Bose-Einstein Condensate

Super Solids

Daniel Scheiermann

December 3, 2020

Contents

1 Bose Einstein Condensate

1.1 Questions

- $\int \frac{1}{r^n} \mathrm{d}^3\mathrm{r}$ divergent just for $n \leq 3$? Yes. Apart from an angular part, the integral goes as $\int dr \frac{1}{r^{n-2}}$. If you integrate till R (which you will tend to infinity), then you can easily see that for $n = 3$ it diverges as $\ln(R)$. For any other n is goes as R^{3-n}, and hence it converges for $n > 3$, and diverges for $n < 3$. Thanks.
- What is an s-wave? s-wave means angular momentum $l = 0$. Thanks.
- if $l < \frac{n-3}{2}$, and like k^{n-2} otherwise (Landau and Lishitz, 1977). For a van der Waals-like potential ($n = 6$), only $l = 0$ (s-wave) matters at low energies. Whats with $l = 0$? Lifschitz is a typo? $l = 0$ is special. For $l = 0$ the centrifugal barrier $\frac{\hbar^2 l(l+1)}{2mr^2}$ vanishes. Intuitively you can see that for short-range potentials, like $1/r^6$ you need to be close to $r = 0$, such that the particles see each other. If the kinetic energy is too low, then the kinetic energy is not enough to overcome the centrifugal barrier. As a result, only $l = 0$, i.e. the s-wave, contributes. Thanks.
- In the notes: eq. 1.8 to 1.9 the commutator $[\psi, \psi^\dagger]$ was used, but from 1.9 to 1.10 Bogoliubov aproximation was used, why not directly on 1.8 ? It could have been done directly in 1.8. Thanks.
- What is the derivation of 1.11? Its a FFT, but what are the exact steps? First $\hat{\psi}(\vec{r}) = \sum_{\vec{p}} \hat{a}_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{\sqrt{V}}$, where V is just a quantization volume, just to have the proper units. We will also Fourier-Transform the potential $V(\vec{r} - \vec{r}') = \sum \tilde{V}(\vec{q}) e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}$. Then, you see that

$$\int d^3r \int d^3r' V(\vec{r} - \vec{r}') \hat{\psi}^\dagger(\vec{r}) \hat{\psi}^\dagger(\vec{r}') \hat{\psi}(\vec{r}') \hat{\psi}(\vec{r})$$

  becomes of the form of Eq. (1.11), after using that $\frac{1}{V} \int d^3r e^{i\vec{k}\cdot\vec{r}} = \delta(\vec{k})$.
- What are the python packages to use operators like $\hat{\psi}$? There is some sympy implementation, but probably there is a better one? I'm unsure what do you mean here.
- how to get from 1.11 to 1.12? Is it a commutator expansion? Why q is disappearing? This is based on the so-called Bogoliubov approximation. Since the system condenses in $\vec{p} = 0$, you approximate $\hat{a}_0$ and $\hat{a}_0^\dagger$ by $\sqrt{N_0}$, with N_0 the number of condensed atoms. This is because $N_0 \gg 1$. Actually $N_0 \simeq N$, and hence the number of non-condensed atoms is much smaller than N_0. Then you will do the following. You will consider up to second order, i.e.

up to terms with at most two operators $a_{\vec{p}}$ with $\vec{p} \neq 0$ (you will see that there are no terms with just one operator with $p \neq 0$). Let's have a look to the interaction term in Eq. (1.11) (I forget here hats and vectors in order to ease the notation) $a^\dagger_{p_1+q} a^\dagger_{p_2-q} a_{p_2} a_{p_1}$. Let's see which combinations you have that have at most 2 operators with a non zero momentum: $(p_1+q, p_2-q, p_2, p_1) = (0,0,0,0), (0,0,x,x), (x,x,0,0), (0,x,x,0), (x,0,0,x), (0,x,0,x), (x,0,x,0)$, where x means that it is not zero. The first term gives you the energy at order zero: $E_0 = \frac{U(0)}{2V} N_0^2$. Note that $N = N_0 + \sum_{p\neq 0} a^\dagger_p a_p$, and hence up to second order $N^2 = N_0^2 + 2N_0 \sum_{p\neq 0} a^\dagger_p a_p$. As a result: $E_0 = \frac{U(0)}{2V} N^2 - U(0) n_0 \sum_{p\neq 0} a^\dagger_p a_p$. The 3rd and 4th terms give $U(0) n_0 \sum_{p\neq 0} a^\dagger_p a_p$ which cancels exactly with the term in E_0. The other terms will give you what you find in Eq. (1.12).

