DEDEKIND ZETA FUNCTIONS AND THE PRIME IDEAL THEOREM

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Abstract. The Prime Number Theorem is a result describing the asymptotic distribution of primes in $\mathbb{Z}$. Here, we prove a generalization due to Landau, which concerns the asymptotic distribution of prime ideals in a general number ring.

Contents

1. Introduction

Prime numbers have been studied since antiquity, and they naturally arise in problems spanning the entirety of mathematics. Among the first results in their study is the following:

Theorem 1.1. *There are infinitely many prime numbers.*

The first proof of this is attributed to Euclid. We give an analytic proof due to Euler, obtained by study of the following function:

Definition 1.2 (Riemann zeta function)**.** Let $s \in \mathbb{C}, \Re(s) > 1$. Then, we write

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This converges uniformly in the given domain, by the integral test.

Proof of Theorem 1.1. Using unique factorization in $\mathbb{Z}$, we obtain the Euler product formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{1}{p^k} \right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

– see Proposition 2.5 for a rigorous proof. Imagine that there are only finitely many primes. Then the product on the right is finite. In particular, it is bounded as s approaches 1. But ζ diverges as s approaches 1, by comparison with the harmonic series. □

In this proof, we use an analytic fact about ζ to obtain information about the set of prime numbers. In this vein, one can study the behavior of (a meromorphic continuation of) ζ along the line $\Re(s) = 1$ and prove the following theorem:

Prime Number Theorem. *For $N \in \mathbb{R}_{\geq 0}$, let $\pi(N)$ denote the number of primes less than N. Then, we have:*

$$\pi(N) \sim \frac{N}{\log N}$$

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This result gives us information about the distribution of prime numbers in $\mathbb{Z}$, which is interesting and useful in its own right. One would then hope to define zeta functions in the more general context of number rings, and prove a corresponding theorem about prime ideals.[1] To formulate this, one must define the norm $\mathfrak{N}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ in a number ring – this is the covolume of the ideal in $\mathcal{O}_K$ (in $\mathbb{Z}$, one simply has $\mathfrak{N}(p\mathbb{Z}) = p$). Then, we may state the following theorem due to Landau:

Prime Ideal Theorem. *Let K a number field. For $N \in \mathbb{R}_{\geq 0}$, let $\pi_K(N)$ denote the number of prime ideals of $\mathcal{O}_K$ whose norm is less than N. Then, we have:*

$$\pi_K(N) \sim \frac{N}{\log N}$$

This clearly implies the Prime Number Theorem, and it is the result we will prove here. We follow Newman's elegant proof of the Prime Number Theorem as exposited in [LZ] or [Z], adapting the methods to the case of a general number ring. The most complicated generalization is the meromorphic extension of the relevant zeta function over the line $\Re(s) = 1$ – we devote all of section 3 to answering this question, by proving the Class Number formula.

We assume some familiarity with commutative algebra and algebraic number theory, although we state most necessary results. K always denotes a number field – i.e. a finite extension of $\mathbb{Q}$ – and $\mathcal{O}_K$ is its ring of integers (these are called *number rings*). Lowercase fraktur letters always denote ideals, and $\mathfrak{p}$ in particular is always a nonzero prime ideal. In analytic number theory, the letter s often denotes the complex input of a function, in the same way that z is used in complex analysis; with some exceptions, we follow this convention here.

2. Dedekind Zeta Functions

We set out to define zeta functions in the context of number rings. Recall that such rings are integrally closed domains of Krull dimension one, and are finitely-generated $\mathbb{Z}$-algebras. We collect some algebraic facts:

Proposition 2.1. *If $\mathcal{O}_K$ a number ring and $0 \neq \mathfrak{p} \leq \mathcal{O}_K$ is prime, then $\mathcal{O}_K/\mathfrak{p}$ is a finite field.*

Proposition 2.2. *If $\mathcal{O}_K$ is a number ring, then every nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ has a unique factorization into prime ideals $\mathfrak{p}_i$:*

$$\mathfrak{a} = \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_n^{\alpha_n}$$

Given $0 \neq \mathfrak{a} \leq \mathcal{O}_K$, we define its *norm* to be the size of the residue ring:

$$\mathfrak{N}(\mathfrak{a}) := |\mathcal{O}_K/\mathfrak{a}|$$

This is the covolume of $\mathfrak{a}$ in $\mathcal{O}_K$. Note that $\mathfrak{N}(\mathfrak{a}\mathfrak{b}) = \mathfrak{N}(\mathfrak{a})\mathfrak{N}(\mathfrak{b})$, by the Chinese Remainder theorem.

