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Magic polygrams

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Magic polygrams, which are extensions of magic squares, can be found with computer programs through exhaustive searches. However, most polygrams are too large for this method. Thus, these possibilities must be limited algorithmically. This paper investigates both a large traditional hexagram and a traditional octagram. Systematic approaches based on the arrangement of even and odd numbers are used to identify solutions.

1. Introduction

Magic squares are arrangements of numbers in which the orientation of the numbers leads to particular properties. Across the world, people have regarded the construction of magic squares as a form of mathematical study or a form of artistic creation, and the squares themselves were often believed to be objects with inherent good or evil powers [Cammann 1960].

Definition 1. In a *magic square*, nonnegative numbers are arranged so that all rows, columns, and main diagonals all sum to the same number. The common sum is known as the *magic constant*. A *traditional magic square* is an $n \times n$ magic square that is filled with the numbers 1 to n^2 [Beck et al. 2003; Benjamin and Yasuda 1999; Xin 2008].

The first magic square is thought to have originated from the *Lo Shu* diagram in the 23rd century BC, which is an orientation of dots supposedly originally revealed on the shell of a sacred turtle [Cammann 1961]. Although the legend is probably a more recent fabrication, this 3×3 traditional magic square was considered by ancient Chinese as a deeply meaningful symbol [Biggs 1979; Cammann 1961]. Scholars in the Middle East and India also studied magic squares, placing greater importance on them than their current classification in fields such as recreational mathematics [Cammann 1969a; Datta and Singh 1992]. Methods of construction of magic squares have varied greatly by culture and often reflected the philosophies

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8	1	6
3	5	7
4	9	2

traditional

$k+a$	$k+a-b$	$k+b$
$k-a+b$	k	$k+a-b$
$k-b$	$k+a+b$	$k-a$

generic form

Figure 1. A filled-in 3×3 magic square and a generic version that works for any k, a, b [Chernick 1938].

of the particular culture [Biggs 1979; Cammann 1969a; 1969b; Datta and Singh 1992]. In India, 4×4 squares have been worn as amulets to bring luck [Datta and Singh 1992]. To some who studied the mysticism of magic squares, the knowledge that no 2×2 magic square exists was thought to reflect the imperfection of the four elements taken alone [Calder 1949]. As a result of the significance placed on magic squares, solutions have long been identified for traditional magic squares for $n = 3, 4, 5, 6, 7, 8, 9, 10$ [Biggs 1979; Cammann 1960].

Figure 1, left, shows a 3×3 traditional magic square with a magic constant of 15. Using the variables a, b, and k, every 3×3 magic square can be represented by a single pattern, as shown in the right half of the figure. There is only one unique solution to the traditional 3×3 magic square [Chernick 1938]. Increasing from a 3×3 magic square to a 4×4 magic square increases both the number of possible arrangements and the number of ordinary solutions. There are 880 unique 4×4 traditional magic squares [Beck et al. 2003].

A spin-off of the magic square is the magic hexagram.

Definition 2. A *hexagram* is a star with six points containing an arrangement of twelve numbers (see Figure 2).

Unlike a magic square, a magic hexagram is considered to contain only "rows". As seen in Figure 2, a hexagram contains six rows of five triangles each, and a total of twelve triangles. If you use the numbers 1 through 12 to fill every triangle in the

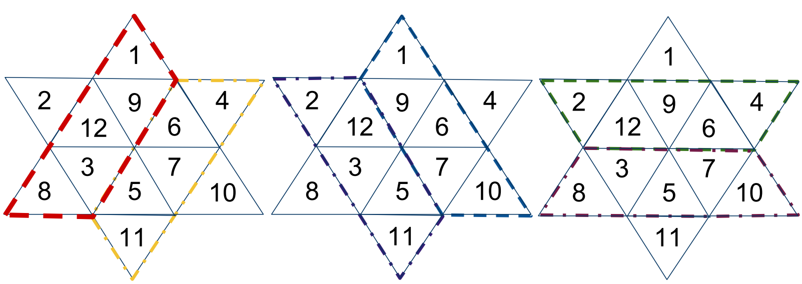


Figure 2. Examples of hexagrams with outlines of every row.

hexagram there are 12! different, but not necessarily unique, arrangements. While a computer can use brute force to find solutions for magic squares, attempting to identify solutions for a hexagram with 12! possible arrangements is computationally challenging. The number of possible arrangements can be reduced algorithmically in order to find solutions; see [Bolt et al. 1991; Gardner 2000].

Due to symmetry, hexagrams that are reflections or rotations of one another are equivalent. Eliminating these equivalent hexagrams, the number of possible unique arrangements is reduced to 11!; see [Bolt et al. 1991; Gardner 2000]. To eliminate the remaining duplicates, complementary solutions can be ignored [Gardner 2000].

Definition 3. [Gardner 2000] A *complementary arrangement* is obtained by subtracting each number of a polygram from the pattern's largest number plus 1.

Using Definition 3, the complement to a 3×3 square can be found be subtracting every number from 10. While the complement of a magic square is just a rotation of the original magic square, this is not the case for many other shapes [Gardner 2000]. To further reduce possibilities, the arrangements of even and odd numbers can be examined. An odd/even arrangement is a pattern of zeros and ones in which the ones represent odd numbers and the zeros represent even numbers [Bolt et al. 1991; Gardner 2000]. Because all rows of a magic hexagram must have a common sum, there must be either an odd number of odds in every row or an even number of odds in every row, limiting possible odd/even patterns. Ignoring transformations, there are only six different ways that even and odd numbers can be arranged throughout the hexagram [Gardner 2000]. However, odd/even patterns that represent complementary solutions are trivial variations of one another and can be ignored, limiting the number of odd/even patterns.

The patterns of many complementary polygrams are opposites. Odd/even patterns of a magic hexagram are considered complements when every number in one pattern is the opposite in the other pattern. While these patterns are different, there exists a complementary magic polygram for every magic polygram, so they are considered trivial variations [Gardner 2000].

