Mathematical Modelling of a System

In Task 0.2, we had discussed about the various steps involved in testing the stability of a system. In this document, we will be discussing the **Euler-Lagrange method** to derive the equations of motion of a given system.

Before you start, you might need to recapitulate a few topics if you want to fully understand what we are going to explain here.

You will need a good understanding of classical mechanics. *"Concepts of Physics by Prof. H.C. Verma"* is a great place to brush up on those concepts.

You will also need to understand mathematical concepts like partial differentiation, jacobians, solving equations with two or more variables etc.

Euler-Lagrange Method

The Euler-Lagrange method states that the equations of motion of a system can be obtained by solving the following equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

(Assuming that there are no non-conservative forces acting on the system)
Here,
L is the Lagrangian which is the difference between the Kinetic energy and Potential energy of the system.
Hence L = K.E - P.E
x and $\dot{x}$ are the state variables (In our case here, position and velocity respectively) in generalized coordinate system.

We will try to understand this using an example.
Consider the following system:

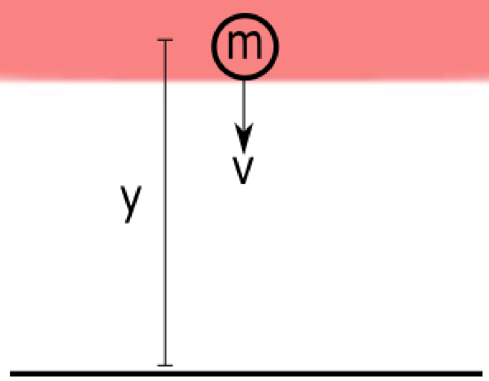

Figure 1: Mass raised to height y

A point mass m is raised to height y. We need to calculate the equations of motion using the Euler-Lagrange method.

Firstly we calculate the KE and PE. Then use those values to calculate the Lagrangian L.

$$PE = mgy \qquad (1)$$
$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2 \qquad (2)$$
$$L = KE - PE = \frac{1}{2}m\dot{y}^2 - mgy \qquad (3)$$

Equation (1) is self explanatory. The mass is raised to height y. So the potential energy stored in mass will by mgy.

Equation (2) is slightly tricky to understand. We know that kinetic energy of a point mass is (1/2)*mass*(velocity)^2. Now velocity v is nothing but rate of change of y with respect to t. Hence v can be written as dy/dt or $\dot{y}$

Equation (3) represents the Lagrangian L which is the difference between the KE and PE of the system.

Now we calculate the Euler Lagrange equations of motion.

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}(\frac{1}{2}m\dot{y}^2 - mgy) = -mg \qquad (4)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}(\frac{1}{2}m\dot{y}^2 - mgy)\right) = m\ddot{y} \qquad (5)$$
$$Hence$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$
$$=> m\ddot{y} - (-mg) = 0$$
$$=> \ddot{y} = -g \qquad (6)$$

Equation (6) gives the final answer. Here $\ddot{y}$ represents the acceleration.

Equation (6) makes sense as only gravitational force is acting on the system. Hence the acceleration of the system is the acceleration due to gravity.

If we had used the Newton's laws of motion, we would have arrived at the same result, albeit in a different way.

Let us consider a somewhat more complex example.

