Report

Master Probability and Finance - Sorbonne University/ Ecole Polytechnique - Research Internship Report - Long-Short Hedge Fund

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The purpose of this report is to present an overview of the main research areas considered during my 6-month internship at a leading multi-strategy hedge fund. However - for confidentiality reasons - this report cannot contain any information related to strategies, assets and other proprietary models and data I may have had access to. In order to comply with these confidentiality requirements and to present results of academic interest, the models and results mentioned in this report are the fruit of independent personal research which does not engage the company's reputation. The first part of this report consists in presenting and extending the results of a paper I co-authored and that had just been submitted to a financial econometrics journal - based mainly on the study of multivariate GARCH models - applied to multivariate risk measures estimation. The model was initally designed to help identify financial institutions that could potentially harm the whole financial system in case of default. However, since the model introduced in this paper is quite flexible, it can easily be used to measure the dependence of several series at the tail levels and therefore be used by practitioners for risk management. We will then present a few illustrative projects related to factor analysis and cointegration for pair trading. This document should not be published or shared to anyone.

Keywords: Financial Econometrics, Multivariate GARCH Models, Factor Analysis, Pair Trading
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1. Introduction

The internship that motivates this report (and which is still ongoing) was completed at a major US multi-strategy fund in London - starting on April 17, 2023 for a period of six months. A multi-strategy fund is a fund that mixes various approaches: e.g. quantitative, fundamental, merger arbitrage and which invests potentially in all asset classes - e.g. equities, bonds, derivatives, commodities - with the aim of achieving absolute performance (i.e. performance that is decorrelated from the markets and that not benchmarked to any index). For my part, I have joined a team of a dozen of people pursuing a long-short approach. Long/short teams have historically been focused on fundamental research, but with the rise of so-called "quantamental" approaches, have now developed an increasing interest for quantitative methods aiming at boosting portfolio performance through better risk management and factor exposure. As mentioned in the abstract - for reasons of confidentiality - this report cannot refer directly to the results established and data used during the internship, but rather presents a synthesis of further personal research, applied to open-source data being linked to the internship by their underlying potential use. The first topic will be the use of multivariate GARCH models to study co-movements of daily log-returns [1] at the tail level. We will present in details the key ideas of the paper Cantin, Francq and Zakoian (2023)[2] [1] which introduces an asymptotic study of conditional systemic risk measures (notably the CoVaR - "Conditional" VaR - that is the Value at Risk (VaR) of a time series conditional to another series exceeding its VaR). The reference

[1] can be generalized to every time series presenting volatility clusters and that may be suited for the use of a GARCH model

[2] most recent public version available but an updated version has been completed and submitted for review since then

paper was originally written to study systemic risks (e.g. to assess the dangerousness of an idiosyncratic shock at the level of a "Systematically Important Financial Institution" (SIFI) - i.e. a major financial institution - on the system of which it is - or is not - a part). However, the model introduced allows us to study co-movements at the level of distribution tails more generally, and may prove useful in portfolio risk management - which will be the focus of this section, with an extension to the CoVaR of a portfolio - once the measure has been carefully redefined. In this section we will introduce carefully the concepts, motivate the use of the model, mention a few backtesting ideas, as well as illustration of the asymptotic theory (confidence intervals). Section 2 deals more generally with factor analysis and its potential use for investment purposes. Finally, section 3 will focus on considerations more specific to long-short investing, describing the basic econometric properties of cointegrated series and dealing with a simple illustrative example using an Ornstein Uhlenbeck movement for optimal trading. Section 4 concludes with a summary of the various topics discussed and potential direct extensions that could have been made with more time.

2. Multivariate Conditional Volatility Model, Tail Co-movements and Applications to Portfolio Management

In this first part, we describe the model of conditional systemic risk measures and analyze the asymptotic properties of the CoVaR (i.e. the VaR of a time series conditional on the event "a second series exceeds its own VaR"), after introducing a few definitions through a short literature review. We will then extend these statistical results, once again initially designed to identify systemically dangerous financial institutions, to portfolio management issues, by introducing the notion of portfolio CoVaR.

2.1. Reminders and Conventions on GARCH Models

2.1.1. GARCH Models in the Literature

While it is well known that investors seek to maximize their profits while minimizing their associated risk, forecasting the direction of a stock price is a complex problem for which no convincing solution has yet been found. The fact that there is no "free lunch" in a complete, frictionless financial market stems from economic theory and the hypothesis of the absence of arbitrage opportunities. Indeed, if there were an available opportunity to make a profit with a positive probability without exposing the investor to any risk, this investment would be instantaneously arbitraged, and the opportunity would disappear. From a statistical point of view, the absence of autocorrelation in returns diminishes the effectiveness of classic SARIMA-type time series models, and has prompted investors to look for answers in the use of more sophisticated models (e.g. deep learning with recurrent neural network (RNN) models, LSTM etc.). One of the major difficulties for a statistical model is to capture the idyosyncratic component of a security. For this reason, the fundamental approach based on anticipating the future through analysts' hyper-specialization on a particular stock/industry seems to remain valid. However, risk management appears to be more suited to statistical and mathematical studies, and is at the heart of modern financial econometrics and mathematical finance theories. ARCH/GARCH models, which are the subject of this first part, have gained a foothold in statistical circles, while theories derived from Brownian motion and statistical calculus have made their mark on derivatives pricing notably. It's worth noting that the theory commonly used to price derivatives is based on the cost of hedging risk. So the study of risk seems central to any quantitative approach. The very definition of risk has evolved considerably over time, having long been confined to the notion of volatility (defined as the squared returns or variance if we consider a model whose returns follow a centered distribution). However, it became clear (notably during the 2008 financial crisis) that empirical extreme risks were greater than theoretical Gaussian risks, and that measuring co-movements by means of correlation (a natural measure for Gaussian processes) was not always ideal for measuring extreme co-movements. Indeed, during a major macroeconomic shock, price stock movements appear to be much more "correlated" than in a standard regime, hence the need to introduce measures capable of accounting for these dependencies. This section adopts a financial econometrics approach, proposing a semi-parametric model (a GARCH model is here prescribed, but we remain agnostic about the distribution of innovations (we recall the definition of a GARCH model in this section). In financial econometrics, volatility has traditionally been estimated using ARCH or GARCH (1) time series models introduced by Engle (1982) [2] and Bollerslev (1986) [3] respectively. We introduce the following conventions for a GARCH(1, 1) model without trend (that could be modeled via an ARMA process if significant):

$$(1)\ \textbf{GARCH(1,1) Model} \begin{cases} \epsilon_t & = \sqrt{h_t}\,\eta_t \\ h_{t+1} & = \omega + \alpha\,\epsilon_t^2 + \beta\,h_t \end{cases} \quad \text{with } \eta_t \sim \mathcal{N}(0,1)$$

- ϵ_t represents the observation at time t of the log-returns of a series (we will only look at the log-returns of stock prices but it is not limited to this framework). For simplicity and clarity sake we assume that $\underline{\epsilon_t} = (\epsilon_1, ..., \epsilon_T)$ follows a GARCH without trend. We can easily generalize the next results adding an ARMA process for a potential trend for instance.
- $\sqrt{h_t}$ represents the conditional volatility at time t ($\mathcal{F}_{t-1}$ measurable) of the series under consideration.
- η_t is the innovation at time t verifying $E[\eta_t] = 0$, $V[\eta_t] = 1$ and the sequence (η_t) is iid, but its distribution is supposed unknown (as we deal with semi-parametric models).
- Let be $\theta = (\omega, \alpha, \beta)$ the parameter of the model to be estimated.

We take here the example of innovations (η_t) following a Gaussian distribution but we actually remain agnostic about its distribution and limit ourselves to the standard assumptions for a white noise (centered, with a unite variance and no autocorrelations). These models have the advantage of verifying stylized features such as volatility clustering - i.e. "large changes tend to be followed by large changes and small changes to be followed by small changes" (Mandelbrot 1963 [4]), resulting in positive auto-correlation of squared returns. The GARCH model also allows for larger distribution tails than in the standard Gaussian Brownian case, which have been observed empirically on stocks log-returns. The major advantage of the GARCH model is that the returns are not assumed to be independent, but only independent conditionally on the information available at date *t-1*. We will call hereafter $\mathcal{F}_{t-1} := \sigma(\epsilon_{-\infty}, ..., \epsilon_0; ..., \epsilon_{t-1})$ the filtration generated by the log-returns until date *t-1*.

The main alternative to the GARCH model (in the financial econometric literature) has been the stochastic volatility (SV) model, whose first introduction is generally attributed to Taylor (1982) [5]. The idea is that, unlike the GARCH model, the (discrete) stochastic volatility model postulates that a second stochastic process directly influences the volatility (2).

$$(2)\ \textbf{Stochastic Volatility Model} \begin{cases} \epsilon_t = \sqrt{h_t}\eta_t \\ log(h_t) = \omega + \beta log(h_{t-1}) + \sigma v_t \end{cases}$$

with $\eta_t \sim \mathcal{N}(0,1)$ and $v_t \sim \mathcal{N}(0,1)$

The major difference with the GARCH model is that conditional on $\mathcal{F}_{t-1}$, the volatility is a stochastic process and no longer a deterministic one, which obviously makes the model much more difficult to estimate. The reason why this model may be interesting to study however, is that it presents a few advantages over GARCH: this model is a natural discretization of the continuous time theory (can be seen as the Euler discretization of an Ornstein Uhlenbeck process).

$$\begin{cases} dlogS_t &= \mu dt + \sigma_t dW_{1t} \\ dlog\sigma_t^2 &= \{\omega + (\beta - 1)log\sigma_{t-1}^2\}dt + \sigma dW_{2t} \end{cases}$$

where (W_{1t}) and (W_{2t}) are two independent Brownian motions and S_t refers to a stock price. Here the log-volatility (σ_t designates the volatility - $\sqrt{h_t}$ in our main notations and μ the "drift"). If we add an intercept μ to the equation (2) of the standard SV of $\epsilon_t = log(\frac{S_t}{S_{t-1}})$ we find the natural discretization of this continuous model.

However - even if they are academically satisfactory - these discrete SV models have never been used extensively in practice as they are lacking simple and robust estimation method even for the most basic discrete stochastic volatility model. I have implemented the simple approach described by Francq, Zakoian (2006) [6] but the result seems still less stable and satifactory than GARCH models. On the contrary, GARCH models are relatively easy to estimate. We will rely on the simple estimation of the θ parameter of a GARCH model (i.e. $\theta = (\omega, \alpha, \beta)$ for a GARCH(1,1)) by QML as described in Francq and Zakoïan (2019) [7].

