

---

# Probabilistic Programming with Gaussian Process Memoization

---

009                   **Anonymous Author(s)**

010                   Affiliation

011                   Address

012                   email

## Abstract

This paper describes the *Gaussian process memoizer*, a probabilistic programming technique that uses Gaussian processes to provides a statistical alternative to memorization. Memoizing a target procedure results in a self-caching wrapper that remembers previously computed values. Gaussian process memoization additionally produces a statistical emulator based on a Gaussian process whose predictions automatically improve whenever a new value of the target procedure becomes available. This paper also introduces an efficient implementation, named `gpmem`, that can use kernels given by a broad class of probabilistic programs. The flexibility of `gpmem` is illustrated via three applications: (i) GP regression with hierarchical hyper-parameter learning, (ii) Bayesian structure learning via compositional kernels generated by a probabilistic grammar, and (iii) a bandit formulation of Bayesian optimization with automatic inference and action selection. All applications share a single 50-line Python library and require fewer than 20 lines of probabilistic code each.

## 1 Introduction

Probabilistic programming could be revolutionary for machine intelligence due to universal inference engines and the rapid prototyping for novel models (Ghahramani, 2015). This levitates the design and testing of new models as well as the incorporation of complex prior knowledge which currently is a difficult and time consuming task. Probabilistic programming languages aim to provide a formal language to specify probabilistic models in the style of computer programming and can represent any computable probability distribution as a program. In this work, we will introduce new features of Venture, a recently developed probabilistic programming language. We consider Venture the most compelling of the probabilistic programming languages because it is the first probabilistic programming language suitable for general purpose use (Mansinghka et al., 2014). Venture comes with scalable performance on hard problems and with a general purpose inference engine. The inference engine deploys Markov Chain Monte Carlo (MCMC) methods (for an introduction, see Andrieu et al. (2003)). MCMC lends itself to models with complex structures such as probabilistic programs or hierarchical Bayesian non-parametric models since they can provide a vehicle to express otherwise intractable integrals necessary for a fully Bayesian representation. MCMC is scalable, often distributable and also compositional. That is, one can arbitrarily chain MCMC kernels to infer over several hierarchically connected or nested models as they will emerge in probabilistic programming.

One very powerful model yet unseen in probabilistic programming languages are Gaussian Processes (GPs). GPs are gaining increasing attention for representing unknown functions by posterior probability distributions in various fields such as machine learning, signal processing, computer vision and bio-medical data analysis. Making GPs available in probabilistic programming is crucial to allow a language to solve a wide range of problems. Hard problems include but are not limited

054 to hierarchical prior construction (Neal, 1997), Bayesian Optimization Snoek et al. (2012) and sys-  
055 tems for inductive learning of symbolic expressions such as the one introduced in the Automated  
056 Statistician project Duvenaud et al. (2013); Lloyd et al. (2014). Learning such symbolic expressions  
057 is a hard problem that requires careful design of approximation techniques since standard inference  
058 method do not apply.

059 In the following, we will present `gpmem` as a novel probabilistic programming technique that solves  
060 such hard problems. `gpmem` introduces a statistical alternative to standard memoization. Our con-  
061 tribution is threefold:

- 063 • we introduce an efficient implementation of `gpmem` in form of a self-caching wrapper that  
064 remembers previously computed values;
- 065 • we illustrate the statistical emulator that `gpmem` produces and how it improves with every  
066 data-point that becomes available; and
- 067 • we show how one can solve hard problems of state-of-the-art machine learning related to  
068 GP using `gpmem` in a Bayesian fashion and with only a few lines of Venture code.

070 We evaluate the contribution on problems posed by the GP community using real world and syn-  
071 thetic data by assessing quality in terms of posterior distributions of symbolic outcome and in terms  
072 of the residuals produced by our probabilistic programs. The paper is structured as follows, we will  
073 first provide some background on memoization. We will explain programming in Venture and pro-  
074 vide a brief introduction to GPs. We introduce `gpmem` and its use in probabilistic programming and  
075 Bayesian modeling. Finally, we will show how we can apply `gpmem` on problems of causally struc-  
076 tured hierarchical priors for hyper-parameter inference, structure discovery for Gaussian Processes  
077 and Bayesian Optimization including experiments with real world and synthetic data.

## 079 2 Background

### 081 2.1 Memoization

083 Memoization is the idea of computing the value of a function given an input once and then store  
084 it. If the function is called again at the same input, we return the stored value without evaluating  
085 the function body again. Research on the Church language (Goodman et al., 2008) pointed out that  
086 although memoization does not change the semantics of a program for deterministic programs it  
087 does for stochastic ones. This implies that one can stochastically decide on each function evaluation  
088 whether to use a stored value or a new function evaluation. Here, memoization induces random  
089 world semantics (Poole, 1993; Sato, 1995) over the probabilistic program: we interpret a specific  
090 mapping from function input to function output as a single possible world (?).

### 091 2.2 Venture

093 Venture is a compositional language for custom inference strategies that comes with a Scheme- and  
094 Java-Script-like front-end syntax. Its implementation is based on on three concepts. (i) sto-  
095 chastic procedure interfaces that specify and encapsulate random variables, analogously to conditional  
096 probability tables in a Bayesian network; (ii) probabilistic execution traces that represent execution  
097 histories and capture conditional dependencies; and (iii) scaffolds that partition execution histories  
098 and factor global inference problems into sub-problems. These building blocks provide a powerful  
099 way to represent probability distributions; some of which cannot be expressed with density func-  
100 tions. For the purpose of this work the most important Venture directives that operate on these  
101 building blocks to understand are ASSUME, OBSERVE, SAMPLE and INFER. ASSUME induces  
102 a hypothesis space for (probabilistic) models including random variables by binding the result of an  
103 expression to a symbol. SAMPLE simulates a model expression and returns a value. OBSERVE  
104 adds constraints to model expressions. INFER instructions incorporate observations and cause Ven-  
105 ture to find a hypothesis that is probable given the data.

106 INFER is most commonly done by deploying the Metropolis-Hastings algorithm (MH) (Metropolis  
107 et al., 1953). Many algorithms used in the MCMC world can be interpreted as special cases of  
MH (Andrieu et al., 2003). We can outline the MH algorithm as follows. For  $T$  steps we sample  $x^*$

108 from a proposal distribution  $q$ :

$$x^* \sim q(x^* | x^{(t)}) \quad (1)$$

109 which we accept  $(x^{t+1} \leftarrow x^*)$  with ratio:

$$\alpha = \min\left\{1, \frac{p(x^*)q(x^t | x^*)}{p(x^{(t)})q(x^* | x^t)}\right\} \quad (2)$$

110 Venture implements an MH transition operator for probabilistic execution traces.

### 116 2.3 Gaussian Processes

117 In the following, we will introduce GP related theory and notations. We will exclusively work on  
 118 two variable regression problems. Let the data be real-valued scalars  $\{x_i, y_i\}_{i=1}^n$  (complete data will  
 119 be denoted by column vectors  $\mathbf{x}, \mathbf{y}$ ). GPs present a non-parametric way to express prior knowledge  
 120 on the space of possible functions  $f$  that we assume to have generated the data.  $f$  is assumed latent  
 121 and the GP prior is given by a multivariate Gaussian  $f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}), k(x_i, x'_i))$ , where  $m(\mathbf{x})$  is  
 122 a function of the mean of all functions that map to  $y_i$  at  $x_i$  and  $k(x_i, x'_i)$  is a kernel or covariance  
 123 function that summarizes the covariance of all functions that map to  $y_i$  at  $x_i$ . We can absorb the  
 124 mean function into the covariance function so without loss of generality we can set the mean to  
 125 zero. The marginal likelihood can be expressed as:

$$p(\mathbf{y}|\mathbf{x}) = \int p(\mathbf{y}|\mathbf{f}, \mathbf{x}) p(\mathbf{f}|\mathbf{x}) d\mathbf{f} \quad (3)$$