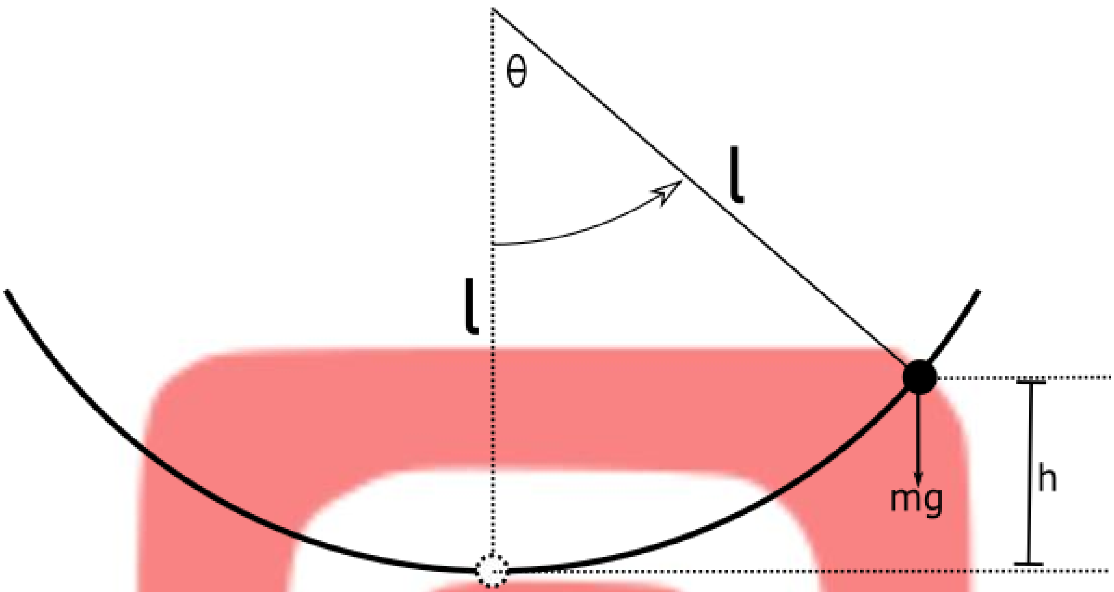

Figure 2: Pendulum

We have our pendulum equation whose equations of motion we demonstrated in Task 0.2 using Newtons Laws of motions. Now we will demonstrate the same using Euler-Lagrangian method.

In this system, we have a pendulum. The mass of the bob is given as m. The length of rod to which the bob is attached to is l. We have assumed the rod to be rigid and have no mass. So all the mass is concentrated to the bob.

While swinging, at any arbitrary point in the pendulum's trajectory, the pendulum can assumed to be at a height h from the bottom. h can be written as a function of θ where θ is the angle the pendulum bob makes with the vertical.

$$h = l - l\cos\theta$$

First, we need to calculate the Lagrangian L for this system. For that we need to compute the kinetic energy and potential energy of this system.

Calculating the potential energy is pretty straightforward.

$$PE = mgh = mg(l - l\cos\theta) = mgl(1-\cos\theta) \qquad (7)$$

The kinetic energy will be defined by (1/2)*mass*velocity^2. Here the velocity is the tangential velocity of the bob. We can take x and y components of velocity v and solve for kinetic energy using those equations

However, we can use rotational mechanics to make our calculations simpler. Since the pendulum bob is oscillating in a circular trajectory, the kinetic energy can be given by

$$KE = \frac{1}{2} I\omega^2$$

Where I is the moment of inertia and ω is the angular velocity.
But we know

$$I = ml^2 \qquad and \qquad \omega = \dot{\theta}$$

Angular velocity ω can be written as rate of change of θ with respect to time. Hence we can write ω= (dθ/dt).

Therefore we have the expression for kinetic energy

$$KE = \frac{1}{2} I\omega^2 = \frac{1}{2} ml^2\dot{\theta}^2 \tag{8}$$

Now we have the expressions for PE and KE we will calculate the Lagrangian L and use it to calculate the equations of motion for this system.

$$L = KE - PE = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1-\cos\theta) \tag{9}$$

Since L is a function of θ, we need to select θ and $\dot{\theta}$ as state variables of the system. Hence the equations of motion can be calculated as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right)\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2} ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right) = 0$$

$$=> \qquad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} = \frac{-g}{l}\sin\theta \tag{10}$$

In Task 0.2, we had used the same pendulum example and calculated the equations of motion for pendulum using Newton’s laws. We can confirm that the same equations have been derived using the Euler-Lagrange method.

Suppose we take $x_1 = \theta$ and $x_2 = \dot{\theta}$

We can express the equations we formed in the following way

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-g}{l}\sin x_1$$

In this way we have a two equations that govern our system.

