Chaos Homework 6

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1 Question 1

$G(x) = 4x(1-x)$
The conjugacy $C(x) = ax + b$
Find the map g which is conjugate to G

$C \circ g(x) = G \circ C(x)$
$g(x) = C^{-1} \circ G \circ C(x)$

$C^{-1}(x) = \frac{x-b}{a}$

$G(C(x)) = (4(ax+b))(1-(ax+b))$
$= -4a^2x^2 - 4abx + 4ax + 4b - 4b^2$

$C^{-1}(G(C(x)) = \frac{-4a^2x^2 - 4abx + 4ax + 4b - 4b^2 - b}{a}$

$g(x) = \frac{-4a^2x^2 - 4abx + 4ax + 4b - 4b^2 - b}{a}$

2 Question 2

<u>Show that each infinite itinerary represents the orbit of exactly 1 point in the tent map</u>
Let $a \neq b \in [0,1]$
Both a and b can be written as **binary** decimals like this:
$0.a_0a_1a_2...$
$0.b_0b_1b_2...$
Since $a \neq b$, there is some $i \in \mathbb{Z}^+$ such that $a_i \neq b_i$
Let i be the least of these points. This means that:
$0.a_0...a_{i-1} = 0.b_0...b_{i-1}$

<u>Brief explanation of how the tent map changes binary numbers:</u>
For each iteration of a point under the tent map, each digit is shifted one to the right with non-decimal part chopped off like this:
$0.x_0x_1x_2... \Rightarrow 0.x_1x_2...$
If the first integer to the right of the decimal is 0, then we're done, but if it is

1, then the bit flips for each index of the number:
$T(0.10011) \rightarrow 0.1100...$
If we iterate 2 numbers under the tent map, the "relationship" between the digits in an index is mantained. By this I mean: if $a_i \neq b_i$, when we iterate the tent map this fact will still hold as both values flip and shift left. Eventually we can iterate until we have a_i and b_i are both the first value to the right of the decimal

For each iteration of the map for $0.x_0x_1...$, if $x_0 = 1$ then append an R and if $x_0 = 0$ then append an L to the itinerary

So we know that the first $i - 1$th items in the itinerarys of a and b will be identical, as $0.a_0...a_{i-1} = 0.b_0...b_{i-1}$

But then we will arrive at $0.a_i...$ and $0.b_i...$

When this happens one itinerary will append an L term and one will append an R term (since $a_i \neq b_i$). This means they do not have the same itinerary

Therefore if two points are distinct, they have unique itinerarys. Done!

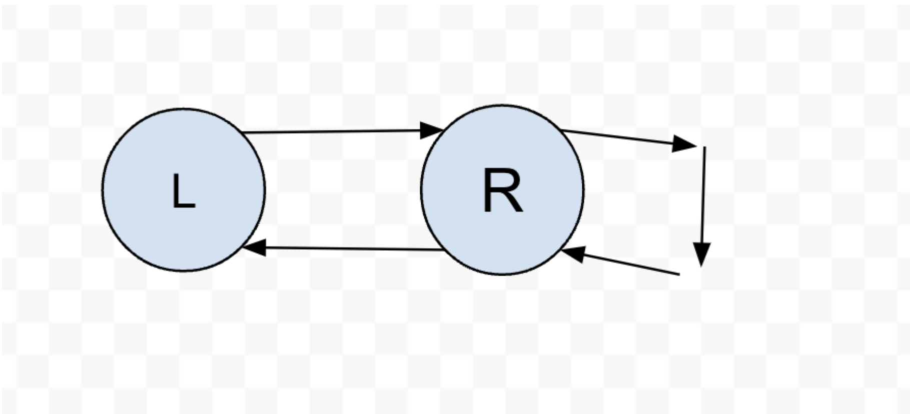
3 Question 3

Part A: devise a scheme to create a sequences $S_1S_2...S_nS_1$ for any positive integer n so it contains no identica
For any positive integer n let $S = L$ repeating $n - 1$ times and then an RL.
For example when n = 3:
$LLRL$
n=6:
$LLLLLRL$
These contain no subsequences!

Part B: Prove that the logistic map G has an orbit for each integer period
Fact: G has a fully connected transition graph
This means that any infinite sequences of L's and R's is possible in G
Because of this, that means any of the sequences defined in part a are possible in G
By collolary 3.18, if we have a sequence in G of the form $S_1S_2...S_nS_1$, then the subinterval contains a fixed point of G^n
And we know that such a sequence exists in G for all positive integers.
Since there is a fixed point of G^n for all $n \in \mathbb{Z}^+$, we can conclude that G has an orbit of every positive integer period!