In the current paper, two polygrams are investigated to find magic arrangements. Both the magic extended hexagram and the magic octagram are polygrams for which a complete list of solutions is difficult to identify. In order to identify these magic polygrams, odd/even arrangements are investigated, and upper and lower bounds of the magic constant are considered as in [Gardner 2000]. To limit the number of arrangements, distinct odd/even patterns are found in which every row has an equivalent number of odds. Methods such as the investigation of odd/even diagrams and magic constant bounds are used to focus the investigation because the number of total arrangements in the polygrams makes computational investigations challenging. The number of arrangements is further limited by finding the possible

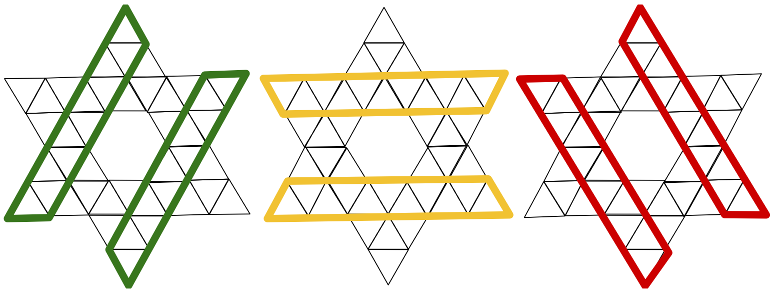


Figure 3. Extended hexagram structure with outlines of every row.

magic constants for each odd/even pattern. The goal of this research is to identify solutions to magic extended hexagrams and to magic octagrams, but not necessarily to find an exhaustive list. Only traditional polygrams are considered.

2. Generalizations about polygrams

The two polygrams that are considered in this study are (1) an extension of the hexagram described in Section 1 [Gardner 2000] and (2) another arrangement we will refer to as an octagram. By taking the magic hexagram described in Section 1 and outlining it, a larger hexagram with more triangles can be formed.

Definition 4. An *extended hexagram* is an arrangement of numbers in the shape of a six pointed star composed of 42 total triangles as shown in Figure 3.

The extended hexagram contains six rows of eleven triangles each. Figure 3 highlights the six different rows of this hexagram.

A similar shape to the hexagram can be formed using an octagon as the center of the diagram instead of a hexagon.

Definition 5. An *octagram* is a star with eight points containing an array of 16 numbers. An octagram has eight rows, containing six numbers each (see Figure 4).

In order to find magic extended hexagrams and magic octagrams, general results that apply to multiple polygrams can be identified. For instance, the odd/even pattern and its complement can be either the same or exact opposites depending on the number of positions in the shape.

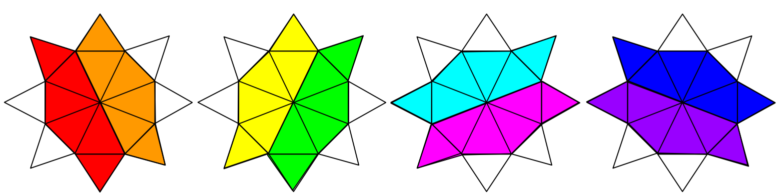


Figure 4. Octagram structure with outlines of the eight rows.

Theorem 1. *If a traditional magic polygram has an odd number of integers arranged within it, then the odd/even pattern of this polygram and its complement are equivalent.*

Proof. Suppose a magic polygram has $2N+1$ numbers arranged throughout it, where N is some integer. Let $O_1, O_2, \dots, O_{2N+1}$ represent the numbers in the original polygram. Let $C_1, C_2, \dots, C_{2N+1}$ represent the numbers in the equivalent positions of the complementary polygram. By Definition 3, for some position P, $C_P = ((2N+1)+1) - O_P$, or $(2N+2) - O_P$. There are two cases for what O_P could be. If O_P is even, $O_P = 2K$ for some integer K. So, $C_P = (2N+2)-(2K)$, or $2(N-K+1)$. This represents an even number, so when O_P is even C_P is also even. For the second case, when O_P is an odd number, it can be represented by $2K+1$. So, $C_P = (2N+2) - (2K+1)$, or $2(N-K)+1$. This represents an odd number, so when O_P is odd, C_P is also odd. As a result, the complementary odd/even pattern is the same as the odd/even pattern for any traditional polygram with an odd number of integers arranged throughout it. □

Theorem 2. *If a traditional magic polygram has an even number of integers arranged within it, then the odd/even pattern of this polygram and its complementary pattern are opposites.*

Proof. Suppose a magic polygram has $2N$ numbers arranged throughout it, where N is some integer. Let $O_1, O_2, \dots, O_{2N}$ represent the numbers in the original polygram. Let $C_1, C_2, \dots, C_{2N}$ represent the numbers in the equivalent positions of the complementary polygram. By Definition 3, $C_P = (2N+1) - O_P$ for some position P. There are two cases for what O_P could be. If O_P is even, $O_P = 2K$ for some integer K. So, $C_P = (2N+1) - (2K)$, or $2(N-K)+1$. This represents an odd number, so when O_P is even, C_P is odd. For the second case, when O_P is an odd number, it can be represented by $2K+1$. So, $C_P = (2N+1) - (2K+1)$, or $2(N-K)$. This represents an even number, so when O_P is odd, C_P is even. As a result, the complementary odd/even pattern is the opposite of the original odd/even pattern for any traditional polygram with an even number of integers arranged throughout it. □

If the number of odds per row of a traditional polygram is fixed, the number of odds in certain positions can be calculated.

Theorem 3. *If a magic polygram containing O odd numbers has N rows, each with K odds, and if a polygram is made up of only $\mathcal{X}$ positions appearing in A rows and $\mathcal{Y}$ positions appearing in B rows, then there are $(BO - NK)/(B - A)$ odds in $\mathcal{X}$ positions and $(NK - AO)/(B - A)$ odds in $\mathcal{Y}$ positions.*