130 where the prior is Gaussian  $\mathbf{f}|\mathbf{x} \sim \mathcal{N}(0, k(\mathbf{x}, \mathbf{x}'))$ . We can sample a vector of unseen data from the  
 131 predictive posterior with

$$\mathbf{y}^* \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (4)$$

132 for a zero mean prior GP with a posterior mean of:

$$\boldsymbol{\mu} = \mathbf{K}(\mathbf{x}, \mathbf{x}^*) \mathbf{K}(\mathbf{x}^*, \mathbf{x}^*)^{-1} \mathbf{y} \quad (5)$$

133 and covariance

$$\boldsymbol{\Sigma} = \mathbf{K}(\mathbf{x}, \mathbf{x}) - \mathbf{K}(\mathbf{x}, \mathbf{x}^*) \mathbf{K}(\mathbf{x}^*, \mathbf{x}^*)^{-1} \mathbf{K}(\mathbf{x}^*, \mathbf{x}). \quad (6)$$

134  $\mathbf{K}$  is a covariance function. The log-likelihood is defined as:

$$\log P(\mathbf{y} | \mathbf{X}) = -\frac{1}{2}\mathbf{y}^\top (\mathbf{K} + \sigma^2 \mathbf{I})^{-1} \mathbf{y} - \frac{1}{2} \log |\mathbf{K} + \sigma^2 I| - \frac{n}{2} \log 2\pi \quad (7)$$

135 with  $n$  being the number of data-points and sigma the independent observation noise. Both log-  
 136 likelihood and predictive posterior can be computed efficiently in a Venture SP with an algorithm  
 137 that resorts to Cholesky factorization(Rasmussen and Williams, 2006, chap. 2) resulting in a com-  
 138 putational complexity of  $\mathcal{O}(n^3)$  in the number of data-points.

139 The covariance function covers general high-level properties of the observed data such as linear-  
 140 ity, periodicity and smoothness. The most widely used type of covariance function is the squared  
 141 exponential covariance function:

$$k(x, x') = \sigma^2 \exp\left(-\frac{(x - x')^2}{2\ell^2}\right) \quad (8)$$

142 where  $\sigma$  and  $\ell$  are hyper-parameters.  $\sigma$  is a scaling factor and  $\ell$  is the typical length-scale. Smaller  
 143 variations can be achieved by adapting these hyper-parameters.

144 Larger variations are achieved by changing the type of the covariance function structure. Note that  
 145 covariance function structures are compositional. We can add covariance functions if we want to  
 146 model globally valid structures

$$k_3(x, x') = k_1(x, x') + k_2(x, x') \quad (9)$$

147 and we can multiply covariance functions if the data is best explained by local structure

$$k_4(x, x') = k_1(x, x') \times k_2(x, x'); \quad (10)$$

148 both,  $k_3$  and  $k_4$  are valid covariance function structures.

### 3 Venture GPs

Given a stochastic process that implements the GP algebra above we can implement a GP sampler (4) to perform GP inference in a few lines of code. We can express simple GP smoothing with fixed hyper-parameters or a prior on hyper-parameters and perform MH on it while allowing users to custom design covariance functions. Throughout the paper, we will use the Scheme-like front-end syntax.

```

69 1 [ASSUME l (gamma 1 3)] ∈ {hyper-parameters}
70 2 [ASSUME sf (gamma 1 3)] ∈ {hyper-parameters}
71 3
72 4  $k(x, x') := \sigma^2 \exp\left(-\frac{(x-x')^2}{2\ell^2}\right)$ 
73 5
74 6 [ASSUME f VentureFunction( $k, \sigma, \ell$ ) ]
75 7 [ASSUME SE make-se (apply-function f l sf) ]
76 8 [ASSUME (make-gp 0 SE) ]
77 9
78 10 [SAMPLE GP (array 1 2 3)] % Prior
79 11 [OBSERVE GP D]
80 12 [SAMPLE GP (array 1 2 3)]
81 13 [INFER (MH {hyper-parameters} one 100) ]
82 14 [SAMPLE GP (array 1 2 3)] % Posterior

```

Listing 1: Bayesian GP Smoothing

The first two lines depict the hyper-parameters. We tag both of them to belong to the set {hyper-parameters}. Every member of this set belongs to the same inference scope. This scope controls the application of the inference procedure used. In this paper, we use MH throughout. Each scope is further subdivided into blocks that allow to do block-proposals. In the following we omit the block notation for readability, since we always choose the block of a certain scope at random.

The ASSUME directives describe the assumptions we make for the GP model, we assume the hyper-parameters  $\ell$  and  $\sigma$  (corresponding to  $\ell, \sigma$ ) to be 1 and 2. The squared exponential covariance function can be defined outside the Venture code with foreign conventional programming languages, e.g. Python. In that way, the user can define custom covariance functions without being restricted to the most common ones. We then integrate the foreign function into Venture as VentureFunction. In the next line this function is associated with the hyper-parameters. Finally, we assume a Gaussian Process SP with a zero mean and the previously assumed squared exponential covariance function.

In the case where hyper-parameters are unknown they can be found deterministically by optimizing the marginal likelihood using a gradient based optimizer. Non-deterministic, Bayesian representations of this case are also known (Neal, 1997).

We have already implemented this in listing 1. We draw the hyper-parameters from a  $\Gamma$ -prior for a Bayesian treatment of hyper-parameters. This is simple using the build in stochastic procedure that simulates drawing samples from a gamma distribution. The program gives rise to a Bayesian representation of GPs, which we will explore in the following.

### 3.1 A Bayesian interpretation

### 3.1.1 Data modelling as a special case of gpmem

From the standpoint of computation, a data set of the form  $\{(x_i, y_i)\}$  can be thought of as a function  $y = f_{\text{restr}}(x)$ , where  $f_{\text{restr}}$  is restricted to only allow evaluation at a specific set of inputs  $x$ . Modelling the data set with a GP then amounts to trying to learn a smooth function  $f_{\text{emu}}$  (“emu” stands for “emulator”) which extends  $f$  to its full domain. Indeed, if  $f_{\text{restr}}$  is a foreign procedure made available as a black-box to Venture, whose secret underlying source code is:

```
12 def f_restr(x):
13     if x in D:
14         return D[x]
15     else:
16         raise Exception('Illegal input')
```

216 Then the OBSERVE code in Listing 1 can be rewritten using gpmem as follows (where here the data  
 217 set D has keys  $x[1], \dots, x[n]$ ):

```
218 [ASSUME (list f_compute f_emu) (gpmem f_restr) ]
219 for i=1 to n:
220   [PREDICT (f_compute x[i])]
221   [INFER {MH {hyper-parameters} one 100} ]
222   [SAMPLE (f_emu (array 1 2 3))]
```

223 This rewriting has at least two benefits: (i) readability (in some cases), and (ii) amenability to active  
 224 learning. As to (i), the statistical code of creating a Gaussian process is replaced with a memoization-  
 225 like idiom, which will be more familiar to programmers. As to (ii), when using gpmem, it is quite  
 226 easy to decide incrementally which data point to sample next: for example, the loop from  $x[1]$  to  
 227  $x[n]$  could be replaced by a loop in which the next index  $i$  is chosen by a supplied decision rule.  
 228 In this way, we could use gpmem to perform online learning using only a subset of the available  
 229 data.

