The Contact Process

Interacting Particle Systems

Michael Markl

December 8, 2020

What is a Contact Process?

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L} f(\eta) = \sum_{x \in S} c(x,\eta) \left(f(\eta^x) - f(\eta) \right)$

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L}f(\eta) = \sum_{x \in S} c(x,\eta)\left(f(\eta^x) - f(\eta)\right)$

$$M = \sup_{x \in S} \sum_{u : x \neq u} \gamma(x,u)$$

$$\gamma(x,u) = \sup_{\eta \in \Omega} |c(x,\eta^u) - c(x,\eta)|$$

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L}f(\eta) = \sum_{x \in S} c(x,\eta)\left(f(\eta^x) - f(\eta)\right)$

$$M = \sup_{x \in S} \sum_{u : x \neq u} \gamma(x,u)$$

$$\gamma(x,u) = \sup_{\eta \in \Omega} |c(x,\eta^u) - c(x,\eta)| = \begin{cases} \lambda & \text{if } u \sim x, \\ 0 & \text{if } u \not\sim x. \end{cases}$$

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0, 1\}^S$ at site $x \in S$:

$$c(x, \eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L} f(\eta) = \sum_{x \in S} c(x, \eta) \left(f(\eta^x) - f(\eta) \right)$

$$M = \sup_{x \in S} \sum_{u : x \neq u} \gamma(x, u) = \lambda \cdot \deg_{\max}$$

$$\gamma(x, u) = \sup_{\eta \in \Omega} |c(x, \eta^u) - c(x, \eta)| = \begin{cases} \lambda & \text{if } u \sim x, \\ 0 & \text{if } u \not\sim x. \end{cases}$$

Let $G=(S,E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L}f(\eta) = \sum_{x \in S} c(x,\eta)\left(f(\eta^x) - f(\eta)\right)$

$$M = \sup_{x \in S} \sum_{u : x \neq u} \gamma(x,u) = \lambda \cdot \deg_{\max}$$

$$\gamma(x,u) = \sup_{\eta \in \Omega} |c(x,\eta^u) - c(x,\eta)| = \begin{cases} \lambda & \text{if } u \sim x, \\ 0 & \text{if } u \not\sim x. \end{cases}$$

$$\epsilon = \inf_{x \in S, \eta \in \Omega} c(x,\eta) + c(x,\eta^x)$$

Let $G = (S, E)$ be a connected graph with bounded degree $\deg_{\max}$.

Flip rate for an infected set $\eta \in \mathcal{P}(S) \hat{=} \{0,1\}^S$ at site $x \in S$:

$$c(x,\eta) := \begin{cases} 1, & \text{if } x \in \eta, \\ \lambda \cdot |\{y \in \eta \mid x \sim y\}|, & \text{if } x \notin \eta. \end{cases}$$

Generator: $\mathcal{L}f(\eta) = \sum_{x \in S} c(x,\eta)\left(f(\eta^x) - f(\eta)\right)$

$$M = \sup_{x \in S} \sum_{u : x \neq u} \gamma(x,u) = \lambda \cdot \deg_{\max}$$

$$\gamma(x,u) = \sup_{\eta \in \Omega} |c(x,\eta^u) - c(x,\eta)| = \begin{cases} \lambda & \text{if } u \sim x, \\ 0 & \text{if } u \nsim x. \end{cases}$$

$$\epsilon = \inf_{x \in S, \eta \in \Omega} c(x,\eta) + c(x,\eta^x) = 1$$

Invariant Measures

Invariant Measures

A contact process is an *attractive* spin system: If $\eta \subseteq \zeta$, then

$$x \in \eta \Rightarrow c(x,\eta) \leqslant c(x,\zeta), \qquad x \notin \zeta \Rightarrow c(x,\eta) \geqslant c(x,\zeta)$$

Invariant Measures

A contact process is an *attractive* spin system: If $\eta \subseteq \zeta$, then

$$x \in \eta \Rightarrow c(x,\eta) \leqslant c(x,\zeta), \qquad x \notin \zeta \Rightarrow c(x,\eta) \geqslant c(x,\zeta)$$