Proof. Suppose a magic polygram has O odd numbers arranged throughout N different rows, and that each row contains K odds. Also, suppose the polygram is made up of $\mathscr{X}$ positions which hold X odds and appear in A rows, as well as $\mathscr{Y}$ positions which hold Y odds and appear in B rows. Because every $\mathscr{X}$ position appears in A rows and every $\mathscr{Y}$ position appears in B rows, $AX + BY$ represents the number of times an odd is a part of a sum of a row. Also, because K is the number of times an odd is part of a sum of each row, and there are N rows, then NK is the total number of times an odd is part of a sum of a row in the total sum. Hence,

$$NK = AX + BY. \tag{1}$$

Every number is placed in either an $\mathscr{X}$ position or a $\mathscr{Y}$ position, so the sum of the number of odds in $\mathscr{X}$ positions and the number of odds in $\mathscr{Y}$ positions is equivalent to the number of odds placed in this hexagram. Hence,

$$X + Y = O. \tag{2}$$

By combining (1) and (2),

$$Y = (NK - AO)/(B - A).$$

By replacing Ys with $(NK - AO)/(B - A)$, Equation (2) becomes

$$X = (BO - NK)/(B - A).$$

Therefore, to have K odd numbers as part of a sum of each row, there must be $(NK - AO)/(B - A)$ odd numbers in $\mathscr{X}$ positions and $(BO - NK)/(B - A)$ odd numbers in $\mathscr{Y}$ positions. □

The lower boundary of a magic constant can be found by placing the smallest numbers in the positions that appear in the largest number of rows. The upper boundary magic constant is found by doing the opposite. A magic polygram containing N rows is made up of only $\mathscr{X}$ positions and $\mathscr{Y}$ positions such that there are M number of $\mathscr{X}$ positions appearing in A rows, and O number of $\mathscr{Y}$ positions appearing in B rows, and the number contained in the i-th $\mathscr{X}$ position is X_i and the number in the i-th $\mathscr{Y}$ position is Y_i. The magic constant of a magic polygram is equivalent to the total sum of all of the rows combined, divided by the number of rows, so the magic constant is equal to

$$\frac{1}{N}\left(A\sum_{i=1}^{M} X_i + B\sum_{i=1}^{O} Y_i\right). \tag{3}$$

3. Traditional magic extended hexagram

If the numbers 1 through 42 are placed throughout the extended hexagram (as shown in Figure 3), there are 42! (total) arrangements possible. In order to reduce this number and make identification of magic traditional extended hexagrams computationally easier, the number of possibilities is reduced by considering odd/even diagrams and by only considering nontrivial variations of a particular scenario.

Throughout the extended hexagram, 24 of the triangles (or numerical positions) appear in two different rows, while the remaining 18 triangles only appear in one row each. There are two ways for an extended hexagram to be magic. The first is that every row has an odd number of odd numbers, and then the magic constant is an odd number. The second is that every row has an even number of odd numbers, resulting in an even magic constant. One specific scenario is when every row has the same number of odd numbers, and the current paper will focus on this particular case.

3.1. ***The possible number of odds in a row.*** To find the possible arrangements of even and odd numbers throughout the hexagram where every row has the same number of odds, the first step is finding the overall number of odd numbers in all of the rows combined. Each number in the extended hexagram can be considered as in an $\mathscr{X}$ position (appearing in two rows) or in a $\mathscr{Y}$ position (appearing in one row). Because there are a total of twenty-one odd numbers placed throughout the six rows in the extended hexagram, O is 21 and N equals 6 in Theorem 3. $\mathscr{X}$ positions appear in two rows and $\mathscr{Y}$ positions appear in one, so A is 2 and B is 1. Using Theorem 3, the number of odds in $\mathscr{X}$ positions can be found by

$$X = 6K - 21 \tag{4}$$

and the number of odds in $\mathscr{Y}$ positions is

$$Y = 42 - 6K. \tag{5}$$

To find all distinct odd/even patterns in which every row has the same number of odds, the possibilities of numbers of odds per row should first be found. $\mathscr{X}$ positions appear in more rows than $\mathscr{Y}$ positions, so the maximum number of odds per row can be found by maximizing the number of odds in $\mathscr{X}$ positions. Because there are twenty-one odds placed throughout $\mathscr{X}$ and $\mathscr{Y}$ positions, when the number of odds in $\mathscr{X}$ positions is maximized, the number in $\mathscr{Y}$ positions is minimized. Because there are twenty-one odds and twenty-four $\mathscr{X}$ positions, the smallest possible number of odds in $\mathscr{Y}$ positions is zero.

By minimizing the number of odds in $\mathscr{Y}$ positions, (5) is set equal to zero. Solving $42 - 6K = 0$ for K shows that $K = 7$ when there are no odds in $\mathscr{Y}$ positions. Replacing K with 7 in (4) and (5) results in $X = 21$ and $Y = 0$. Hence, when there are exactly seven odds per row, twenty-one odd numbers are in $\mathscr{X}$ positions and zero odds are in $\mathscr{Y}$ positions.

To find the minimum possible number of odds per row, the number of odds in $\mathscr{X}$ positions must be minimized. Because there are twenty-one odds and only eighteen $\mathscr{Y}$ positions, the minimum possible number of odds in $\mathscr{X}$ positions is three. Three odds in $\mathscr{X}$ positions ($6K - 21 = 3$) results in $K = 4$. Plugging this into (4) and (5) shows that $X = 3$ and $Y = 18$. So, for there to be exactly four odds in each row, there must be three odds in $\mathscr{X}$ positions and eighteen odds in $\mathscr{Y}$ positions.

Knowing that the number of odds per row must be between four and seven, the possibilities in which K is either five or six must also be investigated. Using (4) and (5), the number of odds in $\mathscr{X}$ and $\mathscr{Y}$ positions can be found. For there to be five odd numbers in each row, there must be nine odds in $\mathscr{X}$ positions and twelve odds in $\mathscr{Y}$ positions. To have exactly six odds per row, there must be fifteen odds in $\mathscr{X}$ positions and six in $\mathscr{Y}$ positions.

3.2. ***Odd/even patterns.*** As determined in the previous section, there are four different cases in which every row of a magic hexagram can have the same number of odds. These cases are

(Case 1) 7 odds per row; 21 odds in $\mathscr{X}$ positions and 0 odds in $\mathscr{Y}$ positions.

(Case 2) 6 odds per row; 15 odds in $\mathscr{X}$ positions and 6 odds in $\mathscr{Y}$ positions.

(Case 3) 5 odds per row; 9 odds in $\mathscr{X}$ positions and 12 odds in $\mathscr{Y}$ positions.

(Case 4) 4 odds per row; 3 odds in $\mathscr{X}$ positions and 18 odds in $\mathscr{Y}$ positions.

