Analysis In Several Variables Homework 10

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1 Question 1

<u>Show $int(\partial A) = \phi$ if A is open or A is closed</u>
If A is open:
Every point in A is an interior point
For some $y \in \partial A$, $\exists x \in B(y)$ such that $\exists B(x) \subseteq A$ (If y is a boundary point, then every open ball contains a point x which is an interior point of A)
If $y \in int(\partial A)$, $\exists B(y)$ such that $B(y) \in \partial A$
There is some open ball which contains only boundary points of A. But we have every open ball contains an interior point (which by definition is not boundary point), therefore there are no points in $int(\partial A)$ and $int(\partial A) = \phi$

If A is closed it is the compliment of an open set $M - A$ and the $\partial A = \partial\{M - A\}$ (I think this proof is trivial! It falls out of the definition of a boundary point)
Therefore $int(\partial A) = int(\partial\{M - A\})$ and since this holds for open sets: $int(\partial\{M - A\}) = \phi$ so $int(\partial A) = \phi$

<u>Give an example where $int(\partial A) = M$</u>
If we consider the metric space $\mathbb{R}$ with euclidean distance function, let our subset be $\mathbb{Q}$
Every point in $\mathbb{R}$ is a boundary point of $\mathbb{Q}$, as every open ball in $\mathbb{R}$ contains rational and irrational points
Therefore $\partial\mathbb{Q} = \mathbb{R}$
And since $\mathbb{R}$ is an open set, it is equal to its interior
So $int(\partial\mathbb{Q}) = \mathbb{R}$
Done!

2 Question 2

Let x_n be a sequence where $x_0 \in (0, 1)$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$
Show that $\{x_n\}$ is decreasing
Proof:
Fact: $x_0 \in (0, 1)$

Show that if $x_n \in (0,1)$, $x_{n+1} \in (0,1) and x_{n+1} \leq x_n$
If $a \in (0,1) : 0 < a < \sqrt{a} < 1$
Also $1-a \in (0,1)$ (was told in class we could use these facts)
Therefore $0 < 1-a < \sqrt{1-a} < 1$
$-1 < -a < -1-\sqrt{1-a} < 0$
$1 > a > 1+\sqrt{1-a} > 0$
Also $\sqrt{1-a}$ is between 0 and 1 therefore: $1 > a > 1-\sqrt{1-a} > 0$
Apply: $x_n > x_{n+1}$
Done!

3 Question 3

If $x_n \to x$ and $y_n \to y$
For some ϵ let $d(x_n, x) < \epsilon/2$ and $d(y_n, y) < \epsilon/2$
$d(x_n, x) + d(y_n, y) < \epsilon$

By the triangle inequality we have:
$d(x, y) < d(x_n, x) + d(y_n, y) + d(x_n, y_n)$
$d(x, y) < d(x_n, y_n) + \epsilon$
$d(x, y) - d(x_n, y_n) < \epsilon$
Therefore $d(x_n, y_n) \to d(x, y)$

4 Question 4

Prove that in a compact metric space: every sequence has a subsequence which converges in S
Since S is compact, it is closed and bounded (in any metric space)
If the range of S_n is finite, it is convergent. This was proved in class
Assume S_n is not finite
Therefore the range of S_n contains infinite points in S
By the bolzano-weirstrauss theorem, Range(S_n) has an accumulation point. Call this point p

Therefore every open ball centered at p contains a point in S_n
We now have $\forall \epsilon > 0, \exists n \in \mathbb{Z}^+ such that d(S_n, p) < \epsilon$
Let a subsequence consist of these S_n's, and we now have a subsequence which converges to p!
Finally we know p is in S because S is closed, and if p were not an element of S it would clearly be a boundary point, and S contains all of its boundary points
Done!

5 Question 5

If a sequence S_n is increasing and bounded above, then it converges to its supremem

Since S_n is a bounded sequence, it has a suprmemum a

By the approximation property, $\forall r < sup(A)$, $\exists x \in S_n$ such that $r < x \leq a$

Since S_n is non decreasing, we know that $\forall n > m \in \mathbb{Z}^+$, $s_n > s_m$

Therefore if x (in the approximation property) is S_m, all previous terms in the sequence are strictly less and all consequent terms are either equal or greater

Fix $\epsilon > 0$

By the approximation property, there is some S_m between ϵ and a and all S_{m+i} where $i \in \mathbb{Z}^+$ are between the supremem and S_m (their distance is less than epsilon)

Therefore $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n > N, d(S_n, a) < \epsilon$ and $S_n \to \epsilon$

Exact same proof for the infemum of a bounded decreasing sequence

If a_n and b_n both converge to 0, then $a_n + b_n$ converges to 0

We showed earlier than $d(a_n, b_n) \to d(a, b)$

$d(a_n, b_n) \leq d(a_n, 0) + d(b_n, 0)$

$a_n + b_n < d(a_n, b_n)$ (I think this is obviously true?)

Therefore $a_n + b_n < d(a_n, 0) + d(b_n, 0)$

And since $\forall \epsilon, \exists N \in \mathbb{Z}^+ s.t. \forall n > N, d(a_n, 0) < \frac{\epsilon}{2}$ and same for b_n

, we have $d(a_n) + d(b_n) < \epsilon$

Therefore $a_n + b_n < \epsilon$

Which is the same as saying they that $d(a_n + b_n, 0) < \epsilon$

Therefore $a_n + b_n \to 0$

6 Question 6

This is a sequence in $\mathbb{R}$

We nknow that all cauchy sequences in $\mathbb{R}$ converge, therefore if S_n is not cauchy, it does not converge

To be cauchy:

$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$ s.t. $\forall m, n > N, d(S_m, S_n) < \epsilon$

Let $\epsilon = \frac{1}{2}$

If N is even, let $n = N$ and $m = N + 1$

$d(S_n, S_m) = 1$

$1 > \epsilon$ Therefore S_n is not cauchy and not convergent

Note: Exact same idea holds if N is odd, because (S_n, S_m) is still 1