Let θ_0 be the true parameter of a GARCH(1,1) model, it takes the form: $\theta_0 = (\omega_0, \alpha_0, \beta_0)$. Let $\hat{\theta}_T$ be a CAN (Consistent and Asymptotically Normal) estimator of θ_0 obtained by Quasi Maximum Likelihood Estimation (QMLE), than it verifies:

$$\hat{\theta}_T := \arg\max_{\theta} \frac{1}{T} \sum_{t=1}^{n} log(h_t(\theta)) + \frac{\epsilon_t^2}{h_t(\theta)} =: \arg\max_{\theta} L_T(\theta)$$

- With $h_t(\theta) = \omega + \alpha\, \epsilon_{t-1}^2 + \beta\, h_{t-1}(\theta)$

The QML estimator is asymptotically normal with the following asymptotic normality property: $\sqrt{n}(\hat{\theta}_T - \theta_0) \sim \mathcal{N}(0, (\kappa_n - 1)J^{-1})$, where:

$$J := E[\frac{\delta^2 l_t(\theta_0)}{\delta\theta\delta\theta'}] = E[\frac{1}{h_t^2(\theta_0)} \frac{\delta h_t(\theta_0)}{\delta\theta} \frac{\delta h_t(\theta_0)}{\delta\theta'}].$$

is a positive definite matrix and let $l_t = l_t(\theta) = h_t(\theta) + \frac{\epsilon_t^2}{h_t(\theta)}$. κ_n is the empirical estimator of $E(\eta^4)$. We will not elaborate any further as the focus of our discussion will be a risk model based on multivariate GARCH model but it is still useful to draw a parallel with the univariate case as the estimation of the CoVaR will result directly from the extension of a well known process used to estimate dynamic VaR with univariate GARCH models.

2.1.2. Mutlivariate GARCH models

The generalization of the univariate GARCH model to the multivariate case has been the subject of numerous papers. The most commonly used models are the Constant Conditional Correlation (CCC) GARCH model, the Dynamic Conditional Correlation (DCC) GARCH model and the BEKK model. Without going into detail, we will focus on a variant of the CCC model, which enables us to verify the assumptions required to establish the risk model we are proposing. For information purposes, however, we offer a few details on the other models mentioned. The DCC model that has been introduced by R. Engle (2002) [8] tends to extend the CCC model (previously introduced by T. Bollerslev in (1990) [9]) by allowing the conditional correlations to change over time. It uses univariate GARCH models to estimate time-varying conditional correlations, making it more flexible and suitable for capturing changing volatility patterns in financial markets. This model is probably the most used in practice but for the purpose of our application we cannot allow the conditional correlations to be time-varying. The BEKK (Baba, Engle, Kraft, and Kroner) [10] model is also an extension of the univariate GARCH model to the multivariate setting and is notable for its flexibility in modeling the time-varying conditional covariances or correlations between multiple financial assets or variables. First we recall the standard form of the CCC-GARCH model:

Usual CCC(m)-GARCH(p,q) Model

$$\begin{cases} \underline{\epsilon_t} & = H_t^{1/2} \underline{\eta_t} \\ H_t & = D_t R_0 D_t, \;\; D_t \, = \, diag\,(\,\sqrt{h_{11,t}}, ..., \sqrt{h_{mm,t}}\,) \\ \underline{h}_t & = \underline{\omega}_0 \, + \, \sum_{i=1}^{q} \mathbf{A}_{0i} \underline{\epsilon}_{t-i}^2 \, + \, \sum_{j=1}^{p} \mathbf{B}_{0j} \underline{h}_{t-j} \end{cases}$$

- is a model describing the dynamics of m assets related by a correlation matrix R_0. We use the natural extension of the notations used for the univariate case: e.g. ϵ_{it} represents the log-returns of the i-th series at time t for i in 1,...,m for m stocks.
- $D_t \, = \, diag\,(\sqrt{h_{11,t}}, ..., \sqrt{h_{mm,t}})$
- $\underline{\epsilon}_t \, = \, (\epsilon_{1t}, ..., \epsilon_{mt})'$
- $\underline{\eta}_t \, = \, (\eta_{1t}, ..., \eta_{mt})'$ a standard multivariate white noise with the identity matrix for correlation matrix.
- $\underline{h}_t \, = \, (h_{11t}, ..., h_{mmt})'$ and $\sqrt{h_{iit}}$ is the conditional volatility of series i, $\forall i \in 1, m$.
- $V[\eta_t] \, = \, Id_m$
- R_0 is a correlation matrix of the m assets, such that: $R_0 \, = \, (\rho_{ij})_{(i,j) \in 1, m^2}$, with $\rho_{ii} \, = \, 1$ and $\rho_{ij} \, \leq \, 1$ for (i,j) $\in 1, m^2$.
- $\mathbf{A}_{0i} = (\alpha_{k,j})_{(k,j) \in 1, m^2} \in \mathcal{M}_m(R)$
- $\mathbf{B}_{0i} = (\beta_{k,j})_{(k,j) \in 1, m^2} \in \mathcal{M}_m(R)$ is assumed diagonal to allow equation by equation estimation.

We can easily show that this model can be rewritten under the following form:

Lemma 1: Every CCC-GARCH can be rewritten under the form:

$$\begin{cases} \underline{\epsilon_t} & = D_t \, R^* \, \underline{\eta_t} \\ H_t & = D_t R^{*2} D_t \\ \underline{h}_t & = \underline{\omega}_0 \, + \, \sum_{i=1}^{q} \mathbf{A}_{0i} \underline{\epsilon}_{t-i}^2 \, + \, \sum_{j=1}^{p} \mathbf{B}_{0j} \underline{h}_{t-j} \end{cases}$$

- $with\, H_t^{1/2} = D_t R^*$
- $and\, R^{*2}\, a\, correlation\, matrix$

which is equivalent to:

***Definition* 1** (Diagonal Form of the CCC Model)**.**

$$\begin{cases} \underline{\epsilon_t} & = D_t \underline{\tilde{\eta}}_t \\ H_t & = D_t R^{*2} D_t \\ \underline{h}_t & = \underline{\omega}_0 \, + \, \sum_{i=1}^{q} \boldsymbol{A}_{0i} \underline{\epsilon}_{t-i}^2 \, + \, \sum_{j=1}^{p} \boldsymbol{B}_{0j} \underline{h}_{t-j} \end{cases}$$

- *with* $\underline{\tilde{\eta}}_t = R^* \, \underline{\eta_t}$
- The expression $\underline{\epsilon_t} = D_t \underline{\tilde{\eta}}_t$ characterizes what we call the "diagonal" form of the CCC-GARCH model.

The reason for this transformation may seem obscure for now but the key idea is that under the standard CCC-GARCH model we have the following relation (for a bivariate model m=2):

$$\begin{cases} \epsilon_{1t} &= \sqrt{h_{11,t}} \, . \, \eta_{1t} \\ \epsilon_{2t} &= \rho_{1,2} \, . \, \sqrt{h_{22,t}} \, . \, \eta_{1t} \, + \, \sqrt{(1 \, - \, \rho_{1,2}^2) \, h_{22,t}} \, . \, \eta_{2t} \end{cases}$$

with $\rho_{1,2}$ and $\rho_{2,1}$ being the non-diagonal coefficients of the correlation matrix of $\underline{(\epsilon_{1t}, \epsilon_{2t})}$ and our transformation allows us to rewrite it under the following form:

$$\begin{cases} \epsilon_{1t} &= \sqrt{h_{11,t}} \, . \, \tilde{\eta_{1t}} \\ \epsilon_{2t} &= \sqrt{h_{22,t}} \, . \, \tilde{\eta_{2t}} \end{cases}$$

We will see after having introduced the risk models why this will be useful. Until now we assumed that there was no trend for the different log-returns. However, in portfolio management and in particular to solve for the coefficients of the portfolio that maximises the Sharpe Ratio it can be useful to consider series with positive means. We will therefore use an AR(1)-CCC(m)-GARCH(1,1) model to model the log-returns of the different series. The model is as follows:

***Definition* 2** (AR-CCC-GARCH Model)**.**

$$\begin{cases} \underline{r_t} &= \underline{\mu} + \underline{\delta} + \underline{\psi} \, \underline{r_{t-1}} + \underline{\epsilon_t}, \\ \underline{\epsilon_t} &= H_t^{1/2} \underline{\eta_t} \\ H_t &= D_t R_0 D_t \\ \underline{h_t} &= \underline{\omega}_0 \, + \, \sum_{i=1}^{q} \boldsymbol{A}_{0i} \underline{\epsilon}_{t-i}^2 \, + \, \sum_{j=1}^{p} \boldsymbol{B}_{0j} \underline{h}_{t-j} \end{cases}$$

- $\underline{r_t} = (r_{1t}, ..., r_{mt})$ *denotes the log-returns.*
- *With* $\underline{\mu} = (\mu_1, ..., \mu_m)$ *the vector of the means of the different series.*
- $\underline{\delta} = (\delta_1, ..., \delta_m)$ *the vector of intercepts of the demeaned processes.*
- $\underline{\psi} = (\psi_1, ..., \psi_m)$ *is the vector of the auto-regressive coefficients.*
- $\underline{\epsilon_t} = (\epsilon_{1t}, ..., \epsilon_{mt})$ *is the vector of residuals following a CCC-GARCH model.*

We have now established and recalled a few results about GARCH models and are ready to introduce the risk models that we can derive from it.

2.2. Reminders and Conventions on Conditional Risk Measures

Estimating risk measures is one of the most natural applications of volatility models. The **Value at Risk** (VaR) is the most commonly used measure to account for the risk of a financial time series. By convention, as we are dealing with potentially asymmetric distributions for the log-returns we wille be considering a loss function that is equal to the opposite of the log-returns.

$$VaR(\alpha) \,=\, inf\{-\epsilon_t \in \mathbb{R} : F_\Upsilon(\epsilon) \,>\alpha\} \,=\, -F_\Upsilon^{-1}(\alpha) =: q_\alpha\left(\epsilon_t\right)$$

- With ϵ_t a log-return in the set of real numbers.
- Υ is the distribution of the ϵ_t for t $\in$ $(1, ..., T)$.
- F_Υ is the cdf of the distribution Υ.
- F_Υ^{-1} is the is the inverse function of F_Υ.
- α is the risk level under consideration.

In order to estimate it, we will focus on what we call the **Conditional** Value at Risk (CVAR at risk level α (conditional to the information set available at *t-1*), which is defined as follows:

$$VaR_{t-1}(\alpha) = -q_\alpha\left(\epsilon_t|\mathcal{F}_{t-1}\right)$$

It is minus the quantile of order α for the log-returns at date t knowing all the past information $\mathcal{F}_{t-1}$.

Its greatest advantage is that it does not require the existence of any moment to exist, which is not the case for some other risk measures such as the (Conditional) **Expected Shortfall** for instance:

$$ES_{t-1}(\alpha) \,=\, \mathbb{E}_{t-1}\,\left(\epsilon_t \,|\, \epsilon_t \,<\, -VaR_{t-1}(\alpha)\right)$$

The estimation of the VaR is relatively easy with a GARCH model estimated by QML by applying, for example, the 2-step method presented by Francq and Zakoïan (2015) [11]. This method relies on the fact that the volatility $\sqrt{h_t}$ is $\mathcal{F}_{t-1}$ measurable and on its positive homogenity property. Indeed, it can be expressed in the form:

$$VaR_{t-1}(\alpha) \,=\, -\sqrt{h_t}(\theta_0)\,\xi_\alpha :$$

- ξ_α refers to the $\alpha - quantile$ of the innovations η_t.

