Chaos Homework 1

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1 Question 1

Part a: Show that a period 2 orbit is of the form (x,y) (y,x)
Let some orbit of period 2 have the 2 points (x, y) and (i, j).
Since they are in an orbit, we know $f(x, y) = (i, j)$
The henon map is defined as $f(x, y) = (a - x^2 + by, x)$
From this we can clearly see that if $f(x, y) = (i, j)$ then $x = j$

We also have $f(i, j) = (x, y)$, as it is a period 2 orbit
Using the same logic, it is clear that $i = y$
Therefore since $f(x, y) = (i, j) = (y, x)$
And $f(i, j) = f(y, x) = (x, y)$
We can conclude that all orbits of period 2 in the Henon map are of the form $(x, y)(y, x)$

Prove that a map has a period 2 orbit iff $4a > 3(1 - b)^2$
To find period 2 orbits we will be looking at the fixed points of the function f^2
$f^2(x, y) = (a - (a - x^2 + by)^2 + bx, a - x^2 + by)$
To find the fixed points we set the equations:
$x = (a - (a - x^2 + by)^2 + bx$
$y = a - x^2 + by$
And use substitution to find:
$0 = (x^2 - a)^2 + (1 - b^3)x - (1 - b)^2 a$
$0 = (x^2 - (1 - b)x - a + (1 - b)^2)(x^2 + (1 - b)x - a)$
We can then take the left factor, as the right factor is for the fixed points of f as shown in the texbook
$(x^2 - (1 - b)x - a + (1 - b)^2) = 0$
Applying quadratic:
$\frac{(1-b)\pm\sqrt{(1-b)^2-4(-a+(1-b)^2)}}{4a}$
Considering when this has real solutions we have
$(1 - b)^2 - 4(-a + (1 - b)^2) > 0$
$(1 - b)^2 + 4a - 4(1 - b)^2 > 0$
$4a > 4(1 - b)^2 - (1 - b)^2$
$4a > 3(1 - b)^2$

We have shown that for there to be real solutions to the equations finding orbits of period 2, $4a > 3(1-b)^2$
Done!

2 Question 2

Find a formula for the inverse of the Henon map
We have $f(x, y) = (a - x^2 + by, x)$

$x_1 = a - x^2 + by$
$y_1 = x$

$x^2 + x_1 - a = by$
$\frac{x^2+x_1-a}{b} = y$
So we can say
$f^{-1}(x, y) = (y, \frac{x^2+x_1-a}{b})$

3 Question 3: Determine if the points are sources sinks or saddles

Part A:
$A = \begin{bmatrix} 4 & 30 \\ 1 & 3 \end{bmatrix}$

$A - \lambda I = 0$

$\begin{bmatrix} 4-\lambda & 30 \\ 1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Find determinant:
$(4-\lambda) * (3-\lambda) - 30 = 0$
$12 - 4\lambda - 3\lambda + \lambda^2 - 30 = 0$
$\lambda^2 - 7\lambda - 18 = 0$
$(\lambda - 9)(\lambda + 2) = 0$
$\lambda = 9$ and -2
Since these values are both greater than 1, the origin is a source

Part B

$B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$
$\begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4}1-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Find the determinant
$(1-\lambda)(\frac{3}{4}-\lambda)-\frac{1}{8}=0$
$(\frac{3}{4}-\lambda-\frac{3}{4}\lambda+\lambda^2-\frac{1}{8}=0$
$\lambda^2-\frac{7}{4}\lambda+\frac{5}{8}=0$
$8\lambda^2-14\lambda+5=0$
$8\lambda^2-4\lambda-10\lambda-5=0$
$4\lambda(2\lambda-1)-5(4\lambda-1)$
$(4\lambda-5)(2\lambda+1)=0$
$\lambda=\frac{1}{2}$ and $\frac{5}{4}$
One eigenvalue is greater than 1 and one is less than 1, therefore the point is a saddle

Part C
$B=\begin{bmatrix}-0.4 & 2.4\\ -0.4 & 1.6\end{bmatrix}$
$\begin{bmatrix}-0.4-\lambda & 2.4\\ -0.4 & 1.6-\lambda\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$
Find the determinant
$(-0.4-\lambda)(1.6-\lambda)-(2.4)(-0.4)=0$
$-0.64+0.4\lambda-1.6\lambda+\lambda^2+0.96=0$
$\lambda^2-1.2\lambda+0.32=0$
Apply quadratic formula:
$\frac{1.2\pm\sqrt{1.44-1.28}}{2}$

$=\frac{1.2\pm 0.4}{2}$
$\lambda=0.4, 0.8$
Since both of the eigenvalues are less than 1, this point is a sink

4 Question 4

We start by finding the fixed points of f by taking the equations
$x=x^2-5x+y$
$y=x^2$
And substituting
$x=x^2-5x+x^2$ $0=2x^2-6x$
$0=(2x)(x-3)$
$x=0,3$
$y=0,9$
Fixed points: $(0,0)$ and $(3,9)$
The Jacobian Matrices for each point are as follows

