}{4} = 2\frac{1}{2}$

B. $\frac{1}{4} \div 2\frac{1}{2} = ?$

C. $? \cdot 2\frac{1}{2} = \frac{1}{4}$

D. $2\frac{1}{2} \div \frac{1}{4} = ?$

Solution

["A", "D"]

Problem 2

Statement

Mai, Clare, and Tyler are hiking from a parking lot to the summit of a mountain. They pass a sign that gives distances.

Parking lot: $\frac{3}{4}$ mile
Summit: $1\frac{1}{2}$ miles

- Mai says: “We are one third of the way there.”
- Clare says: “We have to go twice as far as we have already gone.”
- Tyler says: “The total hike is three times as long as what we have already gone.”

Do you agree with any of them? Explain your reasoning.

Solution

Yes, they are all correct. The total distance in miles from the parking lot to the summit is $\frac{3}{4} + 1\frac{1}{2}$, which is $2\frac{1}{4}$ miles. Mai computed: $\frac{3}{4} = \frac{1}{3} \cdot 2\frac{1}{4}$ or $\frac{3}{4} \div 2\frac{1}{4} = \frac{1}{3}$. Clare computed: $1\frac{1}{2} = 2 \cdot \frac{3}{4}$. Tyler computed: $2\frac{1}{4} \div \frac{3}{4} = 3$.

Problem 3

Statement

Priya's cat weighs $5\frac{1}{2}$ pounds and her dog weighs $8\frac{1}{4}$ pounds. First, estimate the number that would comlpete each sentence. Then, calculate the answer. If any of your estimates were not close to the answer, explain why that may be.

a. The cat is _______ as heavy as the dog.

b. Their combined weight is _______ pounds.

c. The dog is _______ pounds heavier than the cat.

Solution

Answers vary. Sample response:

a. Estimate: The cat weighs less than the dog but more than half as much, so somewhere between $\frac{1}{2}$ and 1. Calculation: $\left(5\frac{1}{2}\right) \div \left(8\frac{1}{4}\right) = \frac{2}{3}$. This matches the estimate.

b. Estimate: Combined, they weigh more than 13 pounds, almost 14 pounds. Calculation: $5\frac{1}{2} + 8\frac{1}{4} = 13\frac{3}{4}$.

c. Estimate: The dog weighs about 3 pounds more than the cat—a little less than 3 pounds. Calculation: $8\frac{1}{4} - 5\frac{1}{2} = 2\frac{3}{4}$.

Problem 4

Statement

Before refrigerators existed, some people had blocks of ice delivered to their homes. A delivery wagon had a storage box in the shape of a rectangular prism that was $7\frac{1}{2}$ feet by 6 feet by 6 feet. The cubic ice blocks stored in the box had side lengths $1\frac{1}{2}$ feet. How many ice blocks fit in the storage box?

A. 270

B. $3\frac{3}{8}$

C. 80

D. 180

Solution

C

(From Unit 4, Lesson 15.)

Problem 5

Statement

Fill in the blanks with 0.001, 0.1, 10, or 1000 so that the value of each quotient is in the correct column.

Close to $\frac{1}{100}$

- $\_\_\_\_\_\_ \div 9$
- $12 \div \_\_\_\_\_\_$

Close to 1

- $\_\_\_\_\_\_ \div 0.12$
- $\frac{1}{8} \div \_\_\_\_\_\_$

Greater than 100

- $\_\_\_\_\_\_ \div \frac{1}{3}$
- $700.7 \div \_\_\_\_\_\_$

Solution

close to $\frac{1}{100}$:

- 0.1
- 1,000

close to 1:

- 0.1
- 0.1

greater than 100:

- 1,000
- 0.001 or 0.1

(From Unit 4, Lesson 1.)

Problem 6

Statement

A school club sold 300 shirts. 31% were sold to fifth graders, 52% were sold to sixth graders, and the rest were sold to teachers. How many shirts were sold to each group—fifth graders, sixth graders, and teachers? Explain or show your reasoning.

Solution

- 93 shirts were sold to fifth graders, because $(0.31) \boldsymbol{\cdot} 300 = 93$.
- 156 shirts were sold to sixth graders, because $(0.52) \boldsymbol{\cdot} 300 = 156$.
- 51 shirts were sold to teachers, because $300 - 93 - 156 = 51$.

(From Unit 3, Lesson 15.)

