Chaos Homework 7

Spencer Brouhard

January 2024

1 Question 1

Part 1: Show that the union of 2 countable sets is countable
Let S and T be 2 countable sets
S can be written as $\{S_1, S_2, S_3...\}$
T can be written as $\{T_1, T_2, T_3..\}$

Let $f : (S \cup T) \to \mathbb{Z}^+$ be defined as follows
If $x \in S$ then it is some element S_i and $f(x) = 2i$
If $x \in T - S$ then it is some element T_i and $f(x) = 2i + 1$

Note: The T - S is to account for an element in both. So if it is in both T and S, default to the S mapping
This is an injective function, which is all that is required to show a set is countable

Part 2: Show that the countable union of countable sets is countable
Let $F = \{F_1, F_2, F_3...\}$ be a countable collection of countable sets
$\forall x \in \bigcup_{f \in F} f$, x is in at least 1 F_i
Each F_i can be written as $\{F_{i1}, F_{i2}, F_{i_3}...\}$

Let P_n represent the n'th prime number greater than 1
For example: $P_1 = 2$, $P_2 = 3$, $P_5 = 7$....
Let $G : \bigcup_{f \in F} f \to \mathbb{N}$ be defined as :
$G(x) = P_i^j$ where $x = F_{i,j}$
To clarify, the jth element of the ith set is mapped to the ith prime to the power of j
This map is injective, therefore the union of countable sets is countable
Note: if an element is in multiple sets, use the mapping of the lowest F_i which it is a member of

2 Question 2

Characterize all members of the middle thirds cantor set in terms of ternary expansions

To start: An element x is a member of K_∞ if it's ternary expansion can be written using only 2's and 0's

All endpoints end with a repeating 0 or a repeating 2

Right endpoints end in a repeating 2, such as $\frac{1}{3} = 0.0\overline{2}$

Left endpoints end in a repeating 0, such as $\frac{2}{3} = 0.2\overline{0}$

Rational numbers have a period longer than 1, consisting only of 0's and 2's
For example: $0.020\overline{02202}$ is a rational number which survives, but is not an endpoint

Irrational numbers are just strings of 0's and 2's which never repeat:
For example: 0.20022020002022202000202020...

3 Question 3

Let the slope 3 tent map be defined as $T(x) =$
$3x$ if $x \leq \frac{1}{2}$
$-3(x-1)$ if $x \geq \frac{1}{2}$
Fix $x \in \{\mathbb{R} - K_\infty\}$

K_∞ contains all numbers who's ternary representation can be written using exculsivly 0's and 2's

Therefore the ternary representation of x must contain a 1 which is not followed by a $\overline{0}$
(For example: $0.1\overline{0} = 0.0\overline{2}$ so this would not be included while $0.12\overline{0}$ can not be written any other way)

Considering the application of T on a point y, we can say the following about how it's ternary representation is affected:

If $y < 1/2$, each digit is shifted to the left once
Ex: $T(0.022) = 0.220$

If $\frac{1}{2} < y < 1$ then each digit is shifted to the left, the digit to the left of the decimal is truncated, and then the 0's and 2's are "flipped". It should be noted that the 1's (not followed by $\overline{0}$) are unaffected
Ex: $T(0.22\overline{0}) = 0.0\overline{2}$

Ex2: $T(0.2120) = 0.10\overline{2}$

If $y > 1$ then $T(y)$ will yield a result less than 0 on the next iteration:
Ex: $T(1.202) = -2.02$

If $y < 0$, then the same rule applies to when $0 < y < \frac{1}{2}$, and it will be shifted to the left. Since it is negative, it will keep being a negative number multiplied by a factor of 3 repeatedly (going off to $-\infty$)
$T(-2.02) = -20.2$
$T(-20.2) = -202.0$
$T(-202) = -2020...$
Off to $-\infty$

So if we can show that since x contains a 1 (in its ternary representation), there is some $k \in \mathbb{N}$ such that $T^k(x) > 1$, we can conclude x is in the basin of $-\infty$

x can be written in ternary as $0.a_1a_2a_3...$ where each a is 0,1, or 2
Since $x \in \mathbb{R} - K_\infty$, $\exists i \in \mathbb{N}$ such that $a_i = 1$
Since 1's are unaffected by the application of T as shown above, every iteration of T shifts a_i to the left by 1

$T^{i-1}(x) = 0.a_ib_1b_2... = 0.1b_1b_2b_3...$ (using b's because we dont know if they have been flipped or not)
$T^{i-1}(x) < \frac{1}{2}$
Therefore $T^i(x) = 1.b_1b_2...$
And $T^i(x) > 1$
Therefore $T^{i+1}(x) < 0$
And based on behavior described above:
$\lim_{j\to\infty} T^j(x) = -\infty$

