1   Derivatives

Fermat's theorem gives us relation between interior extrema and zero derivatives.

Theorem 1.1 (Fermat's theorem). *If $f : (a,b) \to \mathbb{R}$ attains local extremum at some interior point $c \in (a,b)$ and f has derivative at c, then $f'(c)$ must be zero.*

Proof. Assume c is maximum. If $f'(c) > 0$ or ∞, for every x in some $(c, c+\delta)$ we have $f(x) > f(c)$ [contradiction!]. If $f'(c) < 0$ or $-\infty$, for every x in some $(c-\delta, c)$ we have $f(x) > f(c)$ [contradiction!]. So $f'(c) = 0$. Proof is similar when c is minimum. $\blacksquare$

1.1   Mean value theorem

Theorem 1.2 (Rolle's theorem). *Assume real-valued function $f \in C[a,b] \cap D(a,b)$, and $f(a) = f(b)$, then there is $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on compact interval $[a,b]$, we know f attains maximum and minimum values on $[a,b]$. If either of them is interior point of $[a,b]$, then by Fermat's theorem the proof is done. If both of them are end points, then f is constant since $f(a) = f(b)$. $\blacksquare$

We use Rolle's theorem to prove mean value theorem (MVT).

Theorem 1.3 (Cauchy's mean value theorem). *Given $f, g \in C[a,b] \cap D(a,b)$, there is $c \in (a,b)$ such that*

$$\big[f(b) - f(a)\big]g'(c) = \big[g(b) - g(a)\big]f'(c). \tag{1}$$

Proof. Let $h(x) = \big[f(b) - f(a)\big]g(x) - \big[g(b) - g(a)\big]f(x)$, then (1) is equivalent to $h'(c) = 0$. The idea is to use Rolle's theorem on h. Note that $h(b) - h(a) = 0$, and indeed $h \in C[a,b] \cap D(a,b)$ since f and g do. $\blacksquare$

Let g to be $g(x) = x$ and we obtain the usual MVT. MVT and CMVT can be considered as extensions of Rolle's theorem.

1.2   Taylor's theorem

Theorem 1.4 (Taylor's theorem). *Assume $f \in D^n(\alpha, \beta) \cap C^{n-1}[\alpha, \beta]$ and $a \in [\alpha, \beta]$. For $x \in [\alpha, \beta]$ and $x \neq a$, there is $c \in (a, x)$ or (x, a) such that*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

We prove Taylor's theorem by the following extended theorem.

Theorem 1.5 (Taylor's theorem extended). *Assume $f, g \in D^n(\alpha, \beta) \cap C^{n-1}[\alpha, \beta]$ and $a \in [\alpha, \beta]$. For $x \in [\alpha, \beta]$ and $x \neq a$, there is $c \in (a, x)$ or (x, a) such that*

$$\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] g^{(n)}(c) = \left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k\right] f^{(n)}(c). \tag{2}$$

The key idea is to use CMVT. In order to do so, we have to make (2) to match the form $[F(*) - F(*)]G'(\cdot) = [G(*) - G(*)]F'(\cdot)$. Consider the Taylor polynomial:

$$f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}.$$

If we keep to think x as the variable, this polynomial will never produce $f(x)$. Instead if we consider the expansion point a as variable, then expansion at x will give us $f(x)$, and expansion at a will give us the summation part in the square bracket.

Proof. Without loss of generality, assume $x > a$. Let $F(t) = \sum_{k=0}^{n-1} \dfrac{f^{(k)}(t)}{k!}(x-t)^k$ and

$G(t) = \sum_{k=0}^{n-1} \dfrac{g^{(k)}(t)}{k!}(x-t)^k$. Since $f, g \in D^n(\alpha, \beta) \cap C^{n-1}[\alpha, \beta]$, we have $F, G \in C^{n-1}[a, x] \cap$

$D^n(a, x)$. So apply CMVT to F and G, there is some $c \in (a, x)$ such that

$$[F(x) - F(a)]G'(c) = [G(x) - G(a)]F'(c) . \tag{3}$$

Now $F(x) = f(x)$, $G(x) = g(x)$, and we have

$$F(a) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k , \quad G(a) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k ,$$

$$F'(a) = \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1} , \quad G'(a) = \frac{g^{(n)}(a)}{(n-1)!}(x-a)^{n-1} .$$

Substitute into (3), we obtain (2). ∎

By taking $g(x) = (x-a)^n$, we get the usual Taylor's theorem (1.4) with Lagrange form of remainder.