4 Question 4

Find the itineraries which obey the transition graph for this figure:
The transtion graph for this function looks like this:



Which means that any sequence with any number of R's in a row but with each L immediatly followed by an R is valid:
Example:
LRRRL is valid
RRRRRLRRRRRLR is valid

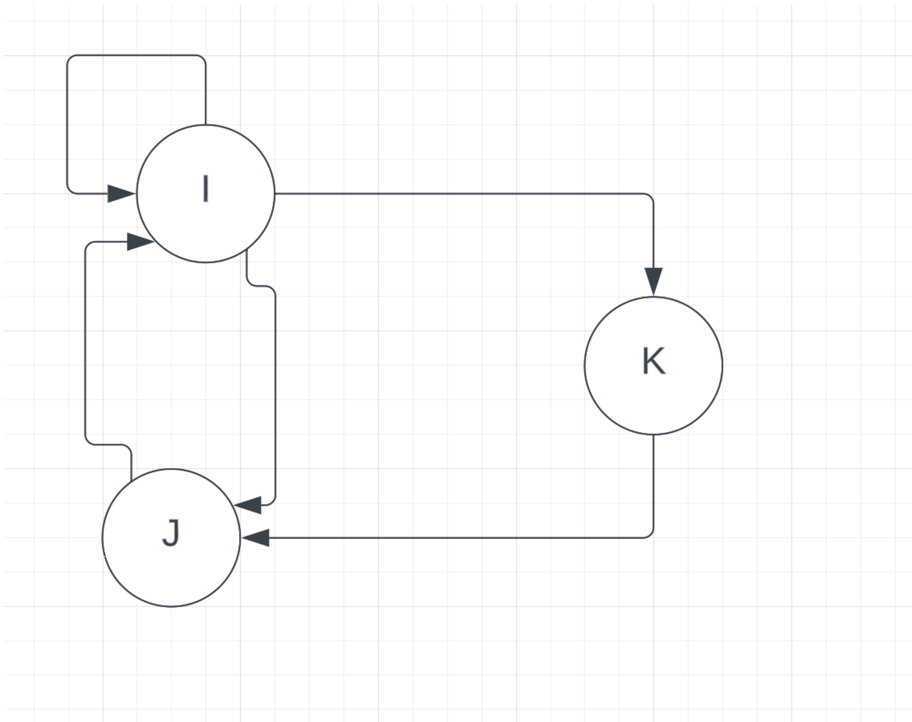
LLLR is not valid

Now using the scheme from question 3, we can assemble a sequence with no perioidic subsequences for any integer n by taking RRR...(n-1 times)LR
Example: n = 3 is $RRLR$
Show the function has an orbit of period n for all integers n
Since we have shown there is a sequence with no subsequences for all integers n, we can use collorary 3.18 to conclude that there exists a fixed point of $f^n(x)$ for all positive integers n.Therefore there is an orbit of period n for all positive integers n!

Why does period 3 imply chaos?
When an orbit of period 3 exists, it implies that it is possible in the transition graph to stay on one side for an indefinite amount of time, and then go to the other side of the function and return. This means we can construct an orbit for every integer period and therefore we can build chaotic orbits! Also as we stay in R (wlog) indefinitely, the lyaponov exponent continues to grow if the slope is greater than 1 and therefore we land on a chaotic orbit

5 Question 4

Draw the transition graph



Fill in the table:

K	Itinerary	Orbit
1	$\overline{I}$	$\{a_1\}$
2	$\overline{IJ}$	$\{a_1, a_2\}$
3	$\overline{IIJ}$	$\{a_2, a_1, a_3\}$
3	$\overline{IKJ}$	$\{a_1, a_3, a_2\}$
4	$\overline{IIKJ}$	$\{a_2, a_1, a_4, a_3\}$
5	$\overline{IIIKJ}$	$\{a_2, a_3, a_1, a_5, a_4\}$
5	$\overline{IJIKJ}$	$\{a_2, a_3, a_1, a_5, a_4\}$

Show there is no period 4 orbit with the itinerary $\overline{IJIJ}$

Find the fixed points of this itinerary:

$f_1(f_2(f_1(f_2(x)))) - x = 0$

$1 - 3(-\frac{1}{3} + (1 - 3(-\frac{1}{3} + x)) - x = 0$

$1 - 5 + 9x - x = 0$

$8x - 4 = 0$

$x = \frac{1}{2}$

But it was stated that $1/2$ is a fixed point of f^2, therefore it is a period 2 orbit. Therefore there is no orbit of period 4!