It will be convenient to consider the relationship between $\mathbb{Z}$ and $\mathcal{O}_K$. We list the relevant facts here:

Proposition 2.3. *Let K a number field, and let $n = [K : \mathbb{Q}] = \dim_{\mathbb{Q}} K$. Let $p \in \mathbb{Z}$ an integer prime. Then:*

- *As a $\mathbb{Z}$-module, $\mathcal{O}_K \cong \mathbb{Z}^n$, and $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Q} \cong K \cong \mathbb{Q}^n$.*
- *$p\mathcal{O}_K$ is an ideal of $\mathcal{O}_K$, and $\mathfrak{N}(p\mathcal{O}_K) = |\mathcal{O}_K/p\mathcal{O}_K| = p^n$.*
- *If $p\mathcal{O}_K$ has the prime decomposition $\mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_k^{\alpha_k}$, then*

  $$\mathcal{O}_K/p\mathcal{O}_K \cong \mathcal{O}_K/\mathfrak{p}_1^{\alpha_1} \oplus \dots \oplus \mathcal{O}_K/\mathfrak{p}_k^{\alpha_k}$$

  In particular,

  $$\mathfrak{N}(\mathfrak{p}_1)^{\alpha_1} \dots \mathfrak{N}(\mathfrak{p}_k)^{\alpha_k} = p^n$$

- *For every prime $\mathfrak{p} \leq \mathcal{O}_K$, $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some integer prime p, and conversely $\mathfrak{p} \mid p\mathcal{O}_K$.*

Definition 2.4 (Dedekind zeta function)**.** Given a number field K, its Dedekind zeta function is:

$$\zeta_K(s) = \sum_{0 \neq \mathfrak{a} \leq \mathcal{O}_K} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

Clearly $\zeta_{\mathbb{Q}}$ reproduces the Riemann zeta function. Now we prove convergence, as well as a useful formula:

[1]In fact, zeta functions and "prime number theorems" arise in many contexts, a notable example being the Selberg zeta functions of certain Riemann surfaces and the corresponding Prime Geodesic Theorem – see Chapter 11 of [DE].

Proposition 2.5. *$\zeta_K(s)$ converges absolutely uniformly on $\{s \in \mathbb{C} \mid \Re(s) > 1\}$. Further, we have the Euler product formula:*

$$\zeta_K(s) = \prod_{\substack{0 \neq \mathfrak{p} \leq \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}}$$

Proof. Remark that since we hope to prove absolute convergence, it suffices to consider $s \in \mathbb{R}$. First, we claim that the Euler product converges exactly where ζ_K does, and that it agrees with ζ_K on this domain. Using the prime decomposition of Proposition 2.2 along with the multiplicativity of norm, we write the following:

$$\sum_{\substack{0 \neq \mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{a}) \leq N}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} \leq \prod_{\substack{0 \neq \mathfrak{p} \leq \mathcal{O}_K \\ \mathfrak{p} \text{ prime} \\ \mathfrak{N}(\mathfrak{p}) \leq N}} \left(\sum_{(\mathfrak{N}(\mathfrak{p}))^k \leq N} \frac{1}{\mathfrak{N}(\mathfrak{p})^{ks}} \right) \leq \sum_{\substack{0 \neq \mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{a}) \leq N!}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}$$

Taking the limit $N \to \infty$, we have proved the claim.

