Chaos Homework 1

Spencer Brouhard

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1 Question 1

Let $f(x) = 3(mod(x))$

Let x_0 be some rational number $\in (0, 1)$ represented in ternary

x_0 must be of the form $0.a_1a_2a_3...a_N\overline{b_1b_2b_3...b_p}$
$f(x_0) = 0.a_2a_3...a_N\overline{b_1b_2b_3...b_p}$
$f^2(x_0) = 0.a_3...a_N\overline{b_1b_2b_3...b_p}$
The pattern here is each iteration of f, all of the digits are shifted over by 1, and the digit in the "1"s place is removed by the mod function"
We then have
$f^{N-1}(x_0) = 0.a_N\overline{b_1b_2b_3...b_p}$
$f^N(x_0) = 0.\overline{b_1b_2b_3...b_p}$
If we continue to consider further iterations we find
$f^{N+1}(x_0) = 0.\overline{b_2b_3...b_pb_1}$
$f^{N+2}(x_0) = 0.\overline{b_3b_4...b_pb_1b_2}$
...
$f^{N+p}(x_0) = 0.\overline{b_1b_2b_3...b_p}$
x_0 is in an orbit of period p, and is therefore eventually periodic!

If x_0 were an irrational number there would be no N such that
$f^N(x_0) = \overline{b_1b_2b_3...b_p}$
as irrational numbers never have repetition (by definition).
Therefore there is no $f^{N+p}(x_0)$ such that $f^N(x_0) = f^{N+p}(x_0)$.
To clarify, since the decimals never repeat, the point will never fall into an orbit.

2 Question 2

Period	Fixed Points	Divisors	Points from period < k	Points only k	Orbits
1	3	1	0	3	3
2	9	1	3	6	3
3	27	1	3	24	8
4	81	1,2	9	72	18
5	243	1	3	240	48
6	729	1,2,3	33	696	116
7	2187	1	3	2174	312
8	6561	1,2,4	81	6480	810
9	19683	1,3	27	19656	2184
10	59049	1,2,5	249	58803	5880

3 Question 3

Part A: Show that any points less than $\frac{1}{6}$ apart have their distance tripled

Fix x, y such that $distance(x, y) < \frac{1}{6}$
$f(x) = 3x(mod1)$
$f(y) = 3y(mod1)$

Case 1: $distance(f(x), f(y)) = |f(x) - f(y)|$
$= |3x(mod1) - 3y(mod1)|$
$= 3|x(mod1) - y(mod1)|$
$= 3|x - y|(mod1)$
$< 3(\frac{1}{6})(mod1)$
$< 0.5mod1$
< 0.5

Case 2: $distance(f(x), f(y)) = 1 - |f(x) - f(y)|$
$= 1 - |3x(mod1) - 3y(mod1)|$
$= 1 - 3|x - y|(mod1)$
$< 1 - 0.5$
< 0.5

Therefore when $distance(x, y) < \frac{1}{6}, distance(f(x), f(y)) < \frac{1}{2}$, or is multiplied by 3
Done!

Part B: Find a pair of points which is not tripled

The pair of points $0, \frac{1}{2}$ does not have their distance tripled, as they are both fixed points.

Part C: Prove sensitive dependence at 0

$x_0 = 0$
Let $k = ln(\frac{d}{|x_0 - x|})/ln3$
$k * ln(3) = ln(\frac{d}{|x_0 - x|})$
$k * ln(3) = ln(\frac{d}{|0 - x|})$
$k * ln(3) = ln(\frac{d}{x})$
$ln(3)^k = ln(\frac{d}{x})$
$3^k = \frac{d}{x}$
$x = \frac{d}{3^k}$

Now apply f(x)
$f(x) = 3\frac{d}{3^k}(mod1) = \frac{d}{3^{k-1}}$
$f^2(x) = 3\frac{d}{3^{k-1}}(mod1) = \frac{d}{3^{k-2}}$
The degree is reduced on every iteration of f so we finally will have
$f^k(x) = \frac{d}{3^0} = d.$
Since there is some iterate k which will bring x and x_0 d apart, 0 must have sensitive dependence on initial conditions.

4 Question 4: Find the left and right endpoints of LLR for the graph $4x(1-x)$

$\frac{1}{2}$ is the divider between L and R, so we will first solve for the values which map to it

$4x(1-x) = \frac{1}{2}$
$-4x^2 + 4x - \frac{1}{2} = 0$
$x = \frac{2 \pm \sqrt{2}}{4}$
And we are interested in $\frac{2-\sqrt{2}}{4}$

Then solve again for the values which map to $\frac{2-\sqrt{2}}{4}$
$4x(1-x) = \frac{2-\sqrt{2}}{4}$
$-4x^2 + 4x - \frac{2-\sqrt{2}}{4} = 0$
$x = \frac{2 \pm \sqrt{2+\sqrt{2}}}{4}$ Note: this was done with quadratic formula

Therefore the left and right endpoints of LLR are $\frac{2-\sqrt{2+\sqrt{2}}}{4}$ and $\frac{2-\sqrt{2}}{4}$ respectively

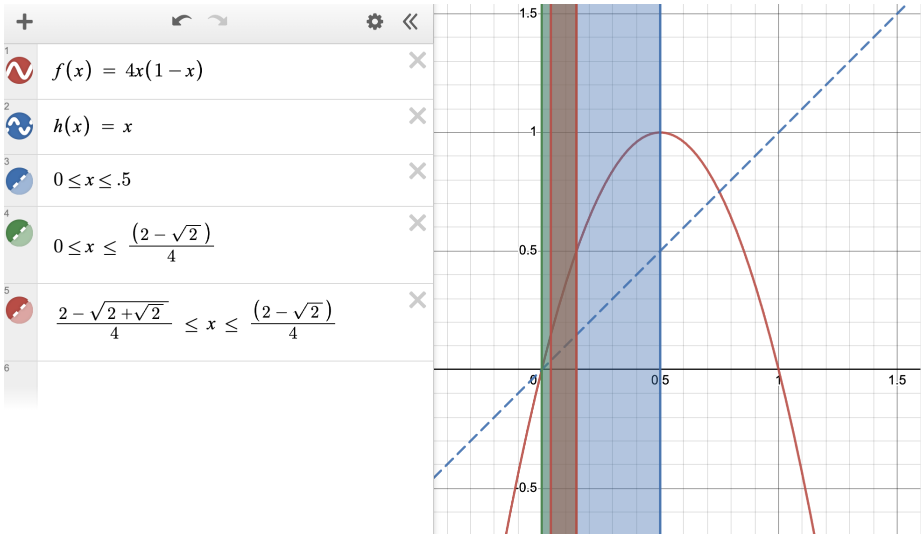

Figure 1: Region LLR is in red

5 Question 5

Matlab file. Explanation of how I found the number also included!
But I found an orbit of $2^{2}6$