Decision Theory - Assignment 1

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Question 1

- $P(S|A_1) = 0.2$
- $P(S|A_2) = 0.1$
- $P(S|A_3) = 0.02$

- $P(S|A_i \cap A_j) = 0 \quad \forall i \neq j \;\wedge\; P(S|A_1 \cap A_2 \cap A_3) = 0$
- $P(S|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) = 0.001$

a) If the GP uses this information alone and the principles of inference a patient showing the syntomp S will always be considered as sick from disease A_1, since its conditional probability is the highest of all.

- $P(A_1) = 0.002$ • $P(A_2) = 0.003$ • $P(A_3) = 0.01$

b) $$P(A_1|S) = \frac{P(S|A_1) \cdot P(A_1)}{P(S)} = \frac{P(S|A_1)P(A_1)}{\sum_{i=1}^{3} P(S|A_i)P(A_i) + P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) \cdot P(S|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)} = \dots$$

$$\dots = \frac{0.2 \cdot 0.002}{[(0.2 \cdot 0.002) + (0.1 \cdot 0.003) + (0.02 \cdot 0.01)] + 0.001 \cdot (0.998 \cdot 0.997 \cdot 0.99)} = 0.2121953$$

$$P(A_2|S) = \frac{P(S|A_2)P(A_2)}{P(S)} = \frac{0.1 \cdot 0.003}{0.0019\dots} = 0.1591465$$

$$P(A_3|S) = \frac{P(S|A_3)P(A_3)}{P(S)} = \frac{0.02 \cdot 0.01}{0.0019\dots} = 0.1060976$$

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 | S) = \frac{[P(\bar{A}_1) \cdot P(\bar{A}_2) \cdot P(\bar{A}_3)] \cdot P(S|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)}{P(S)} = \dots$$

$$\dots = \frac{[0.998 \cdot 0.997 \cdot 0.99] \cdot 0.001}{0.0019\dots} = 0.5225606$$

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Question 2

i) $X \sim Ber(\theta) \Rightarrow p(X|\theta) = \binom{n}{s}\theta^s(1-\theta)^f$ - Binomial sampling

$Y \sim NegB(\theta) \Rightarrow p(Y|\theta) = \binom{n-1}{s-1}\theta^s(1-\theta)^f$ - Pascal sampling

As we can observe from the two probability distributions, X and Y are proportional, which leads to equivalent posteriors (given an equal prior).
This happens because the data, from the perspective of a Bayesian, is fixed. In other words it does not matter if before carrying out the sample we fixed the number of successes (s) or the dimension of the sample (n). What matters is the likelihood functions, which are proportional. With an equal prior, the posteriors will be the same as well.
In conclusion what matters to us is the likelihood principle: if two sampling methods return proportional likelihood, the difference in their models lies in the design of the experiment only, not in the actual random process.
For this reason, the two processes can be considered equivalent.

ii) From the Bayesian perspective the stopping rule is non-informative: even if the dimension of the sample is given by a random variable (as we can state in this case) it does not effect the generative process. Therefore if it returns a sample with the same n and s as a sample generated from a regular Binomial or Pascal sampling, it will be considered equivalent to the others (because of the likelihood principle once more).

NOTE: Since the frequentist approach <u>does not</u> respect fully the likelihood principle, both for question i) and ii) we will have different answers.

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Question 3

$A_1, A_2, A_3, A_4, A_5 \overset{i.i.d.}{\sim} Pois(\theta)$ and $p(\theta) = \begin{cases} 30 & \text{with } \pi_{M_1} = 0.25 \quad (M_1) \\ 40 & \text{with } \pi_{M_2} = 0.5 \quad (M_2) \\ 50 & \text{with } \pi_{M_3} = 0.25 \quad (M_3) \end{cases}$

so $p(A_i) = \frac{\lambda_{M_*}^{a_i}}{a_i!} \cdot e^{-\lambda_{M_*}}$ where λ_{M_*} is the parameter from one of the 3 models.

$X = 380$ with X being $\sum_{i=1}^{5} A_i$ over 2 h $\Rightarrow \begin{cases} A^{M_1} \sim Pois(30 \cdot 10) \\ A^{M_2} \sim Pois(40 \cdot 10) \\ A^{M_3} \sim Pois(50 \cdot 10) \end{cases}$ Exploiting the property of sum of iid Poissons

- M_1: $P(X|A^{M_1}) = \frac{\lambda_{M_1}^{x} \cdot e^{-\lambda_{M_1}}}{x!} \cdot \pi_{M_1} = \frac{300^{380}}{380!} \cdot e^{-300} \cdot 0.25 = 2.7589 \cdot 10^{-7}$

- M_2: $P(X|A^{M_2}) = \frac{\lambda_{M_2}^{x} e^{-\lambda_{M_2}}}{x!} \cdot \pi_{M_2} = \frac{400^{380}}{380!} \cdot e^{-400} \cdot 0.5 = 6.1522 \cdot 10^{-3}$

