Analysis

For the calculations and the analysis of the resulting values the measured anode voltages were reduced by 1% to account for a protection resistor in the leads. For both beam orbit diameters (8 +- 0.05 cm and 10 +- 0.05 cm) Kr was evaluated using Eq. 1 to calculate B_T and B_E using Eq. 2 and Eq. 3. B/B_0 is a ratio given by the manufacturer. The values weren't exact for the chosen ratio of r/R which is why the data sheet which can be seen in Figure 1 was interpolated using Scipy's interp1d. K_r for the 8cm orbit was 7.7103 e-4 +- 3.99e-06 T/A and 7.6837 e-4 +- 3.98e-06 T/A for K_r of the 10cm orbit. B_T varies over each measurement pair while B_E for each pair is around the actual value for the earth's magnetic field. Therefore, B_T values were all used as an array to calculate a series of e/m values with Eq. 4 which were then averaged for an estimate. The resulting B_E values were also averaged to give the best estimate.

B_E as calculated from the taken measurements resulted in 5.09e-05 +- 3.38e-06 T which is within error of the accepted reference value of 5.33375e-05 T [1]. The result of e/m from the B_T values was 2.086 e11 +- 0.075 e11 C/kg which is not within error of the reference value of 175882001076.0 C/kg [2]. The calculated e/m values before they were averaged did not fluctuate around the actual value. Instead, they were all consistently too high. In the end the e/m from the experiment was roughly 118.6% of the reference value. The reason for this discrepancy will be discussed in the discussion section.

To evaluate K to compare to the value given by the manufacturer Eq. 5 was used in the form of Eq. 6. The result of calculating K was 7.709 e-4 +- 2.57e-05 T/A which is within error of the provided 7.73e-4 +- 0.04e-4 T/A.

Details to calculations and sample calculations can be seen in the analysis appendix. The derivation of Eq. 5 can be seen below the equations.

r/R	B/B_0
0	1
0.1	0.99996
0.2	0.99928
0.3	0.99621
0.4	0.98728
0.5	0.96663
0.6	0.92525
0.7	0.85121
0.8	0.73324
0.9	0.56991
1	0.38007

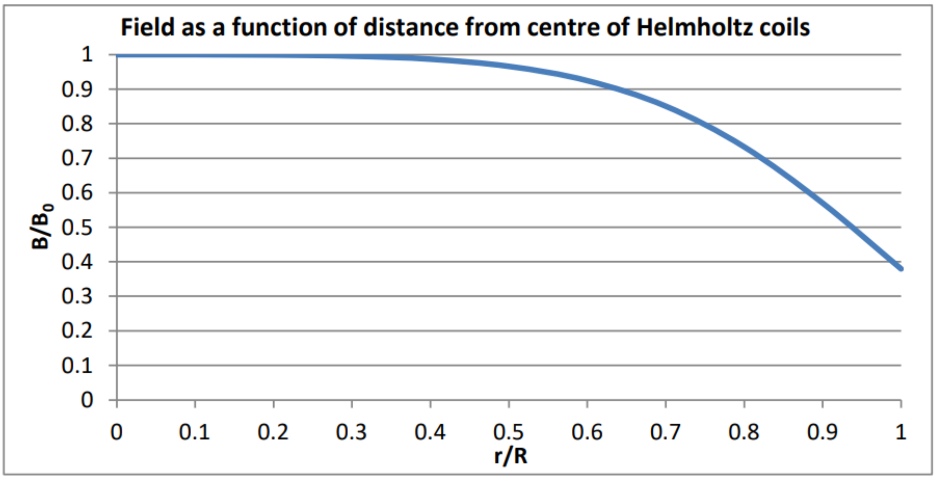


Figure 1: Magnetic field strength as a function of distance from the centre of Helmholtz coils

Equations

$$K_r = \frac{B}{B_0} * K \quad (1)$$

$$B_T = \frac{K_r}{2} * (I_l + I_s) \quad (2)$$

$$B_E = \frac{K_r}{2} * (I_l + I_s) \quad (3)$$

$$\frac{e}{m} = \frac{2 * V}{B^2 * r^2} \quad (4)$$

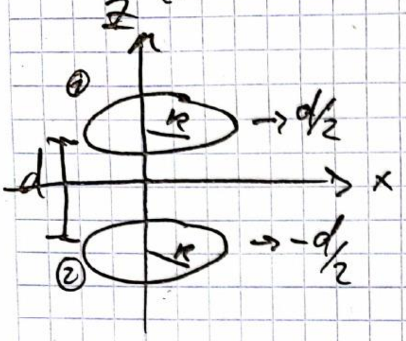
$$B_0 = \frac{\mu_0 * n * I * R^2}{(R^2 + b^2)^{\frac{3}{2}}} \quad (5)$$

$$B_0 = K * I \quad (6)$$

Derivation of Eq. 5

want:

$$B_0 = \frac{\mu_0 n I R^2}{(R^2+b^2)^{3/2}} = \frac{\mu_0 n I R^2}{\left(R^2+\left(\frac{R}{2}\right)^2\right)^{3/2}}$$