Therefore, to estimate the conditional VaR we use the traditional two-step method consisting in obtaining a constant and asymptotically normal (CAN) estimator by Gaussian QML estimation in a first step. And then constructing estimations of the innovations (i.e. $\hat{\eta}_t$) as well as an estimator of the $\alpha - quantile$ (i.e. -$\hat{\xi}_T$). We will then consider the following estimator:

$$\widehat{VaR}_{t-1}(\alpha) \,=\, -\sqrt{h_t}(\hat{\theta}_T)\,\hat{\xi}_T :$$

- $\hat{\theta}_T = (\hat{\omega}, \hat{\alpha}, \hat{\beta})$ refers to the estimated paramater obtained by Gaussian QML.
- $\hat{\xi}_T$ is obtained taking the $\alpha - quantile$ of the residuals $\hat{\eta}_t$ defined as: $\hat{\eta}_t = \frac{\epsilon_t}{\sqrt{h_t(\hat{\theta}_T)}}$

We can then deduct natural expression for the expected shortfall and a direct way for estimation:

$$ES(\alpha) = \frac{1}{\alpha}\int_0^\alpha VaR(u)du$$

The (Conditional) Expected Shortfall (ES) is the average loss incurred when the asset is in "distress" - i.e. when the return at time t is below $-VaR_{t-1}(\alpha)$ for α chosen.

$$\begin{aligned} ES_{t-1}(\alpha) &= \frac{1}{\alpha}\int_0^\alpha VaR_{t-1}(u)du \\ &= -E_{t-1}\left[\,\epsilon_t \,|\, \epsilon_t \,<\, -VaR_{t-1}(\alpha)\,\right] \end{aligned}$$

- We write ES_{t-1} as this is a $\mathcal{F}_{t-1}$ measurable measure.

Although ES seems to be more satisfactory than VaR for measuring the risk of losses encountered in times of market stress, it does not capture the joint effects between the simultaneous distress of several financial assets. Indeed, understanding the co-movements between several assets is a major challenge in times of crisis to ensure the financial stability of the system as a whole and to understand which institutions are most at risk (both to themselves and to the system). CoVaR is the first conditional systemic risk measure that we will consider and which will serve as the basis for the calculation of the others (MES, ΔCoVaR). It can then be rewritten under the following form:

$$\begin{aligned} ES_{t-1}(\alpha) &= -E_{t-1}\left[\,\epsilon_t \,|\, \epsilon_t \,<\, -VaR_{t-1}(\alpha)\,\right] \\ &= -E_{t-1}\left[\,\epsilon_t \,|\, \eta_t \,<\, -\xi_\alpha\,\right] \\ &= -\sqrt{h_t}(\theta_0)\, E\left[\,\eta_t \,|\, \eta_t \,<\, -\xi_\alpha\,\right] \end{aligned}$$

We can then propose the following empirical estimator:

$$\widehat{ES}_{t-1}(\alpha) = -\frac{\sqrt{h_t}(\hat{\theta}_T)}{\tilde{T}} \sum_{t=1}^{T} \eta_t\, 1\{\eta_t \,|\, \eta_t \,<\, -\hat{\xi}_T\}$$

- $\tilde{T} = \sum_{t=1}^T 1\{\eta_t \,|\, \eta_t \,<\, -\hat{\xi}_T\}$

2.2.1. Backtesting Conditional Risk Measures

For sake of completeness, we provide below a very brief summary of the existing tests for these conditional risk measures.

2.2.2. Quick Overview of the Existing Tests

There are different types of tests, but the most frequent are those based on what we call the "Hit Variable" (defined below), which is simply an exceedance/ violation of the VaR. A violation is any moment t at which $\epsilon_t < -VaR_{t-1}(\alpha)$. Let's define the hit variable:

$$Hit_t(\alpha) = 1\{\epsilon_t < -VaR_{t-1}(\alpha)\}$$

The most commonly used tests are the Kupiec (1995) [6] - Christoffersen (1998) [5] tests which overlook the estimation error. They are aiming at checking both that the number of violations corresponds to the risk level α and the independence of these violations.

2.2.3. Unconditional Coverage Test

The idea of the test relies on the fact that if $VaR_{t-1}(\alpha)$ is a VaR measure at a α risk level, then we have:

$P[\epsilon_t < -VaR_{t-1}(\alpha)] = \alpha$ and $Hit_t \sim \mathcal{B}(\alpha)$.

- $\mathcal{B}(\alpha)$ refers to a Bernoulli distribution with paramater α.

Therefore we define the Null hypothesis: $H_0^{UC} : P[Hit_t = 1] = \alpha$ and the test statistics:

$LR_{UC} = 2log\frac{\pi_{exp}^{n_1}(1-\pi_{exp})^{n_0}}{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}}$

- π_{exp} is the expected proportion of violations.
- π_{obs} is the observed proportion of violations.
- n_1 is the number of violations and $n_0 = n - n_1$ is the sample size.

$LR_{UC} \sim \chi_1^2$ under the null hypothesis H_0^{UC}. Hence the rejection area: $\{LR_{UC} > \chi_1^2(1-\underline{\alpha})\}$ at the confidence level $\underline{\alpha}$.

2.2.4. Independence Test

As $VaR_{t-1}(\alpha)$ is a conditional VaR for a risk level α, we have:

$$\begin{aligned} E[Hit_t\,Hit_{t+k}] &= E[E_{t+k-1}[Hit_t\,Hit_{t+k}]] \\ &= E[Hit_t E_{t+k-1}[Hit_{t+k}]] \\ &= \alpha\,E[Hit_t] \\ &= \alpha^2 \end{aligned}$$

Hence, $CoV(Hit_t, Hit_{t+k}) = 0$, which is equivalent to Hit_t and Hit_{t+k} are independents. Eventually, we have:

$(Hit_t)\, iid \sim \mathcal{B}(\alpha)$

Therefore we define the Null Hypothesis: $H_0^{Ind} : P[Hit_t = 1|Hit_{t-1} = 0] = P[Hit_t = 1|Hit_{t-1} = 1]$ and the test statistics:

$$LR_{Ind} = 2log\frac{\pi_{obs}^{n_1}\,(1-\pi_{obs})^{n_0}}{\pi_{01}^{n_{01}}\,(1-\pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}\,(1-\pi_{11})^{n_{10}}}$$

- n_{ij} is the number of indicator i followed by indicator j.
- $\pi_{01} = \frac{n_{01}}{(n_{00}+n_{01})}$ and $\pi_{11} = \frac{n_{11}}{(n_{10}+n_{11})}$

$LR_{Ind} \sim \chi_1^2$ under the null hypothesis H_0^{Ind}. Hence the rejection area: $\{LR_{Ind} > \chi_1^2(1-\underline{\alpha})\}$ at the confidence level $\underline{\alpha}$.

2.2.5. Conditional Coverage Test

The Conditional Coverage Test tests for both effects (consistency and independence). The test statistics take the following form:

$$LR_{cc} = 2log\frac{\pi_{exp}^{n_1}\,(1-\pi_{exp})^{n_0}}{\pi_{01}^{n_{01}}\,(1-\pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}\,(1-\pi_{11})^{n_{10}}}$$

Remark: LR_{cc} can easily been calculated if we already have LR_{UC} and LR_{Ind}, as $LR_{cc} = LR_{UC} + LR_{Ind}$.

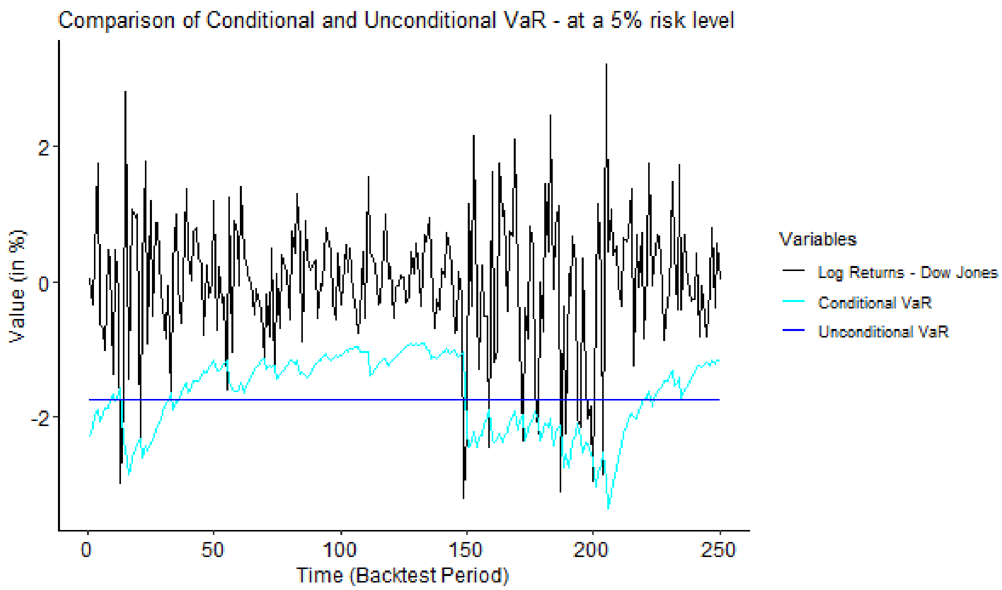
$LR_{cc} \sim \chi_2^2$ under the null hypothesis. Hence the rejection area: $\{LR_{cc} > \chi_2^2(1-\underline{\alpha})\}$ at the confidence level $\underline{\alpha}$.

The three tests proposed by Christoffersen described above (Unconditional Coverage test, Independence Test and Conditional Coverage test) have the drawback that they only test the model for a particular α. However, the model could work for a well-chosen α and perform much worse for another α. This is why tests applied to multiple risk levels have been introduced but that is not really the purpose of this report.

2.2.6. Conditional versus non-conditional Models

Before introducing the CoVaR which is the core of the work on this section, we just want to motivate the use of dynamic VaR versus an unconditional model which is just based on the historical quantile of the log-returns. Of course, practitioners will always prefer considering an unconditional model recomputed on a rolling basis, but this model will always be lagging and will not represent effectively the cluster effect of the volatility. We can see on the following chart that a dynamic model fits much better the data.

Figure 1. Conditional Risk Models Perform Better than Unconditional One



2.3. Asymptotic study of CoVaR and preliminary results for the selected semi-parametric model

Let's introduce now the CoVaR risk measure which will be our main interest in this section. This measure will help us modeling the co-movements between two series at the tail level and propose a natural "causal" interpretation. We can use this concept for a lot of application in portfolio management. We can use it to better understand the effect of a strong stress on a market (for instance an index log-returns exceeding their VaR on a particular stock). This could help us understand which stocks are the most exposed to potential severe losses in case of a sharp and unexpected crash. It can also be interesting to see what would be the impact of some macro data on a stock or a portfolio. For instance, what would be the impact of a shock in oil prices on a portfolio, what about a sharp fluctuation in currency exchange rates, etc. The goal of this measure is not to replace the volatility as measure of risk in order to propose an optimal portfolio theory but rather to provide a sense of the risk that can then be used by a discretionary portfolio manager to take decisions according to the probability assigned to such a major downturn. However, we can imagine - if we propose a way to get clear probability values for the occurrence of these events - an algorithm that will maximize the return with some constraints on extreme risks. Unfortunately, there will not be any closed form formula to solve these problems, i.e. the resolution should then be performed via numerical optimization which is left for further research. Let's start by defining clearly the CoVaR measure that has been first introduced by Adrian and Brunnermeier (2011) [12] and from which we propose a refined version which is more flexible and adapted to the use of GARCH models, inspired from the work of Ergun and Girardi (2013) [13].

