Overfitting: how to fool the linear regression

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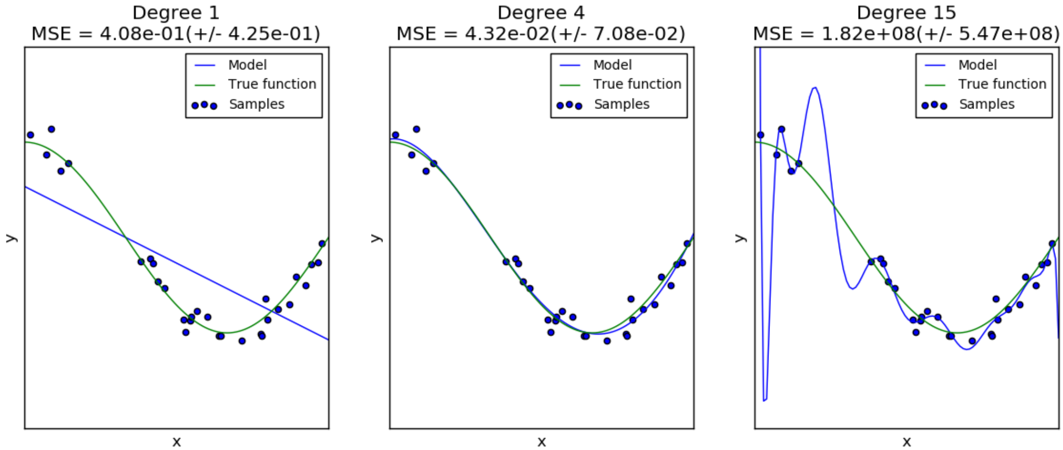
- How well do the regression models perform?

Polynomial fits of different degrees



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- The solution: rely on held-out data to assess model performance

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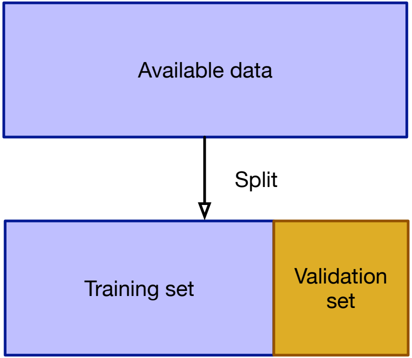
- **May be imprecise:** the holdout estimate of error rate will be misleading if we happen to get an "unfortunate" split



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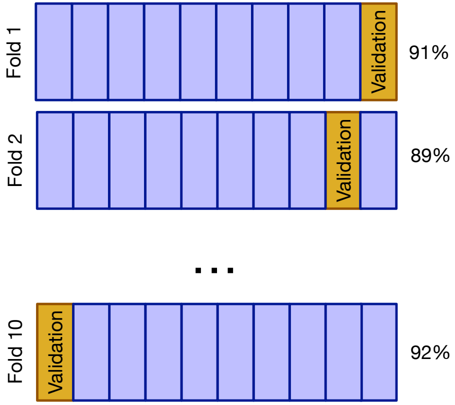
- Leave-one-out cross-validation: $X_k^\ell = \{(\mathbf{x}_k, y_k)\}$ (yes, train ℓ models!)



Cross-validation method: drawbacks

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- Small bias: the estimator will be very accurate
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Few folds:

- Cheap, computationally effective: few experiments
- Small variance: average over many samples
- Large bias: estimated error rate conservative or smaller than the true error rate

Regularization