# 1 Derivatives

## 1.1 Zero and non-zero derivatives

**Theorem 1.1** (non-zero derivatives)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $c$ an interior point of $(a,b)$. If $f'(c) > 0$ or $f'(c) = \infty$, then there is some $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$. For all $x \in (c, c+\delta)$ we have $f(x) > f(c)$ and for all $x \in (c-\delta, c)$ we have $f(x) < f(c)$. Similar statement holds for $f'(c) < 0$ or $f'(c) = -\infty$, with relation between $f(x)$ and $f(c)$ reversed.*

*Proof.* If $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$, there is $B(c, \delta) \subseteq (a,b)$ such that for all $x \in B(c,\delta)$, we have $\dfrac{f(x) - f(c)}{x - c} > 0$. ■

Fermat's theorem gives us relation between interior extrema and zero derivatives.

**Theorem 1.2** (Fermat's theorem)**.** *If $f\colon (a,b) \to \mathbb{R}$ attains local extremum at some interior point $c \in (a,b)$ and $f$ has derivative at $c$, then $f'(c)$ must be zero.*

*Proof.* Assume $c$ is maximum. If $f'(c) > 0$ or $\infty$, for every $x$ in some $(c, c+\delta)$ we have $f(x) > f(c)$ [contradiction!]. If $f'(c) < 0$ or $-\infty$, for every $x$ in some $(c-\delta, c)$ we have $f(x) > f(c)$ [contradiction!]. So $f'(c) = 0$. Proof is similar when $c$ is minimum. ■

## 1.2 Mean value theorem

**Theorem 1.3** (Rolle's theorem)**.** *Assume real-valued function $f \in C[a,b] \cap D(a,b)$, and $f(a) = f(b)$, then there is $c \in (a,b)$ such that $f'(c) = 0$.*

*Proof.* Since $f$ is continuous on compact interval $[a,b]$, we know $f$ attains maximum and minimum values on $[a,b]$. If either of them is interior point of $[a,b]$, then by Fermat's theorem the proof is done. If both of them are end points, then $f$ is constant since $f(a) = f(b)$. ■

We use Rolle's theorem to prove mean value theorem (MVT).

**Theorem 1.4** (Cauchy's mean value theorem)**.** *Given $f, g \in C[a,b] \cap D(a,b)$, there is $c \in (a,b)$ such that*

$$\big[f(b) - f(a)\big]g'(c) = \big[g(b) - g(a)\big]f'(c)\,. \tag{1}$$

*Proof.* Let $h(x) = \big[f(b) - f(a)\big]g(x) - \big[g(b) - g(a)\big]f(x)$, then (1) is equivalent to $h'(c) = 0$. The idea is to use Rolle's theorem on $h$. Note that $h(b) - h(a) = 0$, and indeed $h \in C[a,b] \cap D(a,b)$ since $f$ and $g$ do. ■

Let $g$ to be $g(x) = x$ and we obtain the usual MVT. MVT and CMVT can be considered as extensions of Rolle's theorem.

## 1.3 Taylor's theorem

**Theorem 1.5** (Taylor's theorem)**.** *Assume $f \in D^n(\alpha,\beta) \cap C^{n-1}[\alpha,\beta]$ and $a \in [\alpha,\beta]$. For $x \in [\alpha,\beta]$ and $x \neq a$, there is $c \in (a,x)$ or $(x,a)$ such that*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n\,.$$

We prove Taylor's theorem by the following extended theorem.

**Theorem 1.6** (Taylor's theorem extended)**.** *Assume $f, g \in D^n(\alpha,\beta) \cap C^{n-1}[\alpha,\beta]$ and $a \in [\alpha,\beta]$. For $x \in [\alpha,\beta]$ and $x \neq a$, there is $c \in (a,x)$ or $(x,a)$ such that*

$$\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] g^{(n)}(c) = \left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k\right] f^{(n)}(c)\,. \tag{2}$$

The key idea is to use CMVT. In order to do so, we have to make (2) to match the form $[F(*) - F(*)]G'(\cdot) = [G(*) - G(*)]F'(\cdot)$. Consider the Taylor polynomial:

$$f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\,.$$

If we keep to think $x$ as the variable, this polynomial will never produce $f(x)$. Instead if we consider the expansion point $a$ as variable, then expansion at $x$ will give us $f(x)$, and expansion at $a$ will give us the summation part in the square bracket.

*Proof.* Without loss of generality, assume $x > a$. Let $F(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k$ and $G(t) = \sum_{k=0}^{n-1} \frac{g^{(k)}(t)}{k!}(x-t)^k$. Since $f, g \in D^n(\alpha,\beta) \cap C^{n-1}[\alpha,\beta]$, we have $F, G \in C^{n-1}[a,x] \cap D^n(a,x)$. So apply CMVT to $F$ and $G$, there is some $c \in (a,x)$ such that

$$[F(x) - F(a)]G'(c) = [G(x) - G(a)]F'(c)\,. \tag{3}$$

Now $F(x) = f(x)$, $G(x) = g(x)$, and we have

$$F(a) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k\,, \quad G(a) = \textstyle\sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(x-a)^k\,,$$
$$F'(a) = \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}\,, \quad G'(a) = \frac{g^{(n)}(a)}{(n-1)!}(x-a)^{n-1}\,.$$

Substitute into (3), we obtain (2). ∎

By taking $g(x) = (x-a)^n$, we get the usual Taylor's theorem (1.5) with Lagrange form of remainder. Take $G(t) = t$, we get Cauchy form of remainder

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}(x-a)\,.$$

## 1.4 Limit of derivative

**Theorem 1.7** (Limit of derivative (one-sided))**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $c \in (a,b)$. If $f \in C[c, c+h] \cap D(c, c+h)$ for some $h > 0$, and $\lim_{x\to c+} f'(x)$ exists, then $f'_+(c)$ exists and equals this limit. Similar statement holds for left limit and left derivative.*

*Proof.* Let $\lim_{x\to c+} f'(x) = L$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for } x \in (c, c+\delta)\,.$$

We have

$$f'_+(c) = \lim_{x\to c+} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c+} f'(\xi)\,,$$

where $\xi \in (c,x) \subseteq (c, c+\delta)$, so $|f(\xi) - L| < \varepsilon$. Therefore $\lim_{x\to c+} f'(\xi) = L$. ∎

Intuitively, since $\xi \in (c, x)$, we have $\xi \to c$ as $x \to c$, so $\lim_{x \to c+} f'(\xi)$ can be replaced by $\lim_{\xi \to c+} f'(\xi)$. Another way of seeing this is to consider $\xi$ as a function of $x$ (since the choice of $\xi$ depends on $x$). So $f'(\xi)$ actually means $f'(\xi(x))$. Now apply the limit rule of composite functions, note that: (1) $\lim_{x \to c+} f'(x) = L$; (2) $\lim_{x \to c+} \xi(x) = c$ (and approaches $c$ from above); (3) $\xi$ never attains $c$ when $x$ is near $c$. Therefore $\lim_{x \to c+} f'(\xi(x)) = \lim_{x \to c+} f'(x) = L$.

If limit of derivative exists for both sides, that is limit of derivative exists, then the derivative at this point exists and equals this limit.