1 Supremum

1.1 Sup and arithmetic operations

Definition 1.1 (Operations on sets)**.** *Given two sets A and B of real numbers, and $r \in \mathbb{R}$, define following elementwise operations:*

- *Minkowski addition:* $A + B \coloneqq \{a + b \colon a \in A, b \in B\}$
- *Scalar addition:* $A + c \coloneqq \{a + c \colon a \in A\}$ *for some* $c \in \mathbb{R}$
- *Scalar multiplication:* $cA = \{ca \colon a \in A\}$ *for some* $c \in \mathbb{R}$

Theorem 1.1. *For operations defined above, we have:*

- $\sup(A + B) = \sup A + \sup B$
- $\sup(A + c) = \sup A + c$
- $\sup(cA) = r \sup A$ *for* $c \geqslant 0$

$A + c$ can be seen as a special case $A + \{c\}$.

1.2 Sup and inequalities

Theorem 1.2. *If $A \subseteq B$, then* $\sup A \leqslant \sup B$*.*

Definition 1.2 (Set comparison)**.** *Given two sets A and B. If for every $a \in A$ and every $b \in B$, we have $a \leqslant b$. Then we denote that $A \leqslant B$. If A is bounded above by a number c, that is, for every $a \in A$ we have $a \leqslant c$, then we denote that $A \leqslant c$.*

$A \leqslant c$ can be seen as a special case that $A \leqslant \{c\}$.

Theorem 1.3 (Comparison property)**.** *We can take supremum on both sides.*

1. *If $A \leqslant B$, then* $\sup A \leqslant \sup B$
2. *If $A \leqslant c$, then* $\sup A \leqslant c$

1.3 Sup and functions

Supremum is often applied to a function's range. In this circumstance, we have the following alternative notation of supremum, in which sup can be seen as an operator:

$$\sup_{x \in S} f(x) \coloneqq \sup\{f(x) \colon x \in S\}\,.$$

We ignore the domain and write $\sup f$, when it is clear from the context.

Relations bewteen functions are defined pointwise. For $f, g \colon S \to \mathbb{R}$, the relation $f \leqslant g$ means $f(x) \leqslant g(x)$ for all $x \in S$. Arithmetic operations also perform in pointwise manner. For example $(f + g)(x) = f(x) + g(x)$.

Theorem 1.4. *If $f \leqslant g$, then we can take supremum of both sides, that is*

$$\sup_{x \in S} f(x) \leqslant \sup_{x \in S} g(x)\,.$$

Proof. For every $x \in S$, $f(x) \leqslant g(x) \leqslant \sup g$, this means that $\sup g$ is an upper bound of $\{f(x) \colon x \in S\}$, therefore $\sup f \leqslant \sup g$. ∎

As a special case, if $f \leqslant c$, then $\sup f \leqslant c$.

Theorem 1.5. $\sup(f + g) \leqslant \sup f + \sup g$*.*

Proof. Note that $\{f(x) + g(x) \colon x \in S\} \subseteq \{f(x) \colon x \in S\} + \{g(x) \colon x \in S\}$. ∎