INTRODUCTION

- **Eigen Vector-**

- In linear algebra , an eigenvector or characteristic vector of a square matrix is a vector that does not changes its direction under the associated linear transformation.

- In other words – If V is a vector that is not zero, than it is an eigenvector of a square matrix A if Av is a scalar multiple of v. This condition should be written as the equation:

$$AV= \lambda v$$

Contd….

- **Eigen Value-**
- In above equation λ is a scalar known as the **eigenvalue** or **characteristic value** associated with eigenvector **v.**

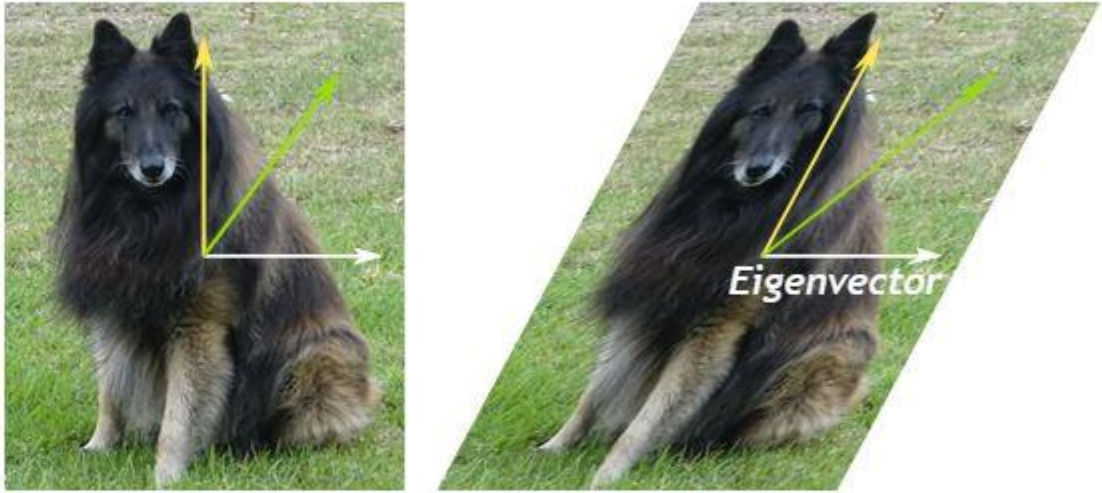
- We can find the eigenvalues by determining the roots of the characteristic equation-

$$|A - \lambda I| = 0$$

Eigenvector and Eigenvalue

They have many uses!

A simple example is that an eigenvector **does not change direction** in a transformation:



How do we find that vector?

The Mathematics Of It

For a square matrix **A**, an Eigenvector and Eigenvalue make this equation true:

$$Av = \lambda v$$

Matrix

Eigenvector

Eigenvalue

Example: For this matrix

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$$

an eigenvector is

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

with a matching eigenvalue of 6

Let's do some matrix multiplies to see if that is true.

Av gives us:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -6\times1+3\times4 \\ 4\times1+5\times4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

λv gives us :

$$6\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

Yes they are equal!

So we get $Av = \lambda v$ as promised.

How do we find these eigen things?

We start by finding the **eigenvalue**. We know this equation must be true:

$$Av = \lambda v$$

Next we put in an identity matrix so we are dealing with matrix-vs-matrix:

$$Av = \lambda I v$$

Bring all to left hand side:

$$Av - \lambda I v = 0$$

If **v** is non-zero then we can (hopefully) solve for λ using just the determinant:

$$| A - \lambda I | = 0$$

Let's try that equation on our previous example:

Example: Solve for λ

Start with $| A - \lambda I | = 0$

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Which is:

$$\begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

Calculating that determinant gets:

$$(-6-\lambda)(5-\lambda) - 3\times4 = 0$$

Which simplifies to this Quadratic Equation:

$$\lambda^2 + \lambda - 42 = 0$$

And solving it gets:

$$\lambda = -7 \text{ or } 6$$

And yes, there are **two** possible eigenvalues.

Why?

What is the purpose of these?

One of the cool things is we can use matrices to do transformations in space, which is used a lot in computer graphics.

In that case the eigenvector is "the direction that doesn't change direction" !

And the eigenvalue is the scale of the stretch:

- **1** means no change,
- **2** means doubling in length,
- **−1** means pointing backwards along the eigenvalue's direction
- etc

Why "Eigen"

Eigen is a German word meaning "own" or "typical"

*"das ist ihnen **eigen**"* is German for *"that is **typical** of them"*

Sometimes in English we use the word "characteristic", so an eigenvector can be called a "characteristic vector".

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad \text{and vectors} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Which are eigenvectors? What are their eigenvalues?

Solution

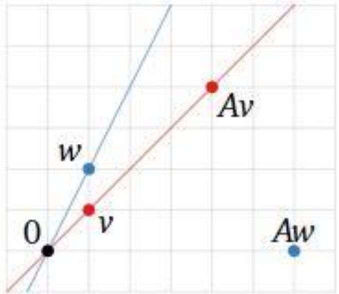
We have

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v.$$

Hence, v is an eigenvector of A, with eigenvalue $\lambda = 4$. On the other hand,

$$Aw = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

which is not a scalar multiple of w. Hence, w is not an eigenvector of A.

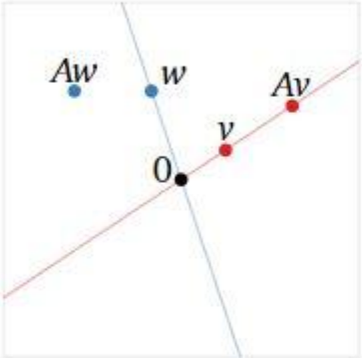


v is an eigenvector

w is not an eigenvector

Example (An eigenvector with eigenvalue 0).

To say that $Av = \lambda v$ means that Av and λv are *collinear with the origin*. So, an eigenvector of A is a nonzero vector v such that Av and v lie on the same line through the origin. In this case, Av is a scalar multiple of v; the eigenvalue is the scaling factor.



v is an eigenvector

w is not an eigenvector

For matrices that arise as the standard matrix of a linear transformation, it is often best to draw a picture, then find the eigenvectors and eigenvalues geometrically by studying which vectors are not moved off of their line. For a transformation that is defined geometrically, it is not necessary even to compute its matrix to find the eigenvectors and eigenvalues.

Examples

- **Two-dimensional matrix example-**

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$$

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Eigenvectors v of this transformation satisfy the equation,

$$Av = \lambda v$$

Rearrange this equation to obtain-

$$(A - \lambda I)v = 0$$

For $\lambda = 1$, Equation becomes,

$$(A - I)v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution, $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 3$, Equation becomes,

$$(A-3I)u = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the vectors $v_{\lambda=1}$ and $v_{\lambda=3}$ are eigenvectors of A associated with the eigenvalues $\lambda = 1$ and $\lambda = 3$, respectively.

Three-dimensional matrix example-

Ex.2 Find the eigenvalue and eigenvector of matrix A.

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -1 & 0 \\ 0 & \lambda+3 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+4)(\lambda+3)(\lambda+2) = 0$$

therefore the eigen values of A are-

$$\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -4$$

For $\lambda = -2$, Equation becomes,

$$(\lambda_1 I - A)v_1 = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has the solution-

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$