Chaos Homework 9

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1 Question 1

For a width w between 0 and 1, the general formula for a skinny baker map is:
$B(x,y) = \begin{cases} wx, 2y & \text{if } 0 \leq y \leq \frac{1}{2} \\ wx(1-w), (2y-1) & \text{if } \frac{1}{2} < y \leq 1 \end{cases}$
If we have a disc with radius r in $\mathbb{R}^2$, the area contraction factor of this map will be the area of the elipse produced by iterating this disc divided by the area of the original disc

$DV(v_0) = \begin{bmatrix} w & 0 \\ 0 & 2 \end{bmatrix}$
So after a single iteration, a disc with radius r will be mapped to an elipse with semi-major axis of length $2r$ and a semi-minor axis of length $w * r$ (as the eigen values of this matrix are 2 and 2)
Therefore the area of the new elipse is $\pi * r^2 * 2w$
So the area contraction factor is:
$\frac{\pi * r^2 * 2w}{2 * \pi * r} = 2w$
Done!

2 Question 2

Let f be a map on $\mathbb{R}^m$ with a constant Jacobian Determinant D
This means that the determinant of the Jacobian is the same at all points in $\mathbb{R}^m$
Since f:$\mathbb{R}^m \to \mathbb{R}^m$, for any x value, it is clear that $\det(Df^k(x)) = D$
So $\det(J_n(x)) = D \quad \forall n \in \mathbb{Z}^+$ and $x \in \mathbb{R}^m$

The kth Lyapunov Number is defined as:
$\lim_{n\to\infty}(r_k^n)^{\frac{1}{n}}$

Each r_k^n is the length of the kth longest orthogonal axis of $J_n N$ where N is the unit disc
From the textbook: we know that we can calculate the length of the r_k^n's by

taking the square root of the eigenvalues of $J_n * (J_n)^T$

Since the determinant of a matrices is multiplicative:
$Det(J_n * (J_n)^T) = Det(J_n) * Det((J_n)^T) = D^2$
Since the determinant of a matrix is equal to the product of its eigenvalues, it is clear that D is equal to the product of the square root of the eigenvalues

Therefore D is equal to the product of the r_k^ns for any n (as the Jacobian determinant is constant everywhere)
Therefore the product of the $lim_{n\to\infty} r_k^n$ is equal to D
And D is equal to the product of the lyapunov numbers of f for any x value

3 Question 3

Lets write the Grahm-Schmidt orthognalization process as a $m \times m$ matrix multiplication where $Z = YR$
We will write the matricies as follows (Each matrix is $m \times m$):

$$\begin{bmatrix} z_1 & z_2 & ... & z_m \\ z_1 & z_2 & ... & z_m \\ ... \\ z_1 & z_2 & ... & z_m \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & ... & y_m \\ y_1 & y_2 & ... & y_m \\ ... \\ y_1 & y_2 & ... & y_m \end{bmatrix} * \begin{bmatrix} 1 & r_{21} & r_{31} & ... & r_{m1} \\ 0 & 1 & r_{32} & ... & r_{m2} \\ 0 & 0 & 1 & ... & r_{m3} \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

This gives us:
$z_1 = y_1$
$z_2 = y_1 * r_{21} + y_2$
$z_3 = y_1 * r_{31} + y_2 * r_{32} + y_3$
...
$z_m = y_m * r_{m1} + y_2 * r_{m2} + ...y_{m-1} * r_{m(m-1)}$
...
So when we solve for the y's we have:
$y_1 = z_1$
$y_2 = z_2 - y_1 * r_{21}$
$y_3 = z_3 - y_1 * r_{31} - y_2 * r_{32}$
...
$y_m = z_m - y_1 * r_{m1} - y_2 * r_{m2} - ...y_{m-1} * r_{m(m-1)}$

For this to follow Grahm-Schmidt orthoganlization, we have: $r_{ij} = \frac{z_i \cdot y_j}{||y_j||^2}$
When we plug this in for the r values we get:
$y_1 = z_1$
$y_2 = z_2 - \frac{z_2 \cdot y_1}{||y_1||^2} y_1$
$y_3 = z_3 - \frac{z_3 \cdot y_1}{||y_1||^2} * y_1 - \frac{z_3 \cdot y_2}{||y_2||^2} y_2$
...
$y_m = z_m - \frac{z_m \cdot y_1}{||y_1||^2} * y_1 - \frac{z_m \cdot y_2}{||y_2||^2} * y_2 - ... \frac{z_m \cdot y_{m-1}}{||y_{m-1}||^2} * y_{m-1}$
Nice!
The determinant of this R matrix is clearly 1, as its left-right diagonal is all 1's

and its right-left diagonal has a 0 in it
This implies that the m-dimensional volume is the same, as the length of the axes of the elipsiods are determined by the eigenvalues of Y and Z. Since their determinants are the same, the product of their eigenvalues (and therefore length of axes) is the same. Since the product of the axis length is the same, their volume must be the same!

4 Question 4

Plot the tinkerbell map and talk about different c_4 values!

When the value of c_4 is -0.5, the orbit diverges to infinity

When it is -0.222, it tends towards a shape which I would describe as a chicken nugget with sharp edges

When it is 0.05555, the edges of the attractor become smoother!

When it is 0.33, the area of the attracting orbit decrereases and the edges become more elongated

At 0.0611, the attracting orbit becomes very ellipse-like with a smaller area! Still quasi-periodic

At 0.8888, they appear to approach a sink at 0

At 1.16, the points approach a sink at [-0.44474723 -1.2305759] (Note: there may be numerical error here and it is actually approaching a sink at 0. It becomes hard to tell!)

Then beyond 1 the iterations go to infinity again

5 Question 5

Below are all of my calculated lyapunov exponents!
Note: For the tinkerbell quasi-periodic, the first one is a number very close to 0!

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spencerbrouhard@Spencers-MBP Homework 9 Code % python3 lyapunov_exponents.py

Lyapunov Exponents for Tinkerbell with Quasi-Periodic Orbit
3.8894548148090155e-05
-0.056871786896727734

Lyapunov Exponents for Henon Map
0.4130927401098868
-1.6170655444358226

Lyapunov Exponents for Tinkerbell with Chaotic
0.1918267825163929
-0.5118082235729449

Lyapunov Exponents for Ikeda Map
0.5013776539138926
-0.7120986852295452
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