Analysis In Several Variables Homework 5

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1 Question 1

If S is an infinite set, prove that S contains a countably infinite subset

Case 1: S is countable
If S is countable it can be written as $\{s_1, s_2, s_. ..\}$
Consider the set $T = S - \{s_1\}$
$T \subset S$
Let $f : T \to \mathbb{Z}^+$ be defined as $f(s_i) = i$
This is injective, therefore T is countable
Cast 2: S is uncountable
Idea: We will inductively build a subset of S which is countable

Base case:
Since S is infinite: $\exists x \in S$
Denote this x as t_0
Since S is infinite: $S - \{t_0\}$ is still an infinite set and therefore contains another element we can select

Induction: Take some set $T = \{s_0, s_1 ... s_{n-1}\}$
Since S is unaccountably infinite: $S - T$ is still an infinite set.
Therefore we can take another element $x \in S$ and title it s_n
Continue infinitely

We now have a countably infinite set $T = \{s_0, s_1, s_2...\}$ Note: consider the map $f(T \to \mathbb{Z}^+)$ where $f(s_n) = n$

T is also a subset of S as $\forall t \in T, t \in S$ based on how we constructed T.
Done!

2 Question 2

Show that every infinite set S has a proper subset which is similar to S

Construct a countably infinite proper subset of S called T and denote the terms $\{t_0, t_1, t_2...\}$ Note: This is done using our proof from Question 1

Now consider the set $S - \{t_0\}$

This is a proper subset of S. We will now find a bijection between $S - \{t_0\}$ and S

Let $f : S \to (S - \{t_0\})$ be defined as follows:

- If $x \in S - T$, then $f(x) = x$
- If $x \in T$, then it can be written as t_n and $f(t_n) = t_{n+1}$

This function is injective: Each element in S is either mapped into a unique element of T or itself

This function is surjective: All elements of S are mapped to other than t_0.

Done!

3 Question 3

Let S be the collection of all sequences who are terms are the integers 0 and 1. Show that S is uncountable

Assume towards contradiction that S is countable

S is comprised of the union of all finite binary sequences and all infinite binary sequences. We shall consider the infinite sequences.

Since S is countable, there is a sequence $S = \{S_n\}$ whose terms comprise all of S

Each S_n can be written in the form:

$v_n 1 v_n 2 v_n 3 v_n 4...$

We can write all of S_n in a table like this:

$v_{11}v_{12}v_{13}v_{14}...$
$v_{21}v_{22}v_{23}v_{24}...$
$v_{31}v_{32}v_{33}v_{34}...$
$v_{41}v_{42}v_{43}v_{44}...$
.
.
.

Let w be a binary sequence where $w_n = w_1 w_2 w_3...$ and

$w_n =$

- 1 if $v_{n,n} = 0$
- 0 if $v_{n,n} = 1$

Since w contains a different integer in some digit for each S_n, w is not in S.
This is a contradiction, as w is an infinite binary string
Therefore S is uncountable

4 Question 4

Let f be a real valued function defined for every x in the interval $0 < x < 1$
Suppose there exists some positive M such that $|f(x_1) + f(x_2) + ...f(x_n)| \leq M$
Note: For any finite set of x's
Let S be the set of all $x \in (0, 1)$ such that $f(x) \neq 0$

Proof Idea: Contradiction. Assume that S is not countable and then pick some function which meets the properties described above. Show that S is countable for this specific function, and therefore contradicts the assumption that it is not countable for all real valued functions with the M property.

Assume towards contradiction that S is not countable
Let $f(x) = \begin{cases} 0 \text{ if x} \neq 0.5 \\ 0.5 \text{ if x} = 0.5 \end{cases}$

This is clearly defined on all points in the interval $(0, 1)$
The M property must hold as the sum of $f(x)x \in (0, 1)$ is either 0 or 0.5 for any finite set of x's.(This meets the criteria of f in the proof claim)
But the set S is comprised of only the element $\{0.5\}$
Since S is finite, it is countable
Contradiction, therefore S is countable

5 Question 5

The problem with this proof is that the function being defined is not injective
Consider the interval (a, b) s.t. $f((a, b)) = 1$
To clarify, consider an interval which contains x_1, which is the least-indexed rational number. Now consider the interval $(a - 1, b)$. $f(a - 1, b) = 1$
Therefore the function is not injective and a 1:1 correspondance has not been established