Problem 7

Statement

Jada has some pennies and dimes. The ratio of Jada's pennies to dimes is 2 to 3.

a. From the information given, can you determine how many coins Jada has?

b. If Jada has 55 coins, how many of each kind of coin does she have?

c. How much are her coins worth?

Solution

a. No, there is not enough information to determine how many pennies and how many dimes Jada has. (We only know that for every 2 pennies, there are 3 dimes.)

b. 22 pennies and 33 dimes. (There are 5 coins total in each group of 2 pennies and 3 dimes. If Jada has 55 coins, that means there are 11 groups, because $55 \div 5 = 11$. There are 22 pennies ($11 \cdot 2 = 22$) and 33 dimes ($11 \cdot 3 = 33$) in total.)

c. \$3.52. (The 22 pennies are worth \$0.22, and the 33 dimes are worth \$3.30. $0.22 + 3.30 = 3.52$.)

(From Unit 2, Lesson 15.)

Lesson 17: Fitting Boxes into Boxes

Goals

- Compare and contrast (orally and using other representations) different ways jewelry boxes could be packed inside larger shipping boxes.
- Determine which size shipping box is least expensive, and present (orally and in writing) a justification.
- Make simplifying assumptions and determine what information is needed to solve a problem about shipping costs.

Learning Targets

- I can use multiplication and division of fractions to reason about real-world volume problems.

Lesson Narrative

In this three-part culminating activity, students use what they have learned to determine the most economical way to ship jewelry boxes using the United States Postal Service (USPS) flat-rate options.

In Part 1, students make sense of the task, outline what they will need to know and do to answer the question, and map out their plan. In Part 2, they model the problem, calculate the number of jewelry boxes that will fit into each shipping box, and determine the associated costs. Students experiment with different orientations for the jewelry boxes to optimize space and minimize cost. In Part 3, they present, reflect, and discuss. Students explain their strategies and reasoning (MP3) and evaluate the decisions about how to fit all 270 jewelry boxes so they ship at the lowest cost (MP4). As a class, students reflect on how the orientation of the jewelry boxes and the size of the shipping boxes affected the unit cost for shipping each box of jewelry.

Depending on the instructional choices made, this lesson could take one or more class meetings. The time estimates are intentionally left blank because the amount of time needed might vary depending on factors such as:

- If students will research the flat-rate options themselves, or be provided with this information.
- If each group will explore all size options or only one option.
- How much organizational support is given to students.
- How student work is ultimately shared with the class (e.g., not at all, informally, or with formal presentations).

Consider defining the scope of work further and setting a time limit for each part of the activity to focus students’ work and optimize class time.

Alignments

Addressing

- 6.G.A.2: Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

- 6.NS.A.1: Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- MLR6: Three Reads
- MLR8: Discussion Supports

Required Materials

Geometry toolkits
For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Measuring tapes
Rulers

Required Preparation

Bring in samples of United States Postal Service flat-rate boxes, or have images of these boxes available.

Student Learning Goals

Let’s use what we learned about fractions to find shipping costs.

17.1 Determining Shipping Costs (Part 1)

In this first part of the project, students make sense of the task at hand and determine what they need to know and do to find the most economical shipping box combination. Students should recognize that they will need to:

- Find out the measurements of the jewelry boxes and shipping boxes, as well as the costs for mailing a shipping box of each size. Asking students to find out this information themselves will increase the modeling demand (MP4).
- Decide on an orientation for the jewelry boxes inside each shipping box and calculate how many jewelry boxes will fit with that particular orientation.
- Test out different orientations and how they affect the number of jewelry boxes to be fitted and the cost.

Students may want to think of a strategy for considering different configurations efficiently, rather than testing all of them, which would be time consuming and repetitive. As students make plans to try out different jewelry box orientations and associated calculations, encourage them to work systematically to minimize omissions and errors. Urge students to create drawings or models of the boxes to show the calculations they will need to make.

Consider supporting students by discussing the different orientations of jewelry boxes in a shipping box (Which orientations are possible? How much empty space would result?) and possible ways to use drawings or diagrams to show the arrangements of jewelry boxes.

Addressing

- 6.G.A.2
- 6.NS.A.1

Instructional Routines

- MLR6: Three Reads

Launch

Give students 1–2 minutes to read the task statement individually and to ask any clarifying questions. Demonstrate the idea of the task by putting a small box inside a larger box in different orientations. Consider displaying USPS flat-rate boxes, or an image of each of the boxes.