### 233 3.1.2 The efficacy of learning hyperparameters

236 The probability of the hyper-parameters of a GP with assumptions as above and given covariance  
 237 function structure  $\mathbf{K}$  can be described as:

$$241 P(\boldsymbol{\theta} | \mathbf{D}, \mathbf{K}) = \frac{P(\mathbf{D} | \boldsymbol{\theta}, \mathbf{K})P(\boldsymbol{\theta} | \mathbf{K})}{P(\mathbf{D} | \mathbf{K})}. \quad (11)$$

246 Let the  $\mathbf{K}$  be the sum of a smoothing and a white noise (WN) kernel. For this case, Neal suggested  
 247 the problem of outliers in data as a use-case for a hierarchical Bayesian treatment of Gaussian  
 248 processes (1997)<sup>1</sup>. The work suggests a hierarchical system of hyper-parameterization (Fig. 1a).  
 249 Here, we draw hyper-parameters from a  $\Gamma$  distributions:

$$254 \ell^{(t)} \sim \Gamma(\alpha_1, \beta_1), \sigma^{(t)} \sim \Gamma(\alpha_2, \beta_2) \quad (12)$$

258 and in turn sample the  $\alpha$  and  $\beta$  from  $\Gamma$  distributions as well:

$$263 \alpha_1^{(t)} \sim \Gamma(\alpha_\alpha^1, \beta_\alpha^1), \alpha_2^{(t)} \sim \Gamma(\alpha_\alpha^2, \beta_\alpha^2), \dots \quad (13)$$

---

268 <sup>1</sup>In (Neal, 1997) the sum of an SE plus a constant kernel is used. We stick to the WN kernel for illustrative  
 269 purposes.

```

270 Assuming the covariance structure is an additive comprised of a smoothing and a white noise
271 kernel, one can represent this kind of model using gpmem with only a few lines of code:
272
1 [ASSUME alpha (mem (lambda (i) (gamma 1 3 )))] ∈ {hyper-parameters-Γ}
2 [ASSUME beta (mem (lambda (i) (gamma 1 3 )))] ∈ {hyper-parameters-Γ}
273
3
274
4 [ASSUME l (gamma (alpha 1) (beta 1))] ∈ {hyper-parameters}
5 [ASSUME sf (gamma (alpha 2) (beta 2))] ∈ {hyper-parameters}
6 [ASSUME sigma (uniform 0 5 )] ∈ {hyper-parameters} % Fig. 2
275
7 % above: structured prior, Fig. 1a
276
8
277
9  $k_1(x, x') := \theta^2 \exp\left(-\frac{(x-x')^2}{2\ell^2}\right)$ 
10  $k_2(x, x') := \sigma^2 \delta_{x,x'}$ 
278
11
279
12 [ASSUME k1 VentureFunction(k1, θ, ℓ) ]
13 [ASSUME k2 VentureFunction(k2, σ) ]
280
14
281
15 [ASSUME SE make-se (apply-function k1 l sf) ]
282 [ASSUME WN make-se (apply-function k1 sigma) ]
283
16
284
17 [ASSUME (list f_compute f_emu) (gpmem f_restr (function-plus SE WN) )]
285 [SAMPLE (f_emu (array 1 2 3))] % prior, Fig. 1b
286
18
287
19
288
20
289
21 for i=1 to n:
22   [PREDICT (f_compute x[i])] % observing with a look-up function
23 [SAMPLE (f_emu (array 1 2 3))] % after observation, Fig. 1c
24
25
26
27
28 [INFER (REPEAT 100
29   (DO (MH {hyper-parameters} one 2)
30     (MH {hyper-parameters-Γ} one 2) ))
31 [SAMPLE (f_emu (array 1 2 3))] % posterior , Fig. 1d
296

```

Listing 2: Hierarchical GP Smoothing

Neal provides a custom inference algorithm setting and evaluates it using the following synthetic data problem. Let  $f$  be the underlying function that generates the data:

$$f(x) = 0.3 + 0.4x + 0.5 \sin(2.7x) + \frac{1.1}{(1+x^2)} + \eta \quad \text{with } \eta \sim \mathcal{N}(0, \sigma) \quad (14)$$

We synthetically generate outliers by setting  $\sigma = 0.1$  in 95% of the cases and to  $\sigma = 1$  in the remaining cases. gpmem can capture the true underlying function within only 100 MH steps on the hyper-parameters to get a good approximation for their posterior (see Fig. 1). Note that Neal devices an additional noise model and performs large number of Hybrid-Monte Carlo and Gibbs steps. We illustrate the hyper-parameter by showing the shift of the distribution on the noise parameter  $\sigma$  (Fig. 2). We see that gpmem learns the posterior distribution well, the posterior even exhibits a bimodal histogram when sampling  $\sigma$  100 times reflecting the two modes of data generation, that is normal noise and outliers<sup>2</sup>.

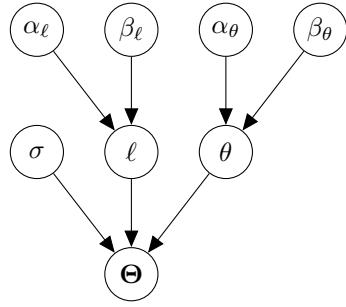
### 3.1.3 Broader applicability of gpmem

More generally, gpmem is relevant not just when a data set is available, but also whenever we have at hand a function  $f_{\text{restr}}$  which is expensive or impractical to evaluate many times. gpmem allows us to model  $f_{\text{restr}}$  with a GP-based emulator  $f_{\text{emu}}$ , and also to use  $f_{\text{emu}}$  during the learning process to choose, in an online manner, an effective set of probe points  $\{x_i\}$  on which to use our few evaluations of  $f_{\text{restr}}$ . This idea is illustrated in detail in Section 4. Before doing this, we will illustrate another benefit of having a probabilistic programming apparatus for GP modelling: the linguistically unified treatment of inference over structure and inference over parameters. This unification makes interleaved joint inference over structure and parameters very natural, and allows us to give a short, elegant description of what it means to “learn the covariance function,” both in prose and in code.

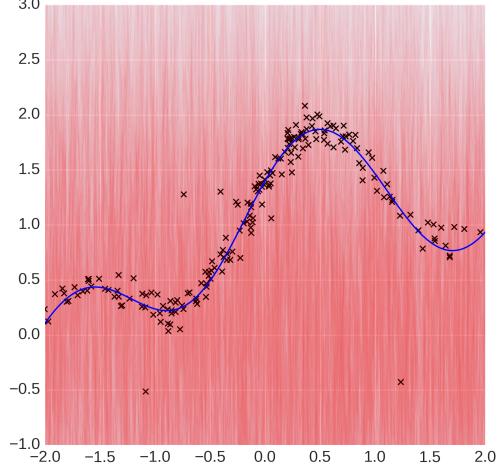
---

<sup>2</sup>For this pedagogical example we have increased the probability for outliers in the data generation slightly from 0.05 to 0.2

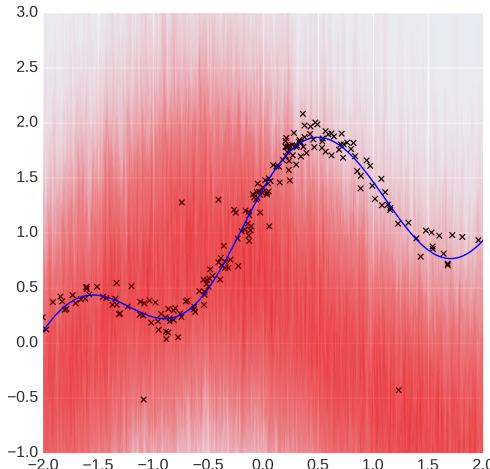
324  
 325  
 326  
 327  
 328  
 329  
 330  
 331  
 332  
 333  
 334  
 335  
 336  
 337  
 338  
 339  
 340  
 341  
 342  
 343  
 344  
 345  
 346



