Chaos Homework 8

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1 Question 1

Let A and B be bounded subsets of $\mathbb{R}$. Show that $bcd(A \times B) = bcd(A) + bcd(B)$

Lets start by establishing what bcd(A) and bcd(B) are
$bcd(A) = \lim_{\epsilon \to 0} \frac{\ln(N_A(\epsilon))}{\ln(\frac{1}{\epsilon})}$
$bcd(B) = \lim_{\epsilon \to 0} \frac{\ln(N_B(\epsilon))}{\ln(\frac{1}{\epsilon})}$

We can assign the same epsilon to both values at any step:
Fix $\epsilon > 0$
We now have $\frac{\ln(N_A(\epsilon))}{\ln(\frac{1}{\epsilon})}$ where $N_A(\epsilon)$ is the number of boxes needed to cover the set A for this episilon and
$\frac{\ln(N_B(\epsilon))}{\ln(\frac{1}{\epsilon})}$ where $N_B(\epsilon)$ is the number of boxes needed to cover the set B for this epislon

Consider the addition of both of these values:
$\frac{\ln(N_B(\epsilon))}{\ln(\frac{1}{\epsilon})} + \frac{\ln(N_A(\epsilon))}{\ln(\frac{1}{\epsilon})}$
$= \frac{\ln(N_B(\epsilon) + \ln(N_A(\epsilon))}{ln(\frac{1}{\epsilon})}$
$= \frac{ln(N_A(\epsilon) * N_B(\epsilon))}{\ln(\frac{1}{\epsilon})}$

$N_A(\epsilon) * N_B(\epsilon) = N_{A \times B}(\epsilon)$
This is because when we cross 2 sets, we are creating an set of ordered pairs. For each $(x, y) \in (A \times B)$, we need one of the boxes from N_A (given some epsilon) and one of the boxes from N_B and there will be no overlap. Therefore we need $N_A(\epsilon) * N_B(\epsilon)$ boxes to cover $A \times B$
Therefore $\frac{\ln(N_B(\epsilon))}{\ln(\frac{1}{\epsilon})} + \frac{\ln(N_A(\epsilon))}{\ln(\frac{1}{\epsilon})} = bcd(A \times B)$
and
$bcd(A) + bcd(B) = bcd(A \times B)$

2 Quesiton 2

State true/false for each statement and either prove or find a counter example:

The box counting dimension of a finite union of sets is the maximum of the box counting dimension of each set
This statement is true!
Proof:
Let F be some finite collection of sets
$F = \{f_1, f_2, f_3 ... f_n\}$ for some $n \in \mathbb{Z}^+$
We will start by considering the union of 2 sets and their box counting dimension:
$bcd(f_1) = \lim_{\epsilon \to 0} \frac{ln(N_{f_1}(\epsilon))}{ln(\frac{1}{\epsilon})}$
$bcd(f_2) = \lim_{\epsilon \to 0} \frac{ln(N_{f_2}(\epsilon))}{ln(\frac{1}{\epsilon})}$
$f_1 \cup f_2$ is the set of all x such that $x \in f_1$ or $x \in f_2$ (or both)

Fact:
$f_1 \subseteq f_1 \cup f_2$ and $f_2 \subseteq f_1 \cup f_2$
Therefore $N_{f_1}(\epsilon) \leq N_{f_1 \cup f_2}(\epsilon)$ and $N_{f_2}(\epsilon) \leq N_{f_1 \cup f_2}(\epsilon)$
So $bcd(f_1) \leq bcd(f_1 \cup f_2)$ and same for f_2

Without Loss of Generality, assume $bcd(f_1) > bcd(f_2)$
We have $bcd(f_2) < bcd(f_1) \leq bcd(f_1 \cup f_2)$

Assume towards contradiction that $bcd(f_1) < bcd(f_1 \cup f_2)$ (That it is strictly greater and not GEQ)
$\lim_{\epsilon \to 0} \frac{ln(N_{f_1}(\epsilon))}{ln(\frac{1}{\epsilon})} < \lim_{\epsilon \to 0} \frac{ln(N_{f_1 \cup f_2}(\epsilon))}{ln(\frac{1}{\epsilon})}$
This implies that for any size epsilon: More boxes are required to cover $f_1 \cup f_2$ than just f_1... a point not covered by the boxes in f_1 must be an element of f_2

But $bcd(f_2) < bcd(f_1)$. This is a contradiction. Therefore $bcd(f_1) = bcd(f_1 \cup f_2)$. Since f_1 was arbitrarily chosen to be the max of $bcd(f_1)$ and $bcd(f_2)$, we have $bcd(f_1 \cup f_2) = max(bcd(f_1), bcd(f_2))$

We can now apply induction, as the union of 2 sets is another just another set. So we can continue to union our new set with the next one for any finite number of sets, and at each iteration, the new bcd will be the max bcd of our new set and the next one