We now claim that the Euler product converges on the required domain. Let $[K : \mathbb{Q}] = n$. By Proposition 2.3, every prime $\mathfrak{q} \leq \mathcal{O}_K$ corresponds to an integral prime p, so that $\mathfrak{N}(\mathfrak{q}) \leq p^n$; further, no more than n primes in $\mathcal{O}_K$ correspond to p. We write:

$$\zeta_K(s) = \prod_{p \in \mathbb{Z}} \left(\prod_{\mathfrak{q} \cap \mathbb{Z} = p\mathbb{Z}} \frac{1}{1 - \mathfrak{N}(\mathfrak{q})^{-s}} \right) \leq \prod_{p \in \mathbb{Z}} \left(\frac{1}{1 - p^{-ns}} \right)^n \leq \prod_{p \in \mathbb{Z}} \left(\frac{1}{1 - p^{-s}} \right)^n = (\zeta_{\mathbb{Q}}(s))^n$$

Now we compute

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^{\infty} \frac{dt}{t^s} = 1 + \frac{1}{s-1} < \infty$$

as required. ∎

We know that ζ_K is defined on $\{s \in \mathbb{C} \mid \Re(s) > 1\}$. We also have that $\lim_{s \to 1^+} \zeta_K(s) = \infty$, so that 1 is not in its domain of holomorphy. In fact, ζ_K admits an analytic extension to $\mathbb{C} \setminus \{1\}$, although we shall prove only the following:

Theorem 3.11. *There exists a constant c_K so that $\zeta_K(s) - \frac{c_K}{s-1}$ admits an analytic continuation to $\Re(s) > 1 - \epsilon$.*

The proof of this result will be the focus of Section 3. For the moment, we take it for granted.

Now, we want to prove that $\zeta_K(s)$ does not vanish along the line $\Re(s) = 1$. To do this, we note that if a function vanishes to order k at a point, its logarithmic derivative has a pole at that point with residue k. To study the logarithmic derivative of ζ_K, we introduce the following function:

$$\Phi_K(s) := \sum_{\mathfrak{p} \leq \mathcal{O}_K} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s}$$

Remark that

$$-\frac{\zeta_K(s)'}{\zeta_K(s)} - \Phi_K(s) = \sum_{\mathfrak{p} \leq \mathcal{O}_K} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^s (\mathfrak{N}(\mathfrak{p})^s - 1)}$$

Note that the sum on the right converges wherever $\zeta_K(2s)$ does, and so is holomorphic for $\Re(s) > \frac{1}{2}$. Therefore Φ_K has the same poles as the logarithmic derivative of ζ_K, with opposite residues.

Lemma 2.5. *$\zeta_K(s) \neq 0$ when $\Re(s) = 1$*

Proof. Imagine that ζ_K had a zero of order k at $1 + iy$. By Schwarz Reflection, it has a zero of the same order at $1 - iy$. Assume that ζ has zeroes of order k' (possibly with $k' = 0$) at $1 \pm 2iy$. By the arguments above, Φ_K has a pole at 0 of residue 1, at $1 \pm iy$ with residue $-k$, and at $1 \pm 2iy$ of residue $-k'$. Write

$$\sum_{n=-2}^{2} \binom{4}{2+n} \Phi_K(1+\epsilon+niy) = \sum_{\mathfrak{p} \leq \mathcal{O}_K} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{1+\epsilon}} (\mathfrak{N}(\mathfrak{p})^{-\frac{iy}{2}} + \mathfrak{N}(\mathfrak{p})^{\frac{iy}{2}})^4$$

$$= \sum_{\mathfrak{p} \leq \mathcal{O}_K} \frac{\log \mathfrak{N}(\mathfrak{p})}{\mathfrak{N}(\mathfrak{p})^{1+\epsilon}} \left(2\cos\left(\frac{y \log \mathfrak{N}(\mathfrak{p})}{2}\right)\right)^4$$

The residue at $\epsilon = 0$ must be positive, as the term on the right is strictly positive for $\epsilon > 0$. From the left-hand side, we see that the residue is $6 - 8k - 2k'$. For this to be positive, $k = 0$. $\square$

3. The Class Number Formula

Our strategy in extending ζ_K across the line $\Re(s) = 1$ involves estimating the number of ideals with a given norm. In $\mathbb{Z}$ or $\mathbb{Z}[i]$, this amounts to counting elements of a given norm, then dividing by the order of the unit group. In a general number ring, there are two complications to this approach. First, $\mathcal{O}_K$ might have an infinite unit group, so that infinitely many elements of $\mathcal{O}_K$ generate the same ideal. Therefore, we must take care to not double-count associate elements. Second, $\mathcal{O}_K$ might not be a PID, in which case counting elements only allows us to assess principal ideals. To access the remainder, we must engage with the *ideal class group*, defined below.

