Analysis In Several Variables Homework 11

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1 Question 1

Let $f:(a,b)\to\mathbb{R}$ and $x\in(a,b)$
Consider the statements:
a) $\lim_{h\to 0}|f(x+h)-f(x)|=0$
b) $\lim_{h\to 0}|f(x+h)-f(x-x)|=0$

<u>Show that a implies b:</u>

Fix $\epsilon_1>0$

Let $\epsilon_2=\frac{\epsilon_1}{2}$
$\exists\delta$ s.t. if $0<|h|<\delta, |f(x+h)-f(x)|<\epsilon_2$

Note that $|-h|=|h|=h$
So for the same δ:
if $0<|-h|<\delta, |f(x-h)-f(x)|<\epsilon_2$
And $|f(x-h)-f(x)|=|f(x)-f(x-h)|$

By the triangle inequality:
$|f(x+h)-f(x)+f(x)-f(x-h)|<|f(x+h)-f(x)|+|f(x)-f(x-h)|<\epsilon_2+\epsilon_2$
$|f(x+h)-f(x-h)|<\frac{\epsilon_1}{2}+\frac{\epsilon_1}{2}$
$|f(x+h)-f(x-h)|<\epsilon_1$

Therefore $\forall\epsilon>0\exists\delta>0$ s.t. if $0<|h|<\delta, |f(x+h)-f(x-h)|<\epsilon$

$\lim_{h\to 0}|f(x+h)-f(x-h)|=0$
Done!
<u>Give an example of a function where b holds, but a does not</u>
Consider the function $f:(0,1)\to\mathbb{R}$
Where $f(x)=0$ if $x\neq 0$ and $f(x)=100$ if $x=0$

$\lim_{h\to 0}|f(x+h)-f(x-h)|=0$
but $\lim_{h\to 0}|f(x+h)-f(x)|=100$

2 Question 2

Let f be continuous at the point $a = (a_1 a_2 ... a_n)$ in $\mathbb{R}^n$. Let $g(x) = f(x, a_2, a3...a_n)$
Show that g is continous at $x = a_1$

Proof:
We know $\forall \epsilon > 0, \exists \delta > 0$ s.t if $0 < ||a - x|| < \delta, ||f(a) - f(x)|| < \epsilon$

For some $x = (x_1, x_2, x_3...x_n), ||a - x|| > ||a - (x_1, a_2, a_3...a_n)||$
Fix $\epsilon > 0$
If $||a - x|| < \delta$ for some $\delta > 0$ (which exists by continuity of f), it is clear that
$||a - (x_1, a_2...a_n)|| < \delta$
$||a - (x_1, a_2...a_n)|| = |a - x_1|$

Now by continuity:
If $|a - x_1| < \delta$, $||f(a) - f((x_1, a_2, a_3...a_n))|| < \epsilon$

Rewrite using g:
$||g(a_1) - g(x_1)|| < \epsilon$

Unfix ϵ to find:
$\forall \epsilon > 0, \exists \delta > 0$ s.t if $0 < |a - x_1| < \delta, ||g(a_1) - g(x_1)|| < \epsilon$

And g is continous at a_1!

3 Question 3

Let f, g, h be defined on $[0, 1]$ as follows:

$f(x) = g(x) = g(x) = 0$ whenever x is irrational
$f(x) = 1$ and $g(x) = x$ whenever x is rational
$h(x) = \frac{1}{n}$ if x is the rational number $\frac{m}{n}$ in lowest terms
$h(0) = 1$

Show that f is not continuous anywhere in $[0, 1]$
Assume towards contradiction that f is continuous at $x \in [0, 1]$
$\forall \epsilon > 0 \exists \delta > 0$ s.t if $0 < |x - p| < \delta, |f(x) - f(p)| < \epsilon$

Let $\epsilon = \frac{1}{2}$
Let $\delta > 0$

If x is rational:
$(x - \delta, x + \delta)$ contains some irrational number y. This is true for any δ selection
We have $|x - y| < \delta$, but $|f(x) - f(y)| = 1$

$1 > \epsilon$
Contradiction

If x is irrational:
Every open ball with radius δ contains a rational number and the identical argument holds from above

<u>Show that g is continuous only at $x = 0$</u>
Fix $\epsilon > 0$
Let $\delta = \epsilon$
If $|x - 0| < \delta$, then $|x| < \delta$
We know $f(0) = 0$

If x is an irrational number:
$|g(0) - g(x)| = 0$ and $0 < \epsilon$

If x is a rational number:
$|g(0) - g(x)| = |x|$ and $|x| < \delta = \epsilon$
$|x| < \epsilon$

Therefore $\forall \epsilon > 0 \exists \delta > 0$ s.t if $0 < |0 - x| < \delta, ||g(0) - g(x)|| < \epsilon$
And g is continuous at 0

Show g is not continuous at any point $x \neq 0$
Let $x \in [0, 1]$

If x is a rational number:
Let $\epsilon = \frac{|x|}{2}$
$\forall \delta > 0, \exists y \in B(x, \delta)$ such that y is irrational
Therefore $\forall \delta > 0$, $\exists y$ such that $0 < |x - y| < \delta$ but
$|g(x) - g(y)| = |x - 0| = |x|$ $|x| > \epsilon$
Therefore if x is rational, g is not continuous at x