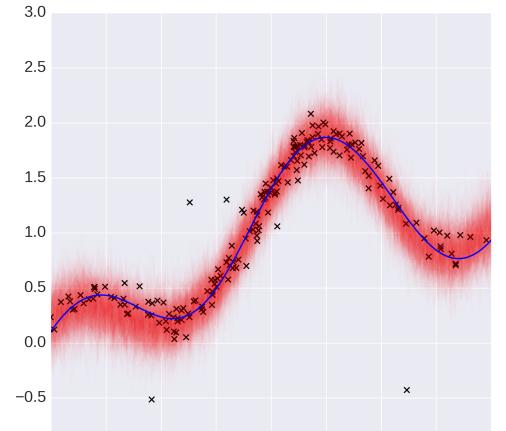
(a) Hierarchical Prior



(b) Prior Inference



(c) Observed



(d) Inferred

367  
 368  
 369  
 370  
 371  
 372  
 373  
 374  
 375  
 376  
 377

Figure 1: (a) depicts the hierarchical structure of the hyper-parameter as constructed in the work by Neal as a Bayesian Network. (b)-(d) shows a Venture GP on Neal’s example. We see that prior renders functions all over the place (a). After gpmem observes a some data-points an arbitrary smooth trend with a high level of noise is sampled. After running inference on the hierarchical system of hyper-parameters we see that the posterior reflects the actual curve well. Outliers are treated as such and do not confound the GP.

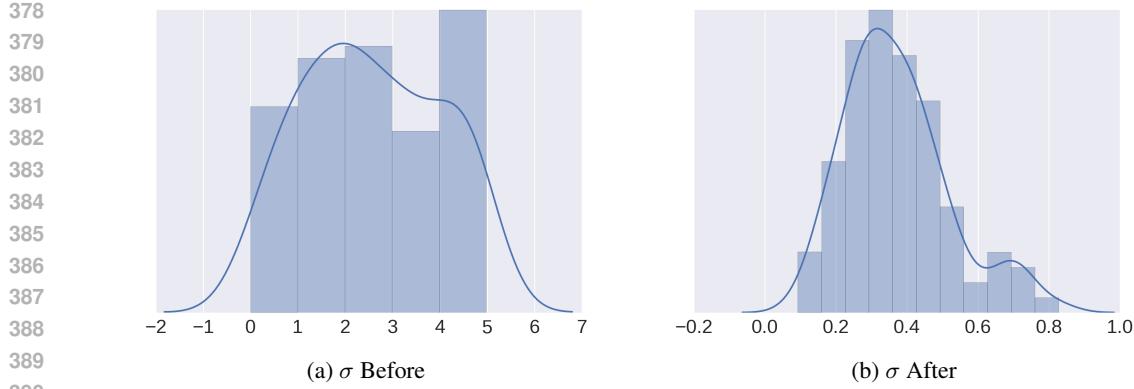


Figure 2: Hyper-parameter inference on the parameter of the noise kernel. We show 100 samples drawn from the distribution on  $\sigma$ . One can clearly recognise the shift from the uniform prior  $\mathcal{U}(0, 5)$  to a double peak distribution around the two modes - normal and outlier.

Furthermore, the example in Section 3.2 below recovers the performance of current state-of-the-art GP-based models.

### 3.2 Structure Learning

The space of possible kernel composition is infinite. Combining inference over this space with the problem of finding a good parameterization that could potentially explain the observed data best poses a hard problem. The natural language interpretation of the meaning of a kernel and its composition renders this a problem of symbolic computation. Duvenaud and colleagues note that sum of kernels can be interpreted as logical OR operations and kernel multiplication as logical AND (2013). This is due to the kernel rendering two points similar if  $k_1$  OR  $k_2$  outputs a high value in the case of a sum. Respectively, multiplication of two kernel results in high values only if  $k_1$  AND  $k_2$  have high values (see Fig. 3 for examples how to interpret global vs. local aspects and its symbolic analog respectively).

In the following, we will refer to covariance functions that are not composite as base covariance functions. Note that this form of composition can be easily expressed in Venture, for example if one wishes to add a linear and a periodic kernel:

```

1 [ASSUME l (gamma 1 3)]
2 [ASSUME sf (gamma 1 2)]
3 [ASSUME a (gamma 2 2)]
4
5  $k_{LIN}(x, x') := \sigma_1^2(x - \ell)(x' - \ell)$ 
6  $k_{PER}(x, x') := \sigma_2^2 \exp(-\frac{2\sin^2(\pi(x-x')/p)}{\ell^2})$ 
7
8 [ASSUME fLIN VentureFunction(kLIN, σ1) ]
9 [ASSUME fPER VentureFunction(kPER, σ2, ℓ, p) ]
10 [ASSUME LIN (make-LIN (apply-function fLIN a)) ]
11 [ASSUME PER (make-PER (apply-function fPER 1 sf)) ]
12 [ASSUME (make-gp 0 (function-times LIN PER)) ]

```

Listing 3: LIN  $\times$  PER

Knowledge about the composite nature of covariance functions is not new, however, until recently, the choice and the composition of covariance functions were done ad-hoc. The Automated Statistician Project came up with an approximate search over the possible space of kernel structures (Duvenaud et al., 2013; Lloyd et al., 2014). However, a fully Bayesian treatment of this was not done before. The case where the covariance structure is not given is even more interesting. Our probabilistic programming based MCMC framework approximates the following intractable integrals of the expectation for the prediction:

$$\mathbb{E}[y^* | x^*, \mathbf{D}, \mathbf{K}] = \iint f(x^*, \boldsymbol{\theta}, \mathbf{K}) P(\boldsymbol{\theta} | \mathbf{D}, \mathbf{K}) P(\mathbf{K} | \Omega, s, n) d\boldsymbol{\theta} d\mathbf{K}. \quad (15)$$

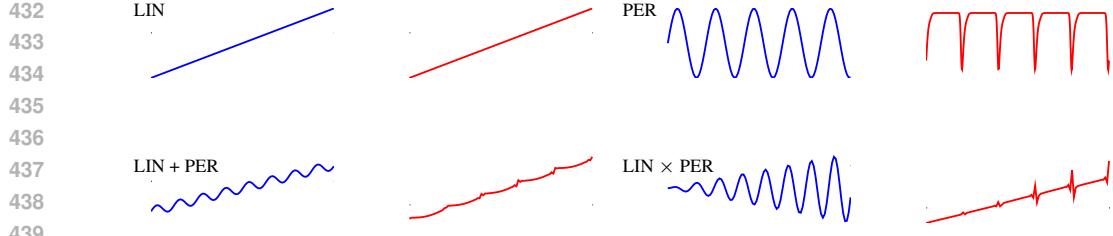


Figure 3: Composition of covariance functions (blue, left) and samples from the distribution of curves they can produce (red, right).