4 Question 4

Let K_∞ be the middle thirds cantor set
<u>Show that K_∞ is closed</u>
A set is closed iff it contains all of its limit points

Assume towards contradiction that K_∞ does not contain all of its limit points

Let x be a limit point of K_∞ where $x \notin K_\infty$

Since $x \notin K_\infty$, it is a member of some removed interval
To be clear, on some iteration of the middle-thirds process, x was a member of a middle third which was removed

This means $\exists a, b \in \mathbb{R}$ such that $x \in (a, b)$ and $(a, b) \cap K_\infty = \phi$

Let $r = \min(|x - a|, |x - b|)$
$B(x, r) \subseteq (a, b)$ (The open ball centered at x of radius r is a subset of (a,b))
Therefore $B(x, r) \cap K_\infty = \phi$
This is a contradiction to the definition of a limit point (of K_∞), which states that: $\forall r \in \mathbb{R}, B(x, r) \cap (K_\infty - \{x\}) \neq \phi$

Therefore all limit points of K_∞ are elements of K_∞ and K_∞ is closed

Show that K_∞ is perfect
Fix $x \in K_\infty$
Fix $r > 0$

$B(x, r) = (x - r, x + r)$ since an open ball is an open interval in $\mathbb{R}$
Fact: K_∞ contains no intervals with measure > 0
Therefore, $(x - r, x + r) \not\subseteq K_\infty$

$\exists y \in (x - r, x + r)$ such that $y \notin K_\infty$

WLOG Assume that $y < x$ (This has symmetry but requires changing the greater thans to less thans etc)

Using the same logic as part a: y was removed in some open interval (a, b)
Since $x \in k_\infty$ it is clear that $x \notin (a, b)$

Therefore $y < b < x$
Since b is the supremum of a removed interval it is also the left endpoint of an interval not removed

Therefore b is an element of K_∞ (Since endpoints survive the removal process)

Since an arbitrary open ball centered at x contains another element of K_∞, x is a limit point
Since x is arbitrary, every point in K_∞ is a limit point
Therefore K_∞ is perfect

Part 3: Find an injective mapping from an arbitrary cantor set S to K_∞
We will do this by building a few injective or bijective functions, and then composing them to build a new injective function, as the composition of injective functions is injective

Let $F : S \to \mathbb{R}$ be defined as:
$F(x) = \frac{1}{1+e^{-x}}$

This is the sigmoid function, and it maps all numbers in $\mathbb{R}$ to the open interval from $(0, 1)$. It is also injective

Let $G : \mathbb{R} \to \mathbb{R}_2$ represent the conversion from a number x in base-10 to a number x in binary
This function is clearly a bijection

Let $H : \mathbb{R}_2 \to K_\infty$ be as follows: $H(x)$ takes the binary number x and flips the every 1 bit to a 2
Example: $H(0.11010) = 0.22020$
Since H(x) contains only 0's and 2's, $H(x)$ is a member of $K\infty$
H is also clearly injective!

Consider $H \circ G \circ F$
This takes an element of S, maps it to $(0, 1)$, maps it to a binary number (between 0 and 1), and then turns the binary number to a ternary number containing only 0's and 2's (also between 0 and 1)

This is the composition of injective functions and therefore is injective
Therefore this is an injective mapping from S to K_∞
Done!

One example for fun:
Let $x = 7.231$ and $x \in S$
$F(x) = 0.9859086$
$G(0.9859086_{10}) = 0.11111100011001_2$
$H(0.11111100011001_2) = 0.22222200022002_3$
This number is in K_∞!

5 Question 5

Part A: What do the black points represent? The black points in the plot represent an approximation of the Julia set. This is found by starting at a fixed point, and performing backwards iterations. Since this is not a well-defined function, we flip a coin to determine which point we pick. We then plot the point which the reverse iteration approaches
This follows from the idea of points on an unstable manifold approach it under backwards iterates
Part B: For which values of c is the set connected, and how do you know?
The set is connected for the values:
-0.5 + 0.3i
0 + i
-1 + 0i
I know that the set is connected because each of these points is a member of the

mandelbrot set, and this means that it's julia set is connected. We discussed this in class

Part 3: For each value of c, what bounded orbits are attracting? $-0.5 + 0.3$ is attracted to a period 1 orbit containing: $-0.3825 + 0.1700i$

$0 + i$ has no bounded orbits which are attracting, as it's boundary is exclusively points which are EP to 0

$-1 + 0i$ is attracted to a period 2 orbit of the points $\{0 + 0i$ and $-1 + 0i\}$

Finally $0 + 1.1i$ has no attracting period