Arrange students in groups of 4. Give students 5 minutes of quiet think time to brainstorm about what information is needed to solve this task and share to it with their group. Give another 5 minutes to plan in groups, followed by time to measure boxes or research box options and dimensions for themselves. Provide access to measuring tools.

USPS flat-rate information:

- Small box: $5\frac{3}{8}$ inches by $8\frac{5}{8}$ inches by $1\frac{5}{8}$ inches. Cost: $6.80.
- Medium box 1: 11 inches by $8\frac{1}{2}$ inches by $5\frac{1}{2}$ inches. Cost: $13.45.
- Medium box 2: $11\frac{7}{8}$ inches by $3\frac{3}{8}$ inches by $13\frac{5}{8}$ inches. Cost: $13.45.
- Large box: 12 inches by 12 inches by $5\frac{1}{2}$ inches. Cost: $18.75.

Support for English Language Learners

Reading, Speaking, Representing: MLR6 Three Reads. Use this routine to orient students to the context of the problem. In the first read, students read the problem with the goal of comprehending the situation (an artist is packing jewelry boxes to ship to a store.). Clarify any unknown language, such as a "flat-rate" box or shipping rates, as needed. For the second read, ask students to identify the quantities and mathematical relationships (number of necklaces ordered, the dimensions of the jewelry box). After the final read, ask students to brainstorm possible strategies they may use to solve the problem.
Design Principle(s): Support sense-making

Student Task Statement

An artist makes necklaces. She packs each necklace in a small jewelry box that is $1\frac{3}{4}$ inches by $2\frac{1}{4}$ inches by $\frac{3}{4}$ inch.

A department store ordered 270 necklaces. The artist plans to ship the necklaces to the department store using flat-rate shipping boxes from the post office.

1. Consider the problem: Which of the flat-rate boxes should she use to minimize her shipping cost?

   What other information would you need to be able to solve the problem?

2. Discuss this information with your group. Make a plan for using this information to find the most inexpensive way to ship the jewelry boxes. Once you have agreed on a plan, write down the main steps.

Student Response

Answers vary. Possible response:

1. We need to know the shipping cost and the size (i.e., the dimensions) for each flat-rate box.
2. Possible plan:
   - Decide which group member or members will work with which shipping box.
   - Find out how many jewelry boxes fit into each shipping box.

    - Find out how many and what combination of shipping boxes we need.
    - Compute the total cost for shipping the jewelry boxes.

Activity Synthesis

Reconvene as a class before continuing with the next part. Ask each group to share a couple of specific steps they have taken toward answering the question and a couple of steps they plan on taking to move forward. Highlight any ideas students might have about making the problem-solving process more efficient and systematic. If not already mentioned by students, suggest that each group divide up the calculations to be done so each person is responsible for one shipping box.

17.2 Determining Shipping Costs (Part 2)

After planning and gathering information in Part 1, students now calculate the cost of shipping jewelry boxes in each of the USPS flat-rate boxes. Each member of the group will select one of the 4 flat-rate shipping box sizes, decide on the best orientation for the jewelry boxes inside the shipping box, and calculate the cost to ship 270 boxes.

Notice groups using different strategies for division with fractions. Ask students to think about the different ways they have used fractions in calculations. If they are stuck, remind students that drawing the boxes, or making them from paper, might help them to visualize what calculations would be most helpful in finding a solution for the task at hand.

Addressing

- 6.G.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Keep students in the same groups of 4. Ask each group member to select a different size of shipping box so that all boxes are represented in each group. Provide access to geometry toolkits and rulers (or tape measures).

Once shipping boxes are assigned, give students quiet time to work. After most students have attempted a few box orientations and made their first calculations, ask the class to pause their work. Use MLR1 Stronger and Clearer Each Time: Students successively share their orientations and reasoning for their particular orientation, and get feedback on both language and orientation choices from their partners before the post-write.

Support for English Language Learners

Conversing: This activity recommends using *MLR1 Stronger and Clearer Each Time* to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. Give students time to meet with 2–3 partners to share and get feedback on their work. Display prompts for feedback that students can use to help their partner strengthen and clarify their ideas. For example, "Your explanation tells me . . .", "Can you say more about why you . . . ?", and "A detail (or word) you could add is _____, because . . . ." Give students with 3–4 minutes to revise their initial draft based on feedback from their peers.
Design Principle(s): Optimize output (for explanation)

Support for Students with Disabilities

Engagement: Internalize Self Regulation. Provide a project checklist that chunks the various steps of this activity into a set of manageable tasks.
Supports accessibility for: Organization; Attention

Student Task Statement

Work with your group to find the best plan for shipping the boxes of necklaces. Each member of your group should select a different type of flat-rate shipping box and answer the following questions. Recall that each jewelry box is $1\frac{3}{4}$ inches by $2\frac{1}{4}$ inches by $\frac{3}{4}$ inch, and that there are 270 jewelry boxes to be shipped.

