

EXERCISES

INTERACTING PARTICLE SYSTEMS

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EXERCISE 1 – DISCRETE-TIME FINITE MARKOV CHAINS

A discrete-time Markov chain with finite state space $S := \{x_1, \dots, x_n\}$ is described by a matrix P with non-negative entries $(P_{i,j})_{i,j \in S}$ with $\sum_j P_{i,j} = 1$ for all $i \in S$. The row $P(\cdot, i)$ can be interpreted as the jump distribution from i , i.e. the distribution of X_{n+1} , if $X_n = i$ holds.

Functions on the state space S are thought of as column vectors $f \in \mathbb{R}^S$, so that the vector $Pf := (Pf(x_1), \dots, Pf(x_n))$ is the expectation of f after one jump. Furthermore a probability distribution μ is a row vector \mathbb{R}^S , so that μP gives the distribution of the position after one step where the initial position is distributed according to μ . The power P^k of P corresponds to advancing k steps, and hence μP^k is the distribution of the position after k steps with initial distribution μ .

A probability distribution is called *invariant* for the chain (X_n) if $\mu P = \mu$ holds.

a) Let ν be any probability distribution on S and consider

$$\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} \nu P^k.$$

Claim 1. For all $x \in S$, it holds $|\mu_n P(x) - \mu_n(x)| \leq \frac{2}{n}$.

Proof. We calculate:

$$\begin{aligned} |\mu_n P(x) - \mu_n(x)| &= \left| \frac{1}{n} \left(\sum_{k=0}^{n-1} \nu P^{k+1}(x) - \sum_{k=0}^{n-1} \nu P^k(x) \right) \right| = \left| \frac{1}{n} (\nu P^n(x) - \nu P^0(x)) \right| \\ &\leq \frac{1}{n} (|\nu P^n(x)| + |\nu(x)|) \leq \frac{2}{n} \end{aligned}$$

□

As the sequence $(\mu_n)_{n \in \mathbb{N}}$ is bounded in $[0, 1]^S$, it follows that it has a convergent subsequence $(n_k)_{k \in \mathbb{N}}$ by the Bolzano-Weierstrass theorem. Now define $\mu := \lim_{k \rightarrow \infty} \mu_{n_k}$.

Claim 2. The limit μ is an invariant distribution.

Proof. It is easy to show, that μ is indeed a probability distribution:

$$\sum_{x \in S} \mu(x) = \sum_{x \in S} \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \nu P^i(x) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \sum_{x \in S} \nu P^i(x) = \lim_{k \rightarrow \infty} \frac{1}{n_k} n_k = 1$$

The invariance now follows by the claim above:

$$|\mu P(x) - \mu(x)| = \left| \lim_{k \rightarrow \infty} (\mu_{n_k} P(x) - \mu_{n_k}(x)) \right| = 0$$

□

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Date: December 15, 2020.

- b) A discrete-time Markov chain is said to be *irreducible*, if for all $i, j \in S$ there is a $k \geq 1$ such that $P^k(i, j) > 0$.

Claim 3. *If μ is an invariant distribution for an irreducible Markov chain, then $\mu(x) > 0$ for all $x \in S$.*

Proof. Let μ be an invariant distribution and $y \in S$ some state in which $\mu(y) > 0$ holds. Let $x \in S$ be arbitrary and $k \geq 1$ such that $P^k(y, x) > 0$ which exists by the irreducibility. As μ is invariant, we have $\mu = \mu P^k$ and hence

$$\mu(x) = \sum_{z \in S} \mu(z) P^k(z, x) \geq \mu(y) P^k(y, x) > 0.$$

□

- c) Now, we can follow the main conclusion:

Claim 4. *An irreducible Markov chain on a finite state space has a unique invariant measure.*

Proof. The existence follows by Claim 2. Let μ_1, μ_2 be two invariant measures and suppose $\mu_1 \neq \mu_2$. By Claim 3, $\mu_1(x)$ and $\mu_2(x)$ are positive for any $x \in S$. Now define

$$\alpha := \min_{x \in S} \frac{\mu_1(x)}{\mu_2(x)} \quad \text{and} \quad \tilde{\mu} := \frac{1}{1 - \alpha} (\mu_1 - \alpha \mu_2).$$

Because μ_1 and μ_2 are both probability distributions and $\mu_1 \neq \mu_2$, there is some x with $\mu_1(x) < \mu_2(x)$ and hence $\alpha \in (0, 1)$ and $\tilde{\mu}$ is well defined.