Definition* 3** (CoVaR).***

$$\begin{aligned} CoVaR^{(i|j)}(\alpha, \alpha') &= -inf\{\epsilon_i \in \Omega_{\epsilon_i} : \\ &\qquad F_{\Upsilon_i|\{\epsilon_j < -VaR^{(j)}(\alpha')\}} > \alpha\} \\ &= -F^{-1}_{\Upsilon_i|\{\epsilon_j < -VaR^{(j)}(\alpha')\}}(\alpha) \end{aligned}$$

- *With ϵ_i a log-return of the series i in the set of taken values $\Omega_{\epsilon i}$.*
- *Υ_i is the distribution of the ϵ_{it} for $t \in 1, T$*
- *F_{Υ_i} is the cdf of the distribution Υ_i.*
- *$F^{-1}_{\Upsilon_i}$ is the is the inverse function of F_{Υ_i}.*
- *α and α' are the risk levels under consideration.*

However, the CoVaR suffers from the same shortcoming as the VaR measure, namely that it fails to take into account the tail distribution of extreme values once this level has been crossed. Just as the ES proposed to correct this weakness for the VaR, the Marginal Expected Shortfall (MES) measure follows suit this time with regard to the CoVaR. We can find a few different definitions for the MES in the literature, we will stick to the following one, that is the perfect counterpart of the ES with respect to the CoVaR (replacing the VaR for ES).

Definition* 4** (Marginal Expected Shortfall).***

$$MES^{(i|j)}(\alpha') = \int_0^1 CoVaR^{(i|j)}(\alpha, \alpha')\, d\alpha$$

Back to CoVaR, we can now understand the usefulness of the ”diagonal” form of the CCC-GARCH model presented previously that allows to simplify the estimation of the CoVaR and isolate the volatility components from a quantile that only depends on the conditional distribution of the innovations and whose quantiles can be estimated empirially:

$$\begin{aligned} &P_{t-1}\,[\,\epsilon_{it} < -CoVaR^{(i|j)}_{t-1}(\alpha, \alpha')\,|\,\epsilon_{jt} < -VaR^{(j)}_{t-1}(\alpha')\,] \\ &= P_{t-1}\,[\,\epsilon_{it} < -CoVaR^{(i|j)}_{t-1}(\alpha, \alpha')\,|\,\tilde{\eta}_{jt} < -\xi^{(j)}_{\alpha'}\,] \\ &= P_{t-1}\,[\,\sqrt{h_{ii,t}}\,\tilde{\eta}_{it} < -\sqrt{h_{ii,t}}\,u^{(i|j)}(\alpha, \alpha')\,|\,\tilde{\eta}_{jt} < -\xi^{(j)}_{\alpha'}\,] \\ &= P\,[\,\tilde{\eta}_{it} < -u^{(i|j)}(\alpha, \alpha')\,|\,\tilde{\eta}_{jt} < -\xi^{(j)}_{\alpha'}\,] \end{aligned}$$

- $\xi^{(j)}_{\alpha'}$ is the opposite of the α'-quantile of the innovations $(\tilde{\eta}_{jt})$ of series j.

- $u(\alpha, \alpha')$ is the opposite of the α-quantile of the $(\tilde{\eta}_{it})$ conditional on $\tilde{\eta}_{jt} < -\xi_{\alpha'}^{(j)}$ (i.e. in practice the conditional quantile $u^{(i|j)}(\alpha, \alpha')$ is calculated over the instants t such that $\tilde{\eta}_{jt} < -\xi_{\alpha'}^{(j)}$ is verified.

I would like to highlight the fact that this $u^{(i|j)}(\alpha, \alpha')$ term can for given values of α and α' be interpreted as a measure of the co-movements of two series at the tail level. This measure can have a causal interpretation as the conditional quantile $u^{(i|j)}$ has no reason to be equal to $u^{(j|i)}$ for two series i and j. The aim of the paper Cantin, Francq and Zakoian (2023) [1] is to propose an asymptotic theory for this term u() and for the CoVaR in general after articulating it with the volatility following the model bivariate model described previously. The proof of the following results are quite technical and rely on a technical result from Francq and Zakoian (2022) [14]. We encourage the reader to refer to these two papers for proofs, I prefer to lay the stress on the practical use of these models here since it is the main purpose of this report. The following characterization of the CoVaR comes directly from the diagonal expression of the CCC-GARCH model:

$$CoVaR_{t-1}^{(i|j)}(\alpha, \alpha') \;=\; \sqrt{h_{ii,t}}(\theta_0)\, u^{(i|j)}(\alpha, \alpha')$$

and the natural estimator associated can be written as:

$$\widehat{CoVaR}_{t-1}^{(i|j)}(\alpha, \alpha') \;=\; \sqrt{h_{ii,t}}(\hat{\theta}_T)\, \hat{u}^{(i|j)}(\alpha, \alpha')$$

We introduce the following estimator of the conditional quantile function u:

$$\widehat{u}(\alpha, \alpha') = \inf \arg \min_{z\in\mathbb{R}} \sum_{t=1}^{n} \rho_\alpha(\widehat{\eta}_{1t} - z)\mathbb{1}_{\widehat{\eta}_{2t} < \widehat{\xi}_{\alpha'}^{(2)}},$$

where $\rho_\alpha(.)$ is the standard "check function" traditionally used to get the quantile of a distribution and can prove the following theorem (please refer to Cantin, Francq and Zakoian (2023) for the complete list of mild assumptions and other details):

$$\sqrt{n}\left\{\widehat{u} - u\right\} = \frac{-1}{\sqrt{n}\alpha' f_1(u|\xi_{\alpha'}^{(2)})} \sum_{t=1}^{n} \left(\mathbb{1}_{\eta_{1t}\le u,\, \eta_{2t}\le \xi_{\alpha'}^{(2)}} - \alpha\alpha'\right)$$
$$+ \frac{1}{f_1(u|\xi_{\alpha'}^{(2)})} \frac{G^{(1)}(u)}{\alpha'} \frac{f_2(\xi_{\alpha'}^{(2)}|u)}{g^{(2)}(\xi_{\alpha'}^{(2)})} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{\mathbb{1}_{\eta_{2t}\le \xi_{\alpha'}^{(2)}} - \alpha'\right\} - \frac{u}{2\sqrt{n}} \boldsymbol{\Omega}_1' \boldsymbol{J}_1^{-1} \sum_{t=1}^{n} (\eta_{1t}^2 - 1)\boldsymbol{D}_{1t},$$

up to some $o_P(1)$. where $\boldsymbol{J}_i = E(\boldsymbol{D}_{it}\boldsymbol{D}_{it}')$ and $\boldsymbol{D}_{it} = \boldsymbol{D}_{it}(\boldsymbol{\theta}_0^{(i)})$. Finally, let $\boldsymbol{\Omega}_i = E(\boldsymbol{D}_{it})$ and $\boldsymbol{J}_{12} = E\boldsymbol{D}_{1t}\boldsymbol{D}_{2t}'$. $G(i)$ refers to the cdf of the i-the series, $g(i)$ represent the pdf of the i-th series and $f_1(.|.)$ represent the pdf of the series 1 conditional to the fact that the series 2 exceedes its quantile and a similar definition for $f_2(.|.)$

With an additional small technical assumption we have,

$$\sqrt{n}\left\{\widehat{u} - u\right\} \;\;\overset{\mathcal{L}}{\to}\;\; \mathcal{N}\left(0, \sigma^2(\alpha, \alpha') = \boldsymbol{\lambda}'\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}\boldsymbol{\lambda}\right),$$

where

$$\boldsymbol{\lambda}' = \left(\frac{-1}{\alpha' f_1(u|\xi^{(2)}_{\alpha'})}, \frac{1}{\alpha'} \frac{G^{(1)}(u)}{f_1(u|\xi^{(2)}_{\alpha'})} \frac{f_2(\xi^{(2)}_{\alpha'}|u)}{g^{(2)}(\xi^{(2)}_{\alpha'})}, \frac{-u}{2} \right).$$

$\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}$ can be characterized as:

$$\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}} = \begin{pmatrix} \alpha\alpha'(1-\alpha\alpha') & \alpha\alpha'(1-\alpha') & \alpha'\varrho_{\alpha,\alpha'} \\ \alpha\alpha'(1-\alpha') & (1-\alpha')\alpha' & \alpha'\nabla_{\alpha,\alpha'} \\ \alpha'\varrho_{\alpha,\alpha'} & \alpha'\nabla_{\alpha,\alpha'} & \kappa_1 - 1 \end{pmatrix}$$

where $\varrho_{\alpha,\alpha'} = E(\eta^2_{1t}\mathbb{1}_{\eta_{1t}\leq u(\alpha,\alpha')}|\eta_{2t} \leq \xi_{\alpha'}) - \alpha$ and $\nabla_{\alpha,\alpha'} = E(\eta^2_{1t}|\eta_{2t} \leq \xi_{\alpha'}) - 1$.

These results coming from the reference paper as well as a result on the co-asymptotic distribution of the conditional quantile and the QML estimators of the GARCH model allows us to derive confidence intervals using the following result:

$$\sqrt{n} \begin{pmatrix} \widehat{\boldsymbol{\theta}}^{(1)} - \boldsymbol{\theta}^{(1)}_0 \\ \widehat{u} - u \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\left\{ \mathbf{0}, \boldsymbol{\Sigma}(\alpha,\alpha') := \begin{pmatrix} \frac{\kappa_1 - 1}{4}\boldsymbol{J}^{-1}_1 & \frac{-1}{2}\boldsymbol{J}^{-1}_1\boldsymbol{\Omega}_1\boldsymbol{\lambda}'\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}\boldsymbol{e}_2 \\ \frac{-1}{2}\boldsymbol{e}'_2\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}\boldsymbol{\lambda}\boldsymbol{\Omega}'_1\boldsymbol{J}^{-1}_1 & \boldsymbol{\lambda}'\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}\boldsymbol{\lambda} \end{pmatrix} \right\}.$$

with $\boldsymbol{e}_2 = (0,1,0)'$.

2.3.1. Illustrations

We propose an illustration below to understand better of to interpret the CoVaR in practice. We look here at the CoVaR of the daily log-returns of AAPL (the model has been calibrated on the 2,000 observations preceeding the depicted period) conditional on the S&P 500 exceeding its VaR. We can see that the exceedances of the CoVaR seem to appear independently from the exceedances of the VaR of the conditioning series, which is exactly the desired quality of our CoVaR model.

