Analysis In Several Variables Homework 6

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1 Question 1

Prove that every non-empty open set S in $\mathbb{R}$ contains both rational and irrational numbers

Finding a rational element:
By the Representation Theorem of Open sets in $\mathbb{R}$, S is the union of component intervals
Let one of these components be denoted (a, b)
Let x be a rational number such that for some $n \in \mathbb{Z}^+$, $nx > a$ and $nx < b$
This can be done using the archimedean property
n is a rational number and x is a rational number therefore nx is rational and in (a, b)
Therefore S contains a rational number

Finding an irrational number:
Suppose towards contradiction that (a, b) contained no irrational numbers
This would mean that $(a, b) \subseteq \mathbb{Q}$
Since $\mathbb{Q}$ is countable, that would mean there is some open interval of the form (a, b) which is countable

This is clearly a contradiction, as cantors argument proves that $(0, 1)$ is uncountable and there is a bijection between (a,b) and an uncountable subset of $(0, 1)$

Note: The sigmoid function $f(x) = \frac{1}{1+e^{-x}}$ with domain (a, b) is injective and maps to a subset of $(0, 1)$

Therefore (a, b) must contain irrational numbers

2 Question 2

Prove that every closed set in $\mathbb{R}$ is the countable intersection of collection of a set of open sets

First consider a closed set of the form $S_1 = \mathbb{R} - I$ Where I is some open interval
Let $I = (a, b)$ $S_1 = \bigcap_{n \in \mathbb{Z}^+} (-\infty, a + \frac{1}{n}) \cup (b - \frac{1}{n}, \infty)$ This is the countable (but

not finite!) intersection of open sets

Now consider any closed set:
Let $S = \mathbb{R} - A$ where A is an open set
A is the union of a countable collection of disjoint component intervals
Let I be the collection of component intervals of A
$A = I_1 \cup I_2 \cup I_3...$
$S = \mathbb{R} - (I_1 \cup I_2...)$
By demorgan's laws:
S = $(\mathbb{R} - I_1) \cap (\mathbb{R} - I_2) \cap ...$
It has been shown above that $\mathbb{R} - I_n$ is the intersection of a countable set of open sets
Therefore S is the countable intersection of the countable intersection of a collection of open sets!

Therefore a closed set S is the countable intersection of a collection of open sets!

3 Question 3

Prove that open n balls and n dimensional open intervals are open sets in $\mathbb{R}^n$

Part 1: Showing open n balls are open:
Let $B(x, r)$ be an n dimensional ball for some point x
Fix $y \in B(x, r)$
If we can show y is an interior point, then we have all points in $B(x, r)$ are interior points, and it is an open set
We know that $||y - x|| < r$
Consider the ball $B_2 = B(y, r - ||y - x||)$
We have for all $z \in B_2$, $||y - z|| < r - ||y - x||$
$||y - z|| + ||y - x|| < r$
$||y - z|| + ||x - y||$
$||y - z + x - y|| \leq ||y - z|| + ||x - y|| < r$ by the triangle inequality
$||x - z|| < r$ Therefore $z \in B_2$ and $B_2 \subseteq B$
Since for all points $y \in B$ there is some open ball which is contained, B is an open set

Part 2: Showing n dimensional open intervals are open
A n dimenensional open interval is of the form $I_1 \times I_2 ... I_n$ where each I is a 1 dimensional open interval
Let S represent some n-dimensional open interval
Fix $x \in S$
x is a vector of the form $(x_1, x_2 ... x_n)$
Each $x_i \in I_i$ for an integer $0 < i \leq n$
Since x_i is an element of I_i, x is an interior point of I_i, and some open ball exists such that $B(x_i, r) \subseteq I_i$

Let $\{B_i(x_i, r)\}$ be the set of open balls contianed in I_i for each x_i
Now consider $B_1 \times B_2 \times ...B_n$
This is an n dimensional open ball which is completely contained in each dimension
Therefore there is an n-dimensional ball which is completely contained in S for each $x \in S$ and S is open

4 Question 4

<u>If $S \subseteq \mathbb{R}^n$ prove that int(S) is the union of all open subsets of $\mathbb{R}^n$ which are contained in S. This is described by saying that int(S) is the largest open subset of S.</u>

Assume towards contradiction that int(S) is not the largest open subset in S
This implies that there exists some open subset of S (call it A) which is "larger" (has more elements)
Therefore there is some element $x \in A - int(S)$
Since A is an open set, all of its points are interior points, and x is an interior point
Let $B(x, r)$ represent an open ball centered at x which is completely contained in A (we know this exists because x is an interior point of A)
$B(x, r) \subseteq A \subseteq S$
$B(x, r) \subseteq S$
Therefore x is an interior point of S and in $int(S)$
This contradicts the claim that $x \in A - int(S)$
Therefore $int(S)$ is the largest open subset in S

5 Question 5

<u>If S and T are subsets of $\mathbb{R}^n$, show that:</u>

Part 1:
$int(S) \cap int(T) = int(S \cap T)$

Fix $x \in int(S \cap T)$
$\exists B(x, r)$ s.t. $B(x, r) \in (S \cap T)$
$B(x, r) \subseteq S$ and $B(x, r) \subseteq T$
Therefore $x \in int(S)$ and $x \in int(T)$ (because there is some open ball centered at x completely contained)
And $x \in int(S) \cap int(T)$
$int(S \cap T) \subseteq int(S) \cap int(T)$

Fix $x \in int(S) \cap int(T)$

$x \in int(S)$
Let $B_1(x, r_1) \subseteq S$ b/c x is an int point of S
Let $B_2(x, r_2) \subseteq T$ b/c x is an int point of T
Consider $B_3 = B_1 \cap B_2$
$B_3 \subseteq B_1 \subseteq S$ and $B_3 \subseteq B_2 \subseteq T$
Therefore $B_3 \subseteq S \cap T$
Since there is some open ball in $S \cap T$ centered at x which is contained:
$x \in int(S \cap T)$
$int(S) \cap int(T) \subseteq int(S) \cap int(T)$

Therefore $int(S) \cap int(T) = int(S) \cap int(T)$
Done!

Part 2: $int(S) \cup int(T) \subseteq int(S \cup T)$
Fix $x \in int(S) \cup int(T)$
w.l.o.g let $x \in int(S)$ (it is in at least 1 or both of them)
x is in the interior of S, so let $B(x, r) \subseteq S$
$B(x, r) \subseteq S \subseteq S \cup T$
Therefore:
$B(x, r) \subseteq S \cup T$
Therefore $x \in int(S \cup T)$
and $int(S) \cup int(T) \subseteq int(S \cup T)$