The patterns resulting from each of these cases can be complementary to each other, because the number of odds in $\mathscr{X}$ positions in one pattern is equivalent to the number of evens in $\mathscr{X}$ positions in the complementary pattern. The number of odds in $\mathscr{Y}$ positions in one pattern is also equivalent to the number of evens in $\mathscr{Y}$ positions in the other pattern. For any pattern in which the Case 1 holds, if all of the evens and odds are switched, Case 4 results. As a result, these two cases are complementary and one of them can be considered trivial. Case 2 and Case 3 are also complementary to each other. By considering Cases 3 and 4 as trivial, there are only two nontrivial cases in which all rows have the same number of odds: when there are either seven odds in each row (Case 1), or six odds in each row (Case 2).

There are many different ways to arrange patterns such that they have the same magic constant and are therefore trivial variations. Numbers that appear in only one row can be switched to other positions only appearing in the same row without changing the magic constant. Arrangements that only switch ones and zeros throughout these positions have equivalent odds in each row. Additionally, there exist pairs of positions that appear in only the same two rows. If the numbers are switched between these two positions, both numbers are still in the same two rows and hence contribute to the same row sum. These switches, as a result, are trivial in both the traditional extended magic hexagram and in the odd/even diagrams.

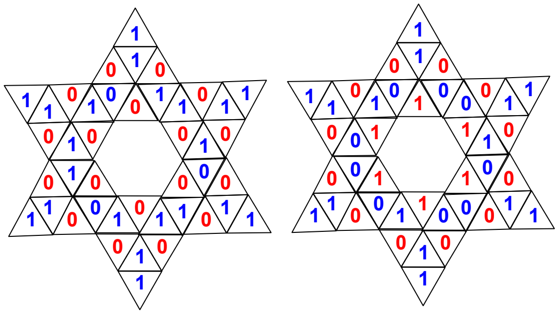


Figure 5. The two distinct odd/even arrangements for which every row in the extended hexagram has the same number of evens and odds.

Considering patterns defined for Case 1 and Case 2 and ignoring trivial variations, there are only two distinct arrangements for magic extended hexagrams for which every row has the same number of odd numbers. These two patterns are shown in Figure 5.

3.3. ***Magic constants.*** The upper and lower bounds of the magic constant can be found for both of these extended hexagrams using (3). Because the extended magic hexagram is made up of six rows, has twenty-four $\mathscr{X}$ positions each appearing in two rows, and has eighteen $\mathscr{Y}$ positions each appearing in only one row, (3) can be rewritten as

$$\frac{1}{6}\left(2\sum_{i=1}^{24} X_i + \sum_{i=1}^{18} Y_i\right). \tag{6}$$

Because $\mathscr{X}$ positions appear in a larger number of rows than $\mathscr{Y}$ positions, the lower bound magic constant is found by placing the lowest possible numbers in the $\mathscr{X}$ positions and the largest numbers in $\mathscr{Y}$ positions. The upper bound magic constant is found by doing the opposite.

For the hexagram with seven odd numbers in each row, all twenty-one odd numbers, along with three even numbers, are placed in positions that appear in two different rows. By placing the lowest possible numbers in the $\mathscr{X}$ positions and the largest in $\mathscr{Y}$ positions, (6) results in a lower bound of 226. By placing the largest numbers in $\mathscr{X}$ positions and smallest in $\mathscr{Y}$ positions, (6) shows the upper bound magic constant is 244.

Similarly, the lower and upper bound magic constants can be found for the hexagram with six odd numbers in each row. This hexagram contains fifteen odds in $\mathscr{X}$ positions and six odds in $\mathscr{Y}$ positions. Equation (6) shows this hexagram has an upper bound magic constant of 269 and a lower bound magic constant of 203.

The possible magic constants for each of the hexagrams can be further limited by taking into account whether the magic constant must be odd or even. The magic constant for the hexagram with seven odds in each row must be a number between 226 and 244. However, because the hexagram has exactly seven odds in each row, the magic constant, or sum of the row, must be an odd number. So, the magic constant for the hexagram with seven odds in each row must be an odd number between 227 and 243.

The magic constant for the hexagram with six odds in each row is a number between 203 and 269. However, if there are exactly six odds in every row, the magic constant must be an even number. So, the magic constant for this hexagram must be an even number between 204 and 268.

Using these limitations, a simple computer program was written in an attempt to find solutions. Initially, each even and odd position was labeled sequentially. The program initially placed each odd number in a position labeled odd, so that 1 was placed in the first odd position, 3 in the second odd position, and so on. Similarly, each even was placed in the appropriate even position. Each time the program looped through, either two of the even numbers or two of the odd numbers were swapped so that every combination of the numbers could be investigated in an attempt to find arrangements where every row summed up to the magic constant. Every magic constant was to be investigated, printing all solutions to a file. However, computer programs written in both Prolog and C were not able to finish running within weeks. A parallelized genetic algorithm was also not able to find any solutions when run for an extended period of time. While the algorithm greatly limited the possible arrangements of this extended hexagram, further reduction is necessary to systematically identify solutions.

3.4. ***Solutions for magic hexagrams.*** Magic extended hexagram solutions can be found for upper and lower bound magic constants by strategically placing numbers in the polygram. The upper bounds of the magic constant for the seven odds per row and six odds per row cases were found by placing the highest numbers in $\mathscr{X}$ positions, but placing the very highest numbers in these locations would not result in a solution. Similarly, lower bounds were identified for the magic constant in both cases by placing the lowest numbers in $\mathscr{X}$ positions, but this arrangement of the lowest numbers would not result in a solution. However, placing most of the high numbers or low numbers in $\mathscr{X}$ positions can lead to solutions through the recognition of patterns in the odd/even diagrams.

Figure 5 contains distinct odd/even arrangements for the seven odds per row and six odds per row cases; numbers in $\mathscr{X}$ positions are blue and numbers in $\mathscr{Y}$ positions are presented in red. In both diagrams, all $\mathscr{X}$ positions appear in diamond pairs, and each number in the pair affects the same rows in the hexagram. The diamonds can be grouped into three categories:

- Pairs of odd numbers at the points of the star, or corner pairs, with sum C,
- Mixed pairs of one even and one odd in the internal hexagon, or mixed internal pairs, with sum M, or
- Pairs of two odds (7 odds per row) or two evens (6 odds per row) in the internal hexagon, or consistent internal pairs, with sum I.