For each type of flat-rate shipping box:

1. Find how many jewelry boxes can fit into the box. Explain or show how the jewelry boxes can be packed in the shipping box. Draw a sketch to show your thinking, if needed.
2. Calculate the total cost of shipping all 270 jewelry boxes in shipping boxes of that type. Show your reasoning and organize your work so it can be followed by others.

Student Response

1. Answers vary. Students should decide how they want to orient the jewelry boxes inside the mailing boxes. Then they can compute how many boxes will fit into the mailing box with that particular orientation. Often there will be a gap left where additional jewelry boxes will fit. Each mailing box allows for several different orientations.

Small box: The jewelry box has dimension $\frac{7}{4} \times \frac{9}{4} \times \frac{3}{4}$ or $\frac{14}{8} \times \frac{18}{8} \times \frac{6}{8}$ and the small shipping box has dimension $\frac{43}{8} \times \frac{69}{8} \times \frac{13}{8}$. At most 18 (3 by 3 by 2) jewelry boxes can fit into the shipping box (other orientations would lead to fewer boxes).

Medium box 1: Different orientations and results are possible. If the jewelry box is oriented with the $\frac{3}{4}$ inch side as the height and lining up the $1\frac{3}{4}$-inch side along the 11-inch side of the mailing box, then we can fit $6 \cdot 3 \cdot 7 = 126$ boxes. This will leave a gap in the back of the box where we can fit an additional $1 \cdot 3 \cdot 7 = 21$ boxes for a total of 147 boxes.

Medium box 2: Different orientations and results are possible. If the jewelry box is oriented with the $\frac{3}{4}$-inch side as the height and lining up the $2\frac{1}{4}$-inch side along the $13\frac{5}{8}$-inch side of the mailing box, then we can fit $6 \cdot 6 \cdot 4 = 144$ boxes. This will leave a gap in the back of the box where we can fit an additional $1 \cdot 1 \cdot 7 = 7$ boxes for a total of 151 boxes.

Large box: Different orientations should be considered. Orientation 1: If the jewelry box is oriented with the $\frac{3}{4}$-inch side as the height, then we can fit 234 boxes: $5 \cdot 6 \cdot 7 = 210$ oriented one way, and an additional $6 \cdot 2 \cdot 2 = 24$ oriented differently, squeezed in the gap left by the first set of boxes. Orientation 2: If the jewelry box is oriented with the $1\frac{3}{4}$-inch side as the height, then we can fit 252 boxes: $16 \cdot 5 \cdot 3 = 240$ oriented one way, and an additional $6 \cdot 1 \cdot 2 = 12$ oriented differently, squeezed in the gap left by the first set of boxes.

2.
    - Small box: We would need 15 small boxes, for a total cost of $102.
    - Medium boxes: We would need 2 medium boxes of either type, for a total cost of $26.90.
    - Large box: We would need 2 large boxes, for a total cost of $37.50.

Activity Synthesis

Small-group and whole-class reflections will occur in the next activity.

17.3 Determining Shipping Costs (Part 3)

In the final phase of the shipping project, students present, reflect on, and revise their work within their small group. They discuss their decisions, evaluate the accuracy of their calculations, and then revise them as needed. In groups, they discuss which shipping box size, or combination of sizes, will be the most economical for shipping 270 jewelry boxes.

Addressing

- 6.NS.A.1

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports

Launch

Keep students in the same groups of 4. Give them 10–12 minutes to share each member's work in small groups and make revisions as needed. Use MLR 2 (Collect and Display) and refer to select student language while engaging in the whole group discussion. Display questions such as the following. Ask students to use them to guide their discussion.

- How many different ways can the jewelry boxes fit into each shipping box?
- How does the orientation of the jewelry boxes affect how they fit within the shipping boxes?
- Do some shipping boxes have more wasted space than others? Why?
- Can you use diagrams to show and compare the unused spaces in different configurations?
- Are there ways to reduce the amount of wasted space when shipping exactly 270 jewelry boxes?
- How does the orientation of the jewelry boxes affect the cost of shipping with each shipping box?
- Is there a way to increase the number of jewelry boxes that will fit into a shipping box? How?