This is done by sampling from the posterior probability distribution of the hyper-parameters and the possible kernel:

$$y^* \approx \frac{1}{T} \sum_{t=1}^T f(x^* | \boldsymbol{\theta}^{(t)}, \mathbf{K}^{(t)}). \quad (16)$$

In order to provide the sampling of the kernel, we introduce a stochastic process to the SP that simulates the grammar for algebraic expressions of covariance function algebra:

$$\mathbf{K}^{(t)} \sim P(\mathbf{K} | \Omega, s, n) \quad (17)$$

Here, we start with a set of possible kernels and draw a random subset. For this subset of size  $n$ , we sample a set of possible operators that operate on the base kernels.

The marginal probability of a kernel structure which allows us to sample is characterized by the probability of a uniformly chosen subset of the set of  $n$  possible covariance functions times the probability of sampling a global or a local structure which is given by a binomial distribution:

$$P(\mathbf{K} | \Omega, s, n) = P(\Omega | s, n) \times P(s | n) \times P(n), \quad (18)$$

with

$$P(\Omega | s, n) = \binom{n}{r} p_{+ \times}^k (1 - p_{+ \times})^{n-k} \quad (19)$$

and

$$P(s | n) = \frac{n!}{|s|!} \quad (20)$$

where  $P(n)$  is a prior on the number of base kernels used which can sample from a discrete uniform distribution. This will strongly prefer simple covariance structures with few base kernels since individual base kernels are more likely to be sampled in this case due to (20). Alternatively, we can approximate a uniform prior over structures by weighting  $P(n)$  towards higher numbers. It is possible to also assign a prior for the probability to sample global or local structures, however, we have assigned complete uncertainty to this with the probability of a flip  $p = 0.5$ .

Many equivalent covariance structures can be sampled due to covariance function algebra and equivalent representations with different parameterization (Lloyd et al., 2014). Certain covariance functions can differ in terms of the hyper-parameterization but can be absorbed into a single covariance function with a different parameterization. To inspect the posterior of these equivalent structures we convert each kernel expression into a sum of products and subsequently simplify expressions using the following grammar:

<sup>1</sup>	SE × SE	→ SE
<sup>2</sup>	{SE, PER, C, WN} × WN	→ WN
<sup>3</sup>	LIN + LIN	→ LIN
<sup>4</sup>	{SE, PER, C, WN, LIN} × C	→ {SE, PER, C, WN, LIN}

Listing 4: Grammar to simplify expressions

For reproducing results from the Automated Statistician Project in a Bayesian fashion we first define a prior on the hypothesis space. Note that, as in the implementation of the Automated Statistician, we upper-bound the complexity of the space of covariance functions we want to explore. We also put vague priors on hyper-parameters.

```

486 1 [ASSUME base_kernels (list K1,K2,...,Kn) ] % defined as above
487 2 [ASSUME pn (uniform_structure n)] % prior on the number of kernels
488 3 [ASSUME SK (subset base_kernels pn) ] % sampling a subset of size n
489 4 [ASSUME composition (lambda (l) % kernel composition
490 5           (if (lte (size l) 1)
491 6             (first l)
492 7               (if (flip)
493 8                 (func_plus (first l) (cov_compo (rest l)))
494 9                 (func_times (first l) (cov_compo (rest l)))
495 10            )
496 11         )
497 12     )
498 13
499 14 [ASSUME K (composition SK) ]
500 15
501 16 [ASSUME (list f_compute f_emu) (gpmem f_restr K )]
502 17
503 18 for i=1 to n:
504 19   [PREDICT (f_compute x[i])] % observing with a look-up function
505 20
506 21 [INFER (REPEAT 2000 (DO
507 22           (MH pn one 1)
508 23           (MH SK one 1)
509 24           (MH K* one 1)
510 25           (MH {hyper-parameters} one 10)) ]
```

Listing 5: Venture Code for Bayesian GP Structure Learning

We defined the space of covariance structures in a way allowing us to reproduce results for covariance function structure learning as in the Automated Statistician. This lead to coherent results, for example for the airline data set. We will elaborate the result using a sample from the posterior (Fig. 4). The sample is identical with the highest scoring result reported in previous work using a search-and-score method (Duvenaud et al., 2013) for the CO<sub>2</sub> data set () and the predictive capability is comparable. However, the components factor in a different way due to different parameterization of the individual base kernels.

We further investigated the quality of our stochastic processes by running a leave one out cross-validation to gain confidence on the posterior. This resulted in 545 independent runs of the Markov chain that produced a coherent posterior: our Bayesian interpretation of GP structure and GPs produced a posterior of structures that is in line with previous results on this data set ( Duvenaud et al., 2013; see Fig. 5).

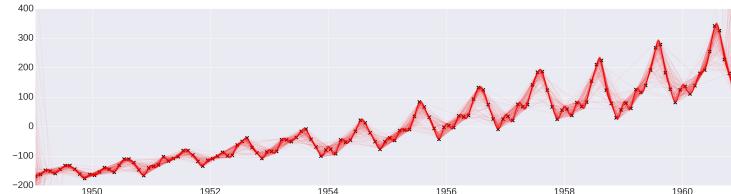
We ran similar evaluation on the airline data set () resulting in a similar structure to what was previously reporte (Fig. 6, residuals and log-score along the Markov chain see Fig. 7).

We found the final sample of multiple runs to be most informative. This kind of Markov Chain seems to produce samples that are highly auto-correlated.

## 4 Bayesian Optimization

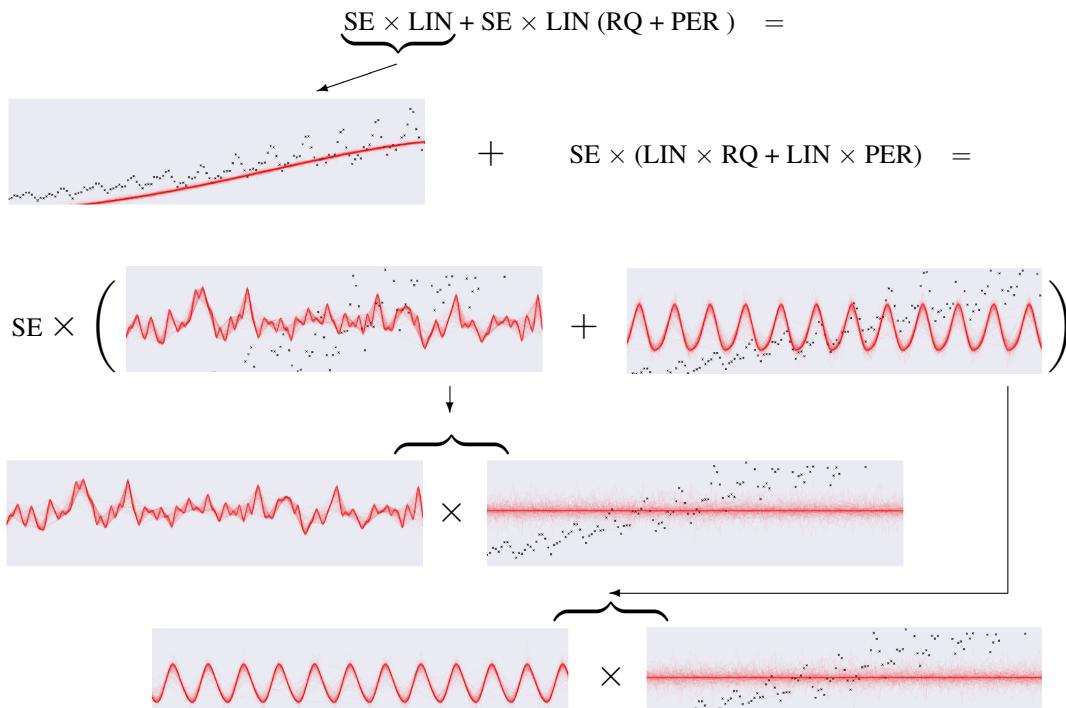
Bayesian optimization casts the problem of finding the global maximum of an unknown function as a hierarchical decision problem (Ghahramani, 2015). Evaluating the actual function may be very expensive, either in computation time or in some other resource. For one example, when searching for the best configuration for the learning algorithm of a large convolutional neural network, a large amount of computational work is required to evaluate a candidate configuration, and the space of possible configurations is high-dimensional. Another common example, alluded to in Section 3.1.3, is data acquisition: for machine learning problems in which a large body of data is available, it is often desirable to choose the right queries to produce a data set on which learning will be most effective. In continuous settings, many Bayesian optimization methods employ GPs (e.g. Snoek et al., 2012).

