THE CONTACT PROCESS
INTERACTING PARTICLE SYSTEMS

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1. Introduction and Preliminaries

Contact processes are special spin-flip systems which can used to model, for example, the spread of an infection. In the general case, we are given a countable, undirected and connected graph $G = (S, E)$ of bounded degree $\deg_{\max} := \sup_{x \in S} \deg(x) < \infty$. The nodes of the graph are usually called *sites* and during the contact process sites are either *infected* or *healthy*. Hence, the state space of the process we are about to define is $\Omega := \{0, 1\}^S$ where 0 should be interpreted as healthy and 1 as infected.
We denote sites by letters $x, y \in S$, configurations by $\eta, \zeta, \xi \in \Omega$. The resulting configuration for flipping site x in configuration is denoted as η^x and two sites x and y are called *neighboring* $(x \sim y)$ if $\{x, y\}$ is an edge in E. We often use η as the set consisting of all sites x with $\eta(x) = 1$.
In a contact process, an infected site becomes healthy after a unit exponential time. On the contrary a healthy site becomes infected at a rate proportional to the number of infected neighbors. This proportionality coefficient $\lambda > 0$ is independent of the site itself. Now we can define the flip rates of a site x in a configuration η by

$$c(x, \eta) := \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \cdot |\{y \sim x \mid \eta(y) = 1\}| & \text{if } \eta(x) = 0. \end{cases}$$

As any other spin system, these spin rates can be translated into a continuous time Markov chain with Q-Matrix of the form $q(\eta, \eta^x) := c(x, \eta)$ and generator

$$\mathcal{L} f(\eta) := \sum_{x \in S} c(x, \eta) \left(f(\eta^x) - f(\eta) \right),$$

if $M := \sup_{x \in S} \sum_{u : x \neq u} \gamma(x, u) < \infty$ holds where $\gamma(x, u) := \sup_{\eta \in \Omega} |c(x, \eta^u) - c(x, \eta)|$. Here, this is indeed the case: For $u \sim x$ we have $\gamma(x, u) = \lambda$, otherwise $\gamma(x, u)$ vanishes implying $M = \lambda \cdot \deg_{\max}$. By looking at the $(M < \epsilon)$-Theorem with

$$\epsilon := \inf_{x \in S, \eta \in \Omega} c(x, \eta) + c(x, \eta^x) = 1,$$

we get the following result: If $\lambda < \deg_{\max}^{-1}$, then η_t is *ergodic*, i.e. there is a unique stationary distribution μ and for every $\eta \in \Omega$ and $f \in C(\Omega)$ it fulfills $\lim_{t \to \infty} S_t f(\eta) = \int_\Omega f \, \mathrm{d}\mu$, where S_t denotes the semigroup generated by $\mathcal{L}$. As the pointmass δ_0 on 0 is always an invariant measure, we discuss in the next sections whether there are more invariant measures for $\lambda \geq \deg_{\max}^{-1}$.
Before that, we note, that any contact process is an *attractive spin system*: If $\eta \leq \zeta$ holds component-wise, we have $c(x, \eta) = \lambda \cdot |\{y \sim x \mid \eta(y) = 1\}| \leq \lambda \cdot |\{y \sim x \mid \zeta(y) = 1\}| = c(x, \zeta)$ for $\eta(x) = 1$ and $c(x, \eta) = 1 \leq 1 = c(x, \zeta)$ for $\zeta(x) = 0$. From the theory of attractive spin systems we know the existence of a lower invariant measure $\underline{\nu} := \lim_{t \to \infty} \delta_0 S_t$ and an upper invariant measure $\overline{\nu} := \lim_{t \to \infty} \delta_1 S_t$. As δ_0 is already invariant, $\underline{\nu} = \delta_0$ follows immediately. The structure of $\overline{\nu}$ is less obvious for $\lambda \geq \deg_{\max}^{-1}$ as we will see in the next sections.

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Date: December 8, 2020.

2. To Be or Not to Be

A central aspect we want to analyze is whether an infection goes extinct at some time in the future or survives "until the end of time". In fact, there are two notions of survival discussed:

Definition 2.1 (Survival)**.** We say, a contact process *survives (weakly)* if there is an $x \in S$ such that $\eta := \{x\}$ fulfills

$$\mathbb{P}_{\{x\}}(\forall t \geq 0 : \eta_t \neq \emptyset) > 0.$$

We say it *strongly survives* if there is an $x \in S$ such that $\eta := \{x\}$ fulfills

$$\mathbb{P}_{\{x\}}(x \in \eta_t \text{ for a sequence of } t\text{'s increasing to } \infty) > 0.$$

Otherwise, the process *dies out.*

We note, that if the probability of weak survival above is positive for some $x \in S$, then it is positive for any $y \in S$ or even for any finite $\eta \in \Omega$ as the graph is assumed to be connected.
We are interested in the possible values λ can attain, such that the process weakly or strongly survives. Using a simple coupling argument we can show that increasing the infection rate of a surviving process results in a surviving process and decreasing the infection rate for a dying process results in a dying process.