- In 1.17 ω_ρ part has a factor 2, but ω_z not, despite beeing symmetric in ψ. Why?This is because of the two directions on the plane.
- What is variable a? Why should $a > 0$ as repulsive short-range interactions stabilize the BEC (p.10)? a is the s-wave scattering length. It characterizes the short-range part of the interaction. As mentioned above the contact interaction is just described by what happens at s-wave. At low energies, this means that the short-range interactions are given by a single parameter, which is the scattering length. A positive scattering length means repulsive interactions, i.e. the interaction energy increases when the density increases, i.e. when the distances decreases (i.e. the particles repel each other). The opposite is true for $a < 0$, for which the particle attract each other. The dipole-dipole interaction is partially atrtractive and partially repulsive. Attraction is dangerous, because the system tends to collapse. Adding repulsive contact interaction may compensate the dipolar attraction, hence preventing collapse.
- "When the atomic density grows due to the attractive interaction, three-body losses predominantly occur in the high-density region. " What does three-body losses mean? Three-body losses means that three particles meet at close distances (which becomes more and more probable for larger and larger densities). When this occurs, two of the particles may fall into a bound state, giving the excess energy to the third one. As a result all three particle are loss from the system, and hence the term three-body loss.
- "As the collapse occurs mainly in the x-y direction due to anisotropy of the DDI (in the absence of inelastic losses, the condensate would indeed become an infinitely thin cigar-shaped cloud along z), and therefore the condensate explodes essentially radially, producing the anisotropic shape of the cloud." Why is the collaps not along z axis? Note that the dipoles attract each other when placed head with tail, i.e. in this case when they are placed aligned on top of each other along the z-direction. Hence collapse produces a thin

"cigar" along z. However, before this "cigar" becomes infinitely thin, three-body losses kick out. When this occurs, the kinetic energy on the xy plane, which is huge (due to the strong compression, recall Heisenberg uncertainty), is released and the BEC explodes. A note here: the key point of all the droplet business in recent works (starting in 2016) is that the collapse in a dipolar condensate may be actually stopped due to the stabilizing effect of quantum fluctuations, but this wasn't known when I wrote the notes!

- How are the regions stable, metastable, unstable derived in Figure 1.5, here Figure 1? It comes from the two solutions of Eq. (1.25).
- Typo in "we obtain a 1D equation similar to the a GP equation", just the or a
- "ground-state wave-function is independent of the in-plane coordinates " Why?
- 1.26 to 1.27, where does the U_{dd} go?
- Typo If: "roton momentum. if this were so"
- Why should a modulation with a finite wavelength allow superfluids?
- Typo repeatance: "the width of the width"
- What are the spin-F matrizes?
- Is the occurance of these spin textures in Figure 3 special?

1.2 Summary

- dipol-dipol interaction (DDI):

$$U(r) = \underbrace{g\delta(r)}_{\frac{4\pi\hbar^2 a(d)\delta(r)}{m}} + \underbrace{U_{dd}(r)}_{\frac{C_{dd}}{4\pi}\frac{(e_1\cdot e_2)r^2 - 3(e_1\cdot r)(e_2\cdot r)}{r^5}} \quad (1)$$

- Use pseudo potential as dipol-dipol interaction is anisotropic and all partial wave (different l) mix
- coupling of different channels generates short-range contribution in the s-channel $s = 0 \Rightarrow$ by changing DDI strength a gets modified too $\Rightarrow$ shape resonances $\Rightarrow$ virtual state transform into a new ground state
- for fermions s-channel does not exists, so just long-range