Hence, we have upper and lower invariant measures:

$$\overline{\nu} := \lim_{t\to\infty} \delta_1 S_t, \qquad \underline{\nu} := \lim_{t\to\infty} \delta_0 S_t$$

Invariant Measures

A contact process is an *attractive* spin system: If $\eta \subseteq \zeta$, then

$$x \in \eta \Rightarrow c(x,\eta) \leqslant c(x,\zeta), \qquad x \notin \zeta \Rightarrow c(x,\eta) \geqslant c(x,\zeta)$$

Hence, we have upper and lower invariant measures:

$$\overline{\nu} := \lim_{t\to\infty} \delta_1 S_t, \qquad \underline{\nu} := \lim_{t\to\infty} \delta_0 S_t = \delta_0$$

Invariant Measures

A contact process is an *attractive* spin system: If $\eta \subseteq \zeta$, then

$$x \in \eta \Rightarrow c(x,\eta) \leqslant c(x,\zeta), \qquad x \notin \zeta \Rightarrow c(x,\eta) \geqslant c(x,\zeta)$$

Hence, we have upper and lower invariant measures:

$$\overline{\nu} := \lim_{t\to\infty} \delta_1 S_t, \qquad \underline{\nu} := \lim_{t\to\infty} \delta_0 S_t = \delta_0$$

The $M < \epsilon$ criterion implies: If $\lambda < \deg_{\max}^{-1}$, then $\mathcal{I} = \{\delta_0\}$.

To Be or Not to Be

Survival or Extinction

Definition 2.1 (Survival)

A contact process $\eta(t)$ *survives (weakly)* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(\forall t \geqslant 0 : \eta_t \neq \varnothing) > 0.$$

Survival or Extinction

Definition 2.1 (Survival)

A contact process $\eta(t)$ *survives (weakly)* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(\forall t \geqslant 0 : \eta_t \neq \emptyset) > 0.$$

It *survives strongly* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(x \in \eta_t \text{ for a sequence } t\text{'s increasing to } \infty) > 0.$$

Survival or Extinction

Definition 2.1 (Survival)

A contact process $\eta(t)$ *survives (weakly)* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(\forall t \geqslant 0 : \eta_t \neq \emptyset) > 0.$$

It *survives strongly* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(x \in \eta_t \text{ for a sequence } t\text{'s increasing to } \infty) > 0.$$

Otherwise the process *dies out*.

Survival or Extinction

Definition 2.1 (Survival)

A contact process $\eta(t)$ *survives (weakly)* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(\forall t \geqslant 0 : \eta_t \neq \varnothing) > 0.$$

It *survives strongly* if there is an $x \in S$ such that

$$\mathbb{P}_{\{x\}}(x \in \eta_t \text{ for a sequence } t\text{'s increasing to } \infty) > 0.$$

Otherwise the process *dies out*.

Proposition 2.2 (Critical Values)

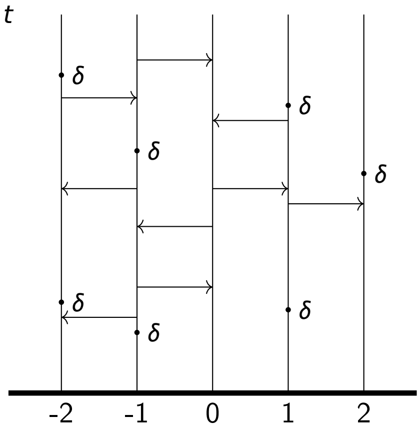
There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

Infection Process in Graphical Representation

A path in $S \times [0, \infty)$ is *active*, if

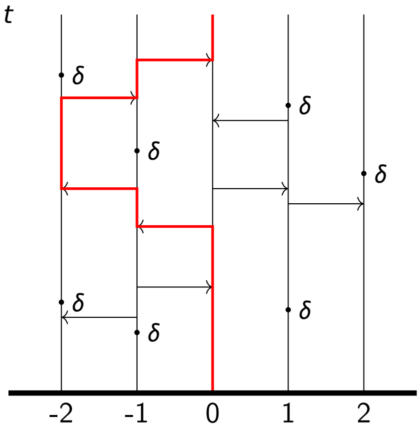
- it only walks upwards in time,
- it switches site only according to infection arrows,
- it does not pass any heal events.