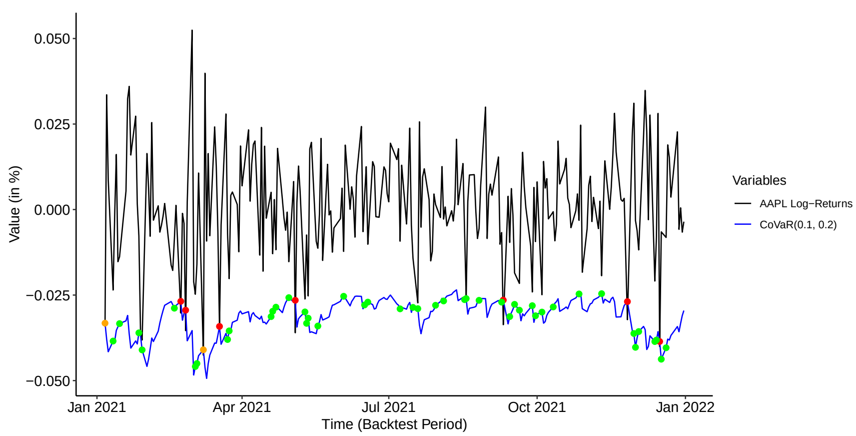


Figure 2. CoVaR(5%, 10%) of AAPL conditionale to S&P 500 log-returns

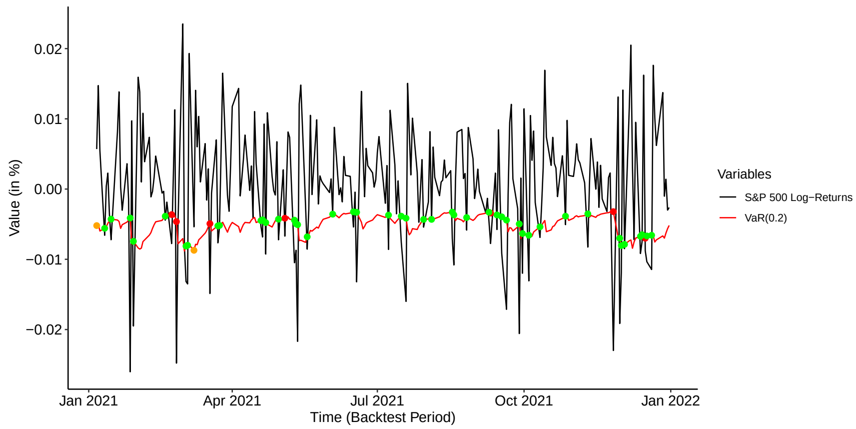
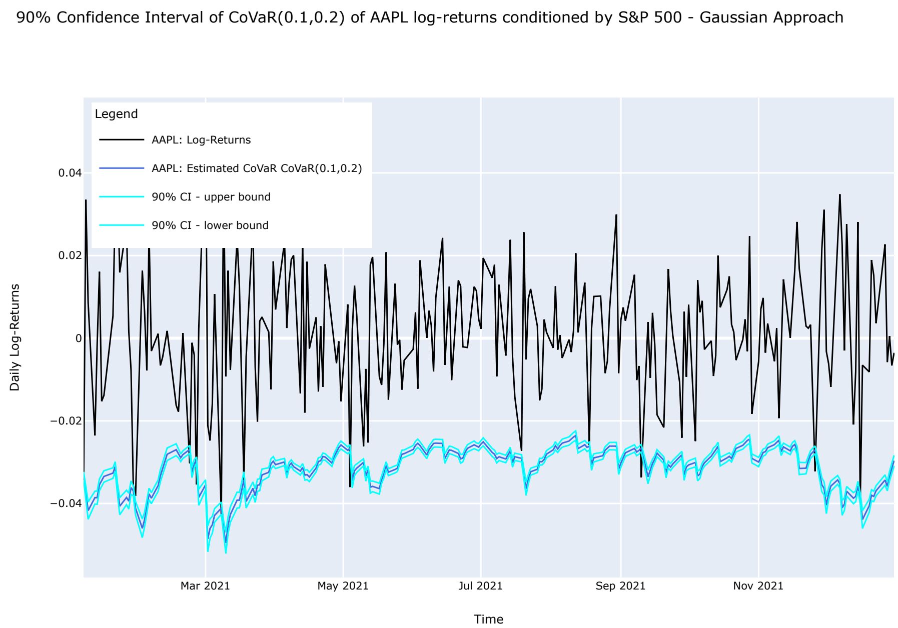
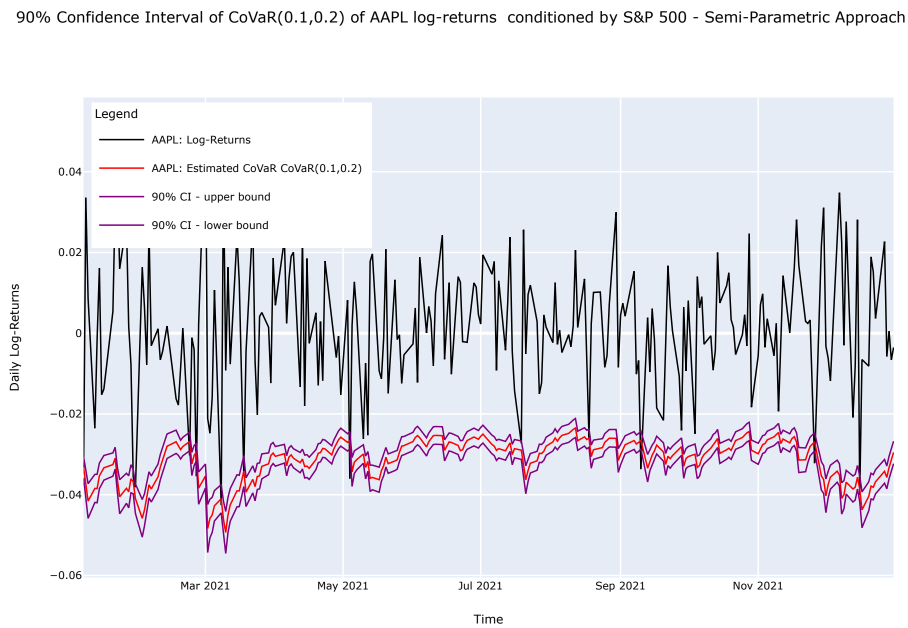


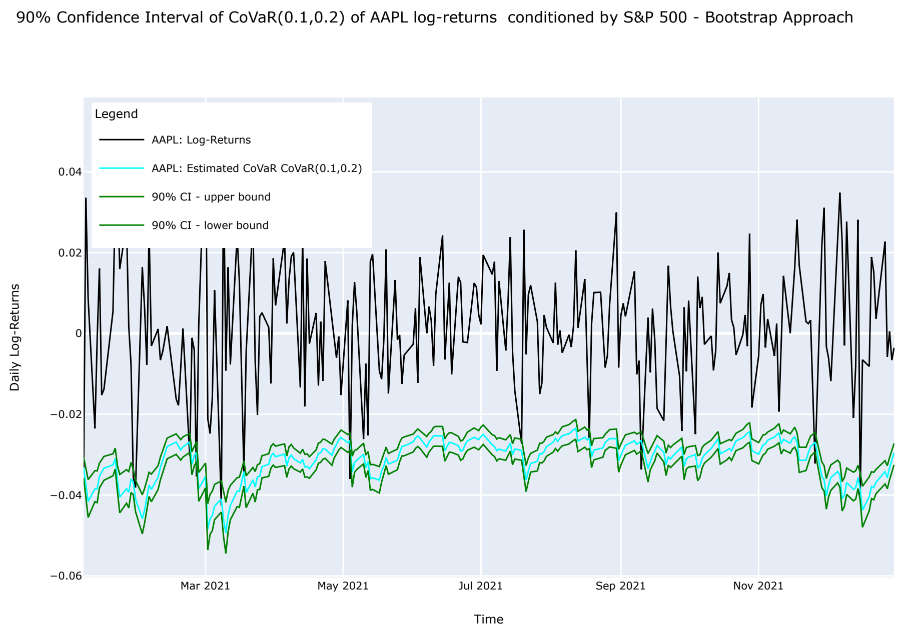
Figure 3. VaR(10%) of the conditioning series, namely S&P 500 log-returns

red: Violations of CoVaR (process 1) and Violations of VaR (process 2).
green: Violations of VaR (process 2) while no Violations of CoVaR (process 1).
orange: Violations of CoVaR (process 1) while no Violations of VaR (process 2).

We then illustrate the asymptotic theory by showing a few confidence intervals for the CoVaR. The first one relies on q parametric assumption on the innovations, namely a Gaussian assumption here. Therefore we can get better approximation than with the semi-parametric model but we take the risk that the assumed distribution does not characterize correctly the empirical one and that in this case we may underestimate the tail risk. The second chart represents a 90% confidence interval using the key result previously introduced. And the third one relies on a bootstrap result taken from the reference paper and that we will not mention here. The interest of the bootstrap is that it can improve the finite-sample performance of the estimator over the asymptotic results that we mentioned.


90% Confidence Interval of CoVaR(0.1,0.2) of AAPL log-returns conditioned by S&P 500 - Gaussian Approach
Legend
AAPL: Log-Returns
AAPL: Estimated CoVaR CoVaR(0.1,0.2)
90% CI - upper bound
90% CI - lower bound
0.04
0.02
0
−0.02
−0.04
Daily Log-Returns
Mar 2021
May 2021
Jul 2021
Sep 2021
Nov 2021
Time


90% Confidence Interval of CoVaR(0.1,0.2) of AAPL log-returns conditioned by S&P 500 - Semi-Parametric Approach
Legend
AAPL: Log-Returns
AAPL: Estimated CoVaR CoVaR(0.1,0.2)
90% CI - upper bound
90% CI - lower bound
Daily Log-Returns
0.04
0.02
0
−0.02
−0.04
−0.06
Mar 2021
May 2021
Jul 2021
Sep 2021
Nov 2021
Time



We can easily extend naturally the Kupiec, Christofersen tests mentioned previously to test the validity of these models but to remain concise we will not develop this topic here. Until now, the focus has been on analyzing the VaR/ CoVaR of a single series conditional on an extreme event relating to a second series. However, from a portfolio management perspective, we are naturally more interested in the effect of a shock to one variable on an entire portfolio and not on a stock. The following subsection deals with this problem.

2.4. Application to portfolio management and introduction of portfolio CoVaR

There are two main approaches here, we will describe one of them. These are a multivariate approach (i.e. the Filtered Historical Simulation approach - FHS) and an approach that treats the portfolio as a univariate variable (i.e. Virtual Historical Simulation - VHS). We will focus on the VHS approach that will be the easiest to be generalized for the portfolio CoVaR.

