Decision Theory - Assignment 2

Stefano Toffol

Question 1

a)

States

Actions	A	B	C	D	E
1	-50	80	20	100	0
2	30	40	70	20	50
3	10	30	-30	10	40
4	-10	-50	-70	-20	200

Payoff table of Q1

We will define this non-stochastic (Deterministic) payoff as $Rd(S(x))$
As stated into the instructions, this method involves two fixed weights, $\pi_{max} = 0.4$ $\pi_{min} = 0.6$. We will now compute the payoff for each action:

$Rd(S(1)) = 100 \cdot 0.4 - 50 \cdot 0.6 = 10$

$Rd(S(2)) = 70 \cdot 0.4 + 20 \cdot 0.6 = 40$

$Rd(S(3)) = 40 \cdot 0.4 - 30 \cdot 0.6 = -2$

$Rd(S(4)) = 200 \cdot 0.4 - 70 \cdot 0.6 = 38$

This deterministic criteria is rather flawed. In fact this method does not take into account the probability of a certain state to occurr since the weights provided depend on the action and have nothing to do with the underlying prior distribution. Moreover all the actions that lead to intermediate payoffs are excluded from the decision making process.
For this reason a prior with uniform weights on all states or one that gives importance to a single one will drastically change the actual outcome ~~[illegible]~~ of a certain action.

Question 1 - continues.

b)

Actions	States I	States II
1	10	4
2	7	9

Now the weights of the deterministic decision rules are $\pi_{max} = 0.8$, $\pi_{min} = 0.2$

$Rd(\delta(1)) = 10 \cdot 0.8 + 4 \cdot 0.2 = 8.8$
$Rd(\delta(2)) = 7 \cdot 0.2 + 9 \cdot 0.8 = 8.6$

For the ER criterion instead we need to know the probability of observing each state (p_I & p_{II}). In general the ER criterion would be:

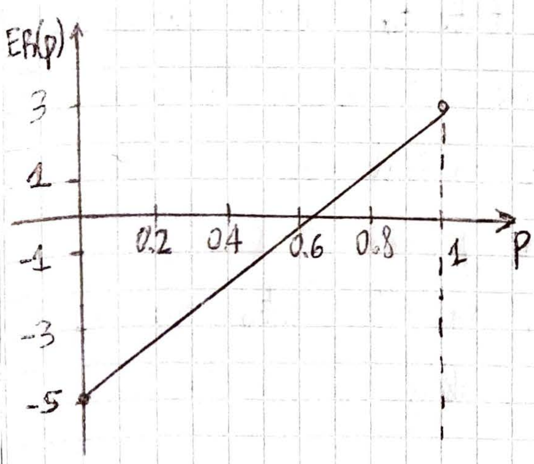
$ER(\delta(1)) = 10p + 4(1-p) = 2(2+3p)$ and $ER(\delta(2)) = 7p + 9(1-p) = 9-2p$

Assuming the two states are equally likely we will have:

$ER(\delta(1)) = 5 + 2 = 7$ and $ER(\delta(2)) = 3.5 + 4.5 = 8$

Here we have a change in the decision-making process according to the unknown probability p. We therefore define a function $ER(p) = ER(\delta(1)) - ER(\delta(2))$

$\Rightarrow ER(p) = 2\cdot(2+3p) - (9-2p) = -5+8p$



As we can see from the figure, decision provided by the ER criterion would be in favour of action 1 only if $P(\theta = I) > 0.625$ (where 0.625 was found solving the equation $ER(p) = 0$).

Question 1 - continues

c) We know that $U(R) = \log(R+71)$ and that $p(\theta) = [0.1; 0.2; 0.25; 0.1; 0.35]$. We want to find the optimal action according to the EU criterion:

$EU[\delta(1)] = [-50, 80, 200, 100, 0] \cdot p(\theta)^T = 4.7494 \Rightarrow \log(4.7494+71) = 4.3274$

$EU[\delta(2)] = [30, 40, 70, 20, 50] \cdot p(\theta)^T = 4.6705 \Rightarrow \log(4.6705+71) = 4.3264$

$EU[\delta(3)] = [10, 30, -30, 10, 40] \cdot p(\theta)^T = 4.2684 \Rightarrow \log(4.2684+71) = 4.3211$

$EU[\delta(4)] = [-10, -50, -70, -20, 200] \cdot p(\theta)^T = 2.7893 \Rightarrow \log(2.7893+71) = 4.3012$

We therefore decide to take action 1 since $EU[\delta(1)] \geq EU[\delta(i)]$; $\forall i = 1,2,3,4$

