Analysis In Several Variables Homework 8

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1 Question 1

Let F be a collection of sets in $\mathbb{R}^n$ and let $S = \bigcup_{A\in F} A$ and $T = \bigcap_{A\in F} A$
Either prove or provide a counterexample to the following:

If x is an accumulation point of T, then x is an accumulation point of each set A in F
This statement is true
Let x be an accumulation point of T
$\forall r \in \mathbb{R}$, $B(x,r) \cap (T - \{x\} \neq \phi$
Therefore for all open balls centered at x, $\exists y \in B(x,r) \cap (T - \{x\}$ which is not equal to x
Since $y \in T$, $y \in A$ for each $A \in T$
Therefore for all r, the open ball centered at x of radius r contains an element y in each A, therefore $x \in A'$ (for each A)
Done!

If x is an accumulation point of S, then x is an accumulation point of at least one set A in F
False!
Consider $F = \{(0,x) | x = \frac{n}{n+1} \text{ for } n \in \mathbb{Z}^+\}$
$F = \{(0,\frac{1}{2}),(0,\frac{2}{3}),(0,\frac{3}{4})...\}$
$\bigcup_{A\in F} A = (0,1)$
1 is an accumulation point of $\bigcup_{A\in F} A$, but is not an accumulation point of any individual $A \in F$

2 Question 2

Prove that the set S of rational numbers in the interval (0,1) can not be expressed as the intersection of a cou
$S = \{x_1, x_2, x_3...\}$

Assume towards contradiction that S can be expressed as the intersection of a countable collection of open sets
$S = \bigcap_{k=1}^{\infty} S_k$

Let x_1 be a rational number in S
$x \in S_1$
S_1 is an open set, therefore it is the union of disjoint component intervals
x_1 is an element of one of these component intervals (a, b)
Let Q_1 be a closed interval from $[c, d]$ s.t. $x_1 \notin Q_1$ and $Q_1 \subseteq (a, b)$
Note: we can do this by making the closed interval $[a + \epsilon, x - \epsilon]$ for some $\epsilon \in \mathbb{R}$

Now let x_2 be a rational number in Q_1. Therefore $x_2 \in S$ and $x_2 \in S_2$
Since $x_2 \in S_2$, we can construct another closed interval $Q_2 \subseteq S_2$ with the same logic
Add the condition that $Q_2 \subseteq Q_1$ (this only requires that the endpoints of Q_2 are greater than/less than the endpoints of Q_1 and a rational number clearly exists in a closed interval which meets this requirement
Note that $Q_2 \subseteq Q_1 \subseteq S_1 \cap S_2$

Continue inductively

By cantor's intersection theorem, $\bigcap_{k=0}^{\infty} Q_k \neq \phi$
This is a contradiction, as this means there exists a rational number which is not contained in any of the $S_k s$
Done!

3 Question 3

Give an example of a set S which is closed but not bounded and exhibit a countable open covering F s.t. no finite subset of F covers S
Let $S = \mathbb{Z}^+$
This is a closed set since it contains all of its accumulation points vacuously
A countable open cover of this set is $\bigcup_{n=1}^{\infty} (0, n)$
This cover has no finite subcover

4 Question 5

Given a set S in $\mathbb{R}^n$ with the propety that for every x in S there is an n-ball $B(x, r)$ such that $B(x, r) \cap S$ is countable
Prove that S is countable

For each $x \in S$, let $A_x = B(x, r)$ where $B(x, r)$ is an n-ball who's intersection with S is countable

Fact: $F = \bigcup_{A \in F} A$ is an open cover of S
Proof: fix $x \in S$

$x \in A_x$ therefore $x \in F$

By the Lindelof covering theorem, there is some countable subcover of F called G which also covers S
Since G is countable, it can be written $\{G_1, G_2, G_3...\}$
$S = S \cap (G_1 \cup G_2 \cup G_3...)$
By demorgans: $S = \bigcup_{G_i \in G}(S \cap G_i)$
And each G_i is an open ball which has countably many points intercepted with S
Therefore S is the countable union of countable sets, which is countable
Therefore S is countable

5 Question 6

This is the contrapositive of question 5 which is already proven