Once each group member has had a chance to share individual work and before discussing this problem as a whole class, give students 4–5 minutes to decide on the best (least expensive) option for shipping 270 jewelry boxes and write down ideas for explaining their strategies.

Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames such as "_____ jewelry boxes can fit into one shipping box because . . . ."
Supports accessibility for: Language; Organization

Student Task Statement

1. Share and discuss your work with the other members of your group. Your teacher will display questions to guide your discussion. Note the feedback from your group so you can use it to revise your work.
2. Using the feedback from your group, revise your work to improve its correctness, clarity, and accuracy. Correct any errors. You may also want to add notes or diagrams, or remove unnecessary information.
3. Which shipping boxes should the artist use? As a group, decide which boxes you recommend for shipping 270 jewelry boxes. Be prepared to share your reasoning.

Student Response

The cheapest option would be to use one small and one large shipping box. This would fit exactly 270 jewelry boxes for $25.55. However, using two medium-sized boxes would cost only slightly more at $26.90 and allows for roomier packing.

Activity Synthesis

After small groups have reached an agreement on shipping box recommendations, discuss as a class so students can see a variety of strategies for orientation and calculation. Depending on the time available, students could just stand and share, groups could create a gallery walk, or each group could present more formally. Select one group to present their findings about each shipping box. Select an additional 1–2 groups to share their recommended shipping box size, or combination of sizes, needed to ship all 270 jewelry boxes.

To help students tie everything together, consider discussing the following questions:

- "How did the choice of jewelry box orientation affect how many would fit into each shipping box?"
- "How did the quantity of jewelry boxes (270) affect the choice of shipping box size?"
- "How did you calculate how many jewelry boxes would fit in a box? Did you multiply the lengths of the jewelry boxes or divide the lengths of the shipping boxes?"
- "Did the size of fractions affect how you performed division? What methods did you use to divide?"
- "How did you confirm or check your calculations?"
- "If you had a chance to solve a similar problem, what might you do differently to improve the efficiency or accuracy of your work?"

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion after each group presents their findings. Invite students to restate what they heard the group present using mathematical language. Consider providing students time to restate what they heard to a partner, before selecting one or two students to share with the class. This will provide additional opportunities for all students to speak.
Design Principle(s): Support sense-making; Maximize meta-awareness

Lesson Synthesis

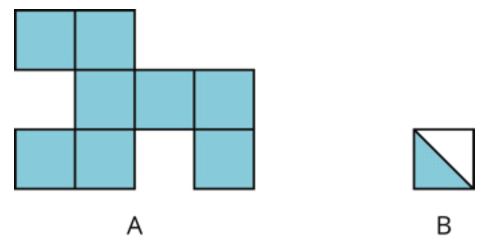
The bulk of students' reflection about the mathematics of the unit should happen in the last task of the lesson. To wrap up this culminating lesson, consider highlighting instances of mathematical modeling in the lesson by asking questions such as:

- “When did you have to make assumptions to make the problem solving possible or more manageable? What assumptions did you make?” (Possible responses: When deciding which shipping boxes to use, I assumed that only one type of shipping box would be used. When deciding how many jewelry boxes would fit into a shipping box, I assumed that no bubble wraps or other packing materials were needed. When deciding how to pack the jewelry boxes, I assumed that using the same orientation within a box was preferable.)
- “Was there any missing information you had to find out before you could proceed?”
- “Were there times when you had to change course or strategy because the approach you had chosen was not productive?”

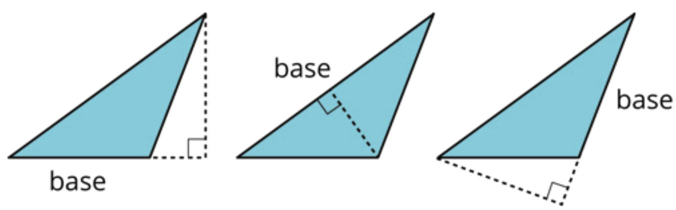
Glossary

area
Area is the number of square units that cover a two-dimensional region, without any gaps or overlaps.

For example, the area of region A is 8 square units. The area of the shaded region of B is $\frac{1}{2}$ square unit.



