THE CONTACT PROCESS

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1. Introduction and Preliminaries

Contact processes are special spin-flip systems which can used to model, for example, the spread of an infection. In the general case, we are given a countable, undirected graph $G = (S, E)$ of bounded degree $\deg_{\max} := \sup_{x\in S} \deg(x) < \infty$. The nodes of the graph are usually called *sites* and during the contact process sites are either *infected* or *healthy*. Hence, the state space of the process we are about to define is $\Omega := \{0,1\}^S$ where 0 should be interpreted as healthy and 1 as infected.

We denote sites by letters $x, y \in S$, configurations by $\eta, \zeta, \xi \in \Omega$. The resulting configuration for flipping site x in configuration is denoted as η^x and two sites x and y are called *neighboring* ($x \sim y$) if $\{x, y\}$ is an edge in E.

In a contact process, an infected site becomes healthy with after a unit exponential time. On the contrary a healthy site becomes infected proportionally to the number of infected neighbors. This proportionality coefficient $\lambda > 0$ is independent of the site itself. Now we can define the flip rates of a site x in a configuration η by

$$c(x,\eta) := \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \cdot |\{y \sim x \mid \eta(y) = 1\}| & \text{if } \eta(x) = 0. \end{cases}$$

As any other spin system, these spin rates can be translated into a continuous time Markov chain with Q-Matrix of the form $q(\eta, \eta^x) := c(x, \eta)$ and generator

$$\mathcal{L}\, f(\eta) := \sum_{x\in S} c(x,\eta)\,(f(\eta^x) - f(\eta)),$$

if $M := \sup_{x\in S} \sum_{u:x\neq u} \gamma(x,u) < \infty$ holds where $\gamma(x,u) := \sup_{\eta\in\Omega} |c(x,\eta^u) - c(x,\eta)|$. Here, this is indeed the case: For $u \sim x$ we have $\gamma(x,u) = \lambda$, otherwise $\gamma(x,u)$ vanishes implying $M = \lambda \cdot \deg_{\max}$. By looking at the $(M < \epsilon)$-Theorem with $\epsilon := \inf_{x\in S,\eta\in\Omega} c(x,\eta) + c(x,\eta^x) = 1$, we get the following result: If $\lambda < \deg_{\max}^{-1}$, then η_t is *ergodic*, i.e. there is a unique stationary distribution μ and for every $\eta \in \Omega$ and $f \in C(\Omega)$ it fulfills $\lim_{t\to\infty} S_t f(\eta) = \int_\Omega f \,\mathrm{d}\mu$, where S_t denotes the semigroup generated by $\mathcal{L}$. As the pointmass δ_0 on 0 is always an invariant measure, we discuss in the next sections whether there are more invariant measures for $\lambda \geq \deg_{\max}^{-1}$.

Before that, we note, that any contact process is an *attractive spin system*: If $\eta \leq \zeta$ holds component-wise, we have $c(x,\eta) = \lambda \cdot |\{y \sim x \mid \eta(y) = 1\}| \leq \lambda \cdot |\{y \sim x \mid \zeta(y) = 1\}| = c(x,\zeta)$ for $\eta(x) = 1$ and $c(x,\eta) = 1 \leq 1 = c(x,\zeta)$ for $\zeta(x) = 0$. From the theory of attractive spin systems we know the existence of a lower invariant measure $\underline{\nu} := \lim_{t\to\infty} \delta_0 S_t$ and an upper invariant measure $\overline{\nu} := \lim_{t\to\infty} \delta_1 S_t$. As δ_0 is already invariant, $\underline{\nu} = \delta_0$ follows immediately. The structure of $\overline{\nu}$ is less obvious for $\lambda \geq \deg_{\max}^{-1}$ as we will see in the next sections.

2. The Graphical Representation

3. Critical Values for Survival

4. The Contact Process on Homogeneous Trees

5. More Results

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Figure 1. Graphical Representation of the contact process