We list some results and definitions from number theory. First, some geometric ideas of Minkowski:

Definition 3.1. Let K a number field. K is in particular an n-dimensional vector space over $\mathbb{Q}$, which acts linearly upon itself by multiplication. This lets us identify K with a subalgebra of $M_n(\mathbb{Q})$, which is itself dense in $M_n(\mathbb{R})$. Let $\overline{K}$ be the closure of K in $M_n(\mathbb{R})$. $\overline{K}$ is a real algebra, and therefore we have $\overline{K} = \mathbb{R}^r \oplus \mathbb{C}^s$ (r is the number of real embeddings, s the number of (conjugate pairs of) complex embeddings). This formula gives $\overline{K}$ a self-dual Haar measure. Note that $\mathcal{O}_K$ is a lattice in $\overline{K}$. We define $\overline{K^\times}$ as the intersection of $\overline{K}$ with $\mathrm{GL}_n(\mathbb{R})$ – this decomposes as $(\mathbb{R}^\times)^r \oplus (\mathbb{C}^\times)^s$, as expected.

Remark 3.2. Unless K is either $\mathbb{Q}$ or an imaginary quadratic field, $\overline{K}$ contains nonzero singular matrices – for example, $\overline{\mathbb{Q}(\sqrt{2})}$ contains the matrix $\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix}$.

Definition 3.3. Given a complete lattice Λ in $\mathbb{R}^n$, denote the volume of a fundamental domain (in some relevant measure μ) $\mu(\Lambda)$. In particular, we define $\mu(\mathcal{O}_K)$ in this manner.

Now, we turn to $\overline{K^\times}$. We have the normalized determinant map $|\det| : \overline{K^\times} \to \mathbb{R}^+$. This is a homomorphism, whose kernel we denote N. Note that $\mathcal{O}_K \cap N$ is exactly the group of units $\mathcal{O}_K^\times$. Now, consider the product structure $\overline{K^\times} = (\mathbb{R}^\times)^r \oplus (\mathbb{C}^\times)^s$. We define another homomorphism:

$$\lambda : \overline{K^\times} \to \mathbb{R}^{r+s} := ((x_1, \ldots, x_r, z_1, \ldots, z_s) \mapsto (\log |x_1|, \ldots, \log |x_r|, \log |z_1|^2, \ldots, \log |z_s|^2))$$

The image of N across this is a codimension 1 additive subgroup of $\mathbb{R}^{r+s}$ – thus, a hyperplane, which we shall call H. The fate of $\mathcal{O}_K^\times$ is as follows:

Theorem 3.4 (Dirichlet's Unit Theorem). *The kernel of $\lambda|_{\mathcal{O}_K^\times}$ is exactly the roots of unity μ_K, and the image of $\mathcal{O}_K^\times$ is a complete lattice in H. Thus it is a projective module, inducing a splitting:*

$$\mathcal{O}_K^\times = \mu_K \oplus \mathbb{Z}^{r+s-1}$$

The volume of this lattice in H is called the *regulator,* denoted R_K.

Definitions 3.5.
- If J is an $\mathcal{O}_K$-submodule of K so that $aJ \subseteq \mathcal{O}_K$ for some $a \in \mathcal{O}_K$, we call J a *fractional ideal.* In particular, every ideal of $\mathcal{O}_K$ is a fractional ideal.
- Let $\mathcal{J}$ be the set of fractional ideals, made into an abelian group by multiplication. Let $\mathcal{P}$ be the subgroup of principal ideals $a\mathcal{O}_K$ for some $a \in K$. Then the quotient $\mathrm{Cl}_K := \mathcal{J}/\mathcal{P}$ is the *ideal class group.*

We can think of ideals of $\mathcal{O}_K$ as belonging to ideal classes in this group; clearly ideals of $\mathcal{O}_K$ belonging to the trivial class are exactly the principal ideals. Thus Cl_K measures the failure of $\mathcal{O}_K$ to be a PID. We have the following theorem due to Minkowski:

