

1 基础知识

1.1 向量记法

向量可以看作是有序 n -元组, 亦可作为维度为 $1 \times n$ 或 $m \times 1$ 的行/列向量. 列向量比行向量更常用, 但两者本质是等价的. 约定本文中未经说明的向量, 默认为列向量. 下面的记法都是常见的.

$$(v_1, v_2, \dots, v_n) \quad \langle v_1, v_2, \dots, v_n \rangle \quad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

1.2 导数

1.2.1 标量对标量求导

下面是常用的导数/偏导数记号:

$$\frac{df}{dx} \equiv \frac{\partial f}{\partial x} \equiv f_x \equiv f'_x \equiv \partial_x f$$

1.2.2 向量对标量求导

向量对标量求导, 即是对向量的每一个分量求导.

设 \mathbf{F} 为 \mathbb{R}^n 中的列向量, 其分量形式为 $\mathbf{F} = \begin{bmatrix} F_1 & F_2 & \cdots & F_n \end{bmatrix}^\top = \langle F_1, F_2, \dots, F_n \rangle$.

$$\frac{\partial \mathbf{F}}{\partial t} = \mathbf{F}_t = \begin{bmatrix} \partial_t F_1 \\ \partial_t F_2 \\ \vdots \\ \partial_t F_n \end{bmatrix} = \left\langle \frac{\partial F_1}{\partial t}, \frac{\partial F_2}{\partial t}, \dots, \frac{\partial F_n}{\partial t} \right\rangle.$$

1.2.3 标量对向量求导

向量对标量求导, 即是标量对向量的每一个分量求导, 并求转置. 向量在分母上结果就要求转置, 在分子上则直接求导, 不用转置. 这种叫做“分子布局” (numerator layout).

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix}.$$

1.2.4 向量对向量求导

\mathbf{y} 为 m 维向量, \mathbf{x} 为 n 维向量, $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ 为一个 $m \times n$ 矩阵, 矩阵中 (i, j) 位置的元素为 $\frac{\partial y_i}{\partial x_j}$, 所以有:

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left[\frac{\partial y_i}{\partial x_j} \right]_{m \times n} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

2 Geometric Interpretation of Line Integral and Surface Integral

Preparation I

Let $\mathbf{v} \in \mathbb{R}^n$; \mathbf{v} can be expressed by its components: $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$, where v_i and \mathbf{e}_i are respectively the i -th component and i -th base of \mathbf{v} . Parameterize \mathbf{v} using t :

$$\mathbf{v}(t) = \sum_{i=1}^n v_i(t) \mathbf{e}_i,$$

$$\mathbf{v}(t + \Delta t) = \sum_{i=1}^n v_i(t + \Delta t) \mathbf{e}_i,$$

$$\Delta \mathbf{v} = \mathbf{v}(t + \Delta t) - \mathbf{v}(t) = \sum_{i=1}^n [(v_i(t + \Delta t) - v_i(t)) \mathbf{e}_i].$$

According to the mean value theorem, for $1 \leq i \leq n$, $\exists t_i^* \in (t, t + \Delta t)$, such that: $v_i(t + \Delta t) - v_i(t) = v_i'(t_i^*) \cdot \Delta t$. When $\Delta t \rightarrow 0$, $t + \Delta t \rightarrow t$, $t_i^* \rightarrow t$. Therefore

$$\lim_{\Delta t \rightarrow 0} [v_i(t + \Delta t) - v_i(t)] = v_i'(t) \Delta t,$$

$$\lim_{\Delta t \rightarrow 0} \Delta \mathbf{v} = \sum_{i=1}^n [v_i'(t) \Delta t \cdot \mathbf{e}_i] = \sum_{i=1}^n [v_i'(t) \mathbf{e}_i] \cdot \Delta t = \mathbf{v}'(t) \Delta t. \quad (*)$$

By convention, when $\Delta t \rightarrow 0$, it can be written in the form of dt .