540  
 541  
 542  
 543  
 544  
 545  
 546  
 547  
 548  
 549  
 550  
 551  
 552  
 553



(a) The predictive posterior using the full grammar structure.

554  
 555  
 556  
 557  
 558  
 559  
 560  
 561  
 562  
 563  
 564  
 565  
 566  
 567  
 568  
 569  
 570  
 571  
 572  
 573  
 574  
 575  
 576  
 577  
 578  
 579  
 580



(b) Compositional Structure

581  
 582  
 583  
 584  
 585  
 586  
 587  
 588  
 589  
 590  
 591  
 592  
 593

Figure 4: a) We see the predictive posterior as a result 1000 nested MH steps on the airline data set. b) depicts a decomposition of this posterior for the structures sampled by Venture. RQ is the rational quadratic covariance function. The first line shows the global trend and denotes the rest of the structure that is shown above. In the second line, the see the periodic component on the right hand side. The left hand side denotes short term deviations both multiplied by a smoothing kernel. The third and fourth lines denote how we reach the second line: both periodic and rational quadratic covariance functions are multiplied by a linear covariance function with slope zero.

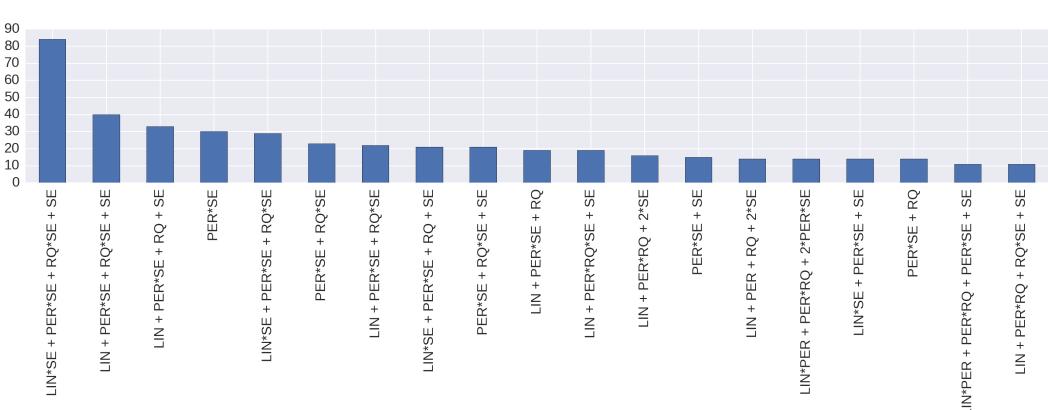


Figure 5: Posterior on structure of the CO<sub>2</sub> data. We have cut the tail of the distribution for space reasons since the number of possible structures is large. We see the final sample of the each of the 545 chains with 2000 nested steps each. Note that Duvenaud et al. (2013) report LIN × SE + PER × SE + RQ × SE.

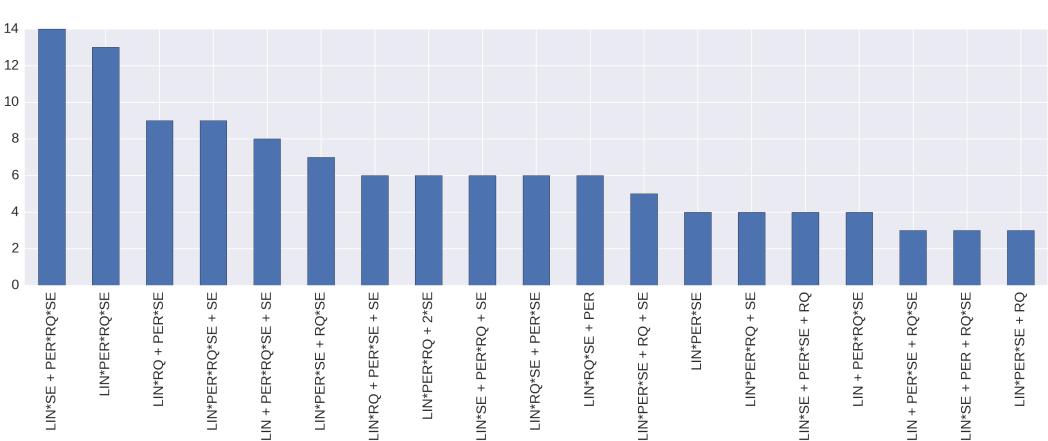


Figure 6: Posterior on structure of airline data set. We have cut the tail of the distribution for space reasons since the number of possible structures is large. We see the final sample of the each of the 144 chains with 2000 nested steps each. Note that Duvenaud et al. (2013) report  $\text{LIN} \times \text{SE} + (\text{PER} + \text{RQ}) \times \text{SE} \times \text{LIN}$

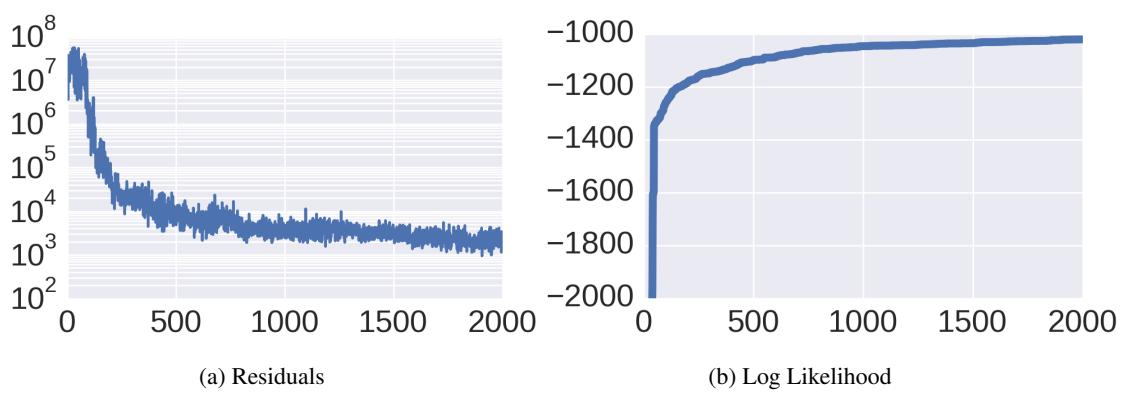


Figure 7: 2000 steps along the Markov Chain.

We have implemented a version of Thompson sampling using GPs in Venture. Thompson sampling Thompson (1933) is a widely-used Bayesian framework for solving exploration-exploitation problems. Our implementation has two notable features: (i) the ability to search over a broader space of contexts than the parametric families that are typically used, and (ii) the parsimony of the resulting probabilistic program.

