Analysis In Several Variables Homework 3

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1 Question 1

Part 1: Prove that $0 \leq a_i \leq k-1$ for each $i = 1, 2...$
We have a_0 is the greatest integer such that $a_0 < x$
We can then write x as $a_0 + f$ where $0 < f < 1$
This can be rewritten as $0 < k * f < k$
Let a_1 denote the largest integer in $k * f$
Since $k * f < k$, the largest integer less than k is at most $k - 1$
Therefore $0 < a_1 < k - 1$
Now f can be written as $a_1 + f_2$ where $0 < f_2 < 1$
So $0 < k * f_2 < k$
If we take a_2 to be the largest integer less than k, $0 < a_2 < k - 1...$ and this continues as each f_n can be written as $a_n + f_{n+1}$ where $0 < f_{n+1} < k$
Done!
Part 2: Show that X is the sup of the set $\{a_0, a_0.a_1, a_0.a_1a_2...\}$
$\forall n \in \mathbb{Z}$: $a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n} \leq x$
This can be rewritten as $\forall n \in \mathbb{Z} : r_n \leq x$
So x is clearly an upper bound of the set of all r_n's.

Show it is the least upper bound:
Assume towards contradiction that x is not the least upper bound. Let α represent this value
$\forall n \in \mathbb{Z}^+$: $x - (a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n}) = \frac{f_{n+1}}{k^{n+1}}$
where f_{n+1} can continue to be broken down
It is clear that $\frac{f_{n+1}}{k^{n+1}} < \frac{1}{k^n}$
There is also some value of n such that $\frac{1}{k^n} < x - \alpha$
$x - (a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n}) < \frac{1}{k^n} < x - \alpha$
$x - (a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n}) < x - \alpha$
$-(a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n}) < -\alpha$
$(a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... \frac{a_n}{k^n}) > \alpha$
Rewrite
$r_n > \alpha$
$\rightarrow\leftarrow$ as α is the least upper bound of the r_n's therefore x is the Sup

2 Question 2

Prove that $r \geq 0$ is a real number which is eventually periodic. Prove that r is rational

Step 1: Reduce to r being periodic, not eventually periodic

We can reduce this problem to showing that a real number r with a periodic decimal representation (defined in the assignment) is rational.

This is because an eventually periodic number takes the form:
$a_0.a_1a_2...b_0b_1b_2...b_nb_0...$
Where the a_0 is an integer, $a_1...a_n$ is the non-repeating component (which is finite) and the b terms are the periodic digits

Since a_0 is an integer, it is rational

$a_1...a_n$ can be written in the form $\frac{a_1}{10^1} + \frac{a_1}{10^2} + ... + \frac{a_n}{10^n}$. Each of the terms of this sum are rational numbers and $\mathbb{Q}$ is closed under addition, therefore this term is rational

Therefore, all we need to show is that a periodic decimal is rational!

Step 2: Show that we can write r as $a_0a_1a_2...a_n * b$ where b is an infinite repeating decimal of 0's and 1's

Examples:
$1234 * 0.\overline{0001} = 0.\overline{1234}$
$12 * 0.\overline{01} = 0.\overline{12}$

For any integer of the form $a_0a_1...a_n$, multiplying it by the decimal $0.\overline{0_10_2...0_{n-1}1}$ yields $0.\overline{a_0a_1...a_n}$

This pattern can be pretty easily seen by performing multiplication by hand! It as long as the pattern repeats, you just keep adding another set of $a_0a_1...a_n$ onto the end of the decimal!

Let this decimal comprised of 0's and 1's be referred to as b

Step 3: Show that we can multiply b by an integer n whose digits are 1 to gain $0.\overline{1}$

We can multiply b by $1_01_1...1_n$ where n is the length of the period of b to get the decimal 0.11111....

Examples: $0.\overline{01} * 11 = 0.\overline{1}$
Examples: $0.\overline{0001} * 1111 = 0.\overline{1}$

Note: This is also pretty easily convinced by doing some examples by hand

Call this value c

Step 4: Show that 9c is rational
$9 * 0.\overline{1} = 0.\overline{9}$
Lets consider the set of rational numbers r_n where $r_1 = 0.9, r_2 = 0.99...$
The Sup of this set is 1, therefore $0.\overline{9}$ is a valid decimal representation of 1
1 is a rational number!
Therefore $9c = 1$ and is rational

Step 5: Explain why this means r is rational
We have shown that r can be written in the form of an integer times b
The rational numbers are closed under multiplication, and all integers are members of $\mathbb{Q}$. This means all we need to show is that b is rational to conclude that r is rational
We showed that there is some integer n such that $b * n = c$
We also showed that $9c$ is a rational number
We can then say that $b * n * 9$ is rational
Since the product of a rational and irrational number is always irrational, we can conclude that b must be rational
Therefore r is rational!

3 Question 3

Show that rational numbers have decimal representations which are eventually periodic

Fix $x \in \mathbb{Q}$
x can be written as $\frac{a}{b}$
Consider long division of $\frac{a}{b}$
Let k represent the greatest integer such that $b * k < a$
Let r represent the remainder of $a - b * k$
If $r = 0$, then the decimal representation is just $k.0000...$, which is periodic
If $r \neq 0$, we then consider how many times b "goes into" 10 * r
$r < a$
$10r < 10a$
Therefore a will at most multiply into 10r 9 times
That means the decimal place is a value 0-9
When repeating infinitely, a number will eventually repeat (as 0-9) is finite
And since the process is multiply by b, subtract, multiply remainder by 10...
Once a number appears for a second time (let this number be called p) it will follow the same pattern it did the first time
This is just due to the steps repeating, which will eventually lead back to p

Therefore a rational number can be represented in a decimal form which is eventually periodic