Theorem 3.6. *Let K a number field. There is a constant M_K so that each ideal class has a representative $\mathfrak{a} \leq \mathcal{O}_K$ with $\mathfrak{N}(\mathfrak{a}) \leq M_K$. In particular, the class group Cl_K is finite.*

Now that we have the necessary number theoretic preliminaries, we take aim at the Class Number Formula. We make use of the following geometric formula – a proof is given in [N] or [Su]:

Lemma 3.7. *Let Λ a complete lattice in $\mathbb{R}^n$, $S \subset \mathbb{R}^n$ a compact neighborhood with Lipschitz boundary. Then:*

$$\sharp(tS \cap \Lambda) = \frac{\mu(S)}{\mu(\Lambda)} t^n + O(t^{n-1})$$

where μ is Lebesgue measure.

So we're looking to find a set S in $\overline{K}$ so that tS contains exactly one representative of each class in $\mathcal{O}_K/\mathcal{O}_K^\times$. We relax this requirement to the $\mathbb{Z}^{r+s-1}$ summand – since μ_K finite, this is all we need.

Construction 3.8. Let $\mathcal{F}$ a fundamental domain of the $\mathbb{Z}^{r+s-1}$ action on H, which contains 0. Let $S := \lambda^{-1}(\mathcal{F} \times \mathbb{R})$, where $\mathbb{R}$ is taken to be the image of $t\mathrm{Id}$ across λ. Define $S_\alpha := \{x \in S \mid |\det(x)| < \alpha\}$. This has two convenient properties: First, $tS_\alpha = S_{t^n\alpha}$. Second, if $\alpha, \beta \in S \cap \mathcal{O}_K$ generate the same ideal, then $\frac{\beta}{\alpha}$ is a unit, so $\lambda\left(\frac{\beta}{\alpha}\right) \in \lambda(\mathcal{O}_K^\times) \cap \mathcal{F} = \{0\}$, and so $\frac{\beta}{\alpha} \in \mu_K$.

Proposition 3.9. $\mu(S_1) = 2^r(2\pi)^s R_K$

Proof. We have $\overline{K^\times} = (\mathbb{R}^\times)^r \oplus (\mathbb{C}^\times)^s$. We further decompose as follows:

$$\overline{K^\times} \xrightarrow{\lambda \times (\operatorname{sgn} x_i) \times (\arg z_i)} \mathbb{R}^{r+s} \oplus (C_2)^r \oplus \mathbb{T}^s$$

Decomposing $\mathbb{R}^{r+s} = H \oplus \mathbb{R}$, the image of S_1 is $\mathcal{F} \times (-\infty, 0] \times (C_2)^r \times \mathbb{T}^s$. By change-of-variables, we have:

$$\mu(S_1) = \int_{\mathbb{T}^s} \int_{(C_2)^r} \int_{\mathcal{F}} \int_{-\infty}^0 e^t \; dt \; dh \; (d\sharp)^r \; (d\theta)^s = 2^r(2\pi)^s R_K$$

□

Now, we have what we need to begin counting ideals. The following theorem is due to Lang:

Theorem 3.10.

$$\sharp\{\mathfrak{a} \leq \mathcal{O}_K \mid \mathfrak{N}(\mathfrak{a}) \leq t\} = \left(\frac{2^r(2\pi)^s R_K \; \sharp(Cl_K)}{\mu(\mathcal{O}_K) \; \sharp(\mu_K)}\right) t + O(t^{1-\frac{1}{n}})$$

Proof. We work one ideal class at a time. In the trivial class, all ideals are principal, so we count elements of $\mathcal{O}_K/\mathcal{O}_K^\times$. Since $\mathfrak{N}((a)) = |\det a|$, we may use the above construction. Noting that $tS_1 = S_{t^n}$, and dividing out by $\sharp(\mu_K)$, we have by Lemma 3.7:

$$\sharp\{a \in \mathcal{O}_K/\mathcal{O}_K^\times \mid \mathfrak{N}((a)) \leq t\} = \left(\frac{2^r(2\pi)^s R_K}{\mu(\mathcal{O}_K) \; \sharp(\mu_K)}\right) t + O(t^{1-\frac{1}{n}})$$