2.4.1. Conditional VaR/ CoVaR for Portfolios

Let's introduce a few notations to deal with portfolio returns:

- Let's denote our "conditional" VaR by $VaR^{(\alpha)}_{t-1}(\epsilon_t)$ for a risk level α.
- $\epsilon^{(p)}_{r,t}$ designates the real return of the portfolio at time t, define as follow: $\epsilon^{(p)}_{r,t} = \sum_{i=1}^{m} w_{it}\,\epsilon_{it}$.
- $\epsilon^{(p)}_{v,t,\tilde{t}}$ refers to the virtual return of the portfolio at time t for a portfolio re-balanced at time $\tilde{t}$, define as follow: $\epsilon^{(p)}_{v,t,\tilde{t}} = \sum_{i=1}^{m} w_{i\tilde{t}}\,\epsilon_{it}$.

and let's remind the form of the bivariate model we consider:

$$\sigma^2_{it} = \omega_i + \alpha_{ii}\epsilon^2_{i,t-1} + \alpha_{ij}\epsilon^2_{j,t-1} + \beta_i\sigma^2_{i,t-1}, \quad i,j = 1,2 \tag{2.1}$$

The Virtual Historical Simulation approach is a univariate approach developed in Francq, Zakoïan (2020) [15] which has the advantage of not having to use a multivariate model and therefore limiting estimation errors. However, this model loses information about the behaviour of each of the stocks in the portfolio in the prediction of the aggregate return. The key idea of this approach, which is relatively basic is that at each instant t (before rebalancing) we should reconstruct the past performance of the portfolio with the new potential weights and treat this new series as a univariate series on which all the theory previously introduced can apply.

$$VaR^{(\alpha)}_{VHS,t-1}(\epsilon^{(p)}_t) \;=\; \sqrt{h^{(p)}_{v,t}}(\theta_0)\,\xi^{(p)}_{\alpha}$$

- Recall that: $\epsilon^{(p)}_{v,t,\tilde{t}} = \sum_{i=1}^{m} w_{i\tilde{t}}\,\epsilon_{it}$.
- $\xi^{(p)}_{\alpha}$ is the opposite of the α-quantile of the virtual innovations $\eta^{(p)}_{v,t}$.
- $\eta^{(p)}_{v,t} = \frac{\epsilon^{(p)}_{v,t}}{\sqrt{h^{(p)}_{v,t}}(\theta_0)}$.
- $h^{(p)}_{v,t}(\theta_0) = \omega + \alpha\,{\epsilon^{(p)}_{v,t-1}}^2 + \beta\,h^{(p)}_{v,t-1}(\theta_0)$.

and we can get the following natural estimator:

$$\widehat{VaR}^{(\alpha)}_{VHS,t-1}(\epsilon^{(p)}_t) \;=\; \sqrt{h^{(p)}_{v,t}}(\hat{\theta}_T)\,\hat{\xi}^{(p)}_{\alpha}$$

The following chart presents an illustration of a portfolio VaR using both the FHS approach (that we did not cover here) and the VHS approach for a simulated series (following the previous model with Gaussian innovations):

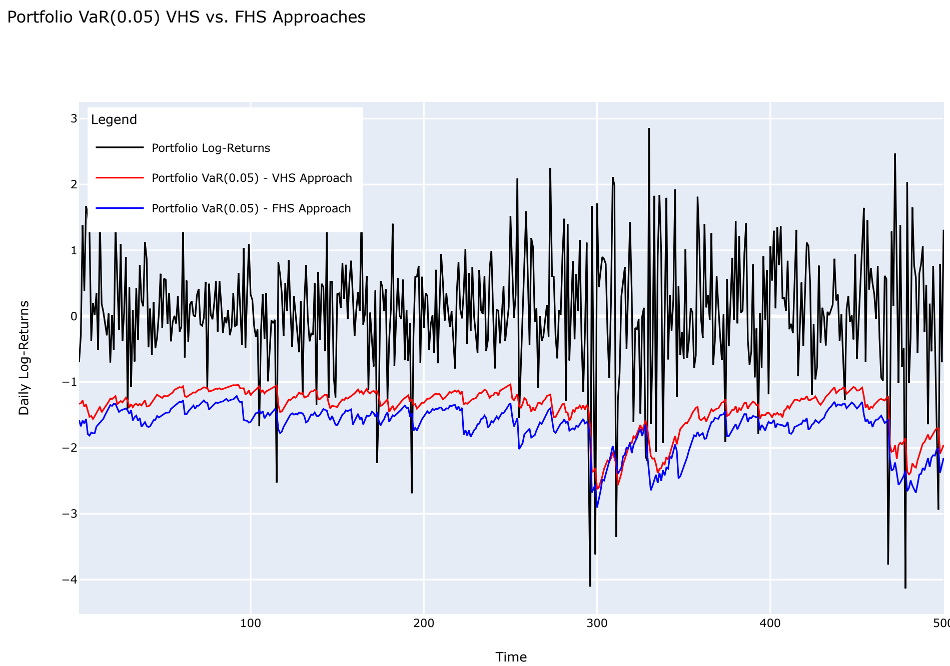


Figure 4. Portfolio VaR(5%) - VHS vs. FHS Approaches

Once we realize that with this approach we can simply treat the portfolio log-returns as a univariate time-series (it can be time-consuming to reconstruct the past returns of all the portfolios with the different potential weights though - and therefore be challenging to be involved in optimization processes), we can easily compute the CoVaR of a portfolio and derive confidence intervals for the CoVaR of the portfolio at each instant t. We will now move to a few other topics trying to get a few on the direction of a potential stock price (or spread if we consider a long/short approach) to complement this first model aiming at hedging extreme risks at a portfolio level.

3. Factor Analysis

In this section, we outline some of the basics of factor analysis as it applies to investment thinking. Factor analysis was popularized by economists E. Fama and KR. who noticed that the CAPM (Capital Asset Pricing Model) was not totally satisfactory, since the residuals remained correlated. They introduced a three- then five-factor model in Fama, French (1993) [16]. The practice of factor research has then been extended with the discovery of numerous factors that can generally be classified in the following categories: momentum, value, growth, quality, size. This section will be divided into two sub-sections, dealing with Fama French's historical approach to determining factors and a more modern approach based on the use of Princiapl Component Analysis. I would like to remind we that here too we will only present illustrative cases recreated using open-source data and well-known public models, for reasons of confidentiality.

3.1. Fama-French approach

The famous and well-renowned economists Fama and French introduced the so-called "Fama-French Three-Factor Model", which expanded upon the Capital Asset Pricing Model (CAPM) by incorporating additional factors to explain stock returns. Their approach to factor investing is grounded in empirical research and the belief that systematic factors, beyond market risk, play a significant role in explaining asset returns. They try to overcome the few shortcomings of the CAPM model sofar. The CAPM is indeed a widely used model in finance that relates the expected return of an asset to its beta, that is a measure of its sensitivity to market movements. According to the CAPM, the expected return of an asset is determined by the risk-free rate, the asset's beta, and the expected market risk premium. The 3-factor models extended this model by adding two additional factors in addition to Market Risk (MKT) that is the same as in the CAPM and represents the return of the overall market: Size (SMB - Small Minus Big) that captures the historical tendency of small-cap stocks to outperform large-cap stocks. And Value (HML - High Minus Low) that captures the historical tendency of value stocks (those with low price-to-book ratios) to outperform growth stocks (those with high price-to-book ratios). They found that size and value factors help explain the cross-sectional variation in stock returns, suggesting that these factors are associated with higher expected returns. Therefore investors can construct portfolios that emphasize exposure to the size and value factors to potentially achieve higher returns. Smaller-cap and value stocks may offer a risk premium, which means they have the potential to outperform the broader market over the long term. Later, Fama and French added two other factors to their inital model: i.e. profitability (RMW - Robust Minus Weak) and investment (CMA - Conservative Minus Aggressive). These factors aim to capture additional dimensions of stock returns beyond market risk, size, and value. One key element of this approach is that when they want to evaluate the performance of a factor through time, they don't just consider the stocks that have the highest loadings but rather consider the spread in performance between a top-quantile and the lowest quantile (regarding to the loadings). This method characterizes the Fama-French method to factor investing.

The rationale for considering the spread (difference) between the highest and lowest stocks within each factor category, rather than solely focusing on the stocks with the highest exposure, is to create a more diversified and robust factor investing strategy. Indeed, by including both the highest and lowest exposure stocks within a factor category, they create a more diversified portfolio. Diversification helps reduce the risk associated with individual stock or factor-specific events. If we only focus on the highest-exposure stocks, the portfolio may become concentrated and more sensitive to the performance of a few individual stocks. It is also useful for risk management (in a broader extend). High exposure stocks may perform well in certain market conditions but poorly in others. Conversely, low exposure stocks may outperform when market dynamics change. Holding both types of stocks can help balance risk and return. It also allows a better factor robustness. Indeed factor exposures can change over time due to market dynamics, economic cycles, and company-specific events. Stocks that have historically had high factor exposure may see their exposure decrease, and vice versa. By considering the entire spread, we are more adaptable to changing factor dynamics. Let's highlight directly the key difference between this approach and a more general approach based on Principal Component Analysis (PCA) that we will mention later. The Fama-French method relies on existing factors, that are based on fundamentals chosen by the practitioner. For this reason, this approach - even if rather basic - is the most natural to support long-short discretionary investing, as analysts may now better which indicators are the most relevant for each stocks or sectors. The PCA approach on the contrary, aims at extracting information from data and summarize it into independent/ uncorrelated components and only after can be subject to economic interpretation. We can quote a few main factors (which is not an extensive list) and their associate financial indicators that can help create the Fama-French factors and other extensions:

- **Value factor**: Price-to-Earnings (P/E) Ratio that compares a stock's current price to its earnings per share. Price-to-Book (P/B) Ratio that compares a stock's market value to its book value (assets - liabilities). Price-to-Sales (P/S) Ratio that compares a stock's market value to its revenue.
- **Quality Factor**: Return on Equity (ROE) that measures a company's profitability in relation to shareholder equity. Return on Assets (ROA) that measures a company's profitability in relation to its total assets. Profit Margin that measures the percentage of revenue that turns into profit.
- **Quality Factor**: Return on Equity (ROE) that measures a company's profitability in relation to shareholder equity. Return on Assets (ROA) that measures a company's profitability in relation to its total assets. Profit Margin that measures the percentage of revenue that turns into profit.
- **Size Factor**: Market Capitalization refers to the total market value of a company's outstanding shares. Total Assets refers to the total assets of a company.
- **Momentum**: The price momentum measures the recent price performance of a stock and earnings momentum measures the recent growth in earnings.
- **Volatility**: log-returns standard deviation measures the historical volatility of a stock's returns.
- **Growth Factor**: earnings growth measures the rate at which a company's earnings are growing and revenue growth measures the rate at which a company's revenue

is growing.

More generally, factor investing relies on the observations that according to some particular economic conditions (that may be repeated over time and follow the well-known economic cycles), factors will have similar performances. These different market conditions for which we observe clear and differentiated performance for the factors are often called "regimes". These regimes can be modeled with quantitative tools (e.g. Markov-switching models - we will not discuss these models here as we prefer a discretionary approach for regime selection but can be the subject of further work.) Let's mention a few well-kown stylized facts for the above factors according to different regimes:

- **Value Factor**:
  - **Expansionary/Recovery**: Value factors often perform well during economic recoveries when interest rates are low, and companies with lower valuations become attractive.
  - **Contractionary/Recession**: On the contrary, value may struggle during recessions when investors flock to safer assets, and companies with higher quality and growth characteristics are favored.
- **Quality Factor**:
  - **Expansionary/Recovery**: Quality factors tend to perform consistently well during economic expansions when investors seek financially stable and well-managed companies.
  - **Contractionary/Recession**: In addition hand, quality may also perform relatively well during recessions as investors prioritize companies with strong balance sheets and low debt.
- **Size Factor**:
  - **Expansionary/Recovery**: Smaller-cap stocks often outperform during economic recoveries as they have more room for growth.
  - **Contractionary/Recession**: Larger-cap stocks may be favored during economic downturns as they are often perceived as more stable.
  - **Momentum Factor**:
  - **Expansionary/Recovery**: Momentum factors can perform well during expansions when trends are strong and investors follow winners.
  - **Contractionary/Recession**: Momentum factors may lose steam during market downturns when trends become less clear, and investors seek safety.

