1 基础知识

1.1 向量记法

向量可以看作是有序 $n-$元组, 亦可作为维度为 $1\times n$ 或 $m\times 1$ 的行/列向量. 列向量比行向量更常用, 但两者本质是等价的. 约定本文中未经说明的向量, 默认为列向量. 下面的记法都是常见的.

$$(v_1, v_2, \dots, v_n) \qquad \langle v_1, v_2, \dots, v_n \rangle \qquad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

1.2 导数

1.2.1 标量对标量求导

下面是常用的导数/偏导数记号:

$$\frac{df}{dx} \equiv \frac{\partial f}{\partial x} \equiv f_x \equiv f'_x \equiv \partial_x f$$

1.2.2 向量对标量求导

向量对标量求导, 即是对向量的每一个分量求导.

设 $\mathbf{F}$ 为 $\mathbb{R}^n$ 中的列向量, 其分量形式为 $\mathbf{F} = \begin{bmatrix} F_1 & F_2 & \cdots & F_n \end{bmatrix}^\top = \langle F_1, F_2, \dots, F_n \rangle$.

$$\frac{\partial \mathbf{F}}{\partial t} = \mathbf{F}_t = \begin{bmatrix} \partial_t F_1 \\ \partial_t F_2 \\ \vdots \\ \partial_t F_n \end{bmatrix} = \left\langle \frac{\partial F_1}{\partial t}, \frac{\partial F_2}{\partial t}, \dots, \frac{\partial F_n}{\partial t} \right\rangle .$$

1.2.3 标量对向量求导

向量对标量求导, 即是标量对向量的每一个分量求导, 并求转置. 向量在分母上结果就要求转置, 在分子上则直接求导, 不用转置. 这种叫做“分子布局” (numerator layout).

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix} .$$

1.2.4 向量对向量求导

$\mathbf{y}$ 为 m 维向量, $\mathbf{x}$ 为 n 维向量, $\dfrac{\partial \mathbf{y}}{\partial \mathbf{x}}$ 为一个 $m \times n$ 矩阵, 矩阵中 (i, j) 位置的元素为 $\dfrac{\partial y_i}{\partial x_j}$, 所以有:

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left[\frac{\partial y_i}{\partial x_j}\right]_{m\times n} = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \dfrac{\partial y_m}{\partial x_2} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

2 Geometric Interpretation of Line Integral and Surface Integral

2.1 Preparation I

Let $\mathbf{v} \in \mathbb{R}^n$; $\mathbf{v}$ can be expressed by its components: $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$, where v_i and $\mathbf{e}_i$ are respectively the i-th component and i-th base of $\mathbf{v}$. Parameterize $\mathbf{v}$ using t:

$$\mathbf{v}(t) = \sum_{i=1}^{n} v_i(t)\mathbf{e}_i\,,$$

$$\mathbf{v}(t+\Delta t) = \sum_{i=1}^{n} v_i(t+\Delta t)\mathbf{e}_i\,,$$

$$\Delta\mathbf{v} = \mathbf{v}(t+\Delta t) - \mathbf{v}(t) = \sum_{i=0}^{n} \big[(v_i(t+\Delta t) - v_i(t))\mathbf{e}_i\big]\,.$$

According to the mean value theorem, for $1 \leqslant i \leqslant n$, $\exists t_i^* \in (t, t+\Delta t)$, such that: $v_i(t+\Delta t) - v_i(t) = v_i'(t_i^*)\cdot\Delta t$. When $\Delta t \to 0$, $t+\Delta t \to t$, $t_i^* \to t$. Therefore

$$\lim_{\Delta t\to 0} \big[v_i(t+\Delta t) - v_i(t)\big] = v_i'(t)\Delta t\,,$$

$$\lim_{\Delta t\to 0} \Delta\mathbf{v} = \sum_{i=1}^{n} \big[v_i'(t)\Delta t\cdot\mathbf{e}_i\big] = \sum_{i=1}^{n} \big[v_i'(t)\mathbf{e}_i\big]\cdot\Delta t = \mathbf{v}'(t)\Delta t\,. \qquad (*)$$

By convention, when $\Delta t \to 0$, it can be written in the form of dt.