Part B: The infinite union of sets each with a BCD is the maximum of their BCDS
False. One counter example is the infinite union of the cantor sets with the middle $\frac{1}{n}$ removed
So the union of the middle thirds, middle fourths, middle fifths, ...
Their union is equivalent to $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ which has a bcd of 1

First I will prove that each middle-n set has no measure, and therefore has dimension less than 1. I will do this by showing that it's compliment has a measure of 1

We'll start by forming an equation for the length of the compliment of the middle-thirds and middle fourths cantor sets, and then define an equation for all of them:
$\mathbb{R} - K_{3\infty} = \frac{1}{3} + (2 * \frac{1}{3} * \frac{2}{3}) + (4 * \frac{1}{3} * \frac{4}{9}) + ...$
The compliment of the middle fourths can be written as:
$\mathbb{R} - K_{4\infty} = \frac{1}{4} + (2 * \frac{1}{4} * \frac{3}{8}) + (4 * \frac{1}{4} * \frac{9}{64}) + ...$
Source: I wrote them out!

We can then generalize (the length of) the compliment of any cantor middle-n set to:
$\frac{1}{n} + (2 * \frac{1}{n} * (\frac{n-1}{2n})) * (4 * \frac{1}{n} * (\frac{n-1}{2n})^2) + ...$
Which can be rewritten as:
$\sum_{i=0}^{\infty} \frac{1}{n} * 2^i * (\frac{n-1}{2n})^i$

$\frac{1}{n} \sum_{i=0}^{\infty} 2^i * (\frac{n-1}{2n})^i$

$\frac{1}{n} \sum_{i=0}^{\infty} (\frac{n-1}{n})^i$

$= \frac{1}{n} * \frac{1}{1 - \frac{n-1}{n}}$ By the rules of geometric sums

$= \frac{1}{n} * \frac{1}{\frac{1}{n}}$

$= \frac{1}{n} * n$

$= 1$
Since for any n, the compliment of the cantor middle-n set has a measure of 1, the cantor middle-n set has a measure of 0
Therefore none of them have dimension 1!
So the BCD of the infinite union is <u>not</u> the max of their bcds

3 Question 3

<u>Part 1: Show that D has a measure of 1/2</u>
We will find the measure of D by doing 1 minus the measure of the compliment of D in $[0, 1]$
D is constructed by removing first $\frac{1}{4}$ from $[0, 1]$ and then 2 copies of $\frac{1}{16}$ and then 4 copies of $\frac{1}{64}$...
We can say that the length of $\{[0, 1] - D\}$ is equal to:
$1 * \frac{1}{4} + 2 * \frac{1}{16} + 4 * \frac{1}{64} + ...$

$\sum_{n=0}^{\infty} 2^n * \frac{1}{4^{n+1}}$

$\sum_{n=0}^{\infty} 2^n * \frac{1}{4^n}^{\frac{1}{4}}$

$\frac{1}{4} * \sum_{n=0}^{\infty} 2^n * \frac{1}{4^n}$

$\frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2}^n$

$\frac{1}{4} * \frac{1}{\frac{1}{2}}$
$\frac{1}{4} * 2$
$= \frac{1}{2}$

Since the compiment of D (in the closed interval from 0 to 1) has measure 1/2 and the entire space has measure 1, we can conclude that D has a measure of 1/2

Part 2: What is the box-counting Dimension of D?
The box-counting Dimension of D is 1
This is because no object which is a subset of a space can have dimension higher than that of the space. Therefore $bcd(D) \leq 1$ as $D \subseteq [0, 1]$

In addition, Theorem 4.16 in the textbook states: Let A be a bounded subset of $\mathbb{R}^m$ where $bcd(A) < m$. Then A is measure 0
The contrapositive of this statement is if a set A is not measure 0, then $bcd(A) \geq m$ in the space $\mathbb{R}^m$
Since D is not measure 0, $bcd(D) \geq 1$ (As $[0, 1]$ has dimension 1)
We have $bcd(D) \leq 1$ and $bcd(D) \geq 1$
Therefore $bcd(D) = 1$

Part 3
No matter what the ϵ size is for the length of the intervals, the lower bound of the boxes will be equal to 0
This is because every interval will contain a point which maps to 0, therefore each box will have height 0
Since every box has height 0, the sum of their areas will be 0

4 Question 4

Part 1: Find the box counting dimension of the invariant set under the Skinny Baker Map
It is stated in the textbook that on the $x-axis$, the invariant set of the skinny baker map is the cantor middle-thirds set
All y values are included in the invariant set under the skinny baker, as the "thinning" only occurs on the x-axis
Let A represent the invariant set under the skinny baker map
$A = K_{\infty} \times [0, 1]$