- M_3: $P(X|A^{M_3}) = \frac{\lambda_{M_3}^{x} e^{-\lambda_{M_3}}}{x!} \cdot \pi_{M_3} = \frac{500^{380}}{380!} e^{-500} \cdot 0.25 = 7.6623 \cdot 10^{-10}$

- Update the prior: normalize the posterior found!

$$P(\theta_{M_1}|x) = \frac{P(X|A^{M_1})}{\sum_{i=1}^{3} P(X|A^{M_i})} = \frac{2.7589 \cdot 10^{-7}}{6.1525 \cdot 10^{-3}} = 4.4841 \cdot 10^{-5}$$

$$P(\theta_{M_2}|x) = \frac{P(X|A^{M_2})}{\sum_{i=1}^{3} P(X|A^{M_i})} = \frac{6.1522 \cdot 10^{-3}}{6.1525 \cdot 10^{-3}} = 0.999955$$

$$P(\theta_{M_3}|x) = \frac{P(X|A^{M_3})}{\sum_{i=1}^{3} P(X|A^{M_i})} = \frac{7.6623 \cdot 10^{-10}}{6.1525 \cdot 10^{-3}} = 1.2454 \cdot 10^{-7}$$

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Question 4

We will first prove that a Beta distribution can be written as an exponential class:

$$X \sim Beta(\alpha,\beta) \Rightarrow p(x) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} = \frac{x^{\alpha-1}\cdot(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

$$\Rightarrow p^{x|\alpha,\beta}(x) = \exp\left\{+\log\left[B(\alpha,\beta)^{-1}\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}\right]\right\} = \exp\left\{-\log\left[B(\alpha,\beta)\right] + (\alpha-1)\log(x) + \dots\right.$$

$$\left.\dots + (\beta-1)\log(1-x)\right\} = \exp\left\{\left[(\alpha-1)\log(x) + (\beta-1)\log(1-x)\right] - \left[\log(B(\alpha,\beta))\right]\right\}$$

$$\Rightarrow p^{x|\alpha,\beta}(x) = e^{\sum_{j=1}^{2} A_j(\theta)B_j(x) + C(x) + D(\theta)}$$ where:

- $\sum_{j=1}^{2} A_j(\theta)B_j(x) = \left[(\alpha-1)\log(x) + (\beta-1)\log(1-x)\right]$
- $C(x) = 0$
- $D(\theta) = -\log\left[B(\alpha,\beta)\right]$

i) Find a prior (conjugate prior) form for a Beta distribution:

$$\theta \sim Beta(\alpha_0,\beta_0) \Rightarrow p(\theta) = \frac{\theta^{\alpha_0-1}(1-\theta)^{\beta_0-1}}{B(\alpha_0,\beta_0)}$$

$$\Rightarrow p(\theta) = \exp\left\{(\alpha_0-1)\log(\theta) + (\beta_0-1)\log(1-\theta) - \log\left[B(\alpha_0,\beta_0)\right]\right\} \propto \exp\left\{\left[\alpha_0\log(\theta) + \beta_0\log(1-\theta)\right] + \dots\right.$$

$$\left.\dots - \left[\log(\theta) + \log(1-\theta)\right]\right\}$$ ($-\log[B(\alpha_0,\beta_0)] = K(\alpha_0,\beta_0)$ is a proportionality constant).

$$\Rightarrow p(\theta) \propto e^{\sum_{j=1}^{2} A_j(\theta)\alpha_j + \alpha_{k+1}\cdot D(\theta)}$$ where:

- $\sum_{j=1}^{2} A_j(\theta)\alpha_j = \left[\alpha_0\log(\theta) + \beta_0\log(1-\theta)\right]$
- $\alpha_{k+1}\cdot D(\theta) = -1\cdot\left[\log(\theta) + \log(1-\theta)\right]$

ii) $$q(\theta|x) \propto p(\theta)\cdot\prod_{i=1}^{5} p(x|\theta) \propto \exp\left\{\left[\alpha_0\log(\theta) + \beta_0\log(1-\theta)\right] - \left[\log(\theta) + \log(1-\theta)\right]\right\}\cdot$$

$$\dots \cdot \exp\left\{\sum_{j=1}^{2} A_j(\theta)\sum_{i=1}^{5} B_j(x_i) + 5\cdot D(\theta)\right\} \propto \dots$$

$$\propto \exp\left\{\sum_{j=1}^{k} A_j(\theta)\left[\sum_{i=1}^{n} B_j(x_i) + \alpha_j\right] + (5 + \alpha_{k+1})\cdot D(\theta)\right\}$$ where:

- $A_1(\theta)\cdot\left[\sum_i B_1(x_i) + \alpha_1\right] = (\alpha-1)\cdot\left[\sum\log(x) + \alpha_0\right]$
- $A_2(\theta)\cdot\left[\sum_i B_2(x_i) + \alpha_2\right] = (\beta-1)\left[\sum\log(x) + \beta_0\right]$
- $(5 + \alpha_{k+1})\cdot D(\theta) = (5-1)\cdot\left[\log(\theta) + \log(1-\theta)\right]$