Let η a nontrivial ideal class, and pick an integral ideal $\mathfrak{a} \in \eta^{-1}$. For any integral $\mathfrak{b} \in \eta$, $\mathfrak{a}\mathfrak{b}$ is a principal ideal in $\mathfrak{a}$ – in fact, this gives a 1-1 correspondence $\mathfrak{b} \in \eta \leftrightarrow a \in \mathfrak{a}/\mathcal{O}_K^\times$. We use the same workup as above, noting that $\mu(\mathfrak{a}) = \mathfrak{N}(\mathfrak{a})\mu(\mathcal{O}_K)$, and write:

$$\begin{aligned}\sharp\{\mathfrak{b} \in \eta \mid \mathfrak{N}(\mathfrak{b}) \leq t\} &= \left(\frac{2^r(2\pi)^s R_K}{\mu(\mathfrak{a}) \; \sharp(\mu_K)}\right) \mathfrak{N}(\mathfrak{a}) t + O(t^{1-\frac{1}{n}}) \\ &= \left(\frac{2^r(2\pi)^s R_K}{\mu(\mathcal{O}_K) \; \sharp(\mu_K)}\right) t + O(t^{1-\frac{1}{n}})\end{aligned}$$

Summing over ideal classes, we obtain the desired formula. □

Theorem 3.11 (Class Number Formula)**.** *There exists a constant c_K so that $\zeta_K(s) - \dfrac{c_K}{s-1}$ admits an analytic continuation to $\Re(s) > 1 - \epsilon$. This constant is:*

$$c_K := \frac{2^r(2\pi)^s R_K \; \sharp(Cl_K)}{\mu(\mathcal{O}_K) \; \sharp(\mu_K)}$$

Proof. Let $a_k := \sharp\{\mathfrak{a} \le \mathcal{O}_K \mid \mathfrak{N}(\mathfrak{a}) = k\}$. Then we have

$$\zeta_K(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}$$

We now write:

$$\begin{aligned}
\zeta_K(s) - \frac{c_K}{s-1} &= \sum_{k=1}^{\infty} \frac{a_k}{k^s} - \int_1^{\infty} \frac{c_K}{x^s} dx \\
&= \sum_{k=1}^{\infty} \frac{c_K}{k^s} - \int_1^{\infty} \frac{c_K}{x^s} dx + \sum_{k=1}^{\infty} \frac{a_k - c_K}{k^s} \\
&= c_K \sum_{k=1}^{\infty} \int_k^{k+1} \left(\frac{1}{k^s} - \frac{1}{x^s} \right) dx + \sum_{k=1}^{\infty} \frac{a_k - c_K}{k^s}
\end{aligned}$$

We have

$$\int_k^{k+1} \left| \left(\frac{1}{k^s} - \frac{1}{x^s} \right) \right| dx \le \frac{1}{k^s} - \frac{1}{(k+1)^s} = \frac{sk^{s-1} + O(k^{s-2})}{k^{2s} + k^s} = \frac{s + O(\frac{1}{k})}{k^{s+1} + k}$$

so that the first sum converges on $\Re(s) > 0$, and by Theorem 3.10 we have

$$\frac{a_k - c_K}{k^s} < \frac{Ck^{-\frac{1}{n}}}{k^s} = \frac{C}{k^{s+\frac{1}{n}}}$$

so that the second sum converges on $\Re(s) > 1 - \frac{1}{n}$. □

Remark 3.12. Theorem 3.10 can be thought of as a version of the Riemann-Roch theorem for Arakelov divisors of $\mathcal{O}_K$ – see Chapter III, section 3 in [N].

4. The Prime Ideal Theorem

We set our sights on the Prime Ideal Theorem. Consider the following function:

$$\theta_K(N) = \sum_{\substack{0 \ne \mathfrak{p} \le \mathcal{O}_K \\ \mathfrak{p} \text{ prime} \\ \mathfrak{N}(\mathfrak{p}) \le N}} \log \mathfrak{N}(\mathfrak{p})$$

This is closely related to the prime-counting function. Our strategy is to show that $N \sim \theta_K(N) \sim \pi_K(N) \log N$ – this immediately implies the Prime Ideal Theorem. First, we show:

