Analysis In Several Variables Homework 9

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1 Question 1

The whole space M is open:
Fix $x \in M$
$\forall r \in \mathbb{R}$, $B(x,r) \subseteq M$ by definition
Therefore $\exists r \in \mathbb{R}$ s.t. $B(x,r) \subseteq M$ and x is an interior point
Since all points are interior points, M is open

ϕ is an open set as it contains no points, therefore "all of its points are interior points" vacuously

M is closed because it is the compliment in M of an open set ϕ

ϕ is closed:
A set S is closed iff $S' \subseteq S$
$\phi' = \phi$ as there are no accumulation points of the empty set
Therefore $\phi' \subseteq \phi$ and ϕ is a closed set in any metric space

2 Question 2

If (M,d) is a metric space: show that $d' = \frac{d(x,y)}{1+d(x,y)}$ is also a metric space
$\underline{\text{d'(x,x)} = 0}$:
$d'(x,x) = \frac{d(x,x)}{1+d(x,x)}$
$d(x,x) = 0$ as (M,d) is a metric space
$d'(x,x) = \frac{0}{1} = 0$
$d'(x,x) = 0$

$\underline{d'(x,y) > 0 if x \neq y}$
$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$
$d(x,y) > 0$
Therefore this is a positive number divived by 1 + a positive number, which is

a positive number and
$d'(x,y) > 0$

$\underline{d'(x,y) = d'(y,x)}$
$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$
Note: we can do this subsitition because (M,d) is given to be a metric space

Triangle Inequality
$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$
$d'(x,y) \leq \frac{d(x,z)+d(y,z)}{1+d(x,z)+d(y,z)}$
$d'(x,y) \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$
$d'(x,y) \leq d'(x,z) + d'(z,y)$
Note: this is just true about fractions:
$\frac{2+3}{1+2+3} = \frac{5}{6}$
$\frac{2}{3} + \frac{3}{4} = \frac{17}{2}$

3 Question 3

Prove that every finite subset of a metric space is closed
Let S be a finite subset of M
By definition, S contains no accumulation points, as an accumulation point x of a set S has the following property:
$\forall r \in \mathbb{R}, B(x,r)$ contains infinite points in S
Since S is finite, there is no open ball which contains infinite points in S. Therefore there are no accumulation points
Since $S' = \phi$ and $\phi \subset S$, $S' \subseteq S$ and S is closed

4 Question 4

Let M be a seperable metric space
Let $A \subseteq M$ and F be an open cover of A. Show that F has a countable subcollection which covers A
Let $A = \{a_1, a_2, a_3...\}$ be a countable subset of M which is dense in M
This means $A \subseteq M \subseteq \overline{M}$
Let $B = \{B_1, B_2...\}$ be the countable set of open balls centered at each $a \in A$
Fix $x \in A$
Since F is an open cover, there is an open set $S \in F$ such that $x \in S$
Let a_x be the point of least index such that $x \in B_x \subset S$

This point must exist, as $A \subseteq M \subseteq \overline{A}$
Therefore every point $x \in M$ is either in A or is an adherent point to A

And there is some open ball centered at a point in A which contains x
Since B is countable and we can pick a B_x for each $x \in M$, where $B_x \subset S$, we have a countable collection of F which covers A

5 Question 5

The intersection of an arbitrary collection of compact sets is compact
Let F be an arbitrary collection of compact sets
Let $X = \bigcap_{F \in F} F$
Since X is the intersection, it is clear that for each $f \in F$, $X \subseteq f$
Also, since each f is compact, each f is closed
Note: This direction of Hiene-Borel holds in arbitrary metric spaces
Since the arbitrary intersection of closed sets is closed, we have X is a closed subset of a compact set
Therefore X is compact (By theorem 3.39)

6 Question 6

Let Q be our metric space with $\mathbb{R}$'s distance

Let S be the set of all rational number in (a, b) where a and b are irrational
S is clearly closed in Q as it is the compliment of an open set $Q - S$ and it is bounded by a (below) and b (above)

Assume towards contradiction that S is compact
Let F be the open cover $\{(a, b - \frac{1}{n}\}$
This open cover clearly has no finite subcover which covers S
Done!