Infection Process in Graphical Representation

A path in $S \times [0, \infty)$ is *active*, if

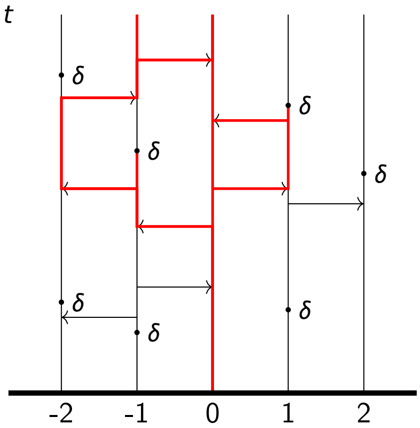
- it only walks upwards in time,
- it switches site only according to infection arrows,
- it does not pass any heal events.



Infection Process in Graphical Representation

A path in $S \times [0, \infty)$ is *active*, if

- it only walks upwards in time,
- it switches site only according to infection arrows,
- it does not pass any heal events.

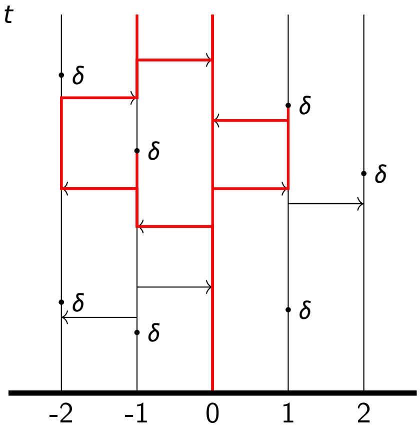


Infection Process in Graphical Representation

A path in $S \times [0, \infty)$ is *active*, if

- it only walks upwards in time,
- it switches site only according to infection arrows,
- it does not pass any heal events.

If η is initially infected, then
$\eta_t = \{x \in S \mid \exists$ active path from $(x, 0)$ to (y,t) for some $x \in \eta\}$



Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ *in* $[0, \infty]$ *such that the contact process*

- *survives weakly iff* $\lambda > \lambda_c$,
- *survives strongly iff* $\lambda > \lambda_s$.

Proof.

Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

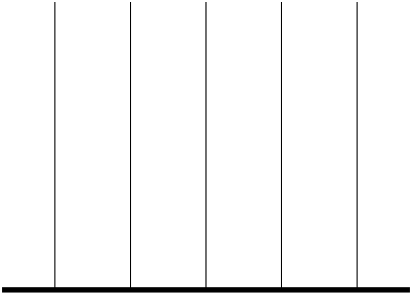
Proof. We show $\mathbb{P}_{\{x\}}(y \in \eta_t)$ is non-decreasing in λ by a coupling argument.

Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

Proof. We show $\mathbb{P}_{\{x\}}(y \in \eta_t)$ is non-decreasing in λ by a coupling argument. Let $\eta(t)$, $\zeta(t)$ be contact processes with $\lambda_\eta \leqslant \lambda_\zeta$.



□

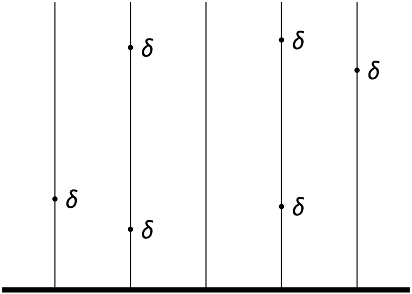
Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

Proof. We show $\mathbb{P}_{\{x\}}(y \in \eta_t)$ is non-decreasing in λ by a coupling argument. Let $\eta(t)$, $\zeta(t)$ be contact processes with $\lambda_\eta \leqslant \lambda_\zeta$.

1) Put δ's at rate 1.



□

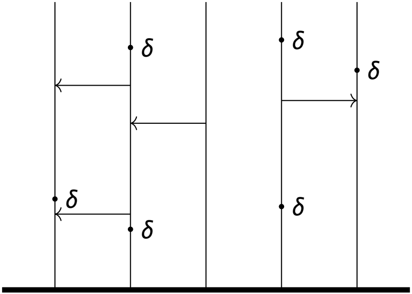
Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

Proof. We show $\mathbb{P}_{\{x\}}(y \in \eta_t)$ is non-decreasing in λ by a coupling argument. Let $\eta(t)$, $\zeta(t)$ be contact processes with $\lambda_\eta \leqslant \lambda_\zeta$.