Proposition 4.1. $\theta_K(N) = O(N)$

Proof. We proceed in two steps. First, we reduce to the case $K = \mathbb{Q}$, by proving $\theta_K(N) = O(\theta_{\mathbb{Q}}(N))$. Let $[K : \mathbb{Q}] = k$. By Proposition 2.3, every prime $\mathfrak{q} \le \mathcal{O}_K$ corresponds to an integral prime p, so that $\mathfrak{N}(\mathfrak{q}) \le p^k$; further, no more than k primes in $\mathcal{O}_K$ correspond to p. We have:

$$\theta_K(N) = \sum_{\substack{\mathfrak{p} \le \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{p}) \le N}} \log \mathfrak{N}(\mathfrak{p}) \le \sum_{\substack{p \in \mathbb{Z} \\ p \le N}} \sum_{\substack{\mathfrak{p} \le \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{p}) \le N \\ \mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}}} k \log p \le k^2 \sum_{\substack{p \in \mathbb{Z} \\ p \le N}} \log p = k^2 \theta_{\mathbb{Q}}(N)$$

Now, we claim that $\theta_{\mathbb{Q}}(N) = O(N)$. Note that if n even,

$$\theta_{\mathbb{Q}}(n) - \theta_{\mathbb{Q}}\left(\tfrac{n}{2}\right) = \sum_{\frac{n}{2} < p \le n} \log p = \log \left(\prod_{\frac{n}{2} < p \le n} p \right)$$

$$\prod_{\frac{n}{2} < p \le n} p \;\Big|\; \binom{n}{\frac{n}{2}} \le \sum_k \binom{n}{k} = (1+1)^n = 2^n$$

If n odd, then possibly $\lceil \frac{n}{2} \rceil$ is prime; this will not divide $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. We then write for general n:

$$\prod_{\lfloor \frac{n}{2} \rfloor < p \leq n} p \;\Bigg|\; \binom{n}{\lfloor \frac{n}{2} \rfloor} \left\lceil \frac{n}{2} \right\rceil \leq 2^n \left\lceil \frac{n}{2} \right\rceil$$

$$\theta_{\mathbb{Q}}(n) - \theta_{\mathbb{Q}}\left(\lfloor \tfrac{n}{2} \rfloor\right) \leq n \log 2 + \log \lceil \tfrac{n}{2} \rceil \leq n(\log 2 + 1)$$

$$\theta_{\mathbb{Q}}(n) = \sum_{i=0}^{\log_2 n} \theta_{\mathbb{Q}}\left(\lfloor \tfrac{n}{2^i} \rfloor\right) - \theta_{\mathbb{Q}}\left(\lfloor \tfrac{n}{2^{i+1}} \rfloor\right) \leq \sum_{i=0}^{\log_2 n} \frac{n(\log 2 + 1)}{2^i} \leq 2n(\log 2 + 1)$$

□

Lemma 4.2. *We have:*

$$\int_1^\infty \frac{\theta_K(x) - x}{x^2} dx < \infty$$

To prove this, we need the following theorem from harmonic analysis:

Theorem 4.3. *Let f a bounded measurable function on $\mathbb{R}_{\geq 0}$. The Laplace transform*

$$\mathcal{L}f(z) := \int_0^\infty f(t) e^{-zt}\ dt$$

is holomorphic on $\{z \in \mathbb{C} \mid \Re(z) > 0\}$. If this admits an analytic continuation to an open set containing $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$, then the improper integral $\displaystyle\int_0^\infty f(t)\ dt$ exists and is equal to $\mathcal{L}f(0)$.

A proof is given in [Z].

Proof of Lemma 4.2. By a change of variables, it suffices to compute

$$\int_0^\infty \frac{\theta_K(e^t)}{e^t} - 1\ dt$$

Note that the integrand is bounded, by Proposition 4.1. With the theorem above in mind, we seek the Laplace transform of this function. We have $\theta_K' = \displaystyle\sum_{\mathfrak{p} \leq \mathcal{O}_K} \delta_{\mathfrak{N}(\mathfrak{p})} \log \mathfrak{N}(\mathfrak{p})$ in the sense of distributions. We then recognize:

$$\Phi_K(s) = \int_1^\infty \frac{\theta_K'(x)}{x^s} dx$$

Using integration by parts, where we use $\Re(s) > 1$ and Proposition 4.1 to ensure $\dfrac{\theta_K(x)}{x^s}$ vanishes as $x \to \infty$, we write:

$$\Phi_K(s) = \int_1^\infty \frac{\theta_K'(x)}{x^s} dx = \int_1^\infty \frac{s\theta_K(x)}{x^{s+1}} dx = s \int_0^\infty \theta_K(e^t) e^{-st}\ dt$$