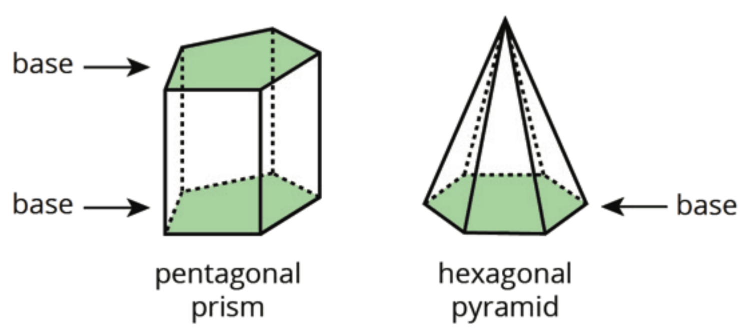
base (of a parallelogram or triangle)
We can choose any side of a parallelogram or triangle to be the shape's base. Sometimes we use the word *base* to refer to the length of this side.



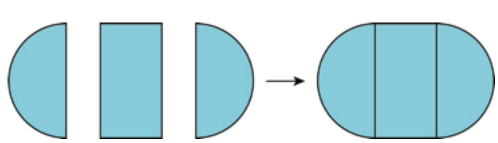
base (of a prism or pyramid)
The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

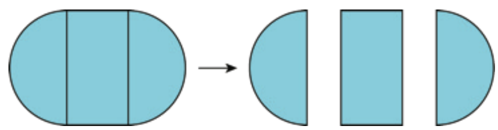


compose
Compose means "put together." We use the word *compose* to describe putting more than one figure together to make a new shape.

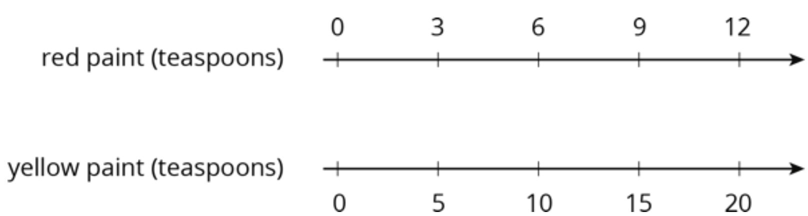


cubed
We use the word *cubed* to mean "to the third power." This is because a cube with side length s has a volume of $s \cdot s \cdot s$, or s^3.

decompose
Decompose means "take apart." We use the word *decompose* to describe taking a figure apart to make more than one new shape.

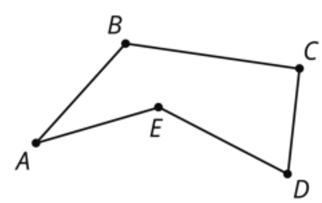


double number line diagram
A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.



edge
Each straight side of a polygon is called an edge.

For example, the edges of this polygon are segments AB, BC, CD, DE, and EA.



equivalent ratios
Two ratios are equivalent if you can multiply each of the numbers in the first ratio by the same factor to get the numbers in the second ratio. For example, $8 : 6$ is equivalent to $4 : 3$, because $8 \cdot \frac{1}{2} = 4$ and $6 \cdot \frac{1}{2} = 3$.

A recipe for lemonade says to use 8 cups of water and 6 lemons. If we use 4 cups of water and 3 lemons, it will make half as much lemonade. Both recipes taste the same, because and are equivalent ratios.

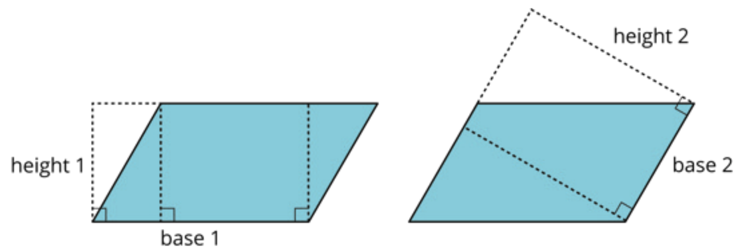
cups of water	number of lemons
8	6
4	3

exponent
In expressions like 5^3 and 8^2, the 3 and the 2 are called exponents. They tell you how many factors to multiply. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

face
Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

height (of a parallelogram or triangle)
The height is the shortest distance from the base of the shape to the opposite side (for a parallelogram) or opposite vertex (for a triangle).

We can show the height in more than one place, but it will always be perpendicular to the chosen base.

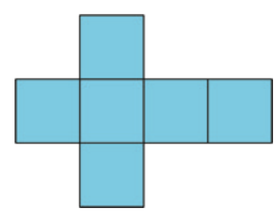


meters per second
Meters per second is a unit for measuring speed. It tells how many meters an object goes in one second.