2.2 Line integral in scalar field

The definition of line integral on scalar field:

$$\int_C f(\mathbf{r})\,ds := \lim_{N\to\infty} \sum_{i=1}^{N} f(\mathbf{r}_i)\Delta s_i\,.$$

Parameterize the curve using $x = x(t), y = y(t), z = z(t), t \in [a, b]$, $\mathbf{r}(x, y, z)$ now becomes $\mathbf{r}(t)$. On the right-hand side, we have

$$\lim_{\Delta t \to 0} \sum_{i=1}^{N} f\big(\mathbf{r}(t)\big)\Delta s_i \,.$$

As we known from last section (equation $*$), when $\Delta t \to 0$, $\Delta s_i = |\mathbf{r}(t_i+\Delta t)-\mathbf{r}(t_i)| = |\mathbf{r}'(t_i)|\, dt$. So the integral

$$\int_C f(\mathbf{r})\, ds = \lim_{\Delta t \to 0} \sum_{i=1}^{N} f\big(\mathbf{r}(t)\big)|\mathbf{r}'(t_i)|\Delta t_i = \int_a^b f\big(\mathbf{r}(t)\big)|\mathbf{r}'(t)|\, dt \,.$$

2.3 Line integral on vector field

The integral of $\mathbf{F}(\mathbf{r})$ along an oriented curve $C \colon \mathbf{r}(t), t \in [a, b]$ is defined by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} := \lim_{N \to \infty} \sum_{i=1}^{n} \left[\mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i\right] \,.$$

Since $N \to \infty$, $\Delta\mathbf{r} \to \mathbf{r}'(t)\, dt$, therefore

$$\lim_{N \to \infty} \sum_{i=1}^{n} \big[\mathbf{F}\big(\mathbf{r}(t_i)\big) \cdot \mathbf{r}'(t_i)\big] dt = \int_a^b \big[\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\big]\, dt \,.$$

Here's another view:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}}\, ds \,.$$

Since $\hat{\mathbf{T}} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)|\, dt$, it's obvious that the second and third parts of the equation above are equivalent. It's common (especially in Physics) to see the notation $d\boldsymbol{\ell}$ or $d\mathbf{s}$. They are defined by $d\boldsymbol{\ell} \equiv d\mathbf{s} := \hat{\mathbf{T}}\, ds$.

Preparation II

Given a plane area A, define $\mathbf{A} := |A|\mathbf{n}$. Here, $|A|$ (or just A) is the area of the plane, $\mathbf{n}$ is the unit normal vector of the plane. $\mathbf{A}$ or $\mathbf{B}$ represents the area of the plane and the direction it towards.

Consider two area in space $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cdot \mathbf{B} = |A||B|\mathbf{n}_A \cdot \mathbf{n}_B = |A||B| \cos\theta$, where θ is the angle between A and B. So it's obvious that $\cos\theta = \hat{\mathbf{A}} \cdot \hat{\mathbf{B}}$. The projection of A's area on B can be expressed as

$$\operatorname{proj}_B A = A_{\parallel B} = \mathbf{A} \cdot \hat{\mathbf{B}} \,.$$

Note that $\hat{\mathbf{B}}$ stands for "unit vector in the same direction of $\mathbf{B}$".

Surface integral on scalar field

The definition of surface integral on a scalar field is given below:

$$\iint_\Omega f(\mathbf{r})\,dS \coloneqq \lim_{N\to\infty}\sum_{i=1}^{N} f(\mathbf{r}_i)\Delta S_i\,.$$

Parameterize the surface:

$$\begin{cases} x = u \\ y = v \\ z = z(u,v) \end{cases} \qquad\qquad \mathbf{r}(x,y,z) \Rightarrow \mathbf{r}\big(u,v,z(u,v)\big)\,,$$

Consider a small patch $\Delta\mathbf{S}$. $\mathbf{r}_u \times \mathbf{r}_v$ is the normal vector of $\Delta\mathbf{S}$. Denote $\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$, then the angle between ΔS and Ouv is

$$\cos\theta = \frac{(\mathbf{r}_u \times \mathbf{r}_v)\cdot\hat{\mathbf{w}}}{|\mathbf{r}_u \times \mathbf{r}_v||\hat{\mathbf{w}}|}\,.$$

