The Contact Process

Interacting Particle Systems

Michael Markl

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What is a Contact Process?

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Invariant Measures

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A contact process is an *attractive* spin system: If $\eta \subseteq \zeta$, then

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The $M < \epsilon$ criterion implies: If $\lambda < \deg_{\max}^{-1}$, then $\mathcal{I} = \{\delta_0\}$.

To Be or Not to Be

Survival or Extinction

Definition 2.1 (Survival)

A contact process $\eta(t)$ *survives (weakly)* if there is an $x \in S$ such that

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Proposition 2.2 (Critical Values)

There exist $\lambda_c \leqslant \lambda_s$ in $[0, \infty]$ such that the contact process

- *survives weakly iff $\lambda > \lambda_c$,*
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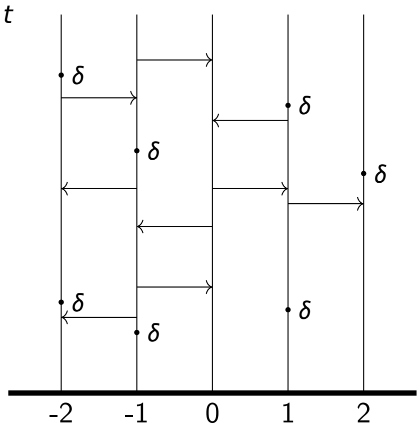
Infection Process in Graphical Representation

A path in $S \times [0, \infty)$ is *active*, if

- it only walks upwards in time,
- it switches site only according to infection arrows,
- it does not pass any heal events.



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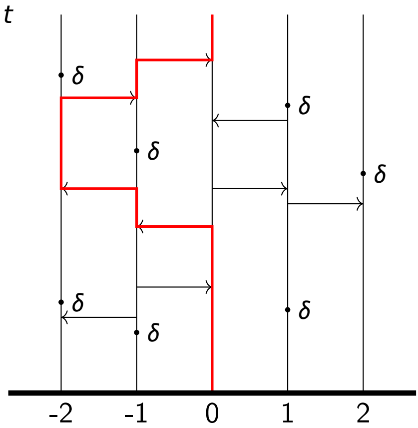

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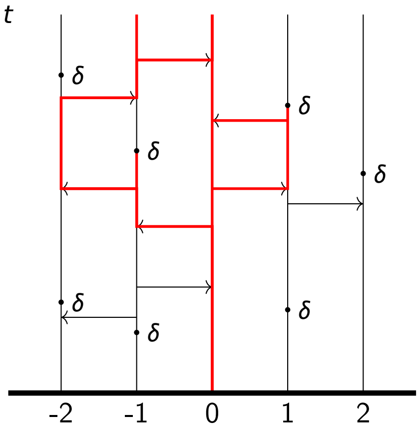

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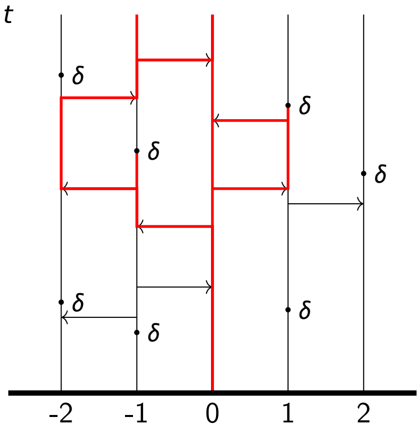
If η is initially infected, then
$\eta_t = \{y \in S \mid \exists$ active path from $(x, 0)$ to (y,t) for some $x \in \eta\}$



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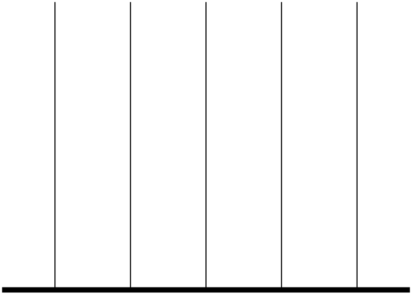

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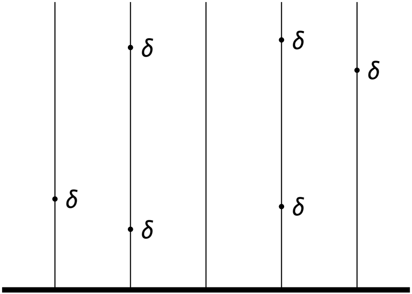

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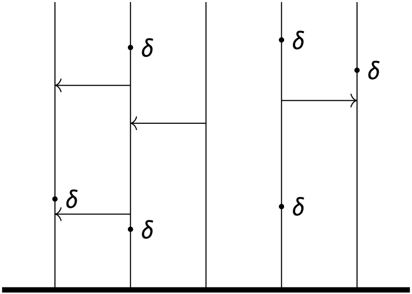

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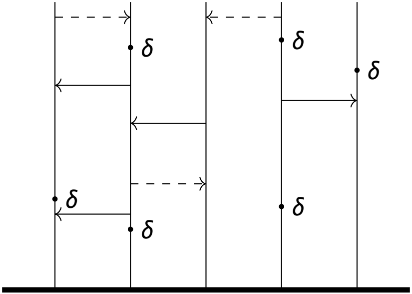

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$\mathbb{P}_\eta(\eta_t \cap \zeta \neq \emptyset) = \mathbb{P}_\zeta(\eta \cap \zeta_t \neq \emptyset)$ *holds for any* $\eta, \zeta \in \{0,1\}^S$.