Furthermore, $\tilde{\mu}$ is an invariant probability distribution, which can be concluded from the following three equations:

$$\begin{aligned} \sum_{x \in S} \tilde{\mu}(x) &= \frac{1}{1 - \alpha} \left(\sum_{x \in S} \mu_1(x) - \alpha \sum_{x \in S} \mu_2(x) \right) = 1, \\ \tilde{\mu}(x) &= \frac{\mu_1(x) - \alpha \mu_2(x)}{1 - \alpha} \geq \frac{\mu_1(x) - \mu_1(x)}{1 - \alpha} = 0 \quad \text{for all } x \in S, \\ \tilde{\mu}P &= \frac{1}{1 - \alpha} (\mu_1 P - \alpha \mu_2 P) = \frac{1}{1 - \alpha} (\mu_1 - \alpha \mu_2) = \tilde{\mu}. \end{aligned}$$

Now, let $x \in S$ be a state such that $\mu_1(x)/\mu_2(x) = \alpha$. Then $\tilde{\mu}(x) = 0$ is a contradiction to Claim 3 applied to $\tilde{\mu}$. Therefore $\mu_1 = \mu_2$ must hold. □

EXERCISE 9 – DISCRETE-TIME VOTER MODEL ON THE COMPLETE GRAPH

The discrete-time version of the voter model on the complete graph K_N works as follows: Start by attributing to each vertex an i.i.d. $\text{Ber}(p)$ opinion and obtain the configuration $\eta_0 \in \{0, 1\}^N$. At each positive integer time, select one vertex $x \in K_N$ uniformly at random and update its opinion by selecting uniformly at random a neighboring vertex y of x and copying its opinion.

a) Consider $X_k := \sum_{x \in K_N} \eta_k(x)$ the number of vertices with opinion 1 at time k .

Claim 5. X_k is a Markov chain in $\{0, \dots, N\}$ with transition rates

$$p(x, x+1) = p(x, x-1) = \frac{(N-x)x}{N(N-1)} =: q(x), \quad p(x, x) = 1 - 2\frac{(N-x)x}{N(N-1)},$$

and 0 everywhere else.

Proof. Let $\eta_k(x)$ be the configuration of opinions of the vertices of K_N at time k . For each time step, there is at most one opinion, that might change. Therefore, the probability for $|X_{k+1} - X_k| > 1$ is 0.

The event $X_{k+1} = X_k + 1$ happens if and only if a vertex with opinion 0 at time k was updated to opinion 1. Such a vertex x is chosen with probability $(N - X_k)/N$ and with probability $X_k/(N-1)$ its chosen neighbor has opinion 1. Therefore the probability for the event $X_{k+1} = X_k + 1$ is $\frac{(N-X_k)X_k}{N(N-1)}$.

Conversely, the event $X_{k+1} = X_k - 1$ happens iff a vertex with opinion 1 at time k is updated to opinion 0. Such a vertex is chosen with probability X_k/N and a neighboring vertex with opinion 0 is chosen with probability $(N - X_k)/(N-1)$. Again, this leads to a probability of $\frac{(N-X_k)X_k}{N(N-1)}$.

The event $X_{k+1} = X_k$ happens iff no vertex was updated. This happens only if the chosen neighbor vertex has the same opinion as the initially chosen vertex. The counter event is one of the events above, hence we get a probability of $1 - 2\frac{(N-X_k)X_k}{N(N-1)}$.

As these probabilities only depend on the value X_k , the Markov chain can be expressed only by the transition probability matrix as declared above. \square

b) Let $\tau := \inf\{k \geq 0 \mid X_k = 0 \text{ or } X_k = N\}$.

Claim 6. Let $h_i := \sum_{j=1}^i 1/j$ for $i \in \mathbb{N}$. For all $k \in \{0, \dots, N\}$ it holds

$$\mathbb{E}_k[\tau] := \mathbb{E}[\tau \mid X_0 = k] = (N-1) \left(k(h_{N-1} - h_{k-1}) + (N-k)(h_{N-1} - h_{N-k}) \right)$$

Proof. We begin by defining $m(k) := \mathbb{E}_k[\tau]$ for convenience. The Markov property implies

$$m(k) = \begin{cases} 0, & \text{if } k \in \{0, N\}, \\ 1 + (m(k+1) + m(k-1)) \cdot q(k) + m(k) \cdot (1 - 2q(k)), & \text{if } k \notin \{0, N\}. \end{cases}$$

This is a system of N linear equations using N variables. For $k \notin \{0, N\}$ we have $q(k) \neq 0$ and we can equivalently express the equation as

$$m(k+1) = \frac{m(k) - 1 - m(k) \cdot (1 - 2q(k))}{q(k)} - m(k-1) = 2m(k) - m(k-1) - \frac{1}{q(k)},$$

which is equivalent to

$$m(k+1) - m(k) = m(k) - m(k-1) - \frac{1}{q(k)}.$$

First we give a representation of all variables using $m(1)$:

Claim 6.1. For all $k \in \{0, \dots, N\}$, $m(k) = k \cdot m(1) - \sum_{i=1}^{k-1} \frac{k-i}{q(i)}$.

