Chaos Homework 10

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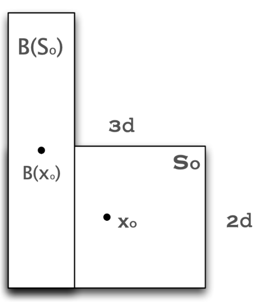
1 Question 1

1. Challenge 5 Step 1: Assume that $B(x_0)$ differs from x_0 by less (*or equal to*) d in each coordinate. In Figure 5.20 we draw a rectangle centered at x_0 with dimensions $3d$ in the horizontal direction and $2d$ in the vertical direction (see below). Assume that the rectangle lies near the bottom left of the unit square, nowhere near the line $y = 1/2$, so that it is not chopped in two by the map. Then its image is the rectangle shown; the center of the rectangle is of course $B(x_0)$, show that the image of the rectangle is guaranteed to "map across" the original rectangle. Explain why there is a fixed point of B in the overlapping region, within $2d$ of x_0.



We know that the skinny baker map contracts area by a factor of 3 horizontally and expands area by a factor of 2 vertically
This means that a square of side lengths a would be mapped to a rectangle of side lengths $\frac{1}{3}a \times 2a$ with the vertical line being longer than the horizontal.
In general, we can not say anything about the overlap between the original square and the image of the square, but it is clear that the new shape is a rectangle due to the continuity of B (locally) and its linearity.

Now consider a $3d \times 2d$ rectangle centered at a point x_0 with $3d$ representing the horizontal lines
We will call this rectangle S
From our hypothesis, we have $||B(x_0) - x_0|| \leq d$

Therefore the center of $B[S]$ is at most d away from x_0

Let's consider the eastern and western Boundaries of S
These are 2 vertical lines who are $1.5d$ to the right and left of x_0 respectively
Likewise, the Northern and Southern Boundaries of S are horizontal lines offset from the y-coordinate of x_0 by d in either direction

Since S has side lengths $3d \times 2d$, based on the expansion and contraction of B, $B[S]$ has side lengths $d \times 4d$

Since $B[S]$ has a horizontal sidelength of d and it's center is at most d offset from x_0, it is clear the that western Boundary of $B[S]$ is at most $1.5d$ to the left of x_0
Therefore the western Boundary of S surrounds the western Boundary of $B[S]$
This argument is mirrored for the eastern Boundary of S and $B[S]$

For the Northern and Southern Boundaries, WLOG consider the Northern Boundary
The Northern Boundary of S lies d above the y-coordinate of x_0
$B[S]$ has a vertical side length of $4d$. Therefore the Northern Boundary of $B[S]$ is at most $2d$ above the y-coordinate of $B(x_0)$
$||B(x_0) - x_0|| \leq d$ therefore the Northern Boundary of $B[S]$ is at most $2d + d$ above x_0
If $||B(x_0) - x_0|| = 0$ then the Northern Boundary for $B[S]$ would be $2d$ offset from x_0. (This is the lowest possible Northern Boundary of $B[S]$)
Since $2d > d$, the Northern Boundary of $B[S]$ is above the Northern Boundary of S
A similar argument applies for the Southern Bound

Therefore the Northern and Southern Bounds of S are surrounded by the N/S Bounds of $B[S]$ and the east-west Bounds of $B[S]$ are surrounded by the E/W Bounds of S

By the Colorado Corollary, there is some fixed point of $B \in S$

Show that the fixed point is at most $2d$ from x_0
Let the fixed point be called y
$y \in S$
Since $y \in S$, the furthest point y can be from x_0 is one of the corners of the rectangle
Apply Pythagorean theorem $||y - x_0|| \leq \sqrt{(1.5d)^2 + d^2} = \sqrt{3.25d^2}|| = 1.802d < 2d$
Done!

2 Question 2

2. Challenge 5 Step 2: Now suppose our computer makes mistakes in evaluating B of size at most 10^{-6}, and it tells us that $B(x_0)$ and x_0 are equal to within 10^{-6}. Prove that B has a fixed point within 10^{-5} of x_0.

We currently know that our computer says that $B(x_0)$ and x_0 are equal within 10^{-6}
Another way to word this is that $||B(x_0) - x_0|| < 10^{-6}$
To clarify, the difference between the 2 points is at most 10^{-6} as the computer said they are equal up until this point

Considering our proof from problem 1, let $d = 10^{-6}$
Our hypothesis condition is met, therefore we can conclude that there is some fixed point y of B within $2 * 10^{-6}$ of x_0

$2 * 10^{-6} < 10^{-5}$, therefore if a point is within $2 * 10^{-6}$ of x_0, it is clearly also within 10^{-5} of x_0
Therefore there is a fixed point of B within 10^{-5} of x_0
Done!