Question 2

To simplify the problem we can reformulate it using only the red chips:

$\theta = \{70\%, 30\%\}$ $A_1 = [1, 0]$ $A_2 = [0, 1]$ $p(\theta) = [0.4, 0.6]$

$c(a_\theta, \theta) = 5$ $c(a_{\bar{\theta}}, \theta) = -3$ $L(\delta(x), \theta) = \max_{\delta}\{R(\delta(x), \theta)\} - R(\delta(x), \theta)$

→ Payoff table:

Action	70%	30%
A_1	+5	-3
A_2	-3	+5

Loss table:

Action	70%	30%
A_1	0	8
A_2	8	0

$ER(A_1) = 5 \cdot 0.4 - 3 \cdot 0.6 = 0.2$ $ER(A_2) = -3 \cdot 0.4 + 5 \cdot 0.6 = 1.8$

$EL(A_2) = 8 \cdot 0.4 + 0 \cdot 0.6 = 3.2$ → We are willing to pay up to 3.2 \$

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Question 2 - continues

Each sample costs is 0.25 \$.

b) What is the ENGS of a sample n=10 using single-stage sampling plan of 10?

Using an R script, I computed the following table:

Y	$P(\theta_{70\%} \mid y)$	$P(\theta_{30\%} \mid y)$	$E''R(\theta_{70\%})$	$E''R(\theta_{30\%})$	a''	a'	VSI	$P(y)$
0	$2.3620 \cdot 10^{-6}$	$1.6999 \cdot 10^{-2}$	-0.0508	0.0847	A2	A2	0	0.0170
1	$5.5112 \cdot 10^{-5}$	$7.2636 \cdot 10^{-2}$	-0.2176	0.3630	A2	A2	0	0.0727
2	$5.7868 \cdot 10^{-4}$	$1.4008 \cdot 10^{-1}$	-0.4173	0.6987	A2	A2	0	0.1407
3	$3.6007 \cdot 10^{-3}$	$1.6010 \cdot 10^{-1}$	-0.4623	0.7897	A2	A2	0	0.1637
4	$1.4703 \cdot 10^{-2}$	$1.2007 \cdot 10^{-1}$	-0.2867	0.5563	A2	A2	0	0.1348
5	$4.1168 \cdot 10^{-2}$	$6.1752 \cdot 10^{-2}$	0.0206	0.1853	A2	A2	0	0.1029
6	$8.0048 \cdot 10^{-2}$	$2.2054 \cdot 10^{-2}$	0.3341	-0.1299	A1	A2	0.4640	0.1021
7	$1.0673 \cdot 10^{-1}$	$5.4010 \cdot 10^{-3}$	0.5175	-0.2932	A1	A2	0.8106	0.1121
8	$9.3390 \cdot 10^{-2}$	$8.6802 \cdot 10^{-4}$	0.4643	-0.2758	A1	A2	0.7402	0.0943
9	$4.8424 \cdot 10^{-2}$	$8.2669 \cdot 10^{-5}$	0.2419	-0.1449	A1	A2	0.3867	0.0485
10	$1.1295 \cdot 10^{-2}$	$3.5429 \cdot 10^{-6}$	0.0565	-0.0339	A1	A2	0.0904	0.0113

$$EVSI = \sum_{Y} VSI \cdot P(y) = 0.2278176$$

$$ENGS = EVSI - n \cdot CS(n) = 0.2278176 - 10 \cdot 0.25 = -2.272182$$

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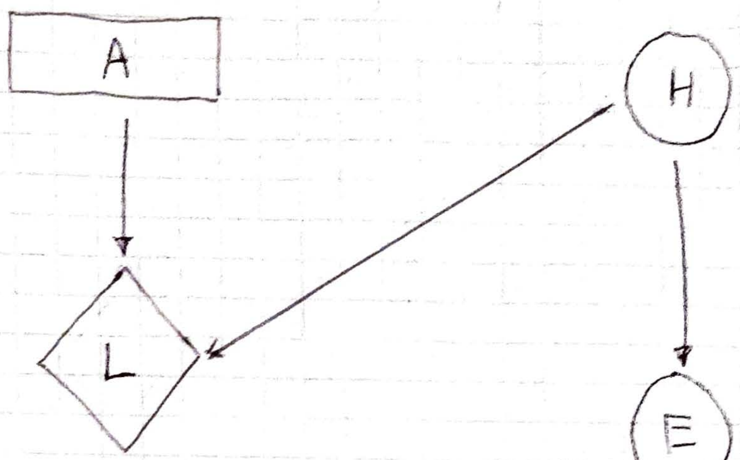
Question 2 - continues

c)