In order to reduce the magic extended hexagram to a problem that is more easily solvable, the sums of the pairs in each category are set to be equal. Under these conditions, the magic constant will be equal to $2C + M + I$ plus the sum of the three numbers in the $\mathcal{Y}$ positions in any particular row. Additionally, by placing the highest or lowest values in particular positions, particular solutions can be identified.

As described in Section 3.3, the upper bound for the magic constant is 243 for the seven odds per row case. The upper bound was identified by placing all twenty-one odds in $\mathcal{X}$ positions and the highest evens in the remaining $\mathcal{X}$ positions. In order to identify magic extended hexagram solutions, the odd numbers from 19 to 41 are placed in corner pair positions such that each pair totals 60. To try to find a solution with a magic constant of 243, the highest evens should be placed in $\mathcal{X}$ positions if possible. Because the corner pairs now contain odds, high evens are placed in the only remaining $\mathcal{X}$ positions: the mixed internal pairs. If consecutive evens are chosen for these three locations, any list of three consecutive odds may be chosen in order to have a consistent sum M. As previously shown in Section 3.3, placing the three highest even numbers (38, 40, and 42) in the mixed internal pair positions will not result in a solution (because the magic constant must be odd in this case). However, placing the consecutive even numbers 36, 38, and 40 in mixed internal pair positions will lead to a solution (although using the numbers 34, 36, and 38 will not). The details of solutions of this form are as follows:

- The odd numbers 19 through 41 are placed in corner pair positions.
- The even numbers 36, 38, and 40 are paired with three consecutive odds (not already in corner pair positions) for the mixed internal pairs.
- The six remaining odds can be ordered $O_1 > O_2 > O_3 > O_4 > O_5 > O_6$ and then grouped into pairs (O_1 and O_6, O_2 and O_5, O_3 and O_4) for the consistent internal pairs.
- The 18 remaining evens are separated into groups of three, each with sum 58, and placed in the $\mathcal{Y}$ positions of six rows.

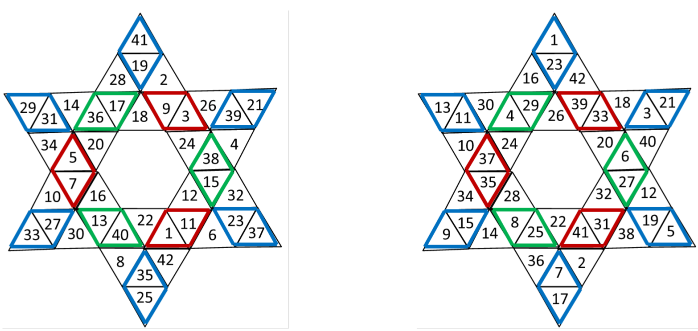
An example of a magic extended hexagram solution with this structure is shown in Figure 6, left.

Solutions for the magic extended hexagram can also be found by placing low numbers in $\mathcal{X}$ positions. The lower bound for the magic constant for the seven odds per row case was found by placing all twenty-one odd numbers in $\mathscr{X}$ positions and the three lowest even numbers in the remaining $\mathscr{X}$ positions. In order to identify solutions with a magic constant of 227, odd numbers from 1 to 21 were placed in corner pair positions such that each pair totals 24. In order to identify a solution with a magic constant of 227, low even numbers were placed in mixed internal pair positions. The structure of existing examples of extended hexagrams with a magic constant of 227 can be identified using a similar method to that used to identify solutions for extended hexagrams with a magic constant of 243. The details of this solution arrangement are as follows:

- The odd numbers 1 through 21 are placed in corner pair positions.
- The even numbers 4, 6, and 8 are paired with three consecutive odds (not already in corner pair positions) for the mixed internal pairs.
- The six remaining odds can be ordered $O_1 > O_2 > O_3 > O_4 > O_5 > O_6$ and then grouped into pairs (O_1 and O_6, O_2 and O_5, O_3 and O_4) for the consistent internal pairs.
- The 18 remaining evens are separated into groups of 3, each with sum 74, and placed in the $\mathscr{Y}$ positions of six rows.

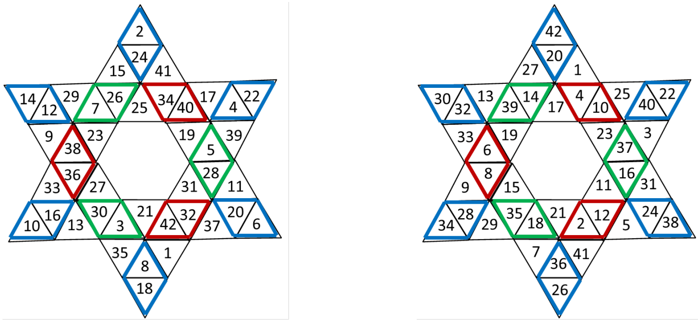
An example of a magic extended hexagram solution with this structure is shown in Figure 6, right. Complements for the magic extended hexagram solutions in Figure 6 are shown in Figure 7.

As shown in Figure 6, solutions to the magic extended hexagram exist for the seven odds per row case. The numbers can be moved around the hexagram and still result in a solution as long as the numbers stay in the same category pair or particular



magic constant = 243 magic constant = 227

Figure 6. Examples of magic extended hexagram solutions for the seven odds per row case when the magic constant is 243 (left) and 227 (right). Corner pairs are outlined in blue, mixed internal pairs in green, and consistent internal pairs in red.



magic constant = 230　　magic constant = 246

Figure 7. Complements of the solutions shown in Figure 6 (color conventions are the same).

set of three evens. Because these changes do not represent a simple rotation of the entire polygram, movement of numbers within category pairs or particular sets results in a new, nontrivial solution. Additionally, different lists of consecutive odds could have been chosen for the mixed internal pairs. Because all odd numbers are placed in one of the pairs (corner, mixed internal, or consistent internal), the odd numbers likely could have been arranged differently to also arrive at a solution. The complements shown in Figure 7 show examples of solutions for the four odds per row case and also reflect the structure of keeping the paired sums equal.