3 Question 3

3. Challenge 5 Step 3: Prove Theorem 5.19. Let B denote the Skinny Baker Map and let $d > 0$. Assume that there is a set of points $\{x_0, x_1, \cdots, x_{k-1}, x_k = x_0\}$ such that each coordinate of $B(x_i)$ and x_{i+1} differ by less than d for $i = 0, 1, \cdots, k-1$. Then there is a periodic orbit $\{z_0, z_1, \cdots, z_{k-1}, z_k = z_0\}$ such that $|x_i - z_i| < 2d$ for $i = 0, 1, \cdots, k-1$. **Note**: The sequence of points $\{x_i\}$ is *not* an orbit of B. **Hint**: Draw a $3d \times 2d$ rectangle S_i centered at each x_i as in Figure 5.21. Show that $B(S_i)$ lies across S_{i+1} by drawing a variant of Figure 5.20, and use Corollary 5.13.

In proof 1, we showed that so long as x_0 and $B(x_0)$ are within d, then the image of a 3d x 2d rectangle centered at x_0 will map across the original rectangle
Since each $B(x_i)$ differs from x_{i+1}, we can conclude that the image of a rectangle centered at x_i lays across the original rectangle and there is some z_i such that $B(z_i)$ is actually falls within the rectangles
Also the final $B(x_{k-1})$ will lie across the a rectangle centered at B_k and by the Colorado corollary, there is a fixed point of period k in the map within $2d$ of each point

4 Question 4

4. Challenge 5 Step 4: Let f be a continuous map, and assume that there is a set of rectangles $S_0, S_1, \cdots, S_k$ such that $f(S_i)$ lies across S_{i+1} for $i = 0, 1, \cdots, k-1$, each with the same orientation. Prove that there is a point x_0 in S_0 such that $f^i(x_0)$ lies in the rectangle S_i for all $i \in \{0, 1, \cdots, k-1\}$. By the way, does k have to be finite?

We will start by considering the base case of our argument, which pertains to S_0, S_1, and S_2
S_0 is a rectangle, and we have that $f[S_0]$ "lies across" S_1

Since $f[S_0]$ lies across S_1, there is a region of the overlap between the two
$f[S_0] \cap S_1 \neq \phi$

Therefore there is a region in S_0 which maps into S_1

Let $Z_0 = f[S_0] \cap S_1$. This is the region of points in the overlap between the two-sets. Visually, think about "chopping off" the edges of $f[S_0]$
Remember that the orientation of $f[S_i]$ when compared to S_{i+1} remains the same for each i
This means that the directions of expansion and contraction are the same for any set in one of these rectangles. This also relies on the continuity of f

It is pretty clear that Z_0 takes up the full-length of S_1 along the direction of expansion. When f is applied, Z_0 will be stretched longer than it's original form
Therefore $f[Z_0]$ lays across S_2
This implies that there is a region of overlap between $f[Z_0]$ and S_2

Note: $Z_0 \subseteq f[S_0]$ therefore $f[Z_0] \subseteq f[f[S_0]]$
Therefore there is a region in S_0 which maps into S_2

Let $Z_1 = f[Z_0] \cap S_2$. This is the set of points in S_2 which are mapped from Z_0, and therefore can be traced back to S_0 under reverse iteration.

Now assume that for some n, $Z_n = f[Z_{n-1}] \cap S_n$ and these Z_i's for $i < n$ have area and can be reverse iterated to find a region in S_0 which maps into S_n.
Note: Z_n also fully occupys S_n along the expansion axis

Show that this implies that Z_{n+1} has area:
$Z_{n+1} = f[Z_n] \cap S_{n+1}$ and $f[Z_n]$ lies across S_n as explained above
Therefore Z_{n+1} is a region with area who maps back to S_0 under $n+1$ reverse iterations!

This means that for any finite n, the property holds

We did not prove it here, but under infinite iteration, the area of these Z_i's will have measure 0 in $\mathbb{R}^2$, but will still have points! Therefore this property will hold in the infinite case, but requires a different proof

5 Question 5

5. Challenge 5 Step 7: Assume that a plot of length one million iterates of the cat map is made on a computer screen, and that the computer is capable of calculating an iteration of the cat map accurately within 10^{-6}. Do you believe that the dots plotted represent a true orbit of the map (to within the pixels of the screen)?

The cat map is a continuous map when we think about it as being on a tourus.
Therefore it meets the criteria for our proof of question 4

What this proof really says is that if we have a rectangle of uncertainty about the position of a point, as we iterate the map and that area of uncertainty grows in 1 direction and shrinks in another, there will be a true orbit which lies within that uncertainty!

A normal computer screen has a resolution of 1920x1080
If we're considering a function on the unit interval, each pixel has a rectangle of $\frac{1}{1920} \times \frac{1}{1080}$. This is about $0.00052083333 \times 0.00092592592$

Our computer can compute the iteration of the cat map to a precision of 10^{-6}
When we iterate, based on question 4 if we make our rectangles $10^{-6} \times 10^{-6}$, we know there is a true orbit within 10^{-5} of our computer calculated orbit

$\frac{1}{1920} \times \frac{1}{1080}$ is a much larger rectangle than a region of $0.0000001 \times 0.0000001$

Therefore the error in our orbit will be contained in a single pixel from the true orbit and we can not see it