- FFT of U_{dd} using sperical harmonics Y_{lm} gives:

$$\tilde{U_{dd}}(k) = \int \mathrm{d}^{\mathrm{r}}\mathrm{r} U_{dd}(r) e^{-ik\cdot r} = \frac{C_{dd}}{3}\left(3\cos^2(\theta_k) - 1\right) \tag{2}$$

- Use DDI in Gross-Pitajevski Equation, FFT, approximate to 2nd order, diagonalize with Bogoliubov transform
- As a result the square root can be imaginary, so the BEC gets dynamically unstable for long-wave length (phonon-instability):

$$\epsilon(p) = \sqrt{\frac{p^2}{2m}\left[\frac{p^2}{2m} + 2n_0\left(g + U_{\tilde{dd}}(p)\right)\right]} \tag{3}$$

$$= pc_s\sqrt{1 + \epsilon_{dd}\left(3\cos^2\theta_p - 1\right)} \tag{4}$$

$$\underset{p\to 0}{=} pc_s\sqrt{1 - \epsilon_{dd}} \tag{5}$$

- For dipolar BEC the trap geometry is crucial (for non-dipolar not)
- "pancake traps" can stabilize the phonon-instability
- qualitative features for $a_{crit}(\lambda)$ by gaussian ansatz, for exact numerical solution non-local Gross-Pitaevskii Equation needed

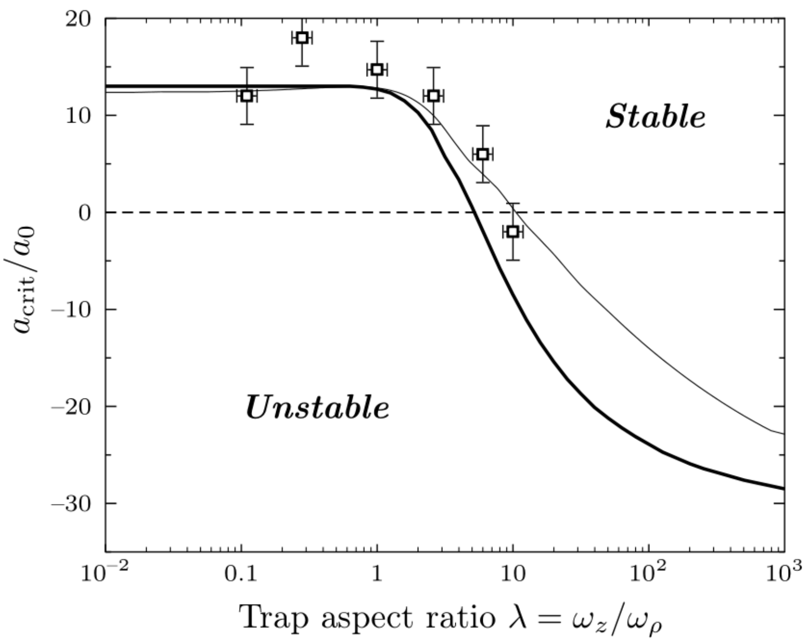

Figure 1: Logo
Santos, *title* (year)

