Note: If it leads to no confusion, italic i and e are used to represent imaginary unit and natural constant (Euler's number) respectively, instead of roman i and e.

1 Complex Number

Definition 1 (imaginary unit)**.** *Define **imaginary unit** i by $i^2 = -1$. Which could be written as $i = \sqrt{-1}$.*

So it's obvious that: for $n \in \mathbb{Z}^+$, $\sqrt{-n} = \sqrt{-1 \cdot n} = \sqrt{-1}\sqrt{n} = \sqrt{n}i$.

Example Solve $x^2 + x + 1 = 0$.

Using **Vieta's formula**, $x_1 = \dfrac{-1 + \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} = \dfrac{-1 + \sqrt{-3}}{2} = \dfrac{-1 + \sqrt{3}i}{2}$, it's the same that $x_2 = \dfrac{-1 - \sqrt{3}i}{2}$.

Because $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = 1$. This induces the periodic equalities: $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$, which hold for all integers k.

By combining real numbers and imaginary unit. We can expand $\mathbb{R}$.

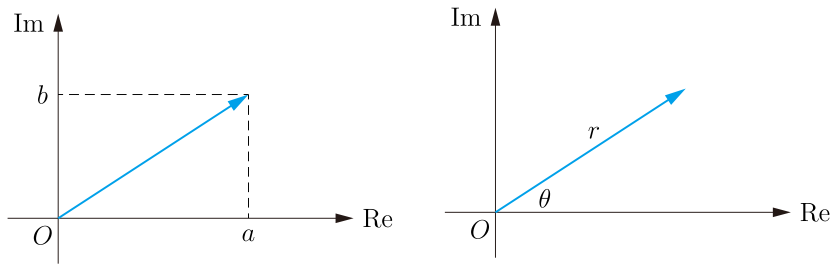
Definition 2. *z is complex number, if it is in the form $a + bi$, where a and b are real numbers. Here a is called the real part and b the imaginary part. The two parts are denoted by $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. The set of all complex numbers is denoted by $\mathbb{C}$.*

Letting b in $a + bi$ equals 0, we have a, a real number. So we can see that $\mathbb{R} \subset \mathbb{C}$. A real number can be regarded a The multipication of b and i, bi is commutative, therefore $a + bi$ may be written as $a + ib$. For example: $1 - 2i$, $3 + i4$, $-3 + i(-2)$.

1.1 Visualization

Since a complex number $a + bi$ is identified with an orderd pair (a, b), we can intepret complex number as a point in a two-dimensional Cartesian plane, with unique coordinates for each complex number and a unique vector pointing to each of them.

The plane, which plots vectors and points representing complex numbers, is called **complex plane** or **Argand diagram**. A complex plane have two perpendicular axes, real axis and imaginary axis.



$z = a + bi$ $\qquad$ $z = r(\cos\theta + i\sin\theta)$

As the figure shows, there're two ways to express a complex number: Cartesian coordinates (left) or polar coordinates (right). $z = r(\cos\theta + i\sin\theta)$ involves trigonometric function, so it can be called the **trigonometric form**.

In polar form, r is called the **modulus** (length) of the complex number, denoted by $|z|$; and θ is called the **argument**, denoted by $\mathrm{Arg}(z)$.
The two forms have the relation:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\dfrac{y}{x}(\pm\pi) \end{cases} .$$

To specify the calculation of θ: when $x > 0$, $\theta = \arctan\frac{y}{x}$; when $x < 0$ and $y \geqslant 0$, $\theta = \arctan\frac{y}{x} + \pi$; when $x < 0$ and $y < 0$, $\theta = \arctan\frac{y}{x} - \pi$. Also when $x = 0$, θ is obvious and the formula above is not applicable.

1.2 Euler's formula

Euler's formula declares that: for any $x \in \mathbb{R}$,

$$\exp ix = e^{ix} = \cos x + i\sin x\,.$$

This gives us another way two express a complex number:

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}\,.$$

Cartesian form	polar form	
	trigonometric	exponential
$a + bi$	$r(\cos\theta + i\sin\theta)$	$re^{i\theta}$

The three forms are equivalent, and can representing any complex number in complex plane. The conversions among them are given in section 1.1.

These are common complex numbers in different forms.

$$1 = \exp i0\,,$$

$$i = \exp\left(i\frac{\pi}{2}\right)\,.$$

$$-1 = \exp(i\pi)\,.$$

$$-i = \exp\left(i\frac{3\pi}{2}\right)\,.$$

And these property are obvious in geometrical view:

$$re^{i(\theta\pm\pi/2)} = \pm rie^{i\theta}\,,$$

$$re^{i(\theta\pm\pi)} = -re^{i\theta}\,.$$

1.3 Operations and relations

1.3.1 Equality

Two complex number $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ meanwhile $b_1 = b_2$.