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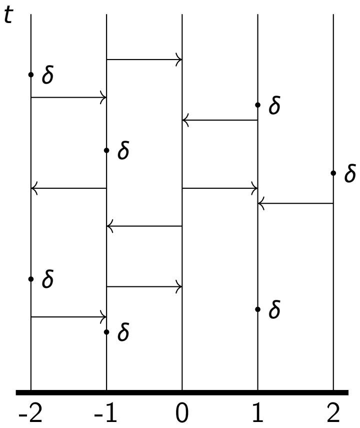

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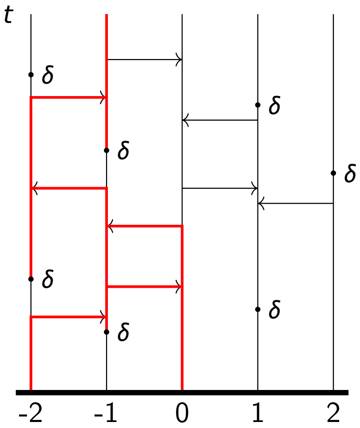

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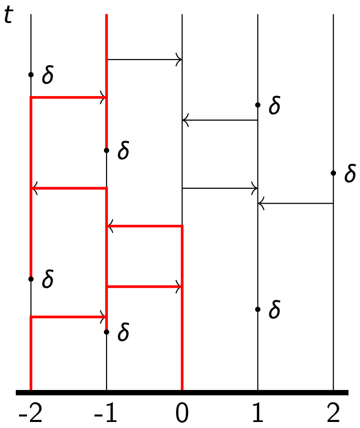

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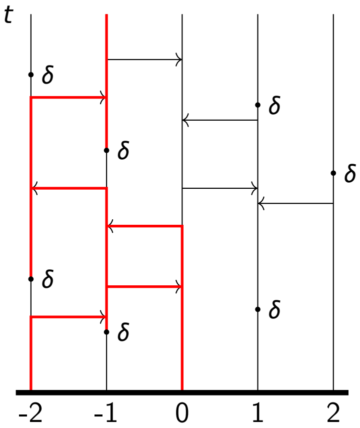
$\hat{\zeta}_{t-s}$ is by distribution equal to ζ_s. $\square$



Corollary 2.4

For finite $\eta \in \{0,1\}^S$: $\mathbb{P}_\eta(\forall t \geqslant 0 : \eta_t \neq \varnothing) = \overline{\nu}(\{\zeta \in \Omega \mid \eta \cap \zeta \neq \varnothing\})$

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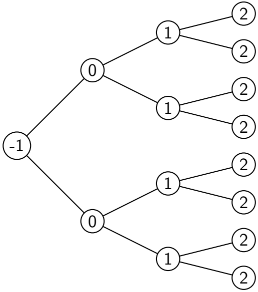
Hence, the process dies iff $\mathcal{I} = \{\delta_0\}$*. Therefore,* $\lambda_c \geqslant \deg_{\max}^{-1} > 0$.

The Contact Process on Homogeneous Trees

Homogeneous Tree



Every node has degree $d+1$.

Results for the Homogeneous Tree

Theorem 3.1 (Weak Survival)

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Theorem 3.2 (Bound for strong survival)

On a homogeneous tree, we have

$$\lambda_s \geqslant \frac{1}{2\sqrt{d}}$$

In particular, $\lambda_c < \lambda_s$ *for* $d \geqslant 6$.

Important Tools

Definition 3.3 (Superharmonicity)

A function $f : \{0,1\}^S \to \mathbb{R}$ is *superharmonic*, if $\mathbb{E}_\eta|f(\eta_t)| < \infty$ and $\mathbb{E}_\eta f(\eta_t) \leqslant f(\eta)$ hold for all $t \geqslant 0$ and $\eta \in \{0,1\}^S$.

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Proposition 3.5

If $f : \{0,1\}^S \to \mathbb{R}$ is bounded and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_\eta f(\eta_t)\big|_{t=0} \leqslant 0$, then $\mathbb{E}_\eta f(\eta_t)$ is decreasing in t.

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Define $f(\eta) := \rho^{|\eta|}$ for some $\rho \in (0,1)$.

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Proposition 3.5 yields weak survival.

$\square$

Literature

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