Chaos Homework 5

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1 Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function where 0 is a sink, $(-1, 1)$ are the boundaries of the largest basin of 0.

Part a: What are the possible trajectories of -1 and 1?

We know that the boundaries of basins of attraction are either points which are eventually periodic to a fixed point, or fixed points themselves. Since -1 and 1 are not in the basin of 0, we know that they are not eventually periodic to it, therefore -1 and 1 must be fixed points. This means that they remain in the same location when iterated.

Part b: What are the possible Lyapunov numbers for 0, 1, -1? We know that the Lyapunov number for the sink at 0 must be between 0 and 1, as a Lyapunov number of less than 1 denotes a sink and 0 is a sink

The points -1 and 1 are non-sink fixed points, as if they were sinks they would have some basin of attraction which would "take a little bit" from the basin of attraction to 0.

Therefore their Lyapunov number must be greater than or equal to 1. Note: if they are equal to 1 then it is a neutral point

2 Question 2

Show that if the Lyapunov orbit of x_0 under a map f is L, then the orbit of x_0 under f^k is L^k

$L_k(x_0) = \lim_{n\to\infty} |f^{k'}(x_0) * f^{k'}(x_k) * f^{k'}(x_{2k})...|^{\frac{1}{n}}$

Which can be rewritten as

$lim_{n\to\infty} \prod_{i=0}^{n} |f^{k'}(x_{i*k})|^{\frac{1}{n}}$

By the chain rule we have:

$f^{k'}(x_{i*k}) = \prod_{j=0}^{k} f'(x_{ik+j})$. Note this is just the chain rule k times.

Substituting into original:

$L_k(x_0) = \lim_{n\to\infty} \prod_{i=0}^{n} \prod_{j=0}^{k} |f'(x_{ik+j})|^{\frac{1}{n}}$

Combining the products (like a double for loop!)
$L_k(x_0) = \lim_{n\to\infty} \prod_{i=0}^{nk} |f'(x_i)|^{\frac{1}{n}}$

Raise to $\frac{k}{k}$
$L_k(x_0) = \lim_{n\to\infty} \prod_{i=0}^{nk} |f'(x_i)|^{\frac{k}{nk}}$

Let $w = nk$. Note: $nk = n$ when taking the limit so we can write:
$L_k(x_0) = (\lim_{w\to\infty} \prod_{i=0}^{w} |f'(x_i)|^{\frac{1}{w}})^k$
$L_k(x_0) = L(x_0)^k$ substituting in based on on definition of $L(x_0)$
Done!

3 Question 3

Let $f(x) = 2.5x(1-x)$
Part a: What are the possible asymptotic behaviors for all $x \in \mathbb{R}$
If $x \in (0,1)$, then x will asymptotically approach the sink at $\frac{3}{5}$. This clear via the cobweb plot, but the fixed point can easily be solved for and found to be a sink
If $x \notin (0,1)$ then it will be sent to negative infinity, for both sides of the interval

Note: There are also EP points 0, 1, and a countably infinite number of points which map to $\frac{3}{5}$ over $k \in \mathbb{Z}^+$ iterates. If they're considered to be AP as well as EP, then they are also in the possible behaviors. I wasn't sure about the phrasing of the question!

Part B: Find the Lyapunov exponent shared by most bounded orbits
Since most bounded orbits asymptotically approach $\frac{3}{5}$, we can take the derivitave of f at the point and take the natural log to find the Lyapunov exponent
$f'(x) = -5x + 2.5$
$f'(\frac{3}{4}) = -0.5$ Therefore the Lyapunov Exponent for most bounded orbits is $\ln(0.5)$

Part C: Do all bounded orbits have the same exponents?
No! 0 is a bounded orbit of f which has a Lyapunov exponent of $\ln(2.5)$! Most of them share the same exponent but points which are EP to 0 don't!

4 Question 4

Part A: Find a conjugacy between $g(x) = 2 - x^2$ and $f(x) = 4x(1-x)$
Step 1: Find the fixed points of both functions

Fixed points of g:
$2 - x^2 = x$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x = -2$ and $x = 1$

Fixed points of f:
$-4x^2 - 3x = 0$
$x(4x-3) = 0$
$x = 0$ and $x = \frac{3}{4}$

Now create the set of points: $(0, -2), (\frac{3}{4}, 1)$ and find the line which passes through both of them:
Slope: $\frac{1+2}{0.75} = 4$
Y-intercept $= -2$
So $C(x) = 4x - 2$

<u>Part B: Use the map to find the fixed points and period 2 orbits of g</u>
Fixed points: $C(0) = -2$ and $C(\frac{3}{4}) = 1$ therefore the fixed points of g are -2 and 1
Their stability can be found by checking the stability of the point pre-conjugation:
$f'(0) = 4$ so 1 is a source in g
$f'(\frac{3}{4}) = -2$ so 1 point is also a source in g

Now we do the same for f^2 to find these points in g. From a previous homework, the fixed points of f^2 are $\frac{5 \pm \sqrt{5}}{8}$
$C(\frac{5 \pm \sqrt{5}}{8}) = \frac{1 \pm \sqrt{5}}{2}$ So these are (new) fixed points of g^2
To find stability: $f'(\frac{5+\sqrt{5}}{8}) * f'(\frac{5-\sqrt{5}}{8}) = (20 + 4\sqrt{5} * 20 - 4\sqrt{5})$ so these points are also unstable (the magnitude of this value is greater than 1)

<u>Show that g has chaotic orbits</u>
We showed in class that the tent map has chaotic orbits for all initial conditions which are irrational numbers. This is because multiplying by 2 and modding only shifts the digits over but if they never repeat, we never land exactly where we started

We then showed a conjugacy between the tent map and $4x(1-x)$
Since we also have a conjugacy between $4x(1-x)$ and g, we know there is a conjugacy between g and the tent map. This is because conjugacy is transitive. To show this we could compose the function mapping the tent map to f and f to g

Since g has a conjugacy with a map who has chaotic orbits, those initial conditions passed through the map will have chaotic orbits in g

5 Question 5

What did I learn plotting the 2 picutures

A couple cool findings: The Lyapunov exponents increase as the a value increases. There is a point where they cross over 1 and become chaotic!

In addition, it seems that the sinks become stronger the closer to the center of the bifurcations the alpha value is. The sink then becomes weaker before a bifurcation occurrs

It also looks like there are interesting drops in the value of the exponent where windows appear in the diagram

Why does it go tangent a few times before breaking through

I believe this happens when there are orbits which contain neutral points. It happens at the moment of bifurcation, and then the exponent decreases as the slopes break through and points settle into their new orbits

Will a new initial condition produce a new figure

Not in spirit. While it is technically possible for slight changes based on the initial condition due to the precsicion of computer calculations, the spirit of the figure will be the same regardless of initial conditions provieded they are in between 0 and 1 and produce bounded orbits