Magic extended hexagram solutions can also be found for the six odds per row case with upper and lower bound magic constants. In order to identify solutions that have a magic constant of 268, the largest odds are placed in corner pair positions. Because the largest even numbers can be placed in either of the internal pair blocks, two different methods are used to find solutions. In the first scenario, the six largest even numbers are placed in consistent internal pairs of the extended hexagram. More details of the construction of this solution are as follows:

- The odd numbers 19 through 41 are placed in corner pair positions.
- The even numbers 32 through 42 are placed in consistent internal pair positions.
- The next highest even numbers (26, 28, 30) are placed in mixed internal pair positions.
- All possible sets of three consecutive (remaining) odd numbers are investigated as potential numbers for the mixed internal pair positions. The only possible list is 11, 13, and 15, which forces the remaining three numbers in each row to total 33.
- The 18 remaining numbers are separated into groups of three (one odd and two evens), each with sum 33, and placed in the $\mathcal{Y}$ positions of six rows.

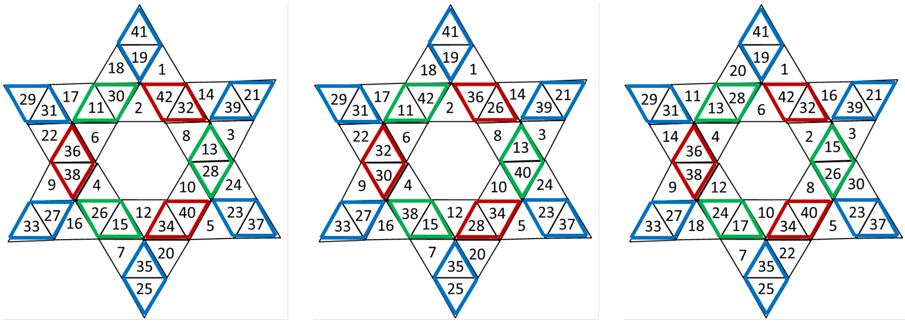


Figure 8. Examples of magic extended hexagram solutions for the six odds per row case when the magic constant is 268. Color conventions are as in Figure 6.

An example of a magic extended hexagram solution with this structure is shown in Figure 8, left. Alternatively, the three highest even numbers are placed in mixed internal pairs of the extended hexagram. More details of solutions of this form are as follows:

- The odd numbers 19 through 41 are placed in corner pair positions.
- The highest even numbers (38, 40, 42) are placed in mixed internal pair positions.
- The even numbers 26 through 36 are placed in consistent internal pair positions.
- All possible sets of three consecutive (remaining) odd numbers are investigated as potential numbers for the mixed internal pair positions. The only possible list is 11, 13, and 15, which forces the remaining three numbers in each row to total 33.
- The 18 remaining numbers are separated into groups of three (one odd and two evens), each with sum 33, and placed in the $\mathscr{Y}$ positions of six rows.

An example of a magic extended hexagram solution with this structure is shown in Figure 8, middle.

Note that the only differences in the solutions presented in the first two parts of Figure 8 are the locations of even numbers in $\mathscr{X}$ positions in the interior of the hexagram. However, this change is not the result of a rotation of the entire polygram or of the interior of the hexagram.

Other solutions to the magic extended hexagram with six odds per row can be found in similar ways. The more generalized process is this:

- The highest (or lowest) odd numbers are placed in corner pair positions.
- Even numbers are selected for the consistent internal pair positions. A set of three consecutive evens are placed in mixed internal pair positions.

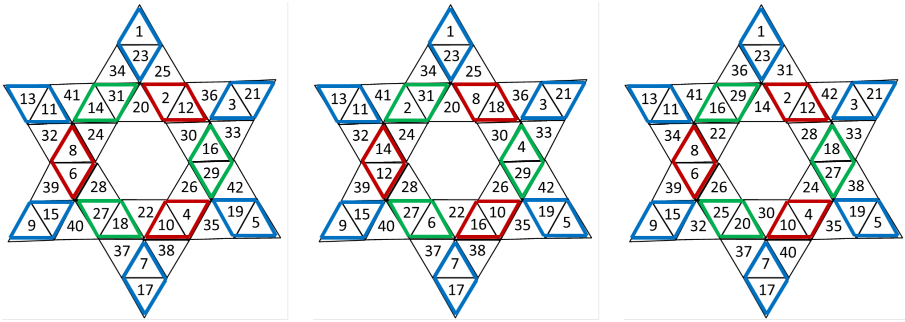


Figure 9. Examples of magic extended hexagram solutions for the six odds per row case when the magic constant is 204.

- All possible sets of three consecutive (remaining) odd numbers are investigated as potential numbers for the mixed internal pair positions. If a viable list is identified, the remaining 18 numbers are separated into groups of three each (one odd and two evens) and placed in the $\mathcal{Y}$ positions.

An additional example of a magic extended hexagram with magic constant 268 is presented in Figure 8, right. Examples of solutions with six odds per row when the magic constant is 204 are shown in Figure 9. Figures 10 and 11 contain complementary solutions to those in Figures 8 and 9, respectively.

As with the solutions presented for the seven odds per row and four odds per row cases, the numbers in the presented solutions for the six odds per row case can be moved around the hexagram within the same category pair and result in a solution. Not only can numbers be rotated within category pairs and result in a different solution, multiple structures are identified for the six odds per row case as shown in Figures 8 and 9. The complements to the six odds per row case represent

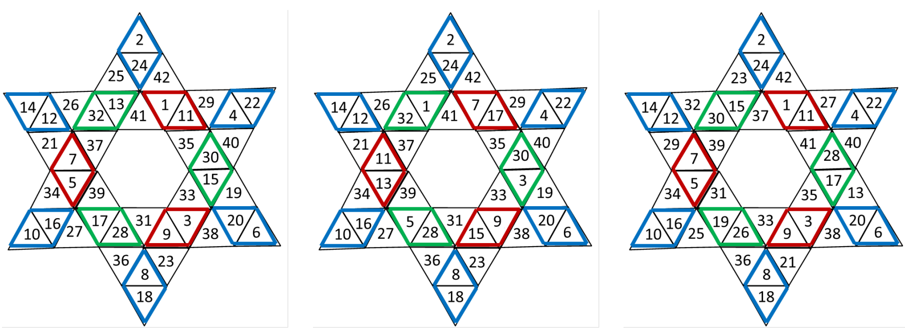


Figure 10. Complements of the magic extended hexagram solutions for the six odds per row case shown in Figure 8. The magic constant is 205.

