Analysis In Several Variables Homework 7

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1 Question 1

Let $S \in \mathbb{R}^n$
Part A: Show that S' is closed
To prove this, we will show that $(S')' \subseteq S'$
This is because a set is closed iff it contains all of its accumulation points
Let $x \in (S')'$
Fix $r \in \mathbb{R}$ and consider $B(x,r)$. This open ball contains infinite points in S'
(Since x is an accumulation point of S')
Fix $y \in B(x,y)$ such that $y \in S'$
Let $r_2 \in \mathbb{R}$ such that $B(y,r_2) \subset B(x,r)$
$B(y,r_2)$ contains infinite points in S, as it is an accumulation point of S
And $B(y,r_2) \subset B(x,r)$
Therefore $B(x,r)$ contains infinite points in S
Therefore any open ball centered at x contains infinite points in S
Therefore x is an accumulation point of S and $x \in S'$

Part B: If $S \subseteq T, S' \subseteq T'$
Let $S \subseteq T$
Fix $x \in S'$
Let $r \in \mathbb{R}$ and consider $B(x,r)$
Since $x \in S'$, $B(x,r)$ contains infinite points in S
$\forall s \in S$, $s \in T$ since $S \subseteq T$
Therefore $B(x,r)$ contains infinite points in T
Therefore x is an accumulation point of T and $x \in T'$
Therefore $S' \subseteq T'$

Part 3: $(S \cup T)' = S' \cup T'$
Show: $(S \cup T)' \subseteq S' \cup T'$

Fix $x \in (S \cup T)'$
Let $r \in \mathbb{R}$
$B(x,r)$ contains infinite points in $S \cup T$
WLOG Assume $B(x,r)$ contains infinite points in S (it contains infinite points

in 1 or both of them)
$B(x, r)$ contains infinite points in S, therefore $x \in S'$
Therefore $x \in S' \cup T'$
Therefore $(S \cup T)' \subseteq S' \cup T'$

Show: $S' \cup T' \subseteq (S \cup T)'$

Fix $x \in S' \cup T'$
Let $r \in \mathbb{R}$ and consider $B(x, r)$
$B(x, r)$ contains either infinite points in either S, T, or both. This is because is is an accumulation point of S, T, or both
WLOG let $B(x, r)$ contain infinite points in S
Therefore $B(x, r)$ contains infinite points in $S \cup T$
Therefore x is an accumulation point of $S \cup T$
$x \in (S \cup T)'$
Therefore $S' \cup T' \subseteq (S \cup T)'$

$(S \cup T)' = S' \cup T'$
Done!

<u>Part 4: Show $(\overline{S})' = S'$</u>
Show: $(\overline{S})' \subseteq S'$
Fix $x \in (\overline{S})'$
Let $r \in \mathbb{R}$
$B(x, r) - \{x\} \cap \overline{S} \neq \phi$ (Since it is an accumulation point of $\overline{S}$)
Let $y \in B(x, r) - \{x\} \cap \overline{S}$
Since $y \in \overline{S}$, all open balls center at y contain a point in S
Let $r_2 = min(||y - x||, ||y - (x + r)||)$
(I know this isn't correct notationally, but its the distance such that the open ball doesn't contain x and is also contained in $B(x, r)$)
$B(y, r_2) \subseteq B(x, r)$ and $x \notin B(y, r_2)$
Since y is an adherent point, $B(y, r_2)$ contains a point in S
Therefore $B(x, r) - \{x\} \cap S \neq \phi$
Therefore $x \in S'$
and $(\overline{S})' \subseteq S'$

Show: $S' \subseteq (\overline{S})'$
Fix $x \in S'$
Let $r \in \mathbb{R}$
$B(x, r) - \{x\} \cap S \neq \phi$
Fact: $S \subseteq \overline{S}$
Therefore $B(x, r) - \{x\} \cap \overline{S} \neq \phi$
So x is an accumulation point of $\overline{S}$ and $S' \subseteq (\overline{S})'$

$(\overline{S})' = S'$

Done!

<u>Part 5: $\overline{S}$ is closed in $\mathbb{R}^n$</u>

Fix $x \in \mathbb{R}^n - \overline{S}$

If we can show that x must be an interior point of $\mathbb{R}^n - \overline{S}$, we can conclude that set is open, and therefore $\overline{S}$ is closed

Assume towards contradiction that x is not an interior point

This means that $\forall r \in \mathbb{R}$, $B(x,r)$ contains a point y not in $\mathbb{R}^n - \overline{S}$

Therefore $\forall r \in \mathbb{R}, B(x,r) - \{x\} \cap \overline{S} \neq \phi$

$x \in S'$

We also have $S' \subseteq \overline{S}$ (this is true for all sets) since $\overline{S} = S \cup S'$

Therefore $x \in \overline{S}$ which is a contradiction since x is in the compliment of $\overline{S}$

Therefore x is an interior point of $\mathbb{R}^n - \overline{S}$,

$\mathbb{R}^n - \overline{S}$ is open,

and $\overline{S}$ is closed

<u>Part 5: $\overline{S}$ is the smallest closed set containing S</u>

Assume towards contradiction that $\overline{S}$ was not the smallest closed set containing S.

That is, there is a smaller one!

Let W be a smaller closed set containing S

We have $S \subseteq W$ and $\exists x \in \overline{S}$ which is not an element of W

Fact: $\overline{S} = S \cup S'$

Since $S \subseteq W$, we know that $\forall s \in S, s \in W$

Therefore the element $x \in \overline{S}$ which is not in W must be an element of S'

But by theorem 3.22, a set is closed iff it contains all of its accumulation points

And since W contains S and is closed, it must also contain all of the accumulation points of S... (S')

But $x \in S'$ and $x \notin W$

Therefore W is not closed

This is a contradiction!

$\overline{S}$ must be the smallest closed set containing S

Done!

2 Question 2

<u>Show: $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$</u>

Fix $x \in \overline{S \cap T}$

Since x is an adherent point to $S \cap T$:

For all $r \in \mathbb{R}$: $B(x,r) \cap (S \cap T) \neq \phi$

Fix $y \in (S \cap T)$

$y \in S$ and $y \in T$

Since every open ball centered at x contains a point (y) in S and T, $x \in \overline{S}$ and $x \in \overline{T}$

Therefore $x \in \overline{S} \cap \overline{T}$
and $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$

Show: if S is open, $S \cap T \subseteq \overline{S \cap T}$
Fix $x \in S \cap T$
Since x is in $S \cap T$, every open ball centered at x contains a point in $S \cap T$.
That point is x
Therefore $x \in \overline{S \cap T}$
So $S \cap T \subseteq \overline{S \cap T}$
Done?