For example, a person walking 3 meters per second is going faster than another person walking 2 meters per second.

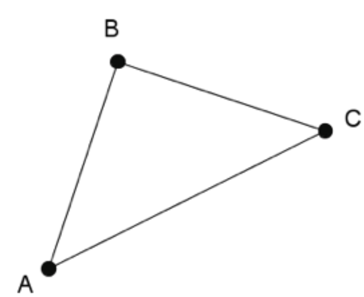
net
A net is a two-dimensional figure that can be folded to make a polyhedron.

Here is a net for a cube.



opposite vertex
For each side of a triangle, there is one vertex that is not on that side. This is the opposite vertex.

For example, point A is the opposite vertex to side BC.

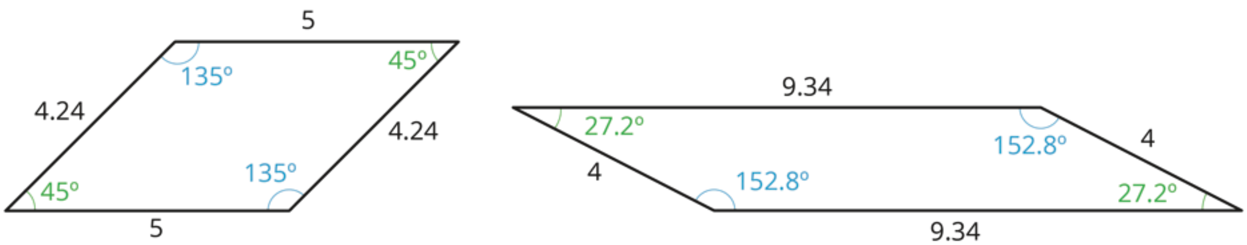


pace
Pace is one way to describe how fast something is moving. Pace tells how much time it takes the object to travel a certain distance.

For example, Diego walks at a pace of 10 minutes per mile. Elena walks at a pace of 11 minutes per mile. Elena walks slower than Diego, because it takes her more time to travel the same distance.

parallelogram
A parallelogram is a type of quadrilateral that has two pairs of parallel sides.

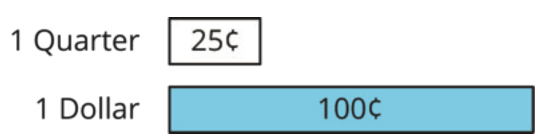
Here are two examples of parallelograms.



per
The word *per* means "for each." For example, if the price is \$5 per ticket, that means you will pay \$5 *for each* ticket. Buying 4 tickets would cost \$20, because $4 \cdot 5 = 20$.

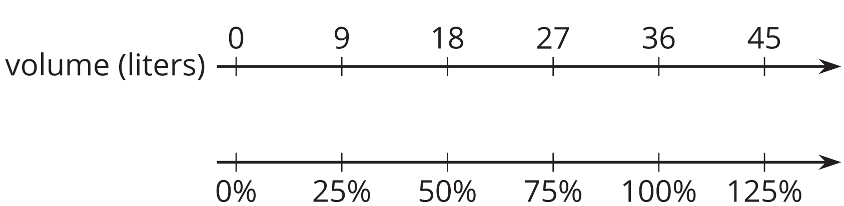
percent
The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



percentage
A percentage is a rate per 100.

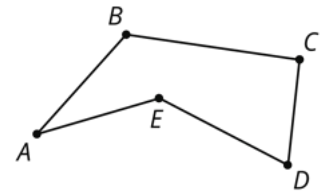
For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.



polygon

A polygon is a closed, two-dimensional shape with straight sides that do not cross each other.

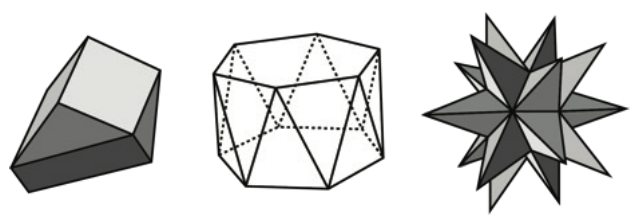
Figure $ABCDE$ is an example of a polygon.



polyhedron

A polyhedron is a closed, three-dimensional shape with flat sides. When we have more than one polyhedron, we call them polyhedra.

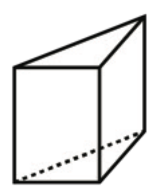
Here are some drawings of polyhedra.



