Numerical Relativity Cheat Sheet

Equations I should remember, but I don't

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Contents

1 Conventions

We denote the 4-dimensional metric as $g_{\alpha\beta}$ and the 3-dimensional one as $\gamma_{\alpha\beta}$.

2 ADM Decomposition

The line element $\mathrm{d}s^2$ is

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -\alpha^2\,\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\,\mathrm{d}t)(\mathrm{d}x^j + \beta^j\,\mathrm{d}t) \tag{1}$$

$$\gamma_{a\beta} = g_{\alpha\beta} + n_\alpha n_\beta \tag{2}$$

$$\gamma^{a\beta} = g^{\alpha\beta} + n^\alpha n^\beta \tag{3}$$

$$n^\alpha \gamma_{\alpha\beta} = 0 \tag{4}$$

$$t^\alpha = \alpha n^\alpha + \beta^\alpha \tag{5}$$

$$n^\alpha = \frac{1}{\alpha}(1, -\beta^i) \tag{6}$$

$$n_\alpha = (-\alpha, 0, 0, 0) \tag{7}$$

2.1 Constraints

Hamiltonian constraint

$$\mathcal{E} = n^\alpha n^\beta T_{\alpha\beta}\,. \tag{8}$$

Momentum constraint

$$S_i = -\gamma_{i\alpha} n_\beta T^{\alpha\beta}\,. \tag{9}$$

2.2 BSSN

Conformal metric

$$\bar\gamma_{ij} = \mathrm{e}^{-4\phi}\gamma_{ij}\,, \tag{10}$$

with

$$\phi = \frac{1}{12}\ln\gamma\,. \tag{11}$$

We can also define χ,

$$\chi = \mathrm{e}^{-4\phi}\,, \tag{12}$$

so that

$$\bar\gamma_{ij} = \chi\gamma_{ij}\,. \tag{13}$$

Often ψ is defined:

$$\psi = \mathrm{e}^{\phi} = \gamma^{1/12} \tag{14}$$

3 Matter and Equations of State

Let ρ_0 be the rest-mass density, and ϵ the specific internal energy density, the total mass-energy ρ measured by an observer comoving with the fluid is

$$\rho = \rho_0(1+\epsilon) = \rho_0 + \rho_0\epsilon = \rho_0 + \varepsilon_{\rm int}\,, \tag{15}$$

with $\varepsilon_{\rm int}$ internal energy density.

The specific enthalpy is $h = (1 + \epsilon + P/\rho_0)$, or $\rho_0 h = \rho + P$.

The Gamma-law equation of state is $P = (\Gamma - 1)\rho_0\epsilon$, or $P = (\Gamma - 1)\varepsilon_{\rm int}$.

3.1 Velocity definitions

$$u_\alpha u^\alpha = -1 \tag{16}$$

Let u^α be the four-velocity of the fluid.

$$W = -n_\alpha u^\alpha \tag{17}$$

$$u^t = \frac{W}{\alpha}\,. \tag{18}$$

`IllinoisGRMHD`:

$$v^i_{\mathrm{IL}} = \frac{u^i}{u^t} \tag{19}$$

Valencia:

$$v^i_{\mathrm{VA}} = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right) \tag{20}$$

Conversion:

$$v^i_{\mathrm{VA}} = \frac{1}{\alpha}\left(v^i_{\mathrm{IL}} + \beta^i\right) \tag{21}$$

$$v^i_{\mathrm{IL}} = \alpha\left(\alpha v^i_{\mathrm{IL}} - \beta^i\right) \tag{22}$$

$$W = \frac{1}{\sqrt{1 - v^i_{\mathrm{VA}} v_{i,\mathrm{VA}}}} \tag{23}$$

Stress-energy tensor of a perfect fluid

$$T^{\alpha\beta} = \rho_0 h u^\alpha u^\beta + P g^{\alpha\beta},. \tag{24}$$

4 Global quantities

Rest-mass

$$M_0 = \int \mathrm{d}^3 x \sqrt{\gamma} W \rho_0 \tag{25}$$

5 Useful identities

$$\sqrt{-g} = \alpha\sqrt{\gamma} \tag{26}$$

6 Numerical values

Length-scale associated to $1\,M_\odot$:

$$[\mathrm{L}] = \frac{GM_\odot}{c^2} = 1.477\,\mathrm{km} \tag{27}$$

Time-scale associated to $1\,M_\odot$:

$$[\mathrm{T}] = \frac{GM_\odot}{c^3} = 4.93\,\mu\mathrm{s} \approx 5\,\mu\mathrm{s} \tag{28}$$

Frequency-scale associated to $1\,M_\odot$:

$$[\mathrm{f}] = \frac{1}{[\mathrm{T}]} = \frac{c^3}{GM_\odot} = 203\,\mathrm{kHz} \approx 200\,\mathrm{kHz} \tag{29}$$