2.1 Line integral in scalar field

The definition of line integral on scalar field:

$$\int_C f(\mathbf{r}) ds := \lim_{N \rightarrow \infty} \sum_{i=1}^N f(\mathbf{r}_i) \Delta s_i.$$

Parameterize the curve using $x = x(t), y = y(t), z = z(t), t \in [a, b]$, $\mathbf{r}(x, y, z)$ now

becomes $\mathbf{r}(t)$. On the right-hand side, we have

$$\lim_{\Delta t \rightarrow 0} \sum_{i=1}^N f(\mathbf{r}(t)) \Delta s_i.$$

As we known from last section (equation *), when $\Delta t \rightarrow 0$, $\Delta s_i = |\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)| = |\mathbf{r}'(t_i)| dt$. So the integral

$$\int_C f(\mathbf{r}) ds = \lim_{\Delta t \rightarrow 0} \sum_{i=1}^N f(\mathbf{r}(t)) |\mathbf{r}'(t_i)| \Delta t_i = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

2.2 Line integral on vector field

The integral of $\mathbf{F}(\mathbf{r})$ along an oriented curve $C: \mathbf{r}(t), t \in [a, b]$ is defined by:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} := \lim_{N \rightarrow \infty} \sum_{i=1}^n [\mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i].$$

Since $N \rightarrow \infty$, $\Delta \mathbf{r} \rightarrow \mathbf{r}'(t) dt$, therefore

$$\lim_{N \rightarrow \infty} \sum_{i=1}^n [\mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)] dt = \int_a^b [\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)] dt.$$

Here's another view:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{T}} ds.$$

Since $\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $ds = |\mathbf{r}'(t)| dt$, it's obvious that the second and third parts of the equation above are equivalent. It's common (especially in Physics) to see the notation $d\ell$ or ds . They are defined by $d\ell \equiv ds := \hat{\mathbf{T}} ds$.

Preparation II

Given a plane area A , define $\mathbf{A} := |A|\mathbf{n}$. Here, $|A|$ (or just A) is the area of the plane, \mathbf{n} is the unit normal vector of the plane. \mathbf{A} or \mathbf{B} represents the area of the plane and the direction it towards.

Consider two area in space \mathbf{A} and \mathbf{B} , $\mathbf{A} \cdot \mathbf{B} = |A||B|\mathbf{n}_A \cdot \mathbf{n}_B = |A||B|\cos\theta$, where θ is the angle between A and B . So it's obvious that $\cos\theta = \hat{\mathbf{A}} \cdot \hat{\mathbf{B}}$. The projection of A 's area on B can be expressed as

$$\text{proj}_B A = A_{\parallel B} = \mathbf{A} \cdot \hat{\mathbf{B}}.$$

Note that $\hat{\mathbf{B}}$ stands for "unit vector in the same direction of \mathbf{B} ".

2.3 Surface integral on scalar field

The definition of surface integral on a scalar field is given below:

$$\iint_{\Omega} f(\mathbf{r}) dS := \lim_{N \rightarrow \infty} \sum_{i=1}^N f(\mathbf{r}_i) \Delta S_i.$$

Parameterize the surface:

$$\begin{cases} x = u \\ y = v \\ z = z(u, v) \end{cases} \quad \mathbf{r}(x, y, z) \Rightarrow \mathbf{r}(u, v, z(u, v)),$$

Consider a small patch $\Delta \mathbf{S}$. $\mathbf{r}_u \times \mathbf{r}_v$ is the normal vector of $\Delta \mathbf{S}$. Denote $\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$, then the angle between ΔS and Ouv is

$$\cos \theta = \frac{(\mathbf{r}_u \times \mathbf{r}_v) \cdot \hat{\mathbf{w}}}{|\mathbf{r}_u \times \mathbf{r}_v| |\hat{\mathbf{w}}|}.$$

$$\hat{\mathbf{w}} = (0, 0, 1), |\hat{\mathbf{w}}| = 1; \mathbf{r}_u = (1, 0, z_u), \mathbf{r}_v = (0, 1, z_v), \text{ so } (\mathbf{r}_u \times \mathbf{r}_v) \cdot \hat{\mathbf{w}} = \begin{vmatrix} 1 & 0 & z_u \\ 0 & 1 & z_v \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