## 4.1 Thompson sampling framework

We now lay out the setup of Thompson sampling for Markov decision processes (MDPs). An agent is to take a sequence of actions  $a_1, a_2, \dots$  from a (possibly infinite) set of possible actions  $\mathcal{A}$ . After each action, a reward  $r \in \mathbb{R}$  is received, according to an unknown conditional distribution  $P_{\text{true}}(r|a)$ . The agent's goal is to maximize the total reward received for all actions. In Thompson sampling, the Bayesian agent accomplishes this by placing a prior distribution  $P(\theta)$  on the possible "contexts"  $\theta \in \Theta$ . Here a context is a believed model of the conditional distributions  $\{P(r|a)\}_{a \in \mathcal{A}}$ , or at least, a believed statistic of these conditional distributions which is sufficient for deciding an action  $a$ . One example of such a sufficient statistic is the conditional mean  $V(a|\theta) = \mathbb{E}[r|a, \theta]$ , which can be thought of as a value function. Thompson sampling thus has the following steps, repeated as long as desired:

1. Sample a context  $\theta \sim P(\theta)$ .
2. Choose an action  $a \in \mathcal{A}$  which (approximately) maximizes  $V(a|\theta) = \mathbb{E}[r|a, \theta]$ .
3. Let  $r_{\text{true}}$  be the reward received for action  $a$ . Update the believed distribution on  $\theta$ , i.e.,  $P(\theta) \leftarrow P_{\text{new}}(\theta)$  where  $P_{\text{new}}(\theta) = P(\theta | a \mapsto r_{\text{true}})$ .

Note that when  $P(\theta)$  has high entropy, the chosen action  $a$  may be far from optimal, but the information gained by probing action  $a$  will improve the belief  $\theta$ . This amounts to "exploration." When  $P(\theta)$  has high entropy and is concentrated on a region of belief space that closely matches the true distributions  $P_{\text{true}}(r|a)$ , exploration will be less likely to occur, but the chosen actions  $a$  will tend to receive high rewards. This amounts to "exploitation."

Typically, when Thompson sampling is implemented, the search over contexts  $\theta \in \Theta$  is limited by the choice of representation. In traditional programming environments,  $\theta$  often consists of a few numerical parameters for a family of distributions of a fixed functional form. With work, a mixture of a few functional forms is possible; but without probabilistic programming machinery, implementing a rich context space  $\Theta$  would be an unworkably large technical burden. In a probabilistic programming language, however, the representation of heterogeneously structured or infinite-dimensional context spaces is quite natural. Any computable model of the conditional distributions  $\{P(r|a)\}_{a \in \mathcal{A}}$  can be represented as a (stochastic-)procedure-valued random variable  $(\lambda(a) \dots)$ . Thus, for computational Thompson sampling, the most general context space  $\widehat{\Theta}$  is the space of program texts. Any other context space  $\Theta$  has a natural embedding as a subset of  $\widehat{\Theta}$ .

## 4.2 Thompson sampling in Venture

Because Venture supports sampling and inference on (stochastic-)procedure-valued random variables (and the generative models which produce those procedures), Venture can capture arbitrary context spaces as described above. To demonstrate, we have implemented Thompson sampling in Venture in which the contexts  $\theta$  are Gaussian processes over the action space  $\mathcal{A} = \mathbb{R}$ . That is,  $\theta = (\mu, K)$ , where the mean  $\mu$  is a computable function  $\mathcal{A} \rightarrow \mathbb{R}$  and the covariance  $K$  is a computable (symmetric, positive-semidefinite) function  $\mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$ . This represents a Gaussian process  $\{R_a\}_{a \in \mathcal{A}}$ , where  $R_a$  represents the reward for action  $a$ . Computationally, we represent a context not as a pair of infinite lookup tables for  $\mu$  and  $K$ , but as a finite data structure  $\theta = (K_{\text{prior}}, \sigma, \ell, \mathbf{a}_{\text{past}}, \mathbf{r}_{\text{past}})$ , where

- $K_{\text{prior}} = K_{\text{prior}, \sigma, \ell}$  is a procedure, with parameters  $\sigma, \ell$ , to be used as the prior covariance function:  $K_{\text{prior}}(a, a') = \sigma^2 \exp\left(-\frac{(a-a')^2}{2\ell^2}\right)$
- $\sigma$  and  $\ell$  are (hyper)parameters for  $K_{\text{prior}}$
- $\mathbf{a}_{\text{past}} = (a_i)_{i=1}^n$  are the previously probed actions

- 702     •  $\mathbf{r}_{\text{past}} = (r_i)_{i=1}^n$  are the corresponding rewards  
 703

704 To simplify the treatment, we take prior mean  $\mu_{\text{prior}} \equiv 0$ . The mean and covariance for  $\theta$  are then  
 705 gotten by the usual conditioning formula:

$$\begin{aligned} 706 \quad \mu(\mathbf{a}) &= \mu(\mathbf{a} | \mathbf{a}_{\text{past}}, \mathbf{r}_{\text{past}}) \\ 707 &= K_{\text{prior}}(\mathbf{a}, \mathbf{a}_{\text{past}}) K_{\text{prior}}(\mathbf{a}_{\text{past}}, \mathbf{a}_{\text{past}})^{-1} \mathbf{r}_{\text{past}} \\ 708 \quad K(\mathbf{a}, \mathbf{a}') &= K(\mathbf{a}, \mathbf{a}' | \mathbf{a}_{\text{past}}, \mathbf{r}_{\text{past}}) \\ 709 &= K_{\text{prior}}(\mathbf{a}, \mathbf{a}') - K_{\text{prior}}(\mathbf{a}, \mathbf{a}_{\text{past}}) K_{\text{prior}}(\mathbf{a}_{\text{past}}, \mathbf{a}_{\text{past}})^{-1} K_{\text{prior}}(\mathbf{a}_{\text{past}}, \mathbf{a}). \end{aligned}$$

711 Note that even in this simple example, the context space  $\Theta$  is not a finite-dimensional parametric  
 712 family, since the vectors  $\mathbf{a}_{\text{past}}$  and  $\mathbf{r}_{\text{past}}$  grow as more samples are taken.  $\Theta$  is, however, quite easily  
 713 representable as a computational procedure together with parameters and past samples, as we do in  
 714 the representation  $\theta = (K_{\text{prior}}, \sigma, \ell, \mathbf{a}_{\text{past}}, \mathbf{r}_{\text{past}})$ .

### 716 4.3 Implementation with `gpmem`

718 As a demonstration, we use Thompson sampling to optimize an unknown function  $V(x)$  (the value  
 719 function) using `gpmem`. (TODO we should not assume  $V$  is deterministic, it would be easy enough  
 720 to make it random or have it give noisy samples.) We assume  $V$  is made available to Venture as a  
 721 black-box. The code for optimizing  $V$  is given in Listing 6. For step 3 of Thompson sampling, the  
 722 Bayesian update, we not only condition on the new data (the chosen action  $a$  and the received reward  
 723  $r$ ), but also perform inference on the hyperparameters  $\sigma, \ell$  using a Metropolis–Hastings sampler.  
 724 These two inference steps take 1 line of code: 0 lines to condition on the new data (as this is done  
 725 automatically by `gpmem`), and 1 line to call Venture’s built-in `MH` operator. The results are shown  
 726 in Figure 8. We can see from the figure that, roughly speaking, each successive probe point  $a$  is  
 727 chosen either because the current model  $V_{\text{emu}}$  thinks it will have a high reward, or because the value  
 728 of  $V_{\text{emu}}(a)$  has high uncertainty. In the latter case, probing at  $a$  decreases this uncertainty and, due to  
 729 the smoothing kernel, also decreases the uncertainty at points near  $a$ . We thus see that our Thompson  
 sampler simultaneously learns the value function and optimizes it.

```

1 [ASSUME sigma (sigma-prior)]
2 [ASSUME l (l-prior)]
3 [ASSUME K (make-squared-exponential sigma l)]
4 [ASSUME (list V_compute V_emu) (gpmem V K)]
5 [ASSUME V_emu_pointwise (lambda (a) (first (V_emu (array a))))]
6 [ASSUME mc_sampler (uniform_sampler -20 20)]
7
8 for i=1 to 15:
9   [PREDICT (V_compute (mc_argmax V_emu_pointwise mc_sampler))]
10  [INFER (MH 'hypers one 50)
11
12 [INFER (extract_stats V_emu)]
```