Thus, we compute

$$\mathcal{L}\left(\frac{\theta_K(e^t)}{e^t} - 1\right)(s) = \frac{\Phi_K(s+1)}{s+1} - \frac{1}{s}$$

By the Class Number Formula and our earlier meditations on Φ_K, this admits an analytic continuation over $\Re(s) > -\epsilon$. By Theorem 4.3, the integral converges. □

Prime Ideal Theorem. *Let K a number field, π_K be its prime-counting function. Then:*

$$\pi_K(N) \sim \frac{N}{\log N}$$

Proof. First, we claim that $\theta_K(N) \sim N$. Imagine not. Then either $\limsup \frac{\theta_K(N)}{N} > 1$ or $\liminf \frac{\theta_K(N)}{N} < 1$. Imagine $\limsup \frac{\theta_K(N)}{N} > 1$, so that for some $\lambda > 1$, there is a sequence $n_i \to \infty$ where $\theta_K(n_i) > \lambda n_i$. Since θ_K is monotone, we have $\theta_K(t) \geq \lambda n_i$ for all $t > n_i$, so that:

$$\int_{n_i}^{\lambda n_i} \frac{\theta_K(t) - t}{t^2} dt \geq \int_{n_i}^{\lambda n_i} \frac{\lambda n_i - t}{t^2} dt = \int_1^\lambda \frac{\lambda - t}{t^2} dt > 0$$

Summing over n_i, this contradicts Lemma 4.2. Likewise, if for some $\lambda < 1$, $n_i \to \infty$, we have $\theta_K(n_i) < \lambda n_i$, then

$$\int_{\lambda n_i}^{n_i} \frac{\theta_K(t) - t}{t^2} dt \leq \int_{\lambda n_i}^{n_i} \frac{\lambda n_i - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2} dt < 0$$

again contradicting Lemma 4.2.

Now, we claim that $\theta_K(N) \sim \pi_K(N) \log N$. Since

$$\theta_K(N) = \sum_{\substack{\mathfrak{p} \leq \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{p}) \leq N}} \log \mathfrak{N}(\mathfrak{p}) \leq \sum_{\substack{\mathfrak{p} \leq \mathcal{O}_K \\ \mathfrak{N}(\mathfrak{p}) \leq N}} \log N = \pi_K(N) \log N$$

we have $\frac{\theta_K(N)}{\pi_K(N) \log N} \leq 1$. On the other hand, we have

$$\begin{aligned} \theta_K(N) &= \theta_K(N^{1-\epsilon}) + \sum_{N^{1-\epsilon} < \mathfrak{p} \leq N} \log \mathfrak{N}(\mathfrak{p}) \\ &\geq \theta_K(N^{1-\epsilon}) + \sum_{N^{1-\epsilon} < \mathfrak{p} \leq N} \log N^{1-\epsilon} \\ &= \theta_K(N^{1-\epsilon}) + (1-\epsilon)\pi_K(N) \log N \end{aligned}$$

Therefore, we have

$$\lim_{N \to \infty} \frac{\theta_K(N)}{\pi_K(N) \log N} \geq \lim_{N \to \infty} \frac{\theta_K(N^{1-\epsilon})}{\pi_K(N) \log N} + \lim_{N \to \infty} \frac{(1-\epsilon)\pi_K(N) \log N}{\pi_K(N) \log N}$$

The second term is clearly $(1-\epsilon)$. The first term goes to zero, as $\theta_K(N^{1-\epsilon}) \sim N^{1-\epsilon}$, and as we already have $\frac{N}{\pi_K(N) \log N} \leq 1$, we must have $\frac{N^{1-\epsilon}}{\pi_K(N) \log N} \to 0$. Thus $\theta_K(N) \sim \pi_K(N) \log N$ as desired, so that $N \sim \pi_K(N) \log N$. $\square$

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