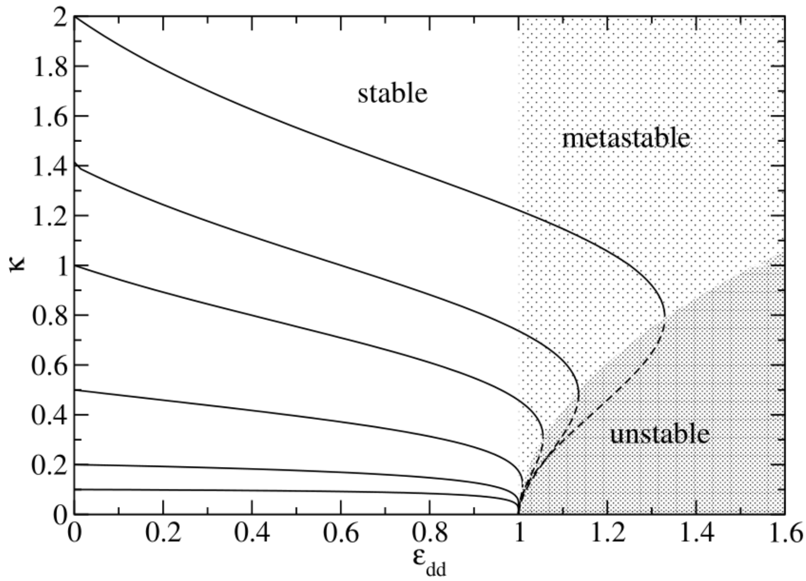

Figure 2: Logo
SANTOS, *title* (year)

- for sufficiently strong interactions, we may neglect quantum pressure, and consider the Thomas-Fermi (TF) regime
- TF solution for the trapped BEC has the same inverted parabola shape (as in non-dipolar case)
- BEC is prolat for $0 < \kappa < 1$ and $1 < \kappa$ oblat
- Bogoliubov-de Gennes Equation shows that the nonlocal character of the DDI causes a momentum dependend coupling constant, leading to a roton-like dispersion law, leading to dynamically instability, when the roton $\beta = \frac{g_d}{g}$ touches zero (experimetally not obeserved yet)
- by varying the density, the frequency of the confinement, and the short-range coupling, one can control the spectrum (roton minimum deeper/shallower)
- sequence of the non-local non-linearity 2D bright solitary waves may become stable under appropriate conditions (Pedri and Santos, 2005)
- two instability regions for 2D solitons (against col- lapse and against unlimited expansion)
- $\tilde{g}_{cr}(\beta) \equiv \frac{gN_{cr}}{2\pi l_z}$, so stable 2D anisotropic self-localised solitons exists just for $N < N_{cr}$
- non-dipolar BECs scatter elastically, the scattering of dipolar solitons is inelastic due to the lack integrability

- The solitons may transfer centre-of-mass energy into internal vibrational modes, resulting in intriguing scattering properties:
  - including soliton fusion (Fig. 1.8)
  - appearance of strong inelastic resonances
  - possibility of observing 2D- soliton spiraling as that already observed in photo-refractive materials
- Dipolar effects in spinor condensates
  - spinor BECs: we focus on an effect which resembles the Einstein-de Haas effect
  - Because of Zeeman sub-levels short-range interactions may occur in different s-wave scattering channels with different total angular momentum (for bosons even number) (spin-1 bosons we have just F = 0 and F = 2)
  - Each scattering channel has an associated s-wave scattering length a_F
  - short-range interactions necessarily preserve the spin projection Sz
  - DDI does not necessarily conserve the spin projection along the quantisation axis as DDI is anisotropic
  - for initially maximally stretched state ($m_F = -F$)
  - short-range interactions cannot induce any spinor dynamics (due to conservation of total magnetisation S_z
  - DDI may induce a transfer to $m_F + 1$
  - for cylindrical symmetry around the quantisation axis, this violation of the spin projection is accompanied by a transfer of angular momentum to the centre of mass, resembling the well known Einstein-de Haas effect $\Rightarrow$ initially spin-polarised dipolar condensate can generate dynamically vorticity
  - Einstein-de Haas effect is destroyed by weak magnetic fields (1 mG)
  - the dominant Larmor precession, and invoking rotating-wave-approximation arguments, the physics must be constrained to manifolds of preserved magnetisation (2D optical lattices could help)
  - Effect of DDI could be even observable under conserved S_z (alkali spinor condensates)
  - spin-changing collisions: collisions that conserve S_z, but do not conserve the relative population of the different Zeeman components

- Spin-changing collisions are characterised by an energy scale proportional to the difference between scattering lengths at different channels
- this difference is very small, so can be significantly modified by the presence of other small energy scales (DDI) ⇒ helical spin textures