H	Probabilities
H_0	0.4
H_1	0.6

		Probabilities	
	H	H_0	H_1
E	E_1	0.70	0.30
	E_2	0.30	0.70

A	
a_1	You bet on 70/30
a_2	You bet on 30/70

H:	H_0		H_1	
A:	a_1	a_2	a_1	a_2
L	0	8	8	0

Where: $\begin{cases} H_0: \text{The distribution of the chips is 70\% red / 30\% blue} \\ H_1: \text{The distribution of the chips is 30\% red / 70\% blue} \end{cases}$

And we are asked to guess (i.e. bet) what is the real distribution

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Stefano Tonoli

Question 3

Payoff table:

Action	Working	Defective
A	-15	-15
B	-14	-24

$\tilde{p}$:= proportion of defective cartridges for B

$\tilde{p} \sim \text{Beta}(2,48)$

$\Rightarrow p(\tilde{p}) = \dfrac{\tilde{p}(1-\tilde{p})^{47}}{\text{Beta}(2,48)}$

a) Having n 500 cartridges to buy, the expected payoffs are:

$ER(\delta(A)) = -15 \cdot 500 = -7500$

$ER(\delta(B)) = n \cdot \{(1-E[\tilde{p}]) \cdot (-14) + E[\tilde{p}] \cdot (-24)\} = 500 \cdot \{(1 - \frac{2}{2+48}) \cdot (-14) + \frac{2}{2+48} \cdot (-24)\}$

$= 500 \cdot \{\frac{48}{50} \cdot (-14) + \frac{2}{50} \cdot (-24)\} = -500 \cdot \{\frac{672}{50} + \frac{48}{50}\} = -500 \cdot \frac{720}{50} = -7200$

$ER[\delta(B)] > ER[\delta(A)]$ therefore our prior decision is to buy from B

b) The amount of money we are willing to pay to know the proportion of defective cartridges is given by the expected loss of the optimal action:

Loss table:

Action	Working	Defective
A	1	0
B	0	9

$EL[\delta(B)] = (0 \cdot \{1-E[\tilde{p}]\} + 9 \cdot \{E[\tilde{p}]\}) \cdot 500$

$= (0 + 9 \cdot \frac{2}{50}) \cdot 500 = 0.36 \cdot 500 = 180$

c) Posterior: $p(\theta) \cdot p(y|\theta) = \dfrac{\theta \cdot (1-\theta)^{47}}{\text{Beta}(2,48)} \cdot \theta^{y}(1-\theta)^{n-y} \cdot \binom{n}{y} = \dfrac{\theta^{y+1}(1-\theta)^{n+47-y}}{\text{Beta}(2,48)} \cdot \binom{n}{y}$

$\theta|y \propto \text{Beta}(y+2, n+48-y)$

$p(y) = \int p(y|\theta)p(\theta)\,d\theta = \int \dfrac{\theta^{y+1}(1-\theta)^{n+47-y}}{\text{Beta}(2,48)} \cdot \binom{n}{y} d\theta = \binom{n}{y}[\text{Beta}(2,48)]^{-1} \cdot \int \theta^{y+1}(1-\theta)^{n+47-y} d\theta$

$= \binom{n}{y} \dfrac{\text{Beta}(y+2, n+48-y)}{\text{Beta}(2,48)} \cdot \int \dfrac{\theta^{y+1}(1-\theta)^{n+47-y}}{\text{Beta}(y+2,n+48-y)} d\theta = \binom{n}{y} \dfrac{\text{Beta}(y+2,n+48-y)}{\text{Beta}(2,48)}$

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Question 3 - continues

$E'R(A) = -15 \cdot 500 = -7500$

$E'R(B) = n \left\{ (1 - E[\tilde{p}|x])(-14) + E[\tilde{p}|x](-24) \right\} = 500 \cdot \left\{ \left(1 - \frac{y+2}{60}\right)(-14) - \frac{y+2}{60}(-24) \right\}$

$= 500 \cdot \left\{ \frac{58-y}{60}(-14) + \frac{y+2}{60}(-24) \right\} = 500 \left\{ -\frac{860+10y}{60} \right\}$

$\Rightarrow ER(A) = ER(B) \qquad -7500 = 500 \cdot \left\{ -\frac{860+10y}{60} \right\} \qquad -900 = -860 - 10y$

