Controller Design

So far we have discussed the basics of state space analysis. In this section, we will discuss different types of controller design. Consider the State Space Equations of a system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

This system can be represented in form of a block diagram as follows:

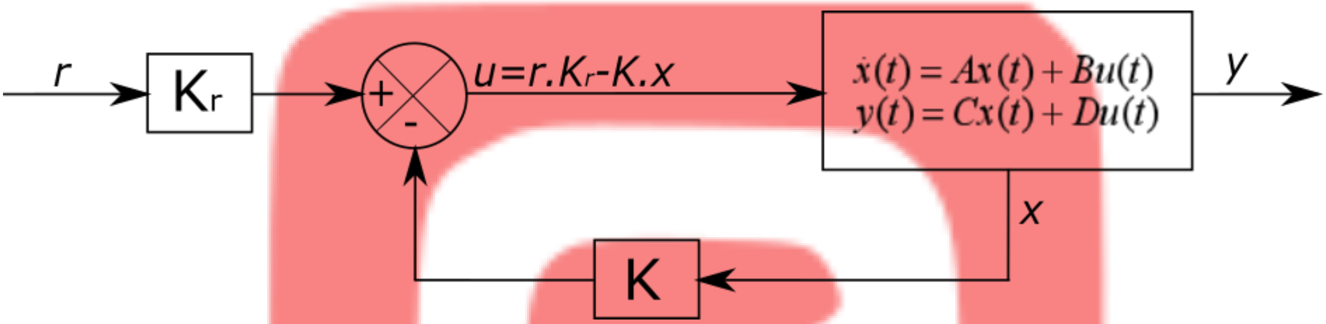

Figure 1: System Block Diagram

Where:
r - is the Reference Signal
K_r - is the Reference Gain
x - State Vector
y - Output Vector
u - System Input
K - Feedback Gain
In this system we have taken the state vector x, multiplied with some gain matrix K and fed that back as feedback input to the system.
By substituting $u = rK_r - Kx$, the State Equation for the system can be written as follows:

$$\dot{x} = Ax + B(rK_r - Kx),$$
$$\implies \dot{x} = Ax - BKx + B_rK_r,$$
$$\implies \dot{x} = \underbrace{(A - BK)}_{\text{New State Matrix}} x + B_rK_r$$

The new state matrix $(A - BK)$ defines the dynamics of the system where $-Kx$ is fed as input and is known as state feedback system. The system stability can be calculated by finding the eigenvalues of the $(A - BK)$ matrix.

Pole Placement

One method to ensure that the system is stable is to select the gain matrix K in such a way so that the eigenvalues of the (A-BK) matrix has all the eigenvalues with negative real part. Placing all the eigenvalues to desired location is only possible when the system is fully controllable. We'll not go into details as to why but you may read the control system books about Controllable system to explore the same. We can select the desired eigenvalues for the system and calculate the K matrix such that (A-BK) has our desired eigenvalues. We again go back to the pendulum(with external torque) system with the following state equation:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u$$

The equilibrium point for this state equation is $(\pi, 0)$. We have already proved that this system is unstable as one of the eigenvalues of A matrix is real and positive. Let the gain matrix $K = [k_1 \ k_2]$. If we assume $u = -Kx$,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} [k_1 \ k_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A - BK$ matrix will be :

$$A - BK = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} [k_1 \ k_2]$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ g/l - k_1/ml^2 & -k_2/ml^2 \end{bmatrix}$$

The characteristic equation for this system will be:

$$|A - BK - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ g/l - k_1/ml^2 & -k_2/ml^2 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{k_2}{ml^2}\lambda + \frac{k_1}{ml^2} - \frac{g}{l} = 0 \tag{1}$$

Let us select some arbitrary eigenvalues.

$$\lambda_1 = -1, \ \lambda_2 = -2$$

These eigenvalues have negative real part and hence if (A-BK) has these eigenvalues, the system will be stable.
If the eigenvalues are as given above, the characteristic equation for the system will be

$$(\lambda + 2)(\lambda + 1) = 0$$

,

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Since 1 and 2 represent the same system, we can calculate the values of k_1 and k_2.

$$\frac{k_2}{ml^2} = 3; \ \frac{k_1}{ml^2} - \frac{g}{l} = 2$$

$$k_2 = 3ml^2; \ k_1 = 2ml^2 + mgl$$

Hence,

$$K = [2ml^2 + mgl \ 3ml^2]$$

Hence we found the K matrix for which the system is stable.

Linear Quadratic Regulator

So far we have seen that if we have a system which is controllable, then we can place its eigenvalues anywhere in the left half plane by choosing appropriate gain matrix K. But the main question is where should we place our eigenvalues?
Till now we have only discussed about the stability of the system. But nowhere have we asked that what is our performance measure?
In this section, we'll see how to optimize the value of gain matrix K to get the desired performance measure from the system.

Linear Quadratic Regulator is a powerful tool which helps us choose the K matrix according to our desired response.Here we use a cost function

$$J = \int_0^\infty x^TQx + u^TRu$$

where Q and R are positive semi-definite diagonal matrices (positive semi-definite matrices are those matrices whose all the eigenvalues are greater than or equal to zero). Also to remind you that for a diagonal matrix, the diagonal entries are its eigenvalues.
x and u are the state vector and input vector respectively.

Let us say that you have your system with four states and one input. Then $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and input u. Let

$$Q = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & Q_4 \end{bmatrix}$$

Then $x^TQx + u^TRu = Q_1x_1^2 + Q_2x_2^2 + Q_3x_3^2 + Q_4x_4^2 + Ru^2$.Thus, you may see that the system taken here is our usual system represented as

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

The controller is of form $u = -Kx$ which is a **Linear** controller and the underlying cost function is **Quadratic** in nature and hence the name **Linear Quadratic Regulator**.
With a careful look at the integrand of the cost function J, we may observe that each Q_i are the weights for the respective states x_i.
So, the trick is to choose weights Q_i for each state x_i so that the desired performance criteria is achieved.Greater the state objective is, greater will be the value of Q corresponding to the said state variable.We can choose $R = 1$ for single input system. In case we have multiple inputs, we could use similar arguments for weighing the inputs as well.
In case of inverted pendulums,we know that angle θ with the vertical and the angular velocity $\dot{\theta}$ is very important and hence the weights corresponding to them should be more as compared to linear position x and linear velocity $\dot{x}$.
LQR minimizes this cost function J based on the chosen matrices Q and R. Its a bit complicated to find out matrix K which minimizes this cost function.This is usually done by solving Algebraic Riccatti Equation (ARE). We'll not go into details of how to solve ARE, as it is not required in our tasks. There is inbuilt lqr command in octave to find K matrix. What is required to be done is to choose the Q and R matrix appropriately to get the desired performance.