

Numerical Relativity Cheat Sheet

Equations I should remember, but I don't

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1 Conventions

We denote the metric as $g_{\alpha\beta}$.

2 ADM Decomposition

The line element ds^2 is

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (1)$$

$$\gamma_{a\beta} = g_{\alpha\beta} + n_\alpha n_\beta \quad (2)$$

$$\gamma^{a\beta} = g^{\alpha\beta} + n^\alpha n^\beta \quad (3)$$

$$n^\alpha \gamma_{\alpha\beta} = 0 \quad (4)$$

$$t^\alpha = \alpha n^\alpha + \beta^\alpha \quad (5)$$

$$n^\alpha = \frac{1}{\alpha}(1, -\beta^i) \quad (6)$$

$$n_\alpha = (-\alpha, 0, 0, 0) \quad (7)$$

2.1 Constraints

Hamiltonian constraint

$$\mathcal{E} = n^\alpha n^\beta T_{\alpha\beta} . \quad (8)$$

Momentum constraint

$$S_i = -\gamma_{i\alpha} n_\beta T^{\alpha\beta} . \quad (9)$$

3 Matter and Equations of State

Let ρ_0 be the rest-mass density, and ϵ the specific internal energy density, the total mass-energy ρ measured by an observer comoving with the fluid is

$$\rho = \rho_0(1 + \epsilon) = \rho_0 + \rho_0\epsilon = \rho_0 + \varepsilon_{\text{int}} , \quad (10)$$

with ε_{int} internal energy density.

The enthalpy is $h = (1 + \epsilon + P/\rho_0)$, or $\rho_0 h = \rho + P$.

The Gamma-law equation of state is $P = (\Gamma - 1)\rho_0\epsilon$, or $P = (\Gamma - 1)\varepsilon_{\text{int}}$.

3.1 Velocity definitions

$$u_\alpha u^\alpha = -1 \quad (11)$$

Let u^α be the four-velocity of the fluid.

$$W = -n_\alpha u^\alpha \quad (12)$$

$$u^t = \frac{W}{\alpha} . \quad (13)$$

IllinoisGRMHD:

$$v_{\text{IL}}^i = \frac{u^i}{u^t} \quad (14)$$

Valencia:

$$v_{\text{VA}}^i = \frac{1}{\alpha} \left(\frac{u^i}{u^t} + \beta^i \right) \quad (15)$$

Conversion:

$$v_{\text{VA}}^i = \frac{1}{\alpha} (v_{\text{IL}}^i + \beta^i) \quad (16)$$

$$v_{\text{IL}}^i = \alpha (v_{\text{VA}}^i - \beta^i) \quad (17)$$

$$W = \frac{1}{\sqrt{1 - v_{\text{VA}}^i v_{i,\text{VA}}}} \quad (18)$$

Stress-energy tensor of a perfect fluid

$$T^{\alpha\beta} = \rho_0 h u^\alpha u^\beta + P g^{\alpha\beta} , . \quad (19)$$

4 Global quantities

Rest-mass

$$M_0 = \int d^3x \sqrt{\gamma} W \rho_0 \quad (20)$$

5 Useful identities

$$\sqrt{-g} = \alpha \sqrt{\gamma} \quad (21)$$