1.3.2 Addition

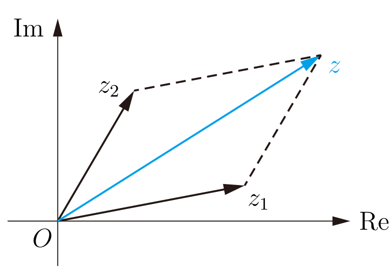
The sum of two complex number is a complex number, which real part and imaginary part is respectively the sum of real parts and the sum of imaginary ones of two complex numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i\,.$$

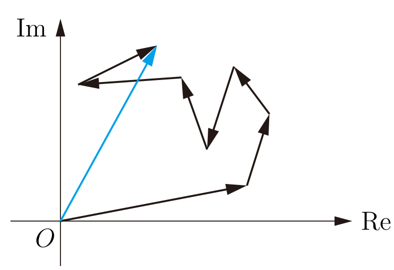
It's convenient to write $a+bi$ in the form of vector (a, b), and use the form regarding complex numbers as vectors.

$$(a, b) + (c, d) = (a + c, b + d)\,.$$

The visualization of addition of two complex numbers is just like that of two vectors.



$z = z_1 + z_2$



$z = \sum_i z_i$

1.3.3 Multipication

Under the rules of the distributive property, we can perform:

$$c \cdot (a + bi) = ca + cbi\,,$$

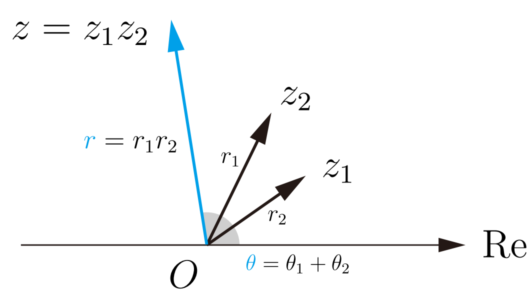
$$c \cdot (a, b) = (ca, cb)\,.$$

and the multipication of two complex numbers:

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + a \cdot di + bi \cdot c + bi \cdot di \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\,.\end{aligned}$$

It will be much more convenient if we use exponential form.

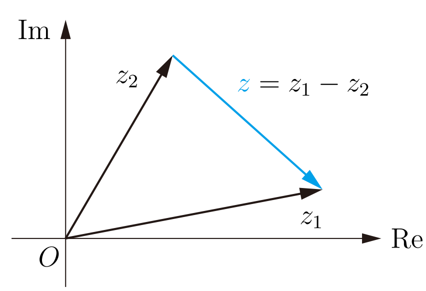
$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1 + \theta_2)}\,.$$



1.3.4 Subtraction

With addition and multipication defined, subtraction are easily understood, we can define $z_1 - z_2 = z_1 + (-1) \cdot z_2$:

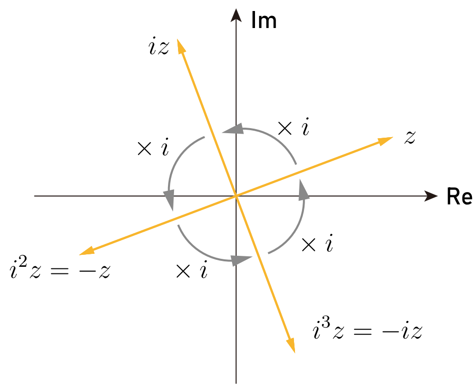
$$(a, b) - (c, d) = (a - c, b - d)\,,$$



These basic operations are very likely the same as real numbers and polynomials. Just use the basic law of arithmetic operations and notice that $i^2 = -1$.

1.3.5 Rotation

We already know from section 1.2 that $i = e^{i\pi/2}$, so if we multiply a complex number $z = re^{i\theta}$ by i: $re^{i\theta} \cdot e^{i\pi/2} = re^{i(\theta+\pi/2)}$. What it leads is the counter-clockwise rotation of original complex number by $90°$.



More generally, if we multiply a complex number z by another one $re^{i\theta}$, we rotate z counter-clockwise by the angle of θ then scale it by r times.

1.3.6 Conjugate

Definition 3. *For $z = a + bi$, $a - bi$ is call the (complex) conjugate of z. It's denoted by either $\bar{z}$ or z^*.*

Geometrically, **reflecting a complex number across real axis**, and you will get its conjugate. So when using exponential form, let $z = re^{i\theta}$, then $\bar{z} = re^{i(-\theta)}$ is obvious.