Proposition 2.2 (Critical Values)**.** *There are values $\lambda_c \leq \lambda_s$ in $[0, \infty]$ such that the process will survive strongly, if $\lambda > \lambda_s$, survive weakly, if $\lambda > \lambda_c$, or die out, if $\lambda < \lambda_c$. We call λ_c and λ_s the* critical values*.*

Immediately, multiple questions arise: Are λ_c and λ_s positive? Is it possible that a process will never survive independent of the infection rate? Is there a graph with $\lambda_c < \lambda_s$, such a process might survive only weakly? The first of these question will be addressed using the graphical representation and the self-duality of contact processes:

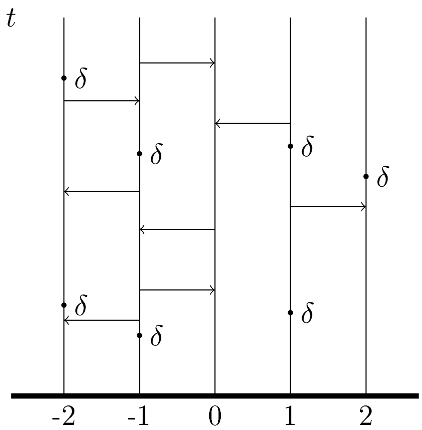

Figure 1. Graphical Representation of the contact process

The contact process can be displayed very intuitively in a *graphical representation*: An example can be seen in Figure 1 for the integer lattice $\mathbb{Z}$: We use the *space-time* picture on $S \times [0, \infty)$. For each site, we generate *heal events* at a unit exponential rate which are denoted as δ in the graph. Additionally we create *infection events* in the form of arrows for each directed edge in the graph at rate λ. Then we call a path from (x, s) to (y, t) in this picture *active*, if it only walks upwards in time at sites or along an infection arrow and never passes a heal event. Given an initial infected set η at time 0, we can deduce the set of infected sites at time t as

$$\eta_t = \{y \in S \mid \exists \text{ active path from } (x, 0) \text{ to } (y, t) \text{ for some } x \in \eta\}.$$

Proposition 2.3 (Self-Duality)**.** *For any $\eta, \zeta \in \Omega$, we have $\mathbb{P}_\eta(\eta_t \cap \zeta \neq \emptyset) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \emptyset)$.*

Proof. We fix some graphical representation of the process and assume, that if we start the process with infected set η, there is a site in ζ that is infected at time t. The dual process is constructed as follows: We look at all possible sites that could have led to an infection state at any site of ζ at time t. This can be done by traversing the graphical representation backwards: We collect in $\hat{\zeta}_{t-s}$ all sites $x \in S$ for which there exists an active path from (x,s) to (y,t) for some $y \in \zeta$. As the rate of infection events from a site x to a site y is the same as for the infection rate from y to x, we observe, that $\hat{\zeta}_{t-s}$ is by distribution equal to ζ_s.
Finally, we observe, that if the infection starting from η leads to an infected site of ζ at time t, then $\hat{\zeta}_0$ must contain an element of η implying $\mathbb{P}_\eta(\eta_t \cap \zeta \neq \emptyset) \leq \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \emptyset)$. □

Applying the duality relation above to finite η and all nodes S, we get for t approaching ∞:

Corollary 2.4. *For finite $\eta \in \Omega$ we have $\mathbb{P}_\eta(\forall t \geq 0 : \eta_t \neq \emptyset) = \overline{\nu}(\{\zeta \in \Omega \mid \eta \cap \zeta \neq \emptyset\})$. Moreover, the process dies out iff δ_0 is the only invariant measure. Hence $\lambda_c \geq \deg_{\max}^{-1} > 0$.*

3. The Contact Process on Homogeneous Trees

Here, G is a homogeneous tree, i.e. a graph in which each node has $d+1$ neighboring nodes. We discuss two main theorems. The first gives bounds for λ_c, in particular implying the possibility of weak survival:

Theorem 3.1 (Weak Survival)**.** *The contact process on a homogeneous tree satisfies*

$$\frac{1}{d+1} \leq \lambda_c \leq \frac{1}{d-1}$$

The second result shows that for large enough d, there might be weak without strong survival:

Theorem 3.2 (Bound for Strong Survival)**.** *The critical value for strong survival on the homogeneous tree satisfies*

$$\lambda_s \geq \frac{1}{2\sqrt{d}}.$$

Therefore $\lambda_s > \lambda_c$ holds for $d \geq 6$.

To convince ourselves of these statements, we have to introduce the notion of superharmonic functions, which play a central in proving weak survival.

Definition 3.3 (Superharmonicity)**.** A function f on Ω is called *superharmonic* if all $t \geq 0, \eta \in \Omega$ satisfy $\mathbb{E}_\eta|f(\eta_t)| < \infty$ and $\mathbb{E}_\eta f(\eta_t) \leq f(\eta)$.