Biot-Savart:

$$B(r) = \frac{\mu_0 I}{4\pi}\int_S \frac{r-R}{|r-R|^3}\times ds$$

Two loops:

$$B_1(r) = \frac{\mu_0 I}{4\pi}\int_{S_1} \frac{r-R}{|r-R|^3}\times ds$$

$$B_2(r) = \frac{\mu_0 I}{4\pi}\int_{S_2} \frac{r-R}{|r-R|^3}\times ds$$

S_1 – line in center of loop ①

S_2 – line in center of loop ②

Superposition:

$S = S_1 + S_2$, r goes along $z \Rightarrow r = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

$ds = R\, d\varphi\, \hat{\varphi}$ → cylindrical coordinates

$$R = \begin{pmatrix} R\cos(\varphi) \\ R\sin(\varphi) \\ \frac{d}{2} \end{pmatrix} \Rightarrow r - R = \begin{pmatrix} -R\cos(\varphi) \\ -R\sin(\varphi) \\ z - \frac{d}{2} \end{pmatrix}$$

$$\Rightarrow |r - R|^3 = \left(R^2 \cos(\varphi)^2 + R^2 \sin(\varphi)^2 + \left(z - \frac{d}{2}\right)^2\right)^{\frac{3}{2}}$$

$$= \left(R^2\left(\cos(\varphi)^2 + \sin(\varphi)^2\right) + \left(z - \frac{d}{2}\right)^2\right)^{\frac{3}{2}} =$$

$$= \left(R^2 + \left(z - \frac{d}{2}\right)^2\right)^{\frac{3}{2}}$$

Then, $(r - R) \times ds = \begin{pmatrix} -R\cos(\varphi) \\ -R\sin(\varphi) \\ z - \frac{d}{2} \end{pmatrix} \times \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} R\, d\varphi =$

a bit of math later $= -R \begin{pmatrix} \left(z - \frac{d}{2}\right)\cos(\varphi) \\ \left(z - \frac{d}{2}\right)\sin(\varphi) \\ R \end{pmatrix} d\varphi$

Integrate over φ from 0 to 2π:

$$\int_0^{2\pi} -R \begin{pmatrix} \left(z - \frac{d}{2}\right)\cos(\varphi) \\ \left(z - \frac{d}{2}\right)\sin(\varphi) \\ R \end{pmatrix} d\varphi = -R \left[\begin{matrix} \left(z - \frac{d}{2}\right)\sin(\varphi) \\ -\left(z - \frac{d}{2}\right)\cos(\varphi) \\ R\cdot\varphi \end{matrix}\right]_0^{2\pi} =$$

$$= -R\left(\begin{pmatrix} \left(z - \frac{d}{2}\right)\sin(2\pi) \\ -\left(z - \frac{d}{2}\right)\cos(2\pi) \\ R\cdot 2\pi \end{pmatrix} - \begin{pmatrix} \left(z - \frac{d}{2}\right)\sin(0) \\ -\left(z - \frac{d}{2}\right)\cos(0) \\ R\cdot 0 \end{pmatrix}\right) = -R\begin{pmatrix} 0 \\ 0 \\ R\cdot 2\pi \end{pmatrix}$$

$$= -R^2\, 2\pi\, \vec{z}$$

→ plug in:

$$B_1(z) = -\frac{\mu_0 I R^2}{2} \left(R^2 + \left(z - \frac{d}{2}\right)^2\right)^{-\frac{3}{2}} \hat{z}$$

for n-windings in the loop:

$$B_1(z) = -\frac{\mu_0 R^2 n I}{2 \cdot \left(R^2 + \left(z - \frac{d}{2}\right)^2\right)^{\frac{3}{2}}} \hat{z}$$

for $B_2(z) = -\frac{\mu_0 R^2 n I}{2} \left(R^2 + \left(z + \frac{d}{2}\right)^2\right)^{-\frac{3}{2}} \hat{z}$

$$B(z) = B_1 + B_2 = -\frac{\mu_0 R^2 n I}{2} \left[\left(R^2 + \left(z - \frac{d}{2}\right)^2\right)^{-\frac{3}{2}} + \left(R^2 + \left(z + \frac{d}{2}\right)^2\right)^{-\frac{3}{2}}\right] \hat{z}$$

simplify denominator: $z = 0$ → middle of the loops

$$\Rightarrow \frac{1}{2}\left[\left(R^2 \left(-\frac{d}{2}\right)^2\right)^{-\frac{3}{2}} + \left(R^2 + \left(\frac{d}{2}\right)^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{d}{2} = \frac{R}{2} = b \Rightarrow \frac{1}{2}\left[\left(R^2 - b^2\right)^{-\frac{3}{2}} + \left(R^2 + b^2\right)^{-\frac{3}{2}}\right] =$$

$$= \frac{1}{2}\left[2\left(R^2 + b^2\right)^{-\frac{3}{2}}\right] = \left(R^2 + b^2\right)^{-\frac{3}{2}}$$

Therefore: $B = \frac{\mu_0 n I R^2}{(R^2 + b^2)^{\frac{3}{2}}}$ ✓

Sources

[1] “Magnetic Declination in Kingston, Canada,” *What is the Magnetic Declination at your location?* https://www.magnetic-declination.com/Canada/Kingston/335754.html (accessed Mar. 24, 2022).

[2] “CODATA Value: electron charge to mass quotient.” https://physics.nist.gov/cgi-bin/cuu/Value?esme (accessed Mar. 24, 2022).