$-40 = -10y \qquad y = 4$

$$\Rightarrow a'' = \begin{cases} A & \text{for } y > 4 \\ B & \text{for } y \leq 4 \end{cases} \qquad \Rightarrow EVSI = E''R(a'') - E''R(a')$$

$E''R(a'') = \sum_{i=0}^{4} ER_i(B) \cdot Beta(i+2, n+48-i) + \sum_{j=5}^{10} ER(A) \cdot Beta(j+2, n+48-j)$

$E''R(a') = \sum_{i=0}^{10} ER_i(B) \cdot Beta(i+2, n+48-i) = \sum_{i=0}^{4} ER_i(B) \cdot Beta(i+2, n+48-i) + \sum_{j=5}^{10} ER_j(B) \, Beta(j+2, n+48-j)$

$\Rightarrow EVSI = \sum_{j=5}^{10} \left[ER(A) \cdot Beta(j+2, n+48-j) - ER_j(B) \cdot Beta(j+2, n+48-j) \right]$

$= \sum_{j=5}^{10} \left[(ER(A) - ER_j(B)) \cdot Beta(j+2, n+48-j) \right]$

d) $y = 1 \quad \Rightarrow \theta | y \propto Beta(3, 57)$

$\Rightarrow ER(B | y=1) = 500 \left\{ -\frac{860+10}{60} \right\} = -500 \left\{ \frac{870}{60} \right\} = -7250$

$ER(B | y=1) > ER(A | y=1) \quad \Rightarrow$ The supplier should deal with firm B.

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Question 1

budget = 120'000 SEK $\qquad X|M \sim N(M, 12'000)$ & $M \sim N(115'000, 9'000)$

$\bar{X} = E[X_{-6}, \dots, X_{-1}] = 121'000$ with no trend

a) We know that the marginal distribution of a Gaussian variable conditioned to another Gaussian is Gaussian itself, with the same mean of M and the variance equal to the sum of the two variances.

$$X \sim N(\mu, \sigma^2) \text{ where } \begin{cases} \mu = E[M] = 115'000 \\ \sigma = \sqrt{\sigma_M^2 + \sigma_{X|M}^2} = \sqrt{12000^2 + 9000^2} = 15'000 \end{cases}$$

Our system of hypothesis will then be:

$$\begin{cases} H_0: \mu < \text{budget} \\ H_1: \mu \geq \text{budget} \end{cases}$$

We can then compute the prior odds ratio: $\frac{P(X > \text{budget} | H_0)}{P(X \leq \text{budget} | H_0)} = \frac{0.3694}{0.6306} = 0.5859$

Since the odds are smaller than one we are in favour of H_1.

b) The bayes factor is given by: $B = \frac{\text{Odds}(H_0 | \text{Data}, I)}{\text{Odds}(H_0 | I)}$

We know that with a normal distribution under a normal prior we get a normal posterior in the form: $P(\theta | x_1, \dots, x_n) \propto N(\theta | \mu_n, \tau_n)$ where:

$$\frac{1}{\tau} = \sqrt{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}} = \sqrt{\frac{6}{(15000)^2} + \frac{1}{(9000)^2}} = 0.0001975 \Rightarrow \tau = 5062,896$$

$$\mu_n = \frac{n/\sigma^2}{n/\sigma^2 + 1/\sigma_0^2} \cdot \bar{X} + \frac{1/\sigma_0^2}{n/\sigma^2 + 1/\sigma_0^2} \mu_0 = 0.6835 \cdot 121000 + 0.3165 \cdot 115'000 = 119'101.3$$

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Question 4 - continues

$$\text{Odds}(H_0 \mid \text{Data}, I) = \frac{P(H_0 \mid \text{Data}, I)}{P(H_1 \mid \text{Data}, I)} = \frac{0.4296}{0.5704} = 0.7530$$

$$\Rightarrow B = \frac{\text{Odds}(H_0 \mid \text{Data}, I)}{\text{Odds}(H_0 \mid I)} = \frac{0.7530}{0.5859} = 1.2852$$

c) The loss table is:

Action	State of the world	
	H_0 is true	H_1 is true
Accept H_0	0	6 000
Reject H_0	4 000	0

The expected posterior losses are:

Action	Expected posterior loss
Accept H_0	$6000 \cdot 0.5704 = 3422.4$
Reject H_0	$4000 \cdot 0.4296 = 1718.4$

$\Rightarrow$ The action that minimizes the posterior loss is rejecting H_0, meaning that we will consider the real mean of the Mocos to be smaller than 120 000.