$\hat{\mathbf{w}} = (0,0,1)$, $|\hat{\mathbf{w}}| = 1$; $\mathbf{r}_u = (1,0,z_u)$, $\mathbf{r}_v = (0,1,z_v)$, so $(\mathbf{r}_u \times \mathbf{r}_v)\cdot\hat{\mathbf{w}} = \begin{vmatrix} 1 & 0 & z_u \\ 0 & 1 & z_v \\ 0 & 0 & 1 \end{vmatrix} = 1.$

$$\cos\theta = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\,.$$

$\Delta S \cdot \cos\theta = \Delta S_{uv} = \Delta A$, we have

$$\begin{aligned} \iint_\Omega f(\mathbf{r})\,dS &= \lim_{N\to\infty}\sum_{i=1}^{N} f(\mathbf{r}_i)\Delta S_i \\ &= \lim_{N\to\infty}\sum_{i=1}^{N} \mathbf{F}\big(\mathbf{r}(u_i,v_i)\big)\cdot\frac{\Delta A}{\cos\theta} \\ &= \lim_{N\to\infty}\sum_{i=1}^{N} \mathbf{F}\big(\mathbf{r}(u_i,v_i)\big)|\mathbf{r}_u \times \mathbf{r}_v|\Delta A \\ &= \iint_{\Omega_{uv}} \mathbf{F}(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \end{aligned}$$

2.4 Surface integral on vector field

The definition of surface integral on a vector field is given below. Note that $\Delta\mathbf{S} = \Delta S \cdot \hat{\mathbf{n}}$, in which $\hat{\mathbf{n}}$ is the unit normal vector of area ΔS. With the conclution we got from last section: $\cos\theta = |\mathbf{r}_u \times \mathbf{r}_v|^{-1}$, θ means the angle between ΔS and ΔA.

$$
\begin{aligned}
\iint_{\boldsymbol{\Omega}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} &= \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{S}_i \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \hat{\mathbf{n}}\, \Delta S_i \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, \Delta S_i \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, \frac{\Delta A_i}{\cos\theta} \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \Delta A_i \\
&= \iint_{\boldsymbol{\Omega}_{uv}} \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times r_v)\, du\, dv
\end{aligned}
$$

Here, $\mathbf{r}_u = \dfrac{\partial \mathbf{r}}{\partial u}$, $\mathbf{r}_v = \dfrac{\partial \mathbf{r}}{\partial v}$.

3 曲线积分和曲面积分计算

3.1 曲线积分

3.1.1 标量场

设标量场 $f(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}$ 中有一条参数化曲线 $\gamma : \mathbf{r}(t), t \in [a,b]$. 则沿该曲线积分的计算如下:

$$
\int_{\gamma} f(\mathbf{r})\, ds = \int_a^b f\big(\mathbf{r}(t)\big) |\mathbf{r}'(t)|\, dt\,.
$$

3.1.2 向量场

设向量场 $\mathbf{F}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ 中有一条定向曲线 $\boldsymbol{\gamma} : \mathbf{r}(t), t \in [a,b]$. 则沿该曲线积分得到标量:

$$
\int_{\boldsymbol{\gamma}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt\,.
$$

使用微分形式计算,

$$
\begin{aligned}
\int_\gamma \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} &= \int_\gamma F_1\,dx + F_2\,dy + F_3\,dz \\
&= \int_a^b F_1\big(\mathbf{r}(t)\big)\,dx(t) + F_2\big(\mathbf{r}(t)\big)\,dy(t) + F_3\big(\mathbf{r}(t)\big)\,dz(t) \\
&= \int_a^b (F_{1t}x_t + F_{2t}y_t + F_{3t}z_t)\,dt
\end{aligned}
$$

此处, $x_t = \dfrac{\partial x}{\partial t}$, $y_t = \dfrac{\partial y}{\partial t}$, $z_t = \dfrac{\partial z}{\partial t}$; F_{it} 表示参数化, 即 $F_{it} = F_i\big(\mathbf{r}(t)\big)$.

3.2 曲面积分

3.2.1 标量场

标量场 $f(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}$ 中有一个参数化曲面 $\Omega : \mathbf{r}\big(x(u,v), y(u,v), z(u,v)\big)$. 这个曲面上的积分计算方法如下:

$$
\iint_\Omega f(\mathbf{r})\,dS = \iint_{\Omega_{uv}} f\big(\mathbf{r}(u,v)\big) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\,dv
$$

3.2.2 向量场

向量场 $\mathbf{F}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ 中有一个可定向曲面 $\boldsymbol{\Omega} : \mathbf{r}\big(x(u,v), y(u,v), z(u,v)\big)$. 这个曲面上的积分方法如下:

$$
\iint_{\boldsymbol{\Omega}} \mathbf{F}(\mathbf{r}) \cdot \mathbf{S} = \iint_{\boldsymbol{\Omega}_{uv}} \mathbf{F}\big(\mathbf{r}(u,v)\big) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\,dv\,.
$$