Proposition 3.4. *If there is a nonconstant bounded superharmonic function $f : \Omega \to \mathbb{R}$ satisfying $f(\emptyset) \geq f(\eta)$ for all $\eta \in \Omega$, the process survives weakly.*

Sketch of Proof. Let $\eta(t)$ be the contact process starting from an $\eta \in \Omega$ and let $\zeta(t)$ be the same process, except that the rate going from $\emptyset$ to η is 1 (instead of 0). Then we can deduce $\mathbb{E}_\eta f(\zeta_t) \leq \mathbb{E}_\eta f(\eta_t)$. Therefore, f is also superharmonic w.r.t. $\zeta(t)$. We observe, that $\zeta(t)$ is an irreducible Markov chain, i.e. $\mathbb{P}_\xi(\xi_t = \xi') > 0$ holds for all $\xi \neq \xi'$ in Ω and $t > 0$. Hence, from the next paragraph, it follows that $\emptyset$ cannot be a recurring state for ζ, what means, that $\mathbb{P}_\eta(\zeta_t = \emptyset$ for an increasing sequence t to ∞ $) < 1$. Therefore, $\emptyset$ cannot be a recurring state for $\eta(t)$ either, implying that $\eta(t)$ survives weakly.
If we suppose, that $\emptyset$ is a recurring state for $\zeta(t)$, then any other state is recurring because $\zeta(t)$ is irreducible. By the martingale theory $\mathbb{E}_\eta f(\zeta_t)$ converges with $\lim_{t\to\infty} \mathbb{E}_\eta f(\zeta_t) \leq f(\eta)$ a.s. Because ζ_t visits any configuration infinitely often for arbitrarily large t almost surely, f is constant. □

Hence, for proving Theorem 3.1 we ought to find a superharmonic function fulfilling $f(\emptyset) \geq f(\eta)$ for all η. Useful for that is the next statement, which follows from the Markov property.

Proposition 3.5. *Suppose $f : \Omega \to \mathbb{R}$ is a bounded function satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_\eta f(\eta_t)\bigg|_{t=0} \leq 0,$$

then $E_\eta f(\eta_t)$ is a decreasing function in t.

With these preliminaries, we can prove weak survival:

Proof of Theorem 3.1. We have already observed $\lambda_c \geq \deg_{\max}^{-1}$ in the last section. As our superharmonic function, we use $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0, 1)$ we will specify later. We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\bigg|_{t=0} = \left(\rho^{-1}|\eta| - \lambda \cdot |E_\eta|\right)(1-\rho)f(\eta),$$

for non-empty η, where E_η is the set of edges connecting nodes in η with nodes in $S \setminus \eta$. As trees are acyclic, the number of edges connecting nodes inside η is at most $|\eta| - 1$. Hence, we can bound $|E_\eta|$ by

$$|E_\eta| \geq |\eta| \cdot (d+1) - 2 \cdot (|\eta| - 1) = |\eta|(d-1) + 2 \geq |\eta|(d-1).$$

Therefore, the derivative of $\mathbb{E}_\eta f(\eta_t)$ is non-positive, if $\rho^{-1} \geq \lambda(d-1)$ holds. For $\lambda > (d-1)^{-1}$, we can choose $\rho = (\lambda(d-1))^{-1}$ for which f is superharmonic due to Proposition 3.5. Applying Proposition 3.4 yields weak survival and thus $\lambda_c \leq (d-1)^{-1}$. $\square$

With the same ideas, we can prove Theorem 3.2:

Sketch of Proof for Theorem 3.2. As strong survival involves not only the cardinality of infection, but also the location of the infection process, the superharmonic function we want to define must also depend on that. It turns out, that $f(\eta) := \sum_{x \in \eta} \rho^{l(x)}$, where $l(x)$ is the level of x w.r.t. some arbitrarily chosen node as root, does the job: We can calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\bigg|_{t=0} \leq \left(\lambda \cdot (d\rho + \rho^{-1}) - 1\right) f(\eta),$$

which is at most 0, if we choose $\rho = \sqrt{d}^{-1}$ and $\lambda \leq (2\sqrt{d})^{-1}$. Applying Proposition 3.5 again, we find that f is superharmonic and thus $\mathbb{E}_\eta f(\eta_t)$ converges almost surely for t to ∞. In the event, that x gets infected infinitely often for arbitrarily large t, this martingale has to change by at least $\rho^{l(x)}$ at arbitrarily large times t. Therefore, this event does almost surely not occur. $\square$

4. Results for Integer Lattices

The same technique for showing weak survival works for the integer lattice $\mathbb{Z}$, but finding a superharmonic function f and value ρ that satisfy the necessary properties is more involved. Nevertheless, this yields the result, that $\mathbb{Z}$ (and hence any graph containing $\mathbb{Z}$) has critical value $\lambda_c \leq 2$. More generally, for a multidimensional integer lattice $\mathbb{Z}^d$ one can bound λ_c by $2/d$.