1) Put δ's at rate 1.
2) Put $\longrightarrow$ at rate λ_η.



□

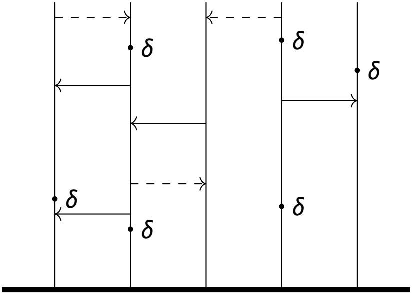
Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
- *survives strongly iff $\lambda > \lambda_s$.*

Proof. We show $\mathbb{P}_{\{x\}}(y \in \eta_t)$ is non-decreasing in λ by a coupling argument. Let $\eta(t)$, $\zeta(t)$ be contact processes with $\lambda_\eta \leqslant \lambda_\zeta$.

1) Put δ's at rate 1.
2) Put $\longrightarrow$ at rate λ_η.
3) Put $\dashrightarrow$ at rate $(\lambda_\zeta - \lambda_\eta)$.



□

Self Duality

Proposition 2.3 (Self Duality)

$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \emptyset) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \emptyset)$ *holds for any* $\eta, \zeta \in \{0, 1\}^S$.

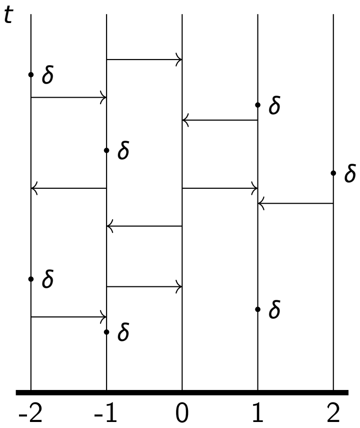
Self Duality

Proposition 2.3 (Self Duality)

$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \varnothing) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \varnothing)$ *holds for any* $\eta, \zeta \in \{0,1\}^S$.

Proof. Construct dual process $\hat{\zeta}$ by

$$\hat{\zeta}_{t-s} = \{x \in S \mid \exists \text{ active path from } (x,s) \text{ to } (y,t) \text{ for some } y \in \zeta\}$$



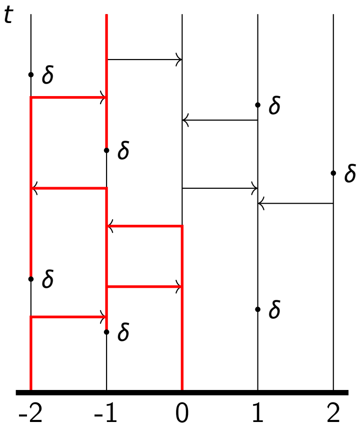
Self Duality

Proposition 2.3 (Self Duality)

$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \emptyset) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \emptyset)$ *holds for any* $\eta, \zeta \in \{0,1\}^S$.

Proof. Construct dual process $\hat{\zeta}$ by

$$\hat{\zeta}_{t-s} = \{x \in S \mid \exists \text{ active path from } (x,s) \text{ to } (y,t) \text{ for some } y \in \zeta\}$$



Self Duality

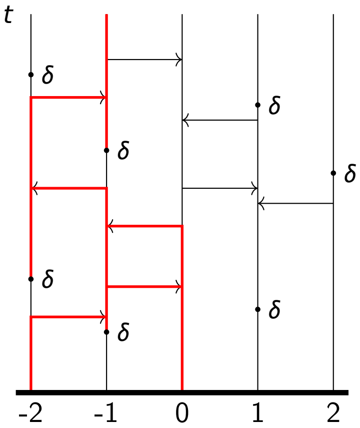
Proposition 2.3 (Self Duality)

$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \varnothing) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \varnothing)$ holds for any $\eta, \zeta \in \{0,1\}^S$.

Proof. Construct dual process $\hat{\zeta}$ by

$$\hat{\zeta}_{t-s} = \{x \in S \mid \exists \text{ active path from } (x,s) \text{ to } (y,t) \text{ for some } y \in \zeta\}$$

Now $\eta_t \cap \zeta \neq \varnothing \iff \hat{\zeta}_0 \cap \eta \neq \varnothing$.