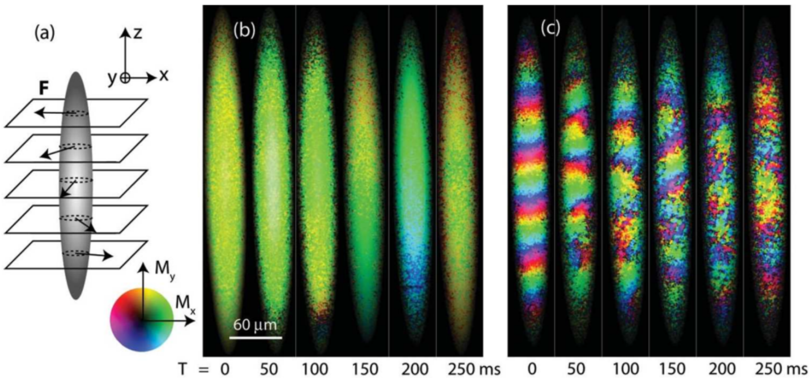

Figure 3: Is the occurance of these textures special?
SANTOS, *title* (year)

2 Supersolids

- supersolid: features both the crystalline structure of a solid and the frictionless flow of a superfluid In this state, every constituent atom is part of the solid and the superfluid simultaneously
- direct observation was limited to systems where the structure formation was mediated by external light fields
- beyond mean-field approximation leads to corrections to the ground state energy stemming from quantum fluctuations of the collective modes in a BEC (LHY-correction)
- In 2018 quantum droplets in a Bose-Bose mixture were observed
- Quantum droplets in Bose-Bose mixtures
  - mean- field energy depends on the difference of the two coupling constants $\delta(g) = |g_{rep}| - |g_{att}|$
  - LHY-correction depends on the individual coupling constants
  - For weakly attractive combination of interactions, a repulsive beyond mean-field correction can stabilize the BEC

– after a peak density increasing the number of particles only leads to an increase in the size of the droplet

– eGPE: kinetic energy, external trapping, and two-body interactions, LHY

– beyond mean-field correction has only been calculated for a homogeneous system and can therefore only be included within a local-density approximation

– QMC calculations in full many-body system verified the formation

– intra-species scattering lengths a_{11} and a_{22} lead to different equilibrium densities $n_0^{(i)}$ for the two components of the mixture.

– droplet forms an intrinsic imbalance in the atom numbers of the two components ($\frac{N1}{N2} = \sqrt{\frac{a_{22}}{a_{11}}}$)

– larger density than in original BEC increases the rate of three-body loss $\Rightarrow$ extra term in eGPE

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r) + \alpha n_0|\psi|^2 + \gamma n_0^{\frac{3}{2}}|\psi|^3\right]\psi \tag{6}$$

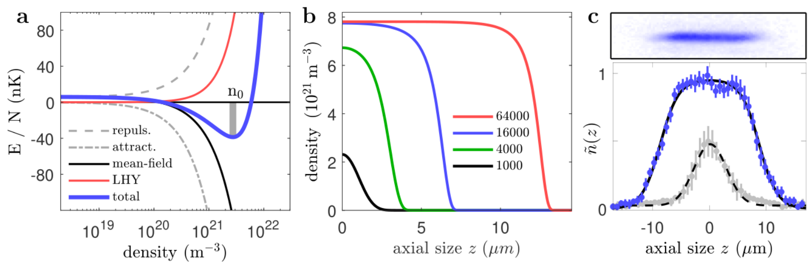

Figure 4: Peak density of the droplet saturates in z-driection
Santos, *title* (year)

- Dipolar Quantum droplets

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(r) + g|\psi|^2 + \phi_{dd} + g_{qf}|\psi|^3\right]\psi \tag{7}$$

- Liquid-like density saturation appears indicating very low compressibility
- a droplet has $2.2 \cdot 10^4$ atoms
- the dynamical state after 5ms of evolution is a Gaussian density distribution