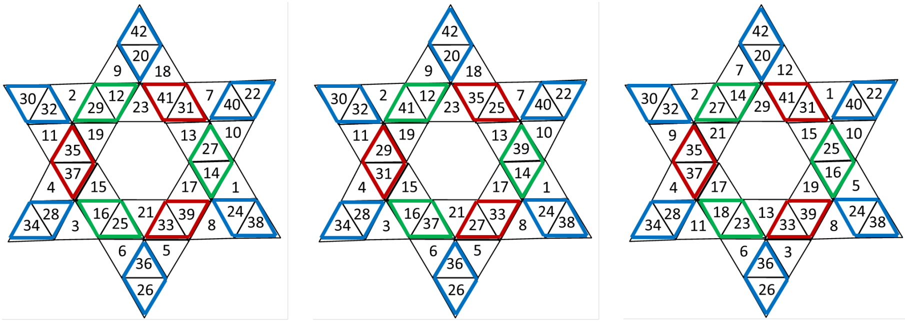


Figure 11. Complements of the magic extended hexagram solutions for the six odds per row case shown in Figure 9. The magic constant is 269.

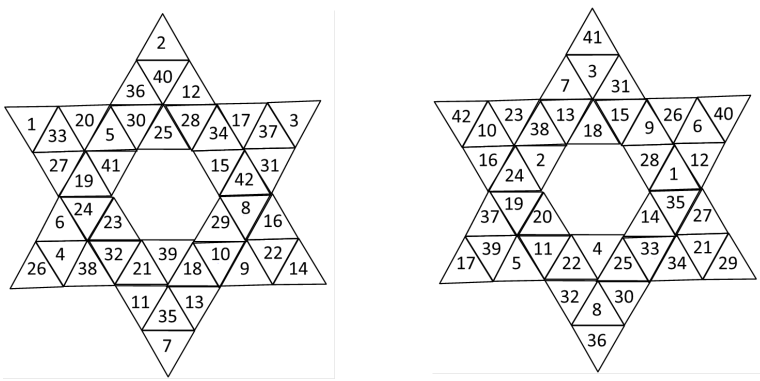


Figure 12. Example of a magic extended hexagram solution when the number of odds per row is not fixed (left), together with its complement (right).

solutions for the five odds per row case and also reflect the same structure. While many solutions have been identified for the cases when there are the same number of odds per row, a definitive list has not been established. Further, solutions for the magic extended hexagram do exist for cases when the number of odds per row is not fixed (as is shown in Figure 12).

4. Traditional magic octagram

The methods used to find magic extended hexagrams can also be applied to find other magic polygrams, such as magic octagrams. If numbers 1 through 16 are placed throughout the octagram (as shown in Figure 4), there are 16! (total) arrangements possible. In order to reduce this number as in the investigation on magic extended hexagrams, the number of possibilities will be reduced by considering odd/even diagrams and by only considering nontrivial variations.

The inner positions of the octagram each appear in four different rows, while the outer positions each appear in two different rows. Similar to the magic hexagram, there are two possible cases in which every row can add up to a single number. The first case is when every row contains an odd number of odds. The magic constant, or common sum of each row, must be an odd number. The second case is when every row contains an even number of odds. In this case, the magic constant would be an even number. As in Section 3, this paper investigates only scenarios in which every row has exactly the same number of odds.

4.1. ***Odd/even patterns.*** As shown in Figure 4, the numbers in the eight positions within the central octagon appear in four rows each (which we will refer to as $\mathscr{X}$ positions), and the numbers in the eight positions that are the points of the star only appear in two rows each (which we will refer to as $\mathscr{Y}$ positions). Because a magic traditional octagram contains eight odd numbers and has eight rows with eight $\mathscr{X}$ positions and eight $\mathscr{Y}$ positions, Theorem 3 shows that there are

$$4K-8 \tag{7}$$

odds in $\mathscr{X}$ positions and

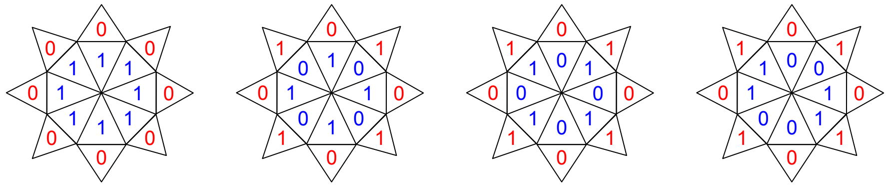
$$16-4K \tag{8}$$

odds in $\mathscr{Y}$ positions.

Setting (8) to zero and solving for K shows that the maximum number of odds per row is four. Plugging this into K for (7) and (8) shows that for there to be exactly four odds in each row, there must be eight odd numbers in $\mathscr{X}$ positions and zero odds in $\mathscr{Y}$ positions. Setting (7) to zero and solving for K shows that the minimum number of odds per row is two. Equations (7) and (8) show that for K, or the number of odds per row, to be equal to two, there must be zero odds in $\mathscr{X}$ positions and eight odd numbers in $\mathscr{Y}$ positions.

Because the maximum possible number of odds per row is four and the minimum is two, the only other possible number of odd numbers per row is three. Substituting 3 for K in (7) and (8) shows that for there to be exactly three odds per row, there must be four odd numbers in $\mathscr{X}$ positions and four odds in $\mathscr{Y}$ positions.

The octagram investigated in this paper has an even number of integers arranged throughout it, so by Theorem 2, two odd/even patterns of this octagram are complements when the patterns are exact opposites of each other. The odd/even arrangement for when there are four odds in a row (all odds in the central octagon) is complementary to the odd/even arrangement when there are two odds in a row. In the process of finding magic octagrams, only one of these two patterns needs to be investigated. Figure 13 shows four of the distinct odd/even patterns for the



(a) 4 odds per row (b) 3 odds per row (c) 3 odds per row (d) 3 odds per row

Figure 13. Four distinct odd/even patterns for the octagram.

magic octagram. There are multiple different patterns with three odds in every row, and all of the possible patterns for this scenario have not been investigated.