Self Duality

Proposition 2.3 (Self Duality)

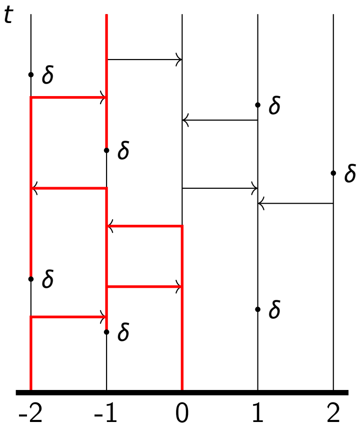
$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \varnothing) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \varnothing)$ *holds for any* $\eta, \zeta \in \{0,1\}^S$.

Proof. Construct dual process $\hat{\zeta}$ by

$$\hat{\zeta}_{t-s} = \{x \in S \mid \exists \text{ active path from } (x,s) \text{ to } (y,t) \text{ for some } y \in \zeta\}$$

Now $\eta_t \cap \zeta \neq \varnothing \iff \hat{\zeta}_0 \cap \eta \neq \varnothing$.

$\hat{\zeta}_{t-s}$ is by distribution equal to ζ_s. $\square$



Corollary 2.4

For finite $\eta \in \{0,1\}^S$*:* $\mathbb{P}_\eta(\forall t \geqslant 0 : \eta_t \neq \varnothing) = \overline{\nu}(\{\zeta \in \Omega \mid \eta \cap \zeta \neq \varnothing\})$

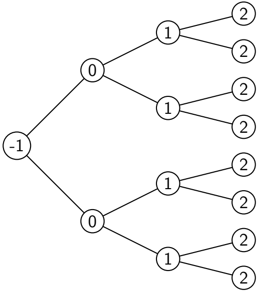
Corollary 2.4

For finite $\eta \in \{0,1\}^S$*:* $\mathbb{P}_\eta(\forall t \geqslant 0 : \eta_t \neq \varnothing) = \overline{\nu}(\{\zeta \in \Omega \mid \eta \cap \zeta \neq \varnothing\})$

Hence, the process dies iff $\mathcal{I} = \{\delta_0\}$*. Therefore,* $\lambda_c \geqslant \mathrm{deg}_{\max}^{-1} > 0$*.*

The Contact Process on Homogeneous Trees

Homogeneous Tree



Every node has degree $d + 1$.

Results for the Homogeneous Tree

Theorem 3.1 (Weak Survival)

On a homogeneous tree, we have

$$\frac{1}{d+1} \leqslant \lambda_c \leqslant \frac{1}{d-1}$$

Results for the Homogeneous Tree

Theorem 3.1 (Weak Survival)

On a homogeneous tree, we have

$$\frac{1}{d+1} \leqslant \lambda_c \leqslant \frac{1}{d-1}$$

Theorem 3.2 (Bound for strong survival)

On a homogeneous tree, we have

$$\frac{1}{d+1} \leqslant \lambda_c \leqslant \frac{1}{d-1}$$

In particular, $\lambda_c < \lambda_s$ *for* $d \geqslant 6$.

Important Tools

Definition 3.3 (Superharmonicity)

A function $f : \{0,1\}^S \to \mathbb{R}$ is *superharmonic*, if $\mathbb{E}_\eta |f(\eta_t)| < \infty$ and $\mathbb{E}_\eta f(\eta_t) \leqslant f(\eta)$ hold for all $t \geqslant 0$ and $\eta \in \{0,1\}^S$.

Important Tools

Definition 3.3 (Superharmonicity)

A function $f : \{0,1\}^S \to \mathbb{R}$ is *superharmonic*, if $\mathbb{E}_\eta|f(\eta_t)| < \infty$ and $\mathbb{E}_\eta f(\eta_t) \leqslant f(\eta)$ hold for all $t \geqslant 0$ and $\eta \in \{0,1\}^S$.

Proposition 3.4

If a nonconstant bounded superharmonic function f satisfies $f(\varnothing) \geqslant f(\eta)$ for all $\eta \in \{0,1\}^S$, the process survives weakly.