What happens if there is any external force acting on the system?

Consider the following system as given in Fig 3.

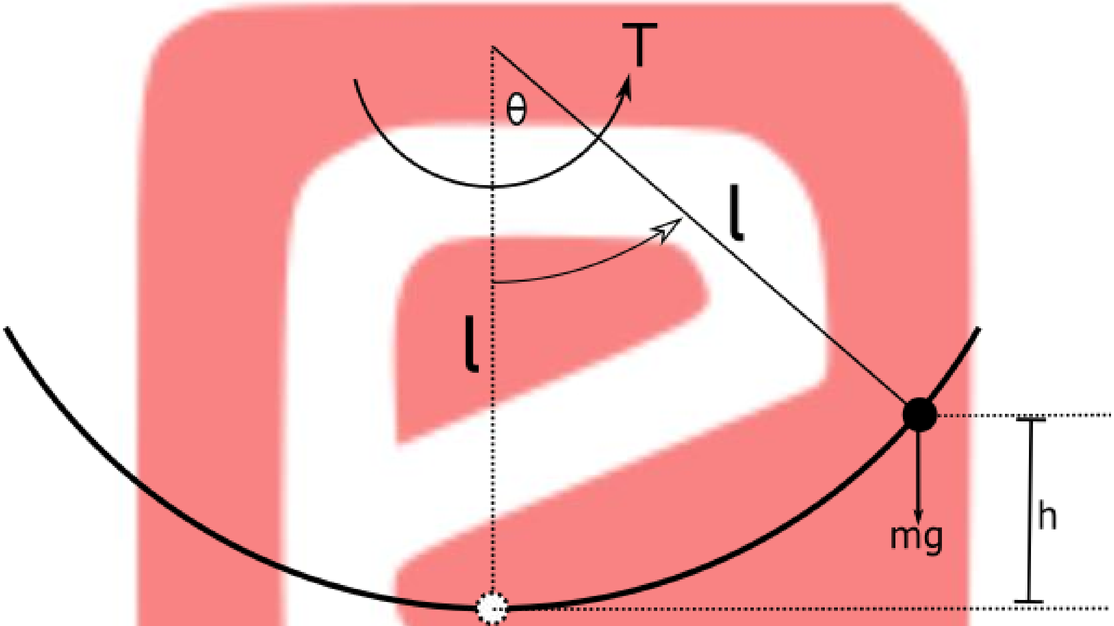

Figure 3: Pendulum with external torque

In the given simple pendulum system, we have applied an external torque to the system.

In this case, the Euler-Lagrange equation formed will be slightly different.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = T$$

Any non-conservative force acting on the system (Since states chosen are angular position and velocity that's why force should also be taken as angular force i.e. Torque. In case we use linear motions as in the first example then we'll use external linear force on the right hand side.) appears on the right side of the Euler-Lagrange equation. Consequently the equations of motion derived for this pendulum system will be as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-g}{l}\sin x_1 + \frac{1}{ml^2}T$$

You can see that there is an additional term in the second equation. We can check if this term is dimensionally correct.

We now x_1 is the angular position θ of the pendulum bob (with respect to vertical) and x_2 is the angular velocity $\dot{\theta}$ of the bob. Hence $\dot{x}_1 = x_2$ will correspond to the angular velocity $\dot{\theta}$ and $\dot{x}_2$ will be the angular acceleration $\ddot{\theta}$. Let the angular acceleration be denoted by α.

The units and dimensions of α are $rad/\sec^2$ and $\left[T^{-2}\right]$ respectively.

We know T=Iα (where T= torque, I = moment of inertia, α=angular acceleration). And I = ml^2.

(T/ml^2) equals angular acceleration α. Hence the last term is an angular acceleration term which is consistent with the equation. We can also calculate the dimensions of this term to verify. It will always come as $\left[T^{-2}\right]$. This method is helpful to verify if or equations are valid.