742 Listing 6: Code for Bayesian optimization using `gpmem`. In the loop, `V_compute` is called to  
 743 probe the value of  $V$  at a new argument. The new argument, `(mc_argmax V_emu_pointwise`  
 744 `mc_sampler)`, is a Monte Carlo estimate of the maximum pointwise sample of  $V_{\text{emu}}$  (itself a  
 745 stochastic quantity), with the Monte Carlo samples being drawn in this case uniformly between  $-20$   
 746 and  $20$ . After each new call to `V_compute`, the Metropolis–Hastings algorithm is used to perform  
 747 inference on the hyperparameters of the covariance function in the GP model in light of the new  
 748 conditioning data. Once enough calls to `V_compute` have been made (in our case we stopped at  
 749 15 calls), we can inspect the full list of probed  $(a, r)$  pairs with `extract_stats`. The answer to  
 750 our maximization problem is simply the maximum  $r$ ; but our algorithm also learns more potentially  
 751 useful information.

## 752 5 Conclusion

753 We have shown Venture GPs. We have introduced novel stochastic processes for a probabilistic  
 754 programming language. We showed how flexible non-parametric models can be treated in Venture

756  
 757  
 758  
 759  
 760  
 761  
 762  
 763  
 764  
 765  
 766  
 767  
 768  
 769  
 770  
 771  
 772  
 773  
 774  
 775  
 776  
 777  
 778  
 779  
 780  
 781  
 782  
 783  
 784  
 785  
 786  
 787  
 788  
 789  
 790  
 791  
 792  
 793  
 794  
 795  
 796  
 797  
 798  
 799  
 800  
 801  
 802  
 803  
 804  
 805  
 806  
 807  
 808  
 809

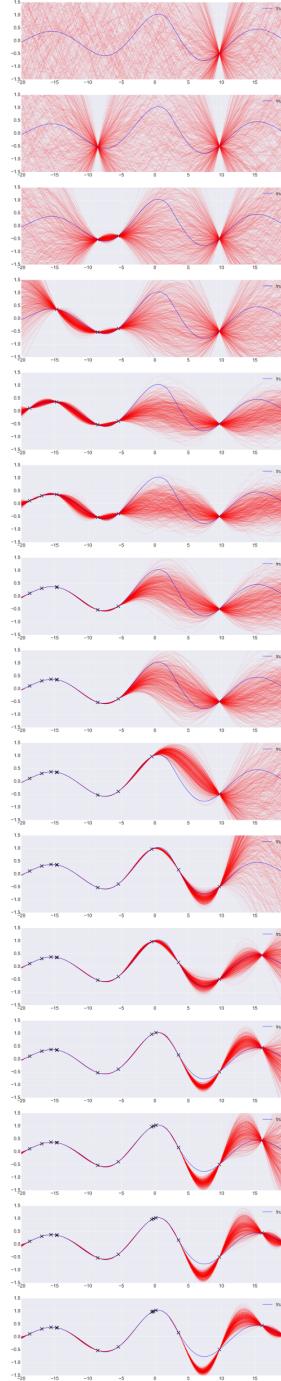


Figure 8: Dynamics of Thompson sampling in Venture. Probe points are marked with an  $\times$ . The blue curve is the true function  $V$ , and the red region is a blending of 100 samples of the curve generated (jointly) by a GP-based emulator  $V_{\text{emu}}$ . Each successive probe point  $a$  is the (stochastic) maximum of  $V_{\text{emu}}$ , sampled pointwise and conditioned on the values of the previously probed points. Note that probes tend to happen at points either where the value of  $V_{\text{emu}}$  is high, or where  $V_{\text{emu}}$  has high uncertainty.

810 in only a few lines of code. We evaluated our contribution on a range of hard problems for state-of-  
811 the-art Bayesian non-parametrics. Venture GPs showed competitive performance in all of them.  
812  
813  
814  
815  
816  
817  
818  
819  
820  
821  
822  
823  
824  
825  
826  
827  
828  
829  
830  
831  
832  
833  
834  
835  
836  
837  
838  
839  
840  
841  
842  
843  
844  
845  
846  
847  
848  
849  
850  
851  
852  
853  
854  
855  
856  
857  
858  
859  
860  
861  
862  
863

864  
865  
**References**  
866  
867

- Andrieu, C., De Freitas, N., Doucet, A., and Jordan, M. I. (2003). An introduction to mcmc for  
machine learning. *Machine learning*, 50(1-2):5–43.
- Duvenaud, D., Lloyd, J. R., Grosse, R., Tenenbaum, J., and Ghahramani, Z. (2013). Structure  
discovery in nonparametric regression through compositional kernel search. In *Proceedings of  
the 30th International Conference on Machine Learning (ICML-13)*, pages 1166–1174.
- Ghahramani, Z. (2015). Probabilistic machine learning and artificial intelligence. *Nature*,  
521(7553):452–459.
- Goodman, N. D .and Mansinghka, V. K., Roy, D., Bonawitz, K., and Tenenbaum, J. (2008). Church:  
A language for generative models. In *Proceedings of the 24th Conference on Uncertainty in  
Artificial Intelligence, UAI 2008*, pages 220–229.
- Lloyd, J. R., Duvenaud, D., Grosse, R., Tenenbaum, J., and Ghahramani, Z. (2014). Automatic  
construction and natural-language description of nonparametric regression models. In *Twenty-  
Eighth AAAI Conference on Artificial Intelligence*.
- Mansinghka, V. K., Selsam, D., and Perov, Y. (2014). Venture: a higher-order probabilistic pro-  
gramming platform with programmable inference. *arXiv preprint arXiv:1404.0099*.
- Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953). Equation  
of state calculations by fast computing machines. *The Journal of Chemical Physics*, 21(6):1087–  
1092.
- Neal, R. M. (1997). Monte carlo implementation of gaussian process models for bayesian regression  
and classification. *arXiv preprint physics/9701026*.
- Poole, D. (1993). Probabilistic horn abduction and bayesian networks. *Artificial Intelligence*,  
64(1):81–129.
- Rasmussen, C. E. and Williams, C. K. I. (2006). *Gaussian Processes for Machine Learning (Adap-  
tive Computation and Machine Learning)*. The MIT Press.
- Sato, T. (1995). A statistical learning method for logic programs with distribution semantics. In *In  
Proceedings of the 12th International Conference on Logic Programming*. Citeseer.
- Snoek, J., Larochelle, H., and Adams, R. P. (2012). Practical bayesian optimization of machine  
learning algorithms. In *Advances in Neural Information Processing Systems*, pages 2951–2959.
- Thompson, W. R. (1933). On the likelihood that one unknown probability exceeds another in view  
of the evidence of two samples. *Biometrika*, pages 285–294.

898  
899  
900  
901  
902  
903  
904  
905  
906  
907  
908  
909  
910  
911  
912  
913  
914  
915  
916  
917