Functions of bounded variation

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1 BV functions

Theorem 1.1. *Suppose $f, g \in \mathrm{BV}[a,b]$, then $f+g, f-g, f \cdot g \in \mathrm{BV}[a,b]$. If g is bounded away from zero, that is $|f| > m$ for some positive m, then $f/g \in \mathrm{BV}[a,b]$.*

Proof. ■

Monotonic functions is of bounded variation.

Theorem 1.2. *If $f\colon [a,b] \to \mathbb{R}$ is monotonic, then $f \in \mathrm{BV}[a,b]$.*

Proof. ■

Continuous functions with bounded derivatives in the interior are of bounded variation. But derivative of a BV function is not necessary bounded. See examples below.

Theorem 1.3. *If $f \in C[a,b]$, and has bounded derivatives on (a,b), that is, $|f'(x)| \leqslant M$ for some $M > 0$, then $f \in \mathrm{BV}[a,b]$.*

Proof. ■

BV functions must be bounded.

Theorem 1.4. *If $f \in \mathrm{BV}[a,b]$, then f is bounded on $[a,b]$.*

Proof. ■

Example (BV functions)**.**

- *$f(x) = x^{1/3}$, then $f \in \mathrm{BV}[-a,a]$ for arbitrary finite a, since f is monotonic on $\mathbb{R}$ (theorem 1.2). Note that $f'(0) = \infty$*
- *$f(x) = x^2 \sin(1/x)$ and $f(0) = 0$. Then $f \in \mathrm{BV}[0,1]$ because of theorem 1.3*

Example (non-BV functions)**.**

- *$f(x) = x\cos(\pi/2x)$ and $f(0) = 0$. Then $f \notin \mathrm{BV}[0,1]$. Consider partition $P = \{0, 1/2n, 1/(2n-1), \ldots, 1/2, 1\}$, as $n \to \infty$, the sum $\sum_{k=1}^{2n} |\Delta f_k| \to \infty$*
- *$f(x) = 1/x$ and $f(0) = 0$. Then $f \notin \mathrm{BV}[0,1]$ since f is not bouned (theorem 1.4)*

2 Total variation

For a BV function f on $[a,b]$, and an arbitrary partition $P = \{x_0 < x_1 < \cdots < x_n\}$, by definition the sum $\sum_{k=1}^n |\Delta f_k|$ is bounded from above (here $\Delta f_k = f(x_k) - f(x_{k-1})$). Therefore for all such sums there must be a least upper bound.

Definition 2.1 (Total variation)**.** *If $f \in \mathrm{BV}[a,b]$, define total variation of f on $[a,b]$ by:*

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_P |\Delta f| \,,$$

where $\sum_P |\Delta f|$ for a partition $P = \{x_0, \ldots, x_n\}$ is the sum $\sum_{k=1}^n |\Delta f_k|$.

Alternative notation: $V(f, [a,b])$.

From definition, we know immediately $V(f) \geqslant 0$, and that $V(f) = 0$ iff. f is constant.

As a functional on real-valued function space, there are inequalities for total variations and function operations.

Theorem 2.1. *Assume $f, g \in \mathrm{BV}[a,b]$, then:*

- $V(f \pm g) \leqslant V(f) + V(g)$
- $V(f \cdot g) \leqslant V(f) \cdot \sup |g| + \sup |f| \cdot V(g)$
- *suppose* $|f| \geqslant M > 0$*, then* $V(1/f) \leqslant V(f)/M^2$

This theorem shows that $\mathrm{BV}[a,b]$, with addition and scalar multiplication, is a vector space, because:

1. $0 \in \mathrm{BV}[a,b]$
2. if $f, g \in \mathrm{BV}[a,b]$, then $f + g \in \mathrm{BV}[a,b]$
3. if $f \in \mathrm{BV}[a,b]$, then $\lambda f \in \mathrm{BV}[a,b]$ for every $\lambda \in \mathbb{R}$

The interval can be divided, and total variation satisfies additive property.

Theorem 2.2 (Additive property)**.** *If $f \in BV[a,b]$, and $c \in (a,b)$, then $f \in BV[a,c] \cap BV[c,b]$ and we have:*

$$V_a^b(f) = V_a^c(f) + V_c^b(f) \,.$$

If we keep f and left endpoint a fixed, consider right endpoint as a variable, we get a new function.

Theorem 2.3. *Let $f \in \mathrm{BV}[a,b]$ and define $V(x)$ to be $V_a^x(f)$ for $x \in (a,b]$, and define $V(a) = 0$. Then:*

- *V is increasing on $[a,b]$*
- *$V - f$ is increasing on $[a,b]$*

This immediately gives us:

Theorem 2.4. *$f \in \mathrm{BV}[a,b]$, iff. f can be expressed as $g - h$ where g, h are increasing or strictly increasing.*

For continuous functions f, it corresponding V function's continuity depends on f.

Theorem 2.5. *Let $\sum_P |\Delta f|$ to denote the sum $\sum_{k=1}^n$*

Theorem 2.6. *Suppose $f \in \mathrm{BV}[a,b]$, then f is continuous at $x \in [a,b]$ iff. V is continuous at x.*

Proof. ∎

3 Rectifiable curves

Let path $f\colon [a,b] \to \mathbb{R}^n$ and $P = \{x_0, \dots, x_n\} \in \mathcal{P}[a,b]$, and define $\Lambda(f,P)$ to be the sum $\sum_{k=1}^{n} \|f(x_k) - f(x_{k-1})\|$.

Definition 3.1 (Rectifiable curves)**.** *If $\{\Lambda(f,P) \colon P \in \mathcal{P}[a,b]\}$ is bounded, then path f is said to be rectifiable and define its arc length by:*

$$\Lambda_a^b(f) = \sup_{P \in \mathcal{P}[a,b]} \Lambda(f,P)\,.$$

For vector $x = (x_1, \dots, x_n)$ we have

$$|x_i| \leqslant \|x\| \leqslant |x_1| + \cdots + |x_n|\,.$$

This fact is used to prove the following similar relation between arc length and toatl variation.

Theorem 3.1. *Let $f = (f_1, \dots, f_n)$ be a path on $[a,b]$. Then f is rectifiable iff. each component f_k is of bounded variation. And we have:*

$$V_a^b(f_k) \leqslant \Lambda_a^b(f) \leqslant V_a^b(f_1) + \cdots + V_a^b(f_n)\,. \tag{1}$$

Like total variation, there are additive property for arc length.

Theorem 3.2 (Additive property)**.** *If f is rectifiable on $[a,b]$, and $c \in (a,b)$, then we have*

$$\Lambda_a^b(f) = \Lambda_a^c(f) + \Lambda_c^b(f)\,.$$

Like total variation, keep f fixed and consider right endpoint as the variable. Define $s(x) = \Lambda_a^x(f)$ and let $s(a) = 0$.

Theorem 3.3. *The function s is increasing and continuous on $[a,b]$. If there is no subinterval of $[a,b]$ on which f is constant, then s is strictly increasing.*