As an illustration we show the performance of the short-term momentum factor calculated on the stocks of the French CAC 40 index. The methodology is as follows: consider a 50-day rolling period (t-50,..,t-1) and compute the performance at date t of a portfolio composed of the spread between the top 5 performers and the bottom 5 (all equally-weighted) and take it as the performance of the factor. It seems that the short-term momentum has significantly under-performed over the considered period, which reveals some potential mean reversion in this market.

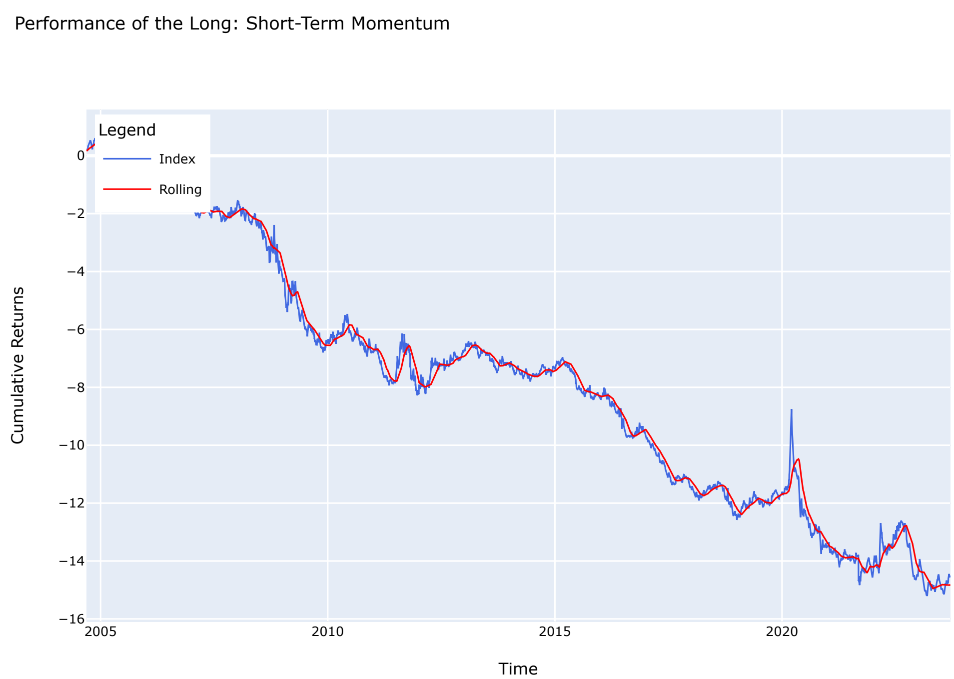


Figure 5. Performance of the Short-term Momentum Factor

3.2. PCA Approach

Without entering into the details, we also ran a quick principal component analysis on the stocks composing the CAC 40 over the same period and show in the following figures the marginal contribution of each factor in the explanation of the total variance of the stock returns, then the exposure of each stock to the first factor and a mapping of the stocks according to the first two components. We will not elaborate on this approach as we focused almost exclusively on the first one but I just wanted to mention an alternative approach and how we could use it in practice briefly.

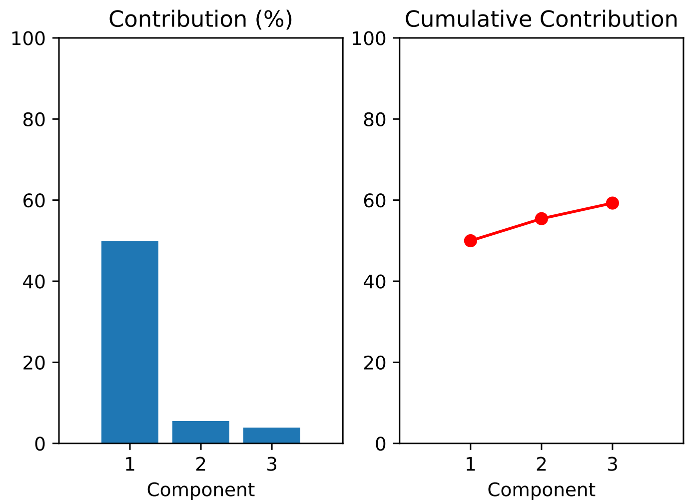


Figure 6. Cumulative Conrtibution of Principal Components

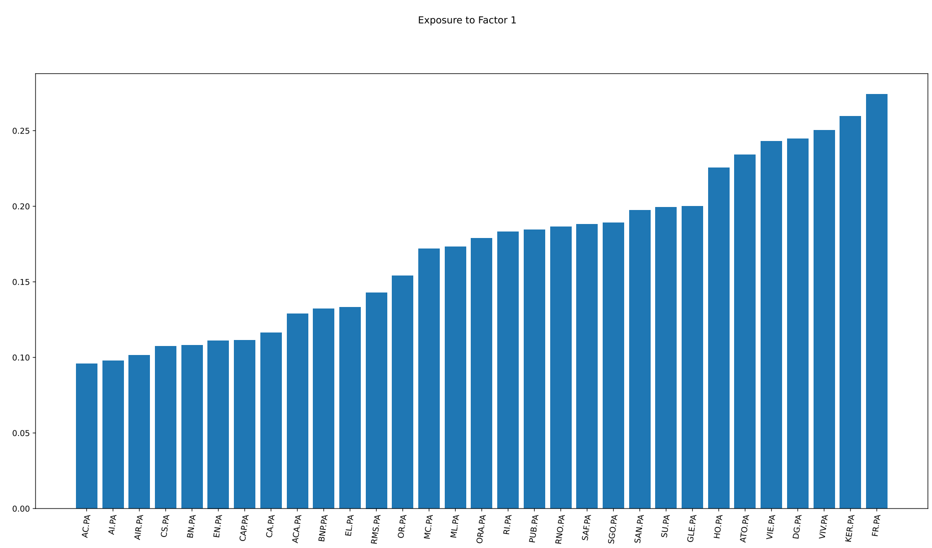


Figure 7. Stocks Exposure to PC1

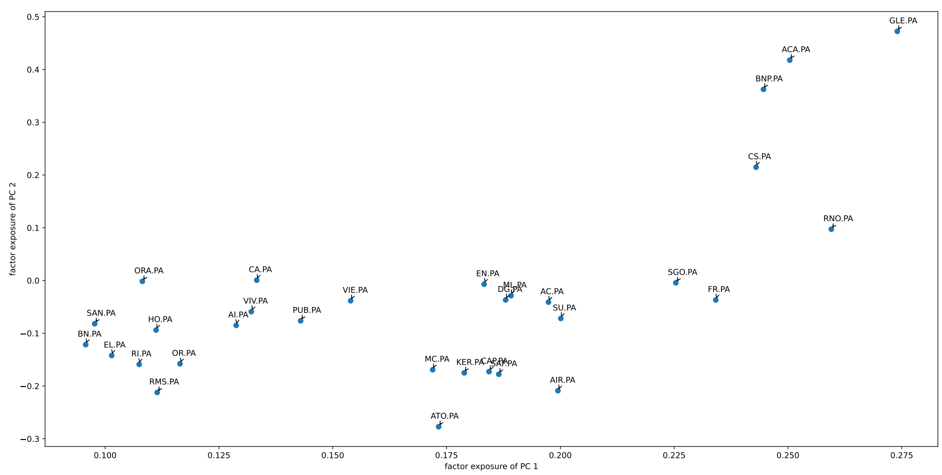


Figure 8. PC1/ PC2 mapping

3.3. Further Studies

As already mentioned above, in order to use the factor analysis previously mentioned for investment purposes, it is common for the investor to identify certain ”regimes” (historical periods with similar macro and market characteristics) during which the performance of the main factors is assumed to follow a similar pattern. The work and analyses carried out in the course of my work focus essentially on discretionary assessment of regimes and their possible shifts. Indeed, while quantitative approaches are very effective for optimization purposes based on the analysis of past patterns, regime changes are often sharp and marked by macro events (an unexpected Fed rate hike, an inflation spike, a health crisis, a major geopolitical conflict, etc.) or market events (financial crisis, real estate market crisis, etc.). The often unprecedented nature of such events makes them unpredictable by most traditional quantitative models, so the assessment of an experienced trader whose job is to assess and anticipate such events may prove more useful in anticipating a potential regime change. However, one area I’d like to study in more detail for the end of my internship is the analysis of regime shifts using a Markov-switching model, and how this can be articulated within a discretionary dominant approach.

4. Pair Trading

In this section, we'll look at the statistical properties of pair trading, which is one of the main strategies adopted by long/short funds to hedge against market fluctuations. As a reminder, long/short funds aim to deliver their clients an absolute performance, decorrelated from the market (or beta neutral - where beta refers to the coefficient associated with the market risk premium in the CAPM model). Having mentioned factor analysis in the previous section, we understand that long-short hedge funds generally wish to hedge against other factors (i.e. growth, value, momentum, size - and not only the market) in order to keep alpha as sole exposure (the intercept of the regression of returns on factors, assuming it is non-zero and that active management can create additional value that the market fails to capture instantaneously). This approach is therefore generally based on the hypothesis of a market with friction, relying on an asymmetry of information created by expertise. However, although the bulk of the performance sought a priori lies in alpha, it seems possible to boost performance by making intelligent use of the combination between the various factors (e.g. certain market conditions lend themselves well to the mean reversion of stocks - it is therefore possible to exploit this by making intelligent use of short/medium-term exposure to momentum factors as part of a strategy that could be described as "smart factor" in reference to popular "smart-beta" strategies). Although beta is one factor among many, it seems very difficult to profit from a smart exposure to it, since its performance can be highly volatile and asymmetrical. Volatile, because it is directly impacted by sensitive macroeconomic data or communications (key rates communicated by Central Banks, employment figures, GDP, CPI print, etc.). And asymmetrical, since it is well known that market premium exposure creates positive long-term earnings expectations, but at the cost of getting exposed to market volatility. Yet, while it's well known that volatility presents series containing clusters (i.e. strong autocorrelation of squared returns - which motivates all the literature on ARCH and GARCH movements discussed in Part 1), it can also be shown that volatility behaves asymmetrically and is proportionally more pronounced after a negative shock (an innovation in GARCH models) than after a positive impact. This effect is known as the leverage effect and is at the origin of numerous refinements of GARCH models, such as the Threshold ARCH (TARCH) introduced and developed in Gourieroux, Monfort (1992) [17], Rabemananjara and Zakoian (1993) [18] and Zakoian (1994) [19] or simply Asymertric GARCH (AGARCH) models.

All this to say that, in general, it's a sound idea to continue hedging beta risk exposure; and for this pair trading (the strategy of hedging a long exposure to one stock with a short exposure to another stock with a similar beta), is the most natural approach. Furthermore, the relationship between long and short can also be interpreted as the short position financing the long position (i.e. so that leverage can be used to enhance exposure). It is this leverage that underpins the main value proposition of long-short hedge funds, which - given their confidence in the ability of their strategies to deliver low-volatility performance - can substantially boost their returns through leverage. In this section, we'll look at the statistical properties of these long-short series and attempt to gain some insight into the underlying statistical and mathematical theory. The first sub-section will deal with the cointegration approach, and also with an index replication method that allows, for example, to gain exposure to a sector or factor with a reduced number of cointegrated stocks. The first sub-section therefore focuses on the introduction of the cointegration concept and its potential use. The second sub-section is based on the use of an Ornstein-Uhlenbeck movement - commonly used to model interest rate reversion, but which can also be useful for representing the spread between a long and a short for an equity strategy (when the two series are cointegrated, for example).