Let $z = a + bi$. It's easy to calculate that:

$$z\bar{z} = a^2 + b^2 = |z|^2 .$$

1.3.7 Reciprocal and division

Since $z\bar{z} = |z|^2$:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} .$$

Let $z = (a, b)$,

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) .$$

Division on complex numbers can be defined by $z_1/z_2 = z_1 \cdot z_2^{-1}$. And to look it in another way:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} \\ &= \frac{z_1\bar{z}_2}{|z_2|^2}\end{aligned}$$

Example $\dfrac{1+2i}{3+4i}$

$\dfrac{1+2i}{3+4i} = \dfrac{(1+2i)(3-4i)}{(3+4i)(3-4i)} = \dfrac{11+2i}{25} = \dfrac{11}{25} + \dfrac{2}{25}i.$

Now introducing a special equality:

$$\frac{1}{i} = -i\,.$$

The equality can be explained in two ways. First, $i = e^{i\pi/2}$, so $i^{-1} = \left(e^{i\pi/2}\right)^{-1} = e^{i(-\pi/2)} = -i$. And secondly, as we know in section 1.3.5, multiply any complex number by i is actually rotating it by 90° counter-clockwise in the complex plane. Then i^{-1}, that is, dividing by i, should be a 90° rotation clockwise. And the effect is exactly the same as $-i$, which stands for "first rotate 90° counter-clockwise then take its opposite".

Therefore, for any complex number:

$$\frac{z}{i} = -iz\,.$$

1.3.8 De Moivre's formula

Theorem 1. *De Moivre's formula (also known as **de Moivre's theorem**) states that: for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ it holds that*

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx\,.$$

This is obvious in exponential form:

$$(\cos x + i\sin x)^n = \left(e^{ix}\right)^n = e^{inx} = \cos nx + i\sin nx\,.$$

Example phasor $A\angle\theta \equiv Ae^{i\theta}$. Therefore $(A\angle\theta)^n = \left(Ae^{i\theta}\right)^n = A^n e^{in\theta} = A^n\angle n\theta$.

1.3.9 Roots of complex numbers

For any non-zero complex number $z = re^{i\theta} = z_0^n$, in order find its n-th root $z_0 = r_0 e^{i\theta_0}$, we write that:

$$re^{i\theta} = r_0 e^{i\theta_0}\,.$$

Two complex numbers $r_1e^{i\theta_1}$, $r_2e^{i\theta_2}$ are equal iff. $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ $(\pi \in \mathbb{Z})$.

Since $z = z_0^n = \left(r_0 e^{i\theta_0}\right)^n = r_0^n e^{in\theta_0}$,

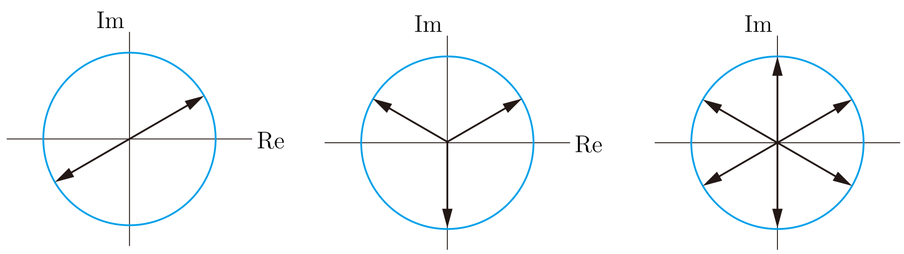
$$\begin{cases} r = r_0^n \\ \theta = n\theta_0 + 2k\pi \end{cases} \qquad k \in \mathbb{Z}\,,$$

Therefore we have $r_0 = \sqrt[n]{r}$, $\theta_0 = \dfrac{\theta - 2k\pi}{n} = \dfrac{\theta + 2k'\pi}{n}$, $k \in \mathbb{Z}$, $k' \in \mathbb{Z}$.

Conclusion 1. *For a non-zero complex number $z = re^{i\theta}$, its n-th root is*

$$z^{-n} = \sqrt[n]{r}\exp\left(i \cdot \frac{\theta + 2k\pi}{n}\right) = \sqrt[n]{r}\exp\left[i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \; k \in \mathbb{Z}\,.$$

Immediately we can see that all roots lies on the circle $|z| = \sqrt[n]{r}$ and are equally spaced every $2\pi/n$ radians, starting with argument θ/n.



square root   cube root   6-th root