If x is irrational
Let $\epsilon = \frac{|x|}{2}$
$\forall \delta > 0, \exists y \in (x, x + \delta) \subset B(x, \delta)$ such that y is rational
Therefore $\forall \delta > 0$, $\exists y$ such that $0 < |x - y| < \delta$ but
$|g(x) - g(y)| = |0 - y| = |y|$ and $y > x$ so $|y| > |x| > \epsilon$
Therefore if x is irrational, g is not continuous at x

Done!
<u>Show that h is continuous only at irrational points</u>
Show that h is not continuous at a rational point x:
Let $\epsilon = \frac{|b|}{2}$ where $x = \frac{a}{b}$
$\forall \delta > 0$, $\exists y \in (x - \delta, x + \delta)$ such that y is irrational

$|x-y|<\delta$, but $|h(x)-h(y)|=|b-0|=|b|$ and $b>\epsilon$
Therefore h is not continuous at rational x

<u>Show that h is continuous at an irrational x</u>
Let x_n be a sequence such that $x_n \to x$
We know that $f(x)=0$
If x_n consists of irrational numbers, we're done, as $f(x_n) \to 0$
For a sequence of rational numbers: in order to get closer and closer to some number x, my fraction $\frac{a}{b}$ must become finer and finer, meaning that the denominator must become larger and larger
Therefore is x_n is a sequence of rational numbers approaching x, then $h(x_n)$ is a decreasing sequence
It is also bounded below by 0, therefore $h(x_n) \to 0 = h(x)$

If x_n contains both rationals and irrationals, it clearly converges to 0 as all irrationals and rationals both converge to 0
Therefore h is continuous on an irrational x

4 Question 4

Show that f is continuous at a point x and additive. Then there is a constant a such that $f(x)=ax$ for all x

Show that it is continuous at all points:
Let $p \in \mathbb{R}$
Let $\epsilon > 0$
Since f is continuous at x
$\exists \delta > 0$ such that if $|x-p|<\delta$, $|f(x)-f(p)|<\epsilon$
Let $y \in \mathbb{R}$ such that $|y-p|<\delta$
Rewrite:
$|x-x+y-p|<\delta$
$|(x+(y-x))(p)|<\delta$
$|f(x+y-x)-f(p)|<\epsilon$
By additivity:
$|f(x)+f(y)-f(x)-f(p)|<\epsilon$
$|f(y)-f(p)|<\epsilon$

Therefore f is continuous at all points in $\mathbb{R}$

Now consider $p \in \mathbb{R}$
$1 = p * \frac{1}{p}$
So $f(1) = f(p * \frac{1}{p})$
$f(1) = p * f(\frac{1}{p})$ by additivity

With $n, m \in \mathbb{Z}^+$, $f(\frac{n}{m}) = n * f(\frac{1}{m}) = \frac{n}{m} * f(1)$
So $f(x) = xf(1)$ when x is rational

Now for any $p \in \mathbb{R}$, we can construct a sequence $\{x_n\}$ where each $x_i \in \mathbb{Q}$ and $x_n \to p$. This is based on the density of $\mathbb{Q}$ in $\mathbb{R}$

Since f is continuous at p: if $(x_n \to p)$, $f(x_n) \to f(p)$
Since $f(x_n) = x_n * f(1)$, $f(p) = p * f(1)$
$f(1)$ is a constant a, so $f(x) = ax$!
Done!

5 Question 5

Let $f(x) = 0$
Show $f(x)$ is continuous

Fix $x \in \mathbb{R}$
Fix $\epsilon > 0$
Let $\delta = 1$
If $0 < |x - p| < \delta$, $|f(x) - f(p)| = |0 - 0| = 0 < \epsilon$
Therefore $\forall \epsilon > 0, \exists \delta > 0$ such that if $0 < |x - p| < \delta, |f(x) - f(p)| < \epsilon$

Let $f(x) = 5x - 2$
Fix $\epsilon > 0$
Let $\delta = \frac{\epsilon}{5}$
If $|x - p| < \delta$, $|f(x) - f(p)| = |(5x - 2) - (5p - 2)| = |5(x - p)|$
$= |5x - 5p| = 5 * |x - p| < 5 * \delta = \epsilon$
Therefore $|f(x) - f(p)| < \epsilon$
So f is continuous

Let $f(x) = x^2$
Fix $\epsilon > 0$

Let $\delta = min(1, \frac{\epsilon}{2*|p|})$
This comes from $|x + p| = |x - p + 2p| \leq |x - p| + |2p|$ so if $\delta = min(1, \frac{\epsilon}{2*p})$, then $|x - p| < 1$ and $(|x + p| < 1 + 2 * |p|)$

If $|x - p| < \delta$, then $|x^2 - p^2| = |(x + p)(x - p)| = |x + p| * |x - p| < \delta * (1 + 2 * |p|)$
$< \frac{\epsilon}{1+2p} * (1 + 2 * |p|)$
$< \epsilon$
Done!