Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

2. Use a direct proof to show that the sum of two even integers is even.

3. Show that the square of an even number is an even number using a direct proof.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?

6. Use a direct proof to show that the product of two odd numbers is odd.

7. Use a direct proof to show that every odd integer is the difference of two squares.

8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

10. Use a direct proof to show that the product of two rational numbers is rational.

11. Prove or disprove that the product of two irrational numbers is irrational.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

13. Prove that if x is irrational, then $1/x$ is irrational.

14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

☞ **16.** Prove that if m and n are integers and mn is even, then m is even or n is even.

17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

18. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

23. Show that at least ten of any 64 days chosen must fall on the same day of the week.

24. Show that at least three of any 25 days chosen must fall in the same month of the year.

25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.

30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.

32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.

33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.

34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.

35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (*2*) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.

36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \to p_4$, $p_3 \to p_1$, $p_4 \to p_2$, $p_2 \to p_5$, and $p_5 \to p_3$ are true.