记 $\mathbf{F}$ 参数化后, 各分量为: $P = F_1\big(\mathbf{r}(u,v)\big)$, $Q = F_2\big(\mathbf{r}(u,v)\big)$, R 同理. $\mathbf{r} = \langle x, y, z\rangle$, 参数化后, $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$, $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$. 由标量三重积的性质, 上面的计算公式可以等价为:

$$
\iint_{\boldsymbol{\Omega}_{uv}} \mathbf{F}\big(\mathbf{r}(u,v)\big) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\,dv = \iint_{\boldsymbol{\Omega}_{uv}} \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\,dv \tag{$*$}
$$

使用微分形式计算向量场中的曲面积分, 注意楔积性质 $dx \wedge dy = -dy \wedge dx$ 以及 $dx \wedge dx = 0$, 同时使用全微分公式 $dx = x_u\,du + x_v\,dv$.

$$
\iint_{\boldsymbol{\Omega}} \mathbf{F}(\mathbf{r}) \cdot \mathbf{S} = \iint_{\boldsymbol{\Omega}} {\color{red} F_1\,dy \wedge dz} + {\color{blue} F_2\,dz \wedge dx} + {\color{olive} F_3\,dx \wedge dy}\,,
$$

参数化后:

$$
\begin{aligned}
F_1\,dy\wedge dz &= P\,(y_u\,du+y_v\,dv)\wedge(z_u\,du+z_v\,dv)\\
&= P(y_uz_u\,du\wedge du+y_uz_v\,du\wedge dv+y_vz_u\,dv\wedge du+y_vz_v\,dv\wedge dv)\\
&= P(y_uz_v\,du\wedge dv+y_vz_u\,dv\wedge du)=P(y_uz_v\,du\wedge dv-y_vz_u\,du\wedge dv)\\
&= P\begin{vmatrix} y_u & z_u\\ y_v & z_v\end{vmatrix}\,du\wedge\,dv\,,
\end{aligned}
$$

同理:

$$
F_2\,dz\wedge dx = Q\begin{vmatrix} z_u & x_u\\ z_v & x_v\end{vmatrix}\,du\wedge dv\,,
$$

$$
F_3\,dx\wedge dy = R\begin{vmatrix} x_u & y_u\\ x_v & y_v\end{vmatrix}\,du\wedge dv
$$

将三项相加, 即得到 $(*)$ 式.

4 梯度, 散度和旋度

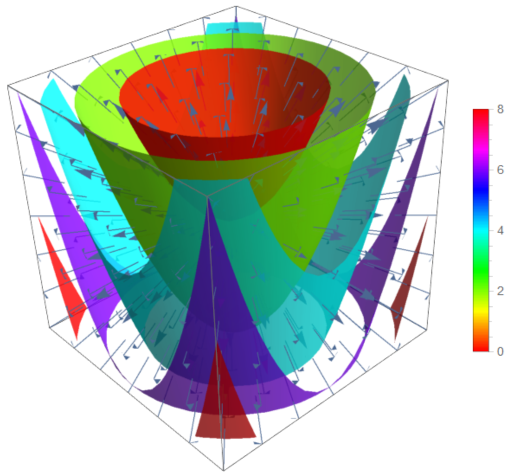
4.1 等高线与水平曲面

对于二元函数, $f(x,y)$ 和常数 c, $f(x,y)=c$ 的曲线被称为该二元函数的等高线 (contour line). 同理, 三元函数取常数 $f(x,y,z)=c$ 时获得的曲面被称为水平曲面 (level surface) 或等值曲面 (isosurface).

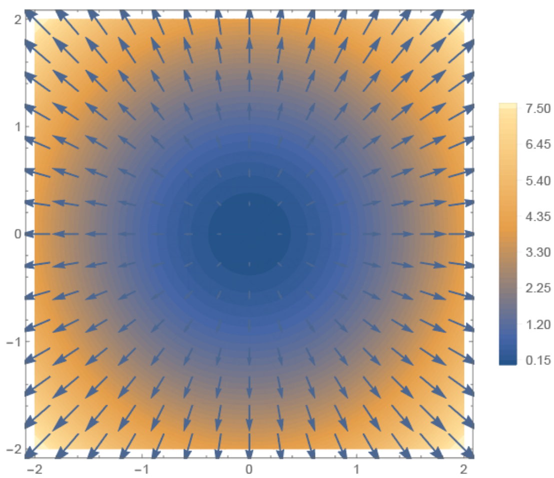
4.2 梯度

设 $f(x,y,z):\mathbb{R}^3\to\mathbb{R}$. 记 $\nabla f=\left(\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y},\dfrac{\partial f}{\partial z}\right)=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$. 也即: $\nabla=\dfrac{\partial}{\partial x}\mathbf{i}+\dfrac{\partial}{\partial y}\mathbf{j}+\dfrac{\partial}{\partial z}\mathbf{k}$.

