Analysis In Several Variables Homework 4

Spencer Brouhard

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1 Question 1

Let X, Y, Z be sets:
$F : X \to Y$
$G : Y \to Z$

a) Show that if F and G are surjective, then $G \circ F$ is surjective
Note: $G \circ F(x) = G(F(x))$
Show that $\forall z \in Z, \exists x \in X$ s.t. $G(F(x)) = z$

Fix $z \in Z$
Since G is surjective, $\exists y \in Y$ s.t. $G(y) = z$
Since F is surjective, $\exists x \in X$ s.t. $F(x) = y$
Substutite:
$G(F(x)) = z$
$G \circ F(x) = z$
ThereFore $G \circ F$ is surjective since For any $z \in Z$, there is some x s.t. $G \circ F(x) = z$

b) IF F is surjective and $G \circ F$ is injective, show that F and G are injective
Assume towards $\rightarrow\leftarrow$ that F is not injective
$\exists (x_1 \neq x_2) \in X$ where $F(x_1) = F(x_2)$
Let $F(x_1) = F(x_2) = y$
Let $G(y) = z$
We now have $G(F(x_1)) = G(F(x_2) = G(y) = z$
So $x_1 \neq x_2$ and $G \circ F(x_1) = G \circ F(x_2)$
This contradicts the objectivity oF $G \circ F$ thereFore F is injective

Assume towards $\rightarrow\leftarrow$ that G is not injective
$\exists (y_1 \neq y_2) \in Y$ where $G(y_1) = G(y_2)$
Since F is bijective (surjective in assumptions and just shown to be injective), we have $\exists x_1, x_2 \in X$ s.t. $F(x_1) = y_1$ and $F(x_2) = y_2$
So $G(F(x_1)) = G(y_1)$ and $G(F(x_2) = G(y_2)$
$G(F(x_1) = G(F(x_2))$ while $x_1 \neq y_1$ due to the injectivity of F.
This contradicts the injectivity of $G \circ F$

Therefore G is injective

2 Question 2

Claims:
a) f is one-to-one on S
b) $f(A \cap B) = f(A) \cap f(B)$
c) $f^{-1}[f(A)] = A$

<u>Prove $a \to b$</u>
Since f is one-to-one (injective) $f(x_1) = f(x_2) \to x_1 = x_2$
Let $A, B \subset S$
Fix $y \in f(A) \cap f(B)$
Since $y \in f(A) \cap f(B)$:
$\exists x_1 \in A$ s.t. $f(x_1) = y$
$\exists x_2 \in B$ s.t. $f(x_2) = y$
Note: This is just using the fact that it's in the intersection of f(A) and f(B).
No claims about surjectivity here
Since F is injective $f(x_1) = f(x_2) \to x_1 = x_2$
Therefore $x_1 \in A \cap B$
And $y \in f(A \cap B)$
Therefore $f(A \cap B) = f(A) \cap f(B)$

<u>Prove $b \to a$</u>
Assumptions: $F(A \cap B) = F(A) \cap F(B)$

Assume towards contradiction that F is not injective
We have $\exists x_1 \neq x_2$ where $f(x_1) = f(x_2)$

Let $A = \{x_1\}$
Let $B = \{x_2\}$

$f(A)\{f(x_1)\}$
$f(B) = \{f(x_2)\}$
Since $f(x_1) = f(x_2)$, $f(A) \cap f(B) = f(x_1)$
But $A \cap B = \{\}$
So $f(A \cap B) \neq f(A) \cap f(b)$
Contradiction! Therefore f must be one-to-one

<u>Prove $a \to c$</u> Assume f is injective
Show: $f^{-1}[f(A)] = A$
Fix $x \in A$
Let $f(x) = y$
Since f is injective x is the unique value in A which maps to y

Due to this $f^{-1}[y] = \{x\}$ Applying this to each item in A we find:
$f^{-1}[f(A)] = A$

These 3 implictions show that the statements are equivalent!

3 Question 3

Show that similarity defines an equivalence relation
$A \sim B$ iff there exists a bijective function from A to B

a) Reflexive $A \sim A$:
Define $f : A \to A$ as the function $f(x) = x$
This is a bijection as $f(x_1) = f(x_2) \to x_1 = x_2$ and all elements of the range are mapped to by the domain

b) Symmetric: $A \sim B \to B \sim A$
Let $f : A \to B$ define an bijective function, as $A \sim B$
Let $g : B \to A$ where $g(b) = a$ such that $f(a) = b$
Note: since f is a bijection, we know it is onto and such an element exists

G is 1:1
Fix $b_1, b_2 \in B$
$g(b_1) = g(b_2) \to b_1 = b_2$ as f is injective and if the elements mapped to b_1 and b_2 by f are equivalent, then b_1 and b_2 must be equivalent

G is surjective:
Fix $a \in A$
Let $b = f(a)$
$g(b) = a$
Therefore b is surjective
Since we have found a bijection from $B \to A$, $B \sim A$

c) Transitive Let $A \sim B$ and $B \sim C$
Let $f : A \to B$ be a bijection
Let $g : B \to C$ be a bijection

Claim: $g \circ f : A \to C$ is a bijection

Injective:
Fix $x_1 \neq x_2 \in X$
$f(x_1) \neq f(x_2)$ By injectivity
Let $f(x_1) = y_1$ and $f(x_2) = y_2$
$g(y_1) \neq g(y_2)$ by Injectivity
Therefore $g(f(x_1)) \neq g(f(x_2))$ and $g \circ f$ is injective

Surjective
Fix $c \in C$
Since g is surjective $\exists b \in B$ such that $g(b) = c$
Since f is surjective $\exists a \in A$ such that $f(a) = b$
Therefore $g(f(a)) = c$
And $\forall c \in C, \exists a \in A$ such that $g \circ f(a) = c$

Since there is a bijection between A and C, $A \sim C$

$A \sim B$ and $B \sim C \to A \sim C$
Done!

4 Question 4

Show that S (the set of all real-valued functions with the domain of $\mathbb{R}$) and $\mathbb{R}$ are not similar
Assume towards contradiction that $S \sim R$
Let $f : R \to S$ be a bijection
For $a \in \mathbb{R}$, let g_a denote $f(a)$
Let $h(a) = 1 + g_x(a)$ where $x \in \mathbb{R}$
$h \in S$, as it is a real valued function
but if $h \in S$, $\exists x \in \mathbb{R}$ s.t. $f(x) = h$, but $f(x) = g_x \forall x \in \mathbb{R}$ and $g_x \neq h$
Therefore f is not surjective onto S, and $\mathbb{R} \nsim S$