$$\cos \theta = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

$\Delta S \cdot \cos \theta = \Delta S_{uv} = \Delta A$, we have

$$\begin{aligned} \iint_{\Omega} f(\mathbf{r}) dS &= \lim_{N \rightarrow \infty} \sum_{i=1}^N f(\mathbf{r}_i) \Delta S_i \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}(u_i, v_i)) \cdot \frac{\Delta A}{\cos \theta} \\ &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}(u_i, v_i)) |\mathbf{r}_u \times \mathbf{r}_v| \Delta A \\ &= \iint_{\Omega_{uv}} \mathbf{F}(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| du dv \end{aligned}$$

2.4 Surface integral on vector field

The definition of surface integral on a vector field is given below. Note that $\Delta \mathbf{S} = \Delta S \cdot \hat{\mathbf{n}}$, in which $\hat{\mathbf{n}}$ is the unit normal vector of area ΔS . With the conclusion we got

from last section: $\cos \theta = |\mathbf{r}_u \times \mathbf{r}_v|^{-1}$, θ means the angle between ΔS and ΔA .

$$\begin{aligned}
\iint_{\Omega} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i) \cdot \Delta \mathbf{S}_i \\
&= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i) \cdot \hat{\mathbf{n}} \Delta S_i \\
&= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \Delta S_i \\
&= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \frac{\Delta A_i}{\cos \theta} \\
&= \lim_{N \rightarrow \infty} \sum_{i=1}^N \mathbf{F}(\mathbf{r}_i) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \Delta A_i \\
&= \iint_{\Omega_{uv}} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv
\end{aligned}$$

Here, $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$, $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.

3 曲线积分和曲面积分计算总结

3.1 曲线积分

3.1.1 标量场

设标量场 $f(\mathbf{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ 中有一条参数化曲线 $\gamma : \mathbf{r}(t), t \in [a, b]$. 则沿该曲线积分的计算如下:

$$\int_{\gamma} f(\mathbf{r}) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

3.1.2 向量场

设向量场 $\mathbf{F}(\mathbf{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ 中有一条定向曲线 $\gamma : \mathbf{r}(t), t \in [a, b]$. 则沿该曲线积分得到标量:

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

使用微分形式计算,

$$\begin{aligned}
\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s} &= \int_{\gamma} F_1 dx + F_2 dy + F_3 dz \\
&= \int_a^b F_1(\mathbf{r}(t)) dx(t) + F_2(\mathbf{r}(t)) dy(t) + F_3(\mathbf{r}(t)) dz(t) \\
&= \int_a^b (F_{1t}x_t + F_{2t}y_t + F_{3t}z_t) dt
\end{aligned}$$

此处, $x_t = \frac{\partial x}{\partial t}$, $y_t = \frac{\partial y}{\partial t}$, $z_t = \frac{\partial z}{\partial t}$; F_{it} 表示参数化, 即 $F_{it} = F_i(\mathbf{r}(t))$.

3.2 曲面积分

3.2.1 标量场

标量场 $f(\mathbf{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ 中有一个参数化曲面 $\Omega : \mathbf{r}(x(u, v), y(u, v), z(u, v))$. 这个面上的积分计算方法如下:

$$\iint_{\Omega} f(\mathbf{r}) dS = \iint_{\Omega_{uv}} f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

3.2.2 向量场

向量场 $\mathbf{F}(\mathbf{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ 中有一个可定向曲面 $\Omega : \mathbf{r}(x(u, v), y(u, v), z(u, v))$. 这个面上的积分方法如下:

$$\iint_{\Omega} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \iint_{\Omega_{uv}} \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv.$$