Therefore $bcd(A) = bcd(K_\infty) + bcd([0,1])$
$bcd(A) = \frac{ln(2)}{ln(3)} + 1$
$bcd(A) = 1.63092\ldots$

<u>Part 2</u>
Skinny-Baker Map: In 2D, the set of attractors is a "comb" of infinitely thin lines who's x axis falls on the cantor middle-thirds set. In 3D, the attractor would be a set of planes on the y-z axis who only fall on the cantor middle thirds set on the x axis
This would be a shape with no volume, but infinite area. (Measure 0 in 3D)

The Serpinski Gasket in 3d is the exact same idea, but instead of a triangle being subdivided, it is a pyramid with a triangular base. Each pyramid has an upside down middle pyramid removed, leaving 4 smaller pyramids instead of 1 large pyramid. Continue recursively!. This will yield a fractal dimension object less than 3D

Serpinski Carpet: One possible (but boring) 3d extension of the Serpinski carpet is to simply map the z values to themselves. In 2D, the serpinski carpet results in a dust of points, so in 3d, we would have a "dust of lines"
An alternative method would be to take a cube and remove a 3-d cross from the center of each cube and continue recursively

<u>Compute these dimensions of these versions from part B</u>
Skinny-Baker:
We already know that the dimension of the skinny baker map is $\frac{ln(2)}{(ln3)} + 1$
Since we are also just mapping z values to themselves, we can say that the dimension is $\frac{ln(2)}{ln(3)} + 1 + 1 = 2.63092...$

Serpinski Gasket (Tetrahedron):
We will use tetrahedrons as our "boxes"
Each iteration, the box length is halfed and the number of tetrahedrons required to cover the set is multiplied by 4
Here is a table with the first few iterates
$\epsilon = 1$, $N(\epsilon) = 1$
$\epsilon = 0.5$, $N(\epsilon) = 4$
$\epsilon = 0.25$, $N(\epsilon) = 16$
$\epsilon = 00.125$, $N(\epsilon) = 64$
...
Therefore we can write that the BCD is equal to:
$lim_{n\to\infty} \frac{ln(4^n)}{ln(2^n)}$
$= 2$
Note: this is the equivalent of epsilon going to 0! On each iterate, epsilon is halfed (going to 0) and we are looking at how $N(\epsilon)$ increases by a factor of 4. I also just wrote $\frac{1}{2^{-n}}$ as 2^n in the denominator

Serpinski Carpet in 3D:
Based on how we extended the serpinski carpet to 3D, we will find the dimension of this object by finding the dimension of the "2d"(ish) serpinski carpet and adding 1

For each iteration, the number of boxes required to cover is multiplied by 4, and the side length is multiplied by $\frac{1}{3}$
Therefore BCD of the 2d carpet is:
$lim_{n\to\infty}\frac{ln(4^n)}{ln(3^n)} = ln(4)/ln(3) = log_3(4) = 1.2618$
Therefore the BCD of our 3d version is 2.2618...
Done!

5 Question 5

<u>Part A: Calculating dimensions by hand</u>
The Koch Curve will be calculated to have a dimension of $\frac{ln(4)}{ln(3)}$. I calculated this by hand as every iteration epsilon is multiplied by 1/3, and we require 4 times as many intervals to cover the line segments. We create 4 lines which have 1/3 the length of the original line. Didn't write it out because I did a very similar table above
$\frac{ln(4)}{ln(3)} = 1.2618...$
Similarly, we can find the Serpinski gasket has dimension $ln(3)/ln(2)$ as we need 3 times as many boxes (triangles) to cover as epsilon is halfed every iteration
$\frac{ln(3)}{ln(2)} = 1.584...$

This disagrees with the numbers which boxcount.m produces because when we generate these fractals, we do so at a certain resolution. This means that there is a finite number of points in each figure, which is not truely reflective of the object. As the box-size decreases, we will eventually reach a point where there is only 1 point in each box, as there is a finite number of points. Beyond this point, as epsilon decreases, the number of boxes required for the size will not change, and our calculation will deviate from the true object for any epsilon size less than that. Specifically, the numerator will stay constant as the denominator increases

<u>Part 2: What is the length of the Koch curve?</u>
The length of the Koch curve is infinite!
It has no area, but it has infinite length. Similar to the coastline of england problem, we find that as resolution increases, the length of the object increases, no matter how fine the resolution already is. Therefore with infinite resolution (which is the "real" Koch curve), we have infinite length

<u>Part 3</u> The serpinski gasket starts by setting the verticies of a triangle

We then consider the origin and randomly place a point on the midpoint of that point and either A,B, or C.
We then do the same thing for the next point, but on the midpoint of that line and the next point
Continue for 50,000 iterates
You end up with random points which lie on the Serpinski Gasket set.
We have enough points that it looks like the drawing!