Analysis In Several Variables Homework 2

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1 Question 1

Show that the sup and inf are uniquely determined:
Let S be a non-empty subset of $\mathbb{R}$
Suppose $\exists x_1, x_2$ such that $x_1 = Sup(S)$ and $x_2 = Sup(S)$ but $x_1 \neq x_2$

Since $x_1, x_2 \in \mathbb{R}$, one of the following holds by Field Axiom 6
a) $x_1 = x_2$:
It is stated that $x_1 \neq x_2 \rightarrow\leftarrow$
b) $x_1 < x_2$:
$x_2 = Sup(S)$ which means that there is no real number less than x_2 which is an upper bound of S, but $x_1 < x_2$ and is an upper bound. $\rightarrow\leftarrow$
c) $x_2 < x_1$:
$x_1 = Sup(S)$ which means that there is no real number less than x_1 which is an upper bound of S, but $x_2 < x_1$ and is an upper bound. $\rightarrow\leftarrow$

Therefore $Sup(S)$ is unique

Suppose $\exists x_1, x_2$ such that $x_1 = Inf(S)$ and $x_2 = Inf(S)$ but $x_1 \neq x_2$

Since $x_1, x_2 \in \mathbb{R}$, one of the following holds by Field Axiom 6
a) $x_1 = x_2$:
It is stated that $x_1 \neq x_2 \rightarrow\leftarrow$
b) $x_1 > x_2$:
$x_2 = Inf(S)$ which means that there is no real number greater than x_2 which is an lower bound of S, but $x_1 > x_2$ and is a lower bound. $\rightarrow\leftarrow$
c) $x_2 > x_1$:
$x_1 = Inf(S)$ which means that there is no real number greater than x_1 which is a lower bound of S, but $x_2 > x_1$ and is an lower bound. $\rightarrow\leftarrow$

Therefore $Inf(S)$ is unique

2 Question 2

Let A, B be positive non-empty subsets of $\mathbb{R}$ where $a = Sup(A)$ and $b = Sup(B)$. Let C be the set of all products of the form xy where $x \in A$ and $y \in B$. Prove that $ab = Sup(C)$

Part 1: Show that ab is an upper bound of C
By the definition of Sup:
$\forall x \in A, x \leq a$ and $\forall y \in B, y \leq b$
$xy \leq ay$
$xy \leq ab$ as $y \leq b$
Since all product $xy \leq ab$, ab is an upper bound of C

Part 2: Show ab is the least upper bound of C
Assume towards $\rightarrow\leftarrow$ that ab is not the least upper bound of C
Let $w = Sup(C)$
We have $w < ab$
$w = x_1 y_1$ where $x_1 \in A$ and $y_1 \in B$ Note: We can do this because we know that $w < ab$, so we must be able to multiply to w from elements in a and b
By the approximation theorem we have:
$\exists x_2 y_2 \in A, B$ where
$x_1 < x_2 \leq a$ and
$y_1 < y_2 \leq b$
And since C is the set of all products of the elements of A and B, $x_2 y_2 \in C$ but $x_2 y_2 > x_1 y_1 \Rightarrow x_2 y_2 > w$
$\rightarrow\leftarrow$
Therefore ab is the least upper bound of C
Done!

3 Question 3

Suppose that $S, T \subset \mathbb{R}$ are nonempty and
$\forall s \in S \; \exists t \in T$ such that $s \leq t$
Show: $Sup(S) \leq Sup(T)$

Assume towards contradiction that $Sup(S) > Sup(T)$
$Sup(S)$ is:
a) An upper bound for S: $\forall s \in S, s \leq Sup(S)$
b) The least upper bound: $\nexists x \in S$ s.t $\forall s \in S, s \leq x \leq Sup(S)$

Consider the value Sup(T)
We have $\forall t \in T, t \leq Sup(T)$ and $Sup(T) < Sup(S)$
But $\forall s \in s, \exists t \in T$ s.t. $s \leq t$
Therefore $Sup(T)$ must be an upper bound of S as well

But $Sup(T) < Sup(S)$ which contradicts $Sup(S)$ being the least upper bound.
Therefore $Sup(S) \leq Sup(T)$

4 Question 4

Let $S, T \subset \mathbb{R}$ be non empty sets where
$\forall s \in S$ and $t \in T$, $s \leq t$
Show $Sup(s) \leq Inf(T)$

Part 1: Show that the inf and sup exist:
Show $Sup(S)$ exists:
Fix $t \in T$
Since $\forall s \in S, s \leq t$, t is an upper bound of S
Therefore S is bounded above
By the completeness axiom, S has a supremum
Show Inf(T) exists:
Fix $s \in S$
Since $\forall t \in T, t \geq S$, s must be a lower bound of T
Since T has a lower bound, it is bounded below
Therefore, T has an infimum. Note: The book claims this is a consequence of the completeness axiom

Part 2: Show that $Sup(S) \leq Inf(T)$
Assume towards a contradiction that $Sup(S) > Inf(T)$
Fix some $a \in \mathbb{R}$ such that $Inf(T) < a < Sup(S)$
By the approximation property:
$\exists x \in S$ s.t. $a < x \leq Sup(S)$
$\exists y \in T$ s.t. $Inf(T) \leq y < a$
From this we have:
$y < a < x \Rightarrow y < x$
But $y \in T$ and $x \in S$
This is a contradiction since every element in S is less than every element in T
Therefore $Sup(S) \leq Inf(T)$
Done!

The reason these properties are stronger:
Question 3 is a stronger form of the comparison property as it does not require the sets to be disjoint from each other. It only requires that there is some t which is greater than every element s, not that every t is greater than every s

Question 4 is stronger as it is showing that not only is Sup(T) greater than Sup(S), but Inf(T) is greater than Sup(S)! Greatest lower bound is still greater than the least upper bound.