记 \mathbf{F} 参数化后, 各分量为: $P = F_1(\mathbf{r}(u, v))$, $Q = F_2(\mathbf{r}(u, v))$, R 同理. $\mathbf{r} = \langle x, y, z \rangle$, 参数化后, $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$. $\mathbf{r}_u = \langle x_u, y_u, z_u \rangle$, $\mathbf{r}_v = \langle x_v, y_v, z_v \rangle$. 由标量三重积的性质, 上面的计算公式可以等价:

$$\iint_{\Omega_{uv}} \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv = \iint_{\Omega_{uv}} \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du dv \quad (*)$$

使用微分形式计算向量场中的曲面积分, 注意楔积性质 $dx \wedge dy = -dy \wedge dx$ 以及 $dx \wedge dx = 0$, 同时使用全微分公式 $dx = x_u du + x_v dv$.

$$\iint_{\Omega} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \iint_{\Omega} F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

参数化后:

$$\begin{aligned}
 F_1 dy \wedge dz &= P(y_u du + y_v dv) \wedge (z_u du + z_v dv) \\
 &= P(y_u z_u du \wedge du + y_u z_v du \wedge dv + y_v z_u dv \wedge du + y_v z_v dv \wedge dv) \\
 &= P(y_u z_v du \wedge dv + y_v z_u dv \wedge du) = P(y_u z_v du \wedge dv - y_v z_u du \wedge dv) \\
 &= P \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} du \wedge dv,
 \end{aligned}$$

同理:

$$\begin{aligned}
 F_2 dz \wedge dx &= Q \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} du \wedge dv, \\
 F_3 dx \wedge dy &= R \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} du \wedge dv
 \end{aligned}$$

将三项相加, 即得到 (*) 式.

4 梯度, 散度和旋度

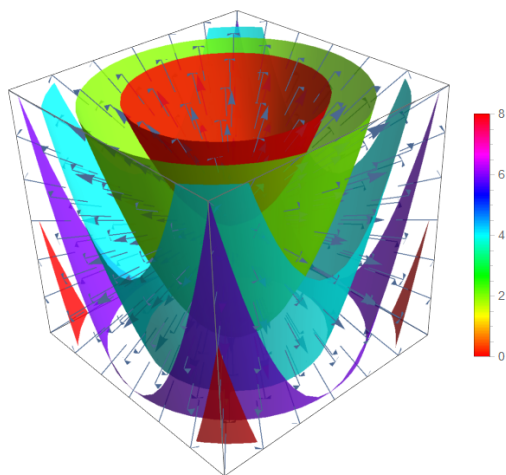
4.1 等高线与水平曲面

对于二元函数, $f(x, y)$ 和常数 c , $f(x, y) = c$ 的曲线被称为该二元函数的等高线 (contour line). 同理, 三元函数取常数 $f(x, y, z) = c$ 时获得的曲面被称为水平曲面 (level surface) 或等值曲面 (isosurface).

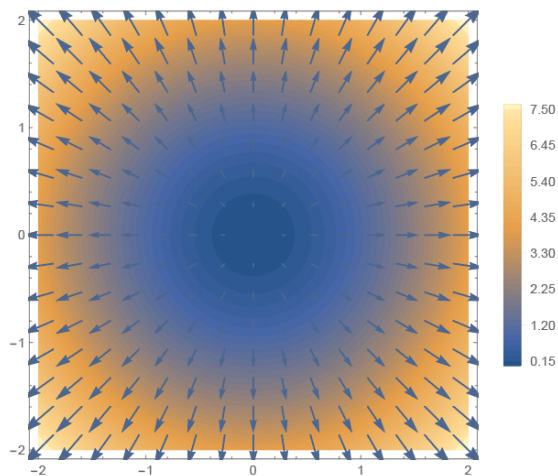
4.2 梯度

设 $f(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$. 记 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$. 也即:
 $\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$.

对于空间中的函数 $f(x, y, z)$, ∇f 是一个 \mathbb{R}^3 向量场. 且 ∇f 是水平曲面 $f(x, y, z) = c$ 的法向量. ∇f 指向函数值增加最快的方向; 梯度的大小为该方向的增长率, 即最大“斜率”.