Important Tools

Definition 3.3 (Superharmonicity)

A function $f : \{0,1\}^S \to \mathbb{R}$ is *superharmonic*, if $\mathbb{E}_\eta |f(\eta_t)| < \infty$ and $\mathbb{E}_\eta f(\eta_t) \leqslant f(\eta)$ hold for all $t \geqslant 0$ and $\eta \in \{0,1\}^S$.

Proposition 3.4

If a nonconstant bounded superharmonic function f satisfies $f(\varnothing) \geqslant f(\eta)$ for all $\eta \in \{0,1\}^S$, the process survives weakly.

Proposition 3.5

If $f : \{0,1\}^S \to \mathbb{R}$ is bounded and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\big|_{t=0} \leqslant 0$, then $\mathbb{E}_\eta f(\eta_t)$ is decreasing in t.

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0, 1)$.

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

Calculate using $E_\eta := \{\{x,y\} \in E \mid x \in \eta, y \notin \eta\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\Big|_{t=0} = (\rho^{-1}|\eta| - \lambda \cdot |E_\eta|)(1-\rho)f(\eta)$$

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

Calculate using $E_\eta := \{\{x,y\} \in E \mid x \in \eta, y \notin \eta\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\Big|_{t=0} = (\rho^{-1}|\eta| - \lambda \cdot |E_\eta|)(1-\rho)f(\eta)$$

Bound $|\mathbb{E}_\eta| \geqslant |\eta| \cdot (d+1) - 2 \cdot (|\eta| - 1) = \eta(d-1) + 2 \geqslant |\eta|(d-1)$

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

Calculate using $E_\eta := \{\{x,y\} \in E \mid x \in \eta, y \notin \eta\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\Big|_{t=0} = (\rho^{-1}|\eta| - \lambda \cdot |E_\eta|)(1-\rho)f(\eta)$$
$$\leqslant |\eta|(\rho^{-1} - \lambda(1-d))(1-\rho)f(\eta)$$

Bound $|\mathbb{E}_\eta| \geqslant |\eta| \cdot (d+1) - 2 \cdot (|\eta|-1) = \eta(d-1) + 2 \geqslant |\eta|(d-1)$

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

Calculate using $E_\eta := \{\{x,y\} \in E \mid x \in \eta, y \notin \eta\}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\Big|_{t=0} &= (\rho^{-1}|\eta| - \lambda \cdot |E_\eta|)(1-\rho)f(\eta) \\ &\leqslant |\eta|(\rho^{-1} - \lambda(1-d))(1-\rho)f(\eta) \end{aligned}$$

Bound $|\mathbb{E}_\eta| \geqslant |\eta| \cdot (d+1) - 2 \cdot (|\eta|-1) = \eta(d-1) + 2 \geqslant |\eta|(d-1)$

For $\rho = \dfrac{1}{\lambda(1-d)}$, f is superharmonic due to Proposition 3.4.

Sketch of proof for $\lambda_c \leqslant (d-1)^{-1}$

Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

Calculate using $E_\eta := \{\{x,y\} \in E \mid x \in \eta, y \notin \eta\}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_\eta f(\eta_t)\Big|_{t=0} &= (\rho^{-1}|\eta| - \lambda \cdot |E_\eta|)(1-\rho) f(\eta) \\ &\leqslant |\eta|(\rho^{-1} - \lambda(1-d))(1-\rho) f(\eta) \end{aligned}$$

Bound $|\mathbb{E}_\eta| \geqslant |\eta| \cdot (d+1) - 2 \cdot (|\eta| - 1) = \eta(d-1) + 2 \geqslant |\eta|(d-1)$

For $\rho = \dfrac{1}{\lambda(1-d)}$, f is superharmonic due to Proposition 3.4.

Proposition 3.5 yields weak survival.

$\square$

Literature

[1] LIGGETT, T. M.
Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.
Springer Berlin Heidelberg, Berlin, Heidelberg, 1999, ch. Contact Processes, pp. 31–137.

[2] LIGGETT, T. M.
Continuous Time Markov Processes - An Introduction.
American Mathematical Soc., Heidelberg, 2010, ch. 4.4 The contact process, pp. 161–175.