Proof. We use induction over k , where $k = 0$ is obvious. For the induction step $k \mapsto k+1$ with $k < N$ we calculate

$$\begin{aligned} m(k+1) &= m(1) + \sum_{i=1}^k (m(i+1) - m(i)) = m(1) + \sum_{i=1}^k \left(m(i) - m(i-1) - \frac{1}{q(i)} \right) \\ &= m(1) + m(k) - \sum_{i=1}^k \frac{1}{q(i)} = (k+1) \cdot m(1) - \sum_{i=1}^{k-1} \frac{k-i}{q(i)} - \sum_{i=1}^k \frac{1}{q(i)} \\ &= (k+1) \cdot m(1) + \sum_{i=0}^k \frac{k+1-i}{q(i)}, \end{aligned}$$

concluding the induction. \square

With the boundary condition $m(N) = 0$, we can compute $m(1)$ as

$$m(1) = \frac{1}{N} \sum_{i=1}^{N-1} \frac{N-i}{q(i)} = \sum_{i=1}^{N-1} \frac{N-1}{i} = (N-1)h_{N-1}.$$

Furthermore, for $k \in \{0, \dots, M\}$ we compute

$$\begin{aligned} \sum_{i=1}^{k-1} \frac{k-i}{q(i)} &= (N-1) \sum_{i=1}^{k-1} \frac{(k-i)N}{(N-i)i} = (N-1) \left(k \cdot \sum_{i=1}^{k-1} \frac{N}{(N-i)i} - \sum_{i=1}^{k-1} \frac{N}{N-i} \right) \\ &= (N-1) \left(k \cdot \sum_{i=1}^{k-1} \left(\frac{1}{N-i} + \frac{1}{i} \right) - N \cdot (h_{N-1} - h_{N-k}) \right) \\ &= (N-1) (k \cdot (h_{N-1} - h_{N-k} + h_{k-1}) - N \cdot (h_{N-1} - h_{N-k})) \\ &= (N-1) (k \cdot h_{k-1} - (N-k) \cdot (h_{N-1} - h_{N-k})). \end{aligned}$$

These two calculations now yield the result using Claim 6.1. \square

Claim 7. We can bound the expected stopping time by

$$\max_k \mathbb{E}_k[\tau] \leq (N-1)^2.$$

Proof. By Claim 6, we can compute

$$\begin{aligned} \mathbb{E}_k[\tau] &= (N-1) \left(\sum_{i=k}^{N-1} \frac{k}{i} + \sum_{i=N-k+1}^{N-1} \frac{N-k}{i} \right) \\ &\leq (N-1) \left((N-1-k+1) + (N-1-(N-k+1)+1) \right) \\ &= (N-1)(N-k+k-1) = (N-1)^2 \end{aligned}$$

\square

c) Now, we look at the probability of stopping in a configuration where everybody has opinion 1.

Claim 8. If we initialize the process according to $\text{Ber}(p)$, we have

$$\mathbb{P}_p[X_\tau = N] = p.$$

Proof. We first observe, that X_k is a martingale:

$$\mathbb{E}[X_{k+1} \mid \mathcal{F}_k] = (X_k - 1)q(k) + X_k(1 - 2q(k)) + (X_k + 1)q(k) = X_k.$$

As for every $k \notin \{0, N\}$ the probability to get to an absorbing state in $\{0, N\}$ in at most N steps is positive, so the stopping time τ is almost surely finite. Therefore, by the martingale property and writing $t \wedge \tau := \min(t, \tau)$, we get (for any $t \in \mathbb{N}$)

$$\mathbb{E}[X_0] = \mathbb{E}[X_{t \wedge \tau}] = \lim_{t \rightarrow \infty} \mathbb{E}[X_{t \wedge \tau}].$$

As X_n is trivially bounded by N , the dominated convergence theorem implies

$$\mathbb{E}[X_0] = \mathbb{E}[\lim_{t \rightarrow \infty} X_{t \wedge \tau}] = \mathbb{E}[X_\tau].$$

As X_τ can only attain the values 0 and N , we have $\mathbb{E}[X_\tau] = N \cdot \mathbb{P}[X_\tau = N]$. Furthermore with the initial configuration initialized according to $\text{Ber}(p)$, we have $\mathbb{E}[X_0] = p \cdot N$. Now, dividing by N yields the result. \square

- d) In the continuous-time voter model, each vertex updates its opinion with rate 1 by copying a neighbor's opinion. Now, we want to bound the maximum stopping time $\tilde{\tau}$ for the continuous-time model:

Claim 9. *For the continuous-time voter model, we have $\max_k \mathbb{E}_k[\tilde{\tau}] \leq N$.*

Proof. We can interpret the continuous-time voter model as follows: At rate 1 each vertex performs an update on its own. This is equivalent to having update events at rate N where at each event some random vertex performs an update. Therefore, we can use the bound for the expectation on the number of update events in the discrete-time model, $\mathbb{E}_k[\tau] \leq (N-1)^2$, of Claim 7 as follows: The expected stopping time $\mathbb{E}_k[\tilde{\tau}]$ of the continuous-time model is equal to the expected time of the discrete-time model divided by the event rate N , i.e. $\mathbb{E}_k[\tilde{\tau}] = \mathbb{E}_k[\tau]/N \leq (N-1)$. \square