4.2. ***Magic constants.*** The upper and lower bounds of the magic constant can be found for distinct octagrams using (3). For the octagram, (3) can be rewritten as

$$\frac{1}{8}\left(4\sum_{i=1}^{8} X_i + 2\sum_{i=1}^{8} Y_i\right). \tag{9}$$

Similar to the process used to find the lower bound magic constant for hexagrams, the lower bound magic constant for each odd/even pattern of this octagram is found by placing the lowest possible numbers in $\mathscr{X}$ positions and the largest numbers in $\mathscr{Y}$ positions, and the upper bound magic constant is found by doing the opposite. For the octagram with four odd numbers in each row, all eight odd numbers are placed in the $\mathscr{X}$ positions and all eight even numbers are placed in the $\mathscr{Y}$ positions. Because there are only eight odd numbers to be placed in the eight $\mathscr{X}$ positions, there is only one possible magic constant rather than a range of possibilities. Using (9), the only possible magic constant for the octagram in which there are four odds per row is 50.

The lower and upper bound magic constants can be found for the octagrams with three odd numbers in each row. This octagram contains four odds in $\mathscr{X}$ positions and four odds in $\mathscr{Y}$ positions. Equation (9) shows that this octagram has an upper bound magic constant of 59 and a lower bound magic constant of 43.

4.3. ***Solutions for magic octagrams.*** By limiting the possible arrangements of numbers 1 through 16 throughout the octagram, the number of possibilities is small enough for a computer program to find magic octagrams. A brute force computer program with restrictions added in was written in C. Similar to the program described in Section 3, this program labeled all of the even and odd positions, initially arranged the numbers throughout their positions, and then stepped through every appropriate arrangement to find solutions. The program investigated each magic constant for every odd/even pattern, and with just these limitations, solutions were found in adequate time. There were hundreds of solutions for each of the four odd/even patterns in Figure 13.

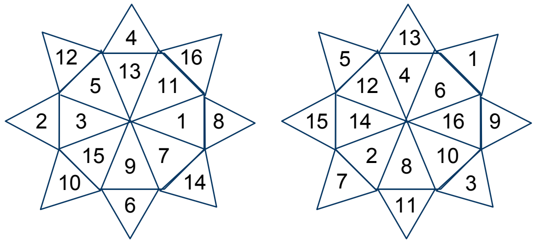


Figure 14. A magic octagram with four odds in each row (left) and a magic octagram with two odds in each row (right).

For the pattern in which all eight odd numbers are in $\mathscr{X}$ positions, as shown in Figure 13, there is only one possible magic constant, 50. For this magic constant, the 1920 different solutions were found computationally. The complementary pattern, in which all eight odd numbers are in $\mathscr{Y}$ positions, has 1920 solutions. Each of the solutions to the pattern with all eight odd numbers in $\mathscr{Y}$ positions is a complement to a solution of the pattern with all eight odd numbers in $\mathscr{X}$ positions.

The data for the other three patterns in Figure 13, each of which have three odds as a part of the sum in each row, is shown in Table 1. Pattern 1 has a total of 736 unique solutions, pattern 2 has 832, and pattern 3 has 1161, all among nine different magic constants. One solution for each pattern is shown on the top row of Figure 15. Each of these three patterns also has a complementary pattern. For every complementary pattern, there is an equivalent number of solutions, each one being the complement of an original solution. The complementary solutions for those shown in on the top row of Figure 15 are shown on the bottom row.

magic constant	pattern 1 Figure 13(b)	pattern 2 Figure 13(c)	pattern 3 Figure 13(d)
43	3	12	43
45	41	42	36
47	83	96	145
49	144	160	199
51	196	224	358
53	144	160	199
55	83	96	145
57	41	42	36
59	4	12	43

Table 1. Number of distinct octagram patterns of the forms shown in Figure 13(b)–(d).

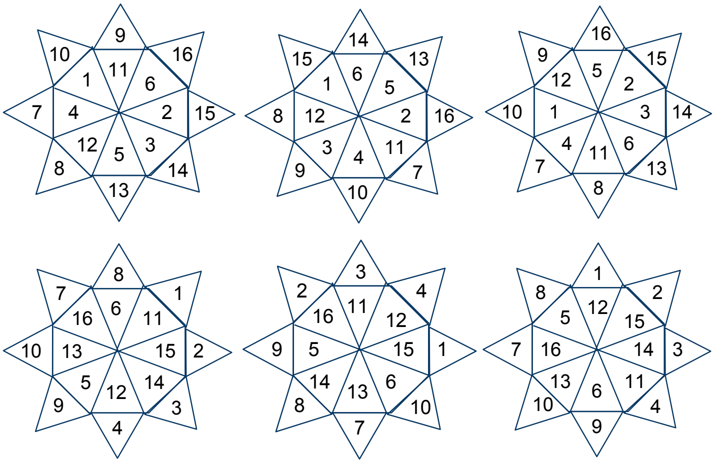


Figure 15. Top: solutions with a magic constant of 45 and three odds per row. Bottom: their complementary solutions, with magic constant 57.

5. Discussion and conclusions

If restrictions are placed on the number of even and odd numbers per row of the extended hexagram, characteristics of solutions can be identified. When there are seven odds per row, the magic constant of this extended hexagram must be an odd number between 227 and 243. The magic constant of the extended hexagram with six odds per row must be an even number between 204 and 268. Although an exhaustive list of solutions has not been established, multiple solutions have been identified.

Using similar restrictions on the number of evens and odds in the rows of the traditional extended hexagram, magic octagrams can also be identified. The possible magic constants for restricted patterns were found to further limit possibilities before using computer programs to identify specific solutions. The magic constant for the octagram with four odd numbers per row must be 50. The octagrams with three odds per row must have a magic constant between 43 and 59. The computationally discovered solutions were categorized by pattern and magic constant, but not all magic octagram solutions have necessarily been identified.

The only odd/even patterns that have been investigated for both polygrams are cases in which there are an equivalent number of odds in each row. The solutions for both polygrams when there are different numbers of odds in each row have not been investigated. Similarly, not every odd/even pattern in which there are three odds in every row of the octagram has been investigated. Additional studies could investigate more focused algorithms to find solutions in the fixed number of odds per row cases. Most hexagram solutions found in this study had the same number of odds per row, but solutions do exist without this restriction as shown in Figure 12. Future investigations could focus on values in internal locations of the hexagram or octagram without the overall expectation of a certain number of odds per row in order to find solutions.

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