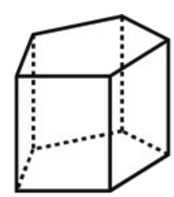
prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

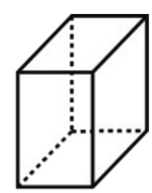
Here are some drawings of prisms.



triangular prism



pentagonal prism

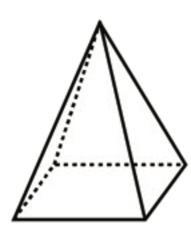


rectangular prism

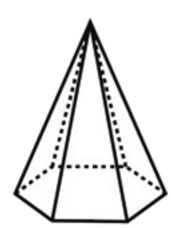
pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

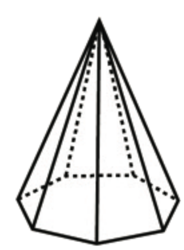
Here are some drawings of pyramids.



rectangular pyramid



hexagonal pyramid

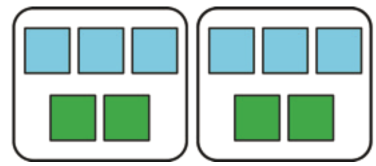


heptagonal pyramid

quadrilateral
A quadrilateral is a type of polygon that has 4 sides. A rectangle is an example of a quadrilateral. A pentagon is not a quadrilateral, because it has 5 sides.

ratio
A ratio is an association between two or more quantities.

For example, the ratio $3:2$ could describe a recipe that uses 3 cups of flour for every 2 eggs, or a boat that moves 3 meters every 2 seconds. One way to represent the ratio $3:2$ is with a diagram that has 3 blue squares for every 2 green squares.



reciprocal
Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

region
A region is the space inside of a shape. Some examples of two-dimensional regions are inside a circle or inside a polygon. Some examples of three-dimensional regions are the inside of a cube or the inside of a sphere.

same rate
We use the words *same rate* to describe two situations that have equivalent ratios.

For example, a sink is filling with water at a rate of 2 gallons per minute. If a tub is also filling with water at a rate of 2 gallons per minute, then the sink and the tub are filling at the same rate.

speed
Speed is one way to describe how fast something is moving. Speed tells how much distance the object travels in a certain amount of time.

For example, Tyler walks at a speed of 4 miles per hour. Priya walks at a speed of 5 miles per hour. Priya walks faster than Tyler, because she travels more distance in the same amount of time.

squared
We use the word *squared* to mean "to the second power." This is because a square with side length s has an area of $s \cdot s$, or s^2.

surface area
The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

table

A table organizes information into horizontal *rows* and vertical *columns*. The first row or column usually tells what the numbers represent.

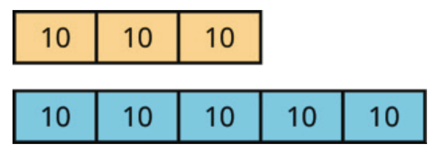
For example, here is a table showing the tail lengths of three different pets. This table has four rows and two columns.

pet	tail length (inches)
dog	22
cat	12
mouse	2

tape diagram

A tape diagram is a group of rectangles put together to represent a relationship between quantities.

For example, this tape diagram shows a ratio of 30 gallons of yellow paint to 50 gallons of blue paint.



If each rectangle were labeled 5, instead of 10, then the same picture could represent the equivalent ratio of 15 gallons of yellow paint to 25 gallons of blue paint.

unit price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

unit rate

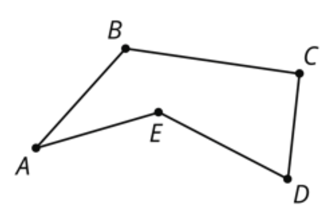
A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

vertex

A vertex is a point where two or more edges meet. When we have more than one vertex, we call them vertices.

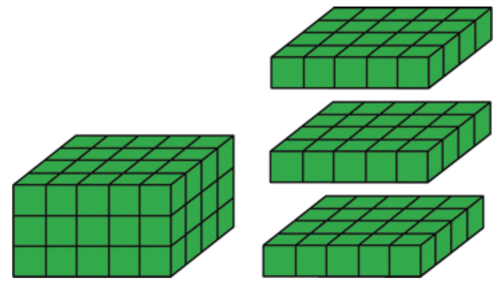
The vertices in this polygon are labeled A, B, C, D, and E.



volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 units^3, because it is composed of 3 layers that are each 20 units^3.



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