对于空间中的函数 $f(x,y,z)$, ∇f 是一个 $\mathbb{R}^3$ 向量场. 且 ∇f 是水平曲面 $f(x,y,z)=c$ 的法向量. ∇f 指向函数值增加最快的方向; 梯度的大小为该方向的增长率, 即最大“斜率”.



对于平面内的二元函数 $f(x,y)$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ 是一个平面向量场, 是等高线的法向量, 且指向函数值增加最快的方向; 梯度的大小也为该方向的增长率, 即最大斜率.



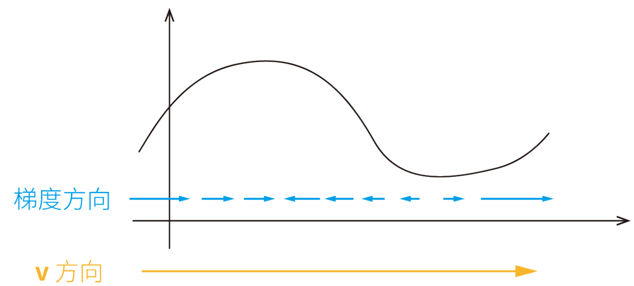
4.2.1 方向导数

借助梯度, 可以计算 $f:\mathbb{R}^n \to \mathbb{R}$ 在任意方向 $\mathbf{v}$ 的导数 $D_{\mathbf{v}}$.

$$D_{\mathbf{v}} f = \nabla f \cdot \hat{\mathbf{v}}\,.$$

如果 $\mathbf{v}$ 指向变化率最大的方向, 即和 ∇f 同向, 则计算得到 $D_{\mathbf{v}} f = \nabla f$. 反向, 即函数值下降最快的方向, 则得到 $D_{\mathbf{v}} f = -\nabla f$.

举例: 一元函数 $f(x)$ 只有一个走向, x 轴正向, 此处 $\mathbf{v}$ 即沿 x 轴正向. 梯度 $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} = \frac{\partial f}{\partial x}\hat{\mathbf{x}}$. 当函数上升时, 梯度方向 ∇f 和 $\mathbf{v}$ 方向一致, 所以方向导数 $D_{\mathbf{v}} f = \frac{\partial f}{\partial x}\mathbf{i} = \frac{\partial f}{\partial x} = \frac{df}{dx}$. 当函数下降时, 梯度方向和所求导数方向相反, 方向导数 $D_{\mathbf{v}} f = -\nabla f = -\frac{df}{dx}$.



4.2.2 柱坐标系中的梯度算子

$$\nabla_{(r,\varphi,z)} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\right) = \left(\partial_r, \frac{\partial_\varphi}{r}, \partial_z\right)$$

4.2.3 球坐标系中的梯度算子

$$\nabla_{(r,\theta,\varphi)} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right) = \left(\partial_r, \frac{\partial_\theta}{r}, \frac{\partial_\varphi}{r\sin\theta}\right)$$

4.2.4 梯度定理

梯度定理是微积分基本定理在曲线积分中的推广.

定理 1. 设有 $\mathbb{R}^n$ 标量场 f 和其梯度场 ∇f. 则任意一条以点 $\mathbf{p}$ 为起点, $\mathbf{q}$ 为终点的曲线 $\gamma[\mathbf{p},\mathbf{q}]$ 上梯度场的积分都是固定的, 即积分与路径无关. 满足这种性质的场, 便称为保守场.

$$\int_{\gamma[\mathbf{p},\mathbf{q}]} \nabla f \cdot d\mathbf{r} = \left[f(\mathbf{r})\right]_{\mathbf{p}}^{\mathbf{q}} = f(\mathbf{q}) - f(\mathbf{p})\,.$$

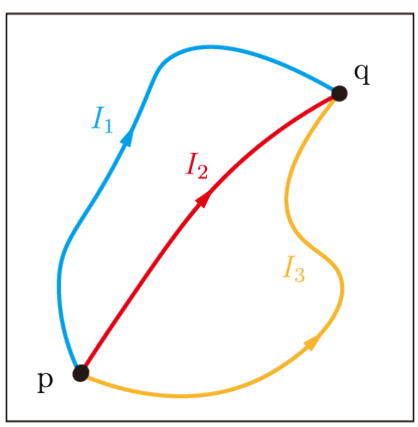


图 1: 梯度定理: 积分与路径无关, $I_1 = I_2 = I_3$