对于平面内的二元函数 $f(x, y)$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ 是一个平面向量场, 是等高线的法向量, 且指向函数值增加最快的方向; 梯度的大小也为该方向的增长率, 即最大斜率.



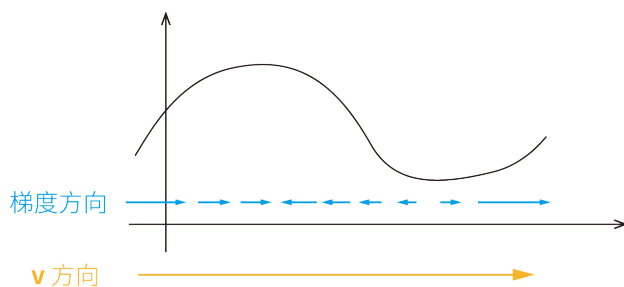
4.2.1 方向导数

借助梯度, 可以计算 $f: \mathbb{R}^n \rightarrow \mathbb{R}$ 在任意方向 \mathbf{v} 的导数 $D_{\mathbf{v}}f$.

$$D_{\mathbf{v}}f = \nabla f \cdot \hat{\mathbf{v}}.$$

如果 \mathbf{v} 指向变化率最大的方向, 即和 ∇f 同向, 则计算得到 $D_{\mathbf{v}}f = \nabla f$. 反向, 即函数值下降最快的方向, 则得到 $D_{\mathbf{v}}f = -\nabla f$.

举例: 一元函数 $f(x)$ 只有一个走向, x 轴正向, 此处 \mathbf{v} 即沿 x 轴正向. 梯度 $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} = \frac{\partial f}{\partial x} \hat{\mathbf{x}}$. 当函数上升时, 梯度方向 ∇f 和 \mathbf{v} 方向一致, 所以方向导数 $D_{\mathbf{v}}f = \frac{\partial f}{\partial x} \mathbf{i} = \frac{\partial f}{\partial x} = \frac{df}{dx}$. 当函数下降时, 梯度方向和所求导数方向相反, 方向导数 $D_{\mathbf{v}}f = -\nabla f = -\frac{df}{dx}$.



4.2.2 柱坐标系中的梯度算子

$$\nabla_{(r,\varphi,z)} = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right) = \left(\partial_r, \frac{\partial_\varphi}{r}, \partial_z \right)$$

4.2.3 球坐标系中的梯度算子

$$\nabla_{(r,\theta,\varphi)} = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right) = \left(\partial_r, \frac{\partial_\theta}{r}, \frac{\partial_\varphi}{r \sin \theta} \right)$$

4.2.4 梯度定理

梯度定理是微积分基本定理在曲线积分中的推广.

定理 1. 设有 \mathbb{R}^n 标量场 f 和其梯度场 ∇f . 则任意一条以点 \mathbf{p} 为起点, \mathbf{q} 为终点的曲线 $\gamma[\mathbf{p}, \mathbf{q}]$ 上梯度场的积分都是固定的, 即积分与路径无关. 满足这种性质的场, 便称为保守场.

$$\int_{\gamma[\mathbf{p}, \mathbf{q}]} \nabla f \cdot d\mathbf{r} = [f(\mathbf{r})]_{\mathbf{p}}^{\mathbf{q}} = f(\mathbf{q}) - f(\mathbf{p}).$$

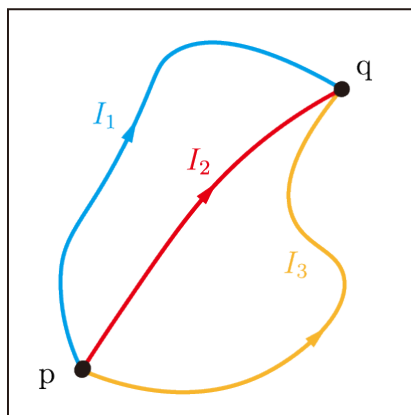


图 1: 梯度定理: 积分与路径无关, $I_1 = I_2 = I_3$