$(0,0)$ Jacobian: $\begin{bmatrix}-5 & 1\\ 0 & 0\end{bmatrix}$

$(3,9)$ Jacobian: $\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$

Considering the origin first:
$\begin{bmatrix} -5-\lambda & 1 \\ 0 & 0-\lambda \end{bmatrix}$
$(-5-\lambda)(-\lambda) = 0$
$5\lambda + \lambda^2 = 0$
$\lambda^2 - 5\lambda = 0$
$\lambda(\lambda+5) = 0$
Eigenvalues: $0, -5$
Since the magnitude of one eigenvalue is greater than 1 and one is less than 1, this is a saddle

Now considering $(3,9)$
$\begin{bmatrix} 1-\lambda & 1 \\ 6 & 0-\lambda \end{bmatrix}$
$(1-\lambda)(-\lambda) - 6 = 0$
$-\lambda + \lambda^2 - 6 = 0$
$\lambda^2 - \lambda - 6 = 0$
$(\lambda-3)(\lambda+2) = 0$
So the Eigenvalues are 3,-2 and since their magnitudes are both greater than, $(3,9)$ is a source.

5 Question 5

a) No, the picture is not different if the y-coordinate is plotted instead. This is because the bifurcations actually take a 3d form and we are simply looking at a 2d visualization of the diagram. Due to this, which axis we "slice" the model does not impact the graph. I have uploaded a comparison of the graphs

b) Are there any periodic windows when viewing a from 1.925 to 1.975? Yes! There is a notable window from a = 1.965 to 1.967 which appears to be of period 11, and there is also a window from 1.965 to 1.97 which seems to be between period 18 and 20. The image got granular, but we were told in class a graphical approach was fine!

c) I used Matlab to find the immediate iterate after the bifurcations to determine these values for estimating the fiegenbaum constant:
Points of bifurcation:
1.812200, 1.921600, 1.945200, 1.950300, 1.951400, 1.951600
$\frac{1.945200-1.812200}{1.945200-1.921600} = 0.133/0.0236 = 5.63$
$1.945200 - 1.921600/1.950300 - 1.945200 = 4.62745098039$
$(1.950300 - 1.945200)/(1.951400 - 1.950300) = 4.63636363636$
$1.951400 - 1.950300/1.951600 - 1.951400 = 5.5$ This last one is wrong because

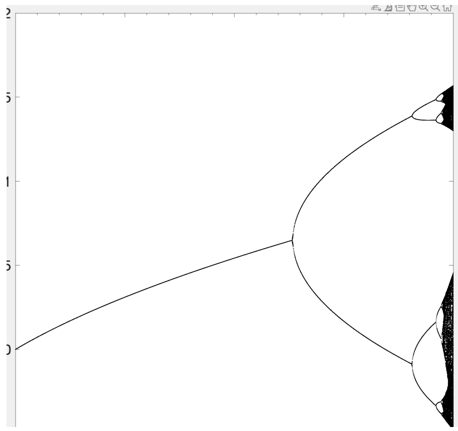

Figure 1: Enter Caption

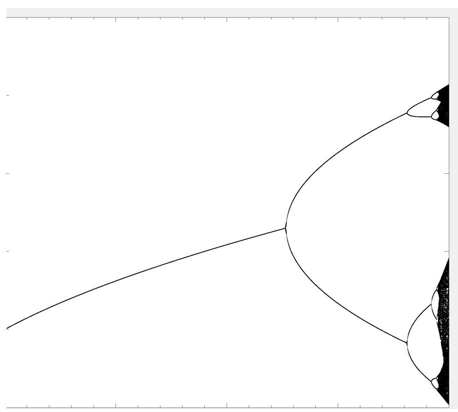

Figure 2: Enter Caption

it is a jump from a 2^7 orbit to a 2^9 orbit. I increased df to 10^{-5} and matlab still couldnt catch the 2^8
Overall it seems to be pretty close to the 4.669 value we're looking for, but it is only truely approaches that value as the bifurcations go to infinity.

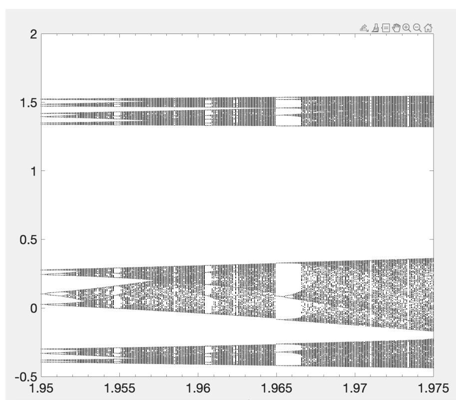

Figure 3: Diagram to show windows