Chaos Homework 4

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1 Question 1

Prove that the map $4x(1-x)$ has an orbit of period k for all $k \in \mathbb{N}$

Strategy: If we can show that the function f^k has some fixed point which does not belong to any of the f^i where $0 < i < k$, then we can conclude that f^k has an orbit of period k

We know that f^k has 2^k fixed points, as shown in class
Let a_i represent the number of unique fixed points of the map f^i where $i \in \mathbb{N}$ and $i|k]$
Each a_i is at most $2^i - 2$ as we already know f has 2 fixed points and $1|i$
From this we can claim:
$a_i < 2^i$

Now define the set of all a_i to be $\{a_1 ... a_n\}$:
$\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} 2^i$

And since each $i < k$
$\sum_{i=1}^{n} 2^i < 2^k$
Therefore f^k has at least 1 unique fixed point which does not belong to any periods prior. This means there is an orbit of period k

2 Question 2

For the general matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ what conditions cause $f = Mv(mod 1)$ to be continuous

This function is continuous if a, b, c, d are integers
This is due to the fact that if $x \in \mathbb{Z}, x(mod 1) = 0$, while if $x \in \{\mathbb{R} - \mathbb{Z}\}, x(mod 1) \neq 0$

A f is continuous if $f(<x,0>) = f(<x,1>)$ and $f(<0,y>) = f(<1,y>)$, as these are the points where the axis' are "stitched" together in the torus representation of f

X case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \mod 1 = \begin{bmatrix} ax \\ cx \end{bmatrix} \mod 1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \mod 1 = \begin{bmatrix} ax+b \\ cx+d \end{bmatrix} \mod 1$$

$ax(\mod 1) = ax + b(\mod 1)$ IFF $b \mod 1 = 0$ Therefore $b \in \mathbb{Z}$
$cx(\mod 1) = cx + d(\mod 1)$ IFF $d \mod 1 = 0$ Therefore $d \in \mathbb{Z}$

Y case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} \mod 1 = \begin{bmatrix} by \\ dy \end{bmatrix} \mod 1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} \mod 1 = \begin{bmatrix} by+a \\ dy+c \end{bmatrix} \mod 1$$

$by(\mod 1) = by + a(\mod 1)$ IFF $a \mod 1 = 0$ Therefore $a \in \mathbb{Z}$
$dy(\mod 1) = dy + c(\mod 1)$ IFF $c \mod 1 = 0$ Therefore $c \in \mathbb{Z}$

Therefore for both of these seams to be continuous, a, b, c, d must be integers

3 Question 3

Period	Fixed Points	Divisors	Points from period < k	Points only k	Orbits
1	1	[]	0	1	1.0
2	5	[1]	1	4	2.0
3	16	[1]	1	15	5.0
4	45	[1, 2]	5	40	10.0
5	121	[1]	1	120	24.0
6	320	[1, 2, 3]	20	300	50.0
7	841	[1]	1	840	120.0
8	2205	[1, 2, 4]	45	2160	270.0
9	5776	[1, 3]	16	5760	640.0
10	15125	[1, 2, 5]	125	15000	1500.0
11	39601	[1]	1	39600	3600.0
12	103680	[1, 2, 3, 4, 6]	360	103320	8610.0
13	271441	[1]	1	271440	20880.0

4 Question 4

Let $f(x, y) = (2x + y, a - y^2)$

<u>Solve for this fixed points of f</u>
System of equations:
$2x + y = x$
$a - y^2 = y$

$x = -y$

$y^2 + y - a = 0$
Quadratic:
$y = \frac{-1 \pm \sqrt{1+4a}}{2}$

Fixed points:
$(\frac{1+\sqrt{1+4a}}{2}, \frac{-1+\sqrt{1+4a}}{2})$ and $(\frac{1-\sqrt{1+4a}}{2}, \frac{-1-\sqrt{1+4a}}{2})$

To find when f has real fixed points:
$1 + 4a \geq 0$
$4a \geq -1$
$a \geq \frac{-1}{4}$
Done!

<u>Fix $a = 0$ and for each of the fixed points determine the stability</u>
Fixed points:
$(1, 0)$ and $(0, -1)$
Jacobian Matrix: $\begin{bmatrix} 2 & 1 \\ 0 & -2y \end{bmatrix}$

Evaluate at (1,0):
$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$
Find EigenValues:
$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 0-\lambda \end{bmatrix} = 0$
$(2 - \lambda)(-\lambda) = 0$
$\lambda = -2$ and 0
This point is a saddle

Evaluate at $(0, -1)$
$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$
$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$
$(2 - \lambda)(2 - \lambda) = 0$

$\lambda = 2$ This point is a source
Done!

For what values of a are both of the points the same (source in this case)
Fixed points:
$(\frac{1+\sqrt{1+4a}}{2}, \frac{-1+\sqrt{1+4a}}{2})$ and $(\frac{1-\sqrt{1+4a}}{2}, \frac{-1-\sqrt{1+4a}}{2})$

Evaluate the Jacobian at both points:
$\begin{bmatrix} 2 & 1 \\ 0 & 1-\sqrt{1+4a} \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 1+\sqrt{1+4a} \end{bmatrix}$

Find eigenvvalues part 1:
$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 1+\sqrt{1+4a}-\lambda \end{bmatrix}$ $(2-\lambda)(1+\sqrt{1+4a}-\lambda) = 0$
$\lambda = 1+\sqrt{1+4a}$
Find values of a where $\lambda > 1$
$1+\sqrt{1+4a} > 1$
$\sqrt{1+4a} > 0$
$a > \frac{-1}{4}$
Find eigenvalues part 2:
$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\sqrt{1+4a}-\lambda \end{bmatrix}$
$(2-\lambda)(1-\sqrt{1+4a}-\lambda) = 0$
$1-\sqrt{1+4a} = \lambda$
$1-\sqrt{1+4a} > 1$
$a > \frac{1}{4}$

a must be greater than 0.25 for both points to be sources!

5 Question 5

I have included a zip containing images of each of the figures plotted in matlab.
Numbered going top-left, top-right...
Then I included an orbit of period 32 with a values 1.950300 and b value -0.3
Verified by checking the output of the function in command line
I took this value from the last homework bifurcation points
Thanks!