4.1. Cointegration and Index Replication

First, let's recall the concept of cointegration. It was introduced by Granger (1981) [20] to extend the study of stationary process to processes which shared a common trend and for which a linear combination of them resulted in a stationary process. This is particularly useful for us, since the stock prices processes are generally not stationary and therefore a lot of financial econometric tool do not apply but since we are focused on the difference between two processes, we can retrieve these stationarity conditions. We will briefly discuss in the next subsection how these spreads can be modeled via an Ornstein-Uhlenbeck proecss but we will first focus on a paper offering a methodology for portfolio replication relying on cointegration. As we mentioned the importance of factors in last section, we might also want to get an exposure to a particular sector or index (because we belief that the regime under which we are operating is favourable for the performance of this sector). But the transaction costs often prevent the average trader to replicate a perfect exposure to an index (let's assume that there is no ETF tracking this index). Therefore, it might be useful to use the cointegration approach to create an index replication strategy - as discussed C. Alexander and A. Dimitriu (2002) [21]. Note that here I use index replication to gain directional exposure to certain factors or sectors, but do not wish to implement the long-short strategy presented in the paper, which consists of recreating two synthetic indexes from a base index - one structurally outperforming it and the other continuously underperforming it. The idea is to take a long position on the "positive" index and a short position on the "negative" index, in order to gain exposure to a supposedly widening spread between the two. However, this strategy ignores the effects of mean reversion and fails to create a satisfactory performance after adjusting for transaction costs. We will now introduce an extension of the model presented above to the portfolio case. In particular, we will show how we can calculate and obtain a confidence interval for the COVAR of a portfolio. How can this be used in practice? This kind of model can be useful if we want to assess the impact of an extreme shock to a macro variable (e.g. the price of oil) on your portfolio. For example, one question we might consider is: what will be the impact on my portfolio's VaR if the price of oil exceeds its own VaR? The same question can be asked about the impact of a particular exchange rate shock on an equity portfolio... Or for a particular sector - for example, what is the impact of a freight rate shock (spot rates carried by certain shipping companies and airlines for cargo transport) on the sector. In this last example, big shocks might follow events such as the congestion of major shipping routes as it has been the case recently in across the Panama Canal following droughts.

In this section we will simply recall the key elements of the theory of cointegration that will be useful in this report. Two series are said to be cointegrated if a linear combination of them is stationary. Let $index_t$ be the price of the index at date $t \in 1, ...n$ for a series composed of n observations. Let $p_{k,t}$ be the price the k^{th} component of the index (for $k \in 1, .., N_k$)[3] at date t. Then, let's introduce coefficients $c'_1, ..., c'_{N_a}$ verifying the following equation (4.1):

$$log(index_t) = \sum_{k=1}^{N_a} c'_k log(p_{k,t}) + \epsilon'_t \text{ for } t \in 1, ..., n \tag{4.1}$$

with ϵ'_t being the innovation and we use the logarithm of the quantity for practical reasons [4].

A Granger cointegration test should be performed on a regression without intercept. To test the cointegration hypothesis, which corresponds to the null hypothesis:

$$\mathcal{H}_0 = (\epsilon_t)_t \text{ has a unique root}$$

that we reject at a α_0 nominal level $= 5\%$ for $\mathcal{H}_1 = (\epsilon_t)$ is stationary, we use the Augmented Dickey Fuller (ADF) test applied to $\hat{\epsilon'_t}$, i.e. the residuals, being the empirical counterpart to the ϵ'_t. Once the cointegration relation has been confirmed, we build the weights of the replicating portfolio by taking the coefficients $c_1, ..., c_{N_a}$, and normalising[5] them to get $w_1, ..., w_{N_a}$ in equation:

$$log(index_t) = c_0 + \sum_{k=1}^{N_a} c_k log(p_{k,t}) + \epsilon_t \text{ for } t \in 1, ..., n \tag{4.2}$$

As practitioners cannot trade weights directly, one must compute the number of stocks of each asset to have in portfolio at each date t and are given by 4.3.

[3] $N_a \leq 40$ represents the number of assets we consider for replication

[4] we tested by taking the quantities without the log and found comparable results

[5] centered and add up to 1, we added a constant term in this regression

$$\pi_{k,t} = \frac{w_k pf_{t-1}}{p_{k,t}} \text{ for } t \in 1, ..., n \tag{4.3}$$

with pf_0 being the initial capital which is available and pf_t the price of the replicating portfolio at date t. We then get the prices of our replicating portfolio given by equation (4.4):

$$pf_t = \sum_{k=1}^{N_a} \pi_{k,t} p_{k,t} \text{ for } t \in 1, ..., n \tag{4.4}$$

As timing convention, we assume that that the prices $p_{k,t}$ and pf_t are the prices at the end of the trading day, while the portfolio compositions $\pi_{k,t}$ are determined at the end of the date $t-1$ (which is equivalent to beginning of date t). Let's formulate two remarks with respect to the paper.

We introduced pf_0 as the initial capital (AUM) available for the investor. The authors do not mention it, but depending on the available AUM the number of shares detained of each stock may differ. Especially if we consider only integers as stock numbers, the above stock number calculation must be rounded and if the initial capital is small, it can have a significant effect on the effectiveness of the replication. If pf_0 is large enough, we can accept the standard assumption for complete markets that one can detain an infinitesimally small portion of stocks. This assumption is implicitly made in this paper.

Concerning the practical implementation of the procedure, we rely on the one well described in the paper, namely:

1. Choose a calibration period length to perform the estimation of the number of shares $\pi_{k,t}$ (e.g. 1,2,3,4,5 years)
2. Fix a number of days (e.g. 10 days) between two portfolio rebalancing[6].
3. Choose the number of assets in the replicating portfolio (we followed the simple method given by the authors to take the assets whose true weights in the index are the largest and which is highly debatable[7]). Before each rebalancing date the cointegration relationship is checked and new weights are computed over the rolling period.
4. Before each rebalancing we compute the implied transaction costs by the updating the portfolio composition and spread them uniformly over the period between two rebalancing (to avoid jumps at rebalancing dates) and the new value of the portfolio is computed net of these transaction costs. We take 20 basis points of the traded volume as suggested in the paper.

The following chart provide an illustration of the replication method applied to the CAC 40 index using 5 stocks (recalibrated every 10 days) and we can observe a very good fit.

[6] a change in the composition of the portfolio

[7] especially as we do not know the composition of the index at each date t but just the one at the last day

Figure 9. Replication of the CAC 40 Index using 5 stocks with daily 10-day rebalancing

We have seen that the concept of cointegration can be used to replicate an index by means of a reduced number of stocks cointegrated with it. However, the main advantage of these cointegration tests is to find ideal candidates for pair-trading. Let's not forget that the aim of pair-trading is generally to trade simultaneously, with a long and a short position, two assets with a common trend, in order to hedge against market movements or factors over which we have no view. When a significant spread is created between two co-integrated assets, it can be interesting to bet on a mean-reversion. In particular, this spread can be modeled with a Ornstein-Uhlenbeck process, which can account for this mean-reversion and propose a natural statistical arbitrage approach.

4.2. Ornstein Uhlenbeck Process and Optimal Control

As previously mentioned, the long-short trading approach is well suited to pair trading and the use of statistical arbitrage, notably through an Ornstein Uhlenbenck process. To ensure the model's validity, we can perform an ADF (Augmented Dickey Fuller) test to verify the absence of unit root and therefore the model's stationarity.

We recall the natural formulation of an Ornstein-Uhlenbeck process:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t.$$

The parameters are:

- $\kappa > 0$: mean reversion coefficient
- $\theta \in \mathbb{R}$: long term mean
- $\sigma > 0$: volatility coefficient

and whose well-known solution takes the following form:

$$X_t = \theta + (X_0 - \theta)e^{-\kappa t} + \int_0^t \sigma\, e^{\kappa(s-t)} dW_s.$$

In this part we will put aside the estimation considerations and focus solely on the trading strategies that can adopted once the OU model has been fitted to the data. We can for instance rely on ML estimation and state-space methods applied to the discretized process as presented in Shumway and Stoffer (1982) [22] for instance.

The basic idea of a simple mean-reversion trading strategy based on an Ornstein-Uhlenbeck process could consist in calculating the spread between the current process and its long-term mean parameter: θ. Then, the trader can open a position when the spread significantly deviates from zero, indicating that the asset's price has moved away from its mean. We can use a threshold to trigger an entry. For example, if the spread exceeds 2, we can consider it as a good entry signal. Then, we could consider closing the position when the spread approaches zero or starts to move back towards zero, indicating mean reversion. This can be determined using another threshold, such as when the spread falls below 0.5. This simple approach, coupled with potential other signals and the discretionary sentiment of the trader can help her boosting her performance and is a good example of how quantitative and fundamental approach can collaborate to create alpha. This brief section, which is more an invitation to future research than a result in itself, completes the overview of the different approaches I wanted to present in this report. The aim was indeed to see how quantitative methods could help discretionary practitioners in their decision-making within a long-short framework. Projects using tools similar to those mentioned in this report coupled with access to larger datasets present for me one of the main development axes for this investment approach in the near future.

5. Conclusion

In this report, we present a list of research topics tackled during the last five months that have a potential interest in optimizing trading strategies through alpha generation or risk management (including extreme risk). Although the study of these topics has represented a substantial part of my reflection and time over the period, I have also had the opportunity to assist the teams in their daily tasks through the processing and analysis of data or by conducting more qualitative analyses on particular sectors. This positive experience enabled me to develop a good understanding of the major dynamics of the equity markets and the link between fundamental and positioning issues as the main drivers of short and medium-term share price movements. This strong exposure to directional trading questions was complemented by my own analysis of long-short fund specific problematic (still poorly studied by theoreticians° and which explain a large part of the low-frequency movements. Finally, it was an opportunity for me to implement the models I'd been working on in the past and to extend them, emphasizing the practical dimension as a complement to the purely academic considerations I'd been focusing on until now. Finally, I believe that the top-down approach discussed in this paper should be seen as complementary, rather than rival, to the bottom-up approach more generally and almost exclusively adopted by long-short hedge funds. Indeed, in a constantly changing environment, prone to major disruptions (e.g. the health crisis was an example of an unprecedented economic and market situation), purely quantitative strategies - based exclusively on the recognition and exploitation of past patterns - generally fail to anticipate sudden major shifts in trends, and may expose themselves to extreme losses because they underestimate the occurrence of rare events. The almost exclusive use of Gauissan models until the global financial crisis of 2008 also demonstrated to another extent the underestimation of the potential danger of extreme events - as described by the American journalist F. Salmon in his article Salmon (2009) [23]. On the other hand, bottom-up fundamental analysis, which is by definition more concentrated, focused on alpha and more often than not on anticipating a trend reversal, is often less optimized and does not seek to take advantage of abnormal statistical movements in factors or relationships between stocks (e.g. cointegration). That's why I'm convinced that the future of short- to medium-term equity investing lies in the collaboration of these two approaches. Meanwhile, quantitative funds still have a clear advantage over the trading of more complex, high-frequency financial products.

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