* **3-body term from the HP formalism**

$$\frac{\Delta E_{LHY}}{L} - \frac{1}{2} n \frac{\partial}{\partial n}\left(\frac{\Delta E_{LHY}}{L}\right) = \chi(n)$$

$$\chi(n) = \frac{1}{2} \sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{\nu, n_r} \left(E_{n_r m}(k) - \varepsilon_{\nu m}(k) \right) \left(\overline{v}_\nu\right)^2_{n_r}$$

$$\varepsilon_\nu u_{\nu n_r} = E_{n_r} u_{\nu n_r} + \sum_{n_r'} U_{n_r n_r'} \left(u_{\nu n_r'} + v_{\nu n_r'}\right)$$

$$\varepsilon_\nu v_{\nu n_r} = -E_{n_r} v_{\nu n_r} + \sum_{n_r'} U_{n_r n_r'} \left(u_{\nu n_r'} + v_{\nu n_r'}\right)$$

$$\varepsilon_\nu f^+_{\nu n_r} = E_{n_r} f^-_{\nu n_r} \longrightarrow f^+_{\nu n_r} = \frac{E_{n_r}}{\varepsilon_\nu} f^-_{\nu n_r}$$

$$v_{\nu n_r} = \frac{1}{2}\left(f^+_{\nu n_r} + f^-_{\nu n_r}\right) = \frac{1}{2}\left(\frac{E_{n_r} - \varepsilon_\nu}{\varepsilon_\nu}\right) f^-_{\nu n_r}$$

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* For $n_{1D} \to 0$ we can assume $U_{n_r n_r'} \simeq U_{n_r} \delta_{n_r n_r'}$ (we don't admix transversal modes)

Then:

$$\left.\begin{aligned} \varepsilon_{n_r} u_{n_r} &= E_{n_r} u_{n_r} + U_{n_r}\left(u_{n_r} + v_{n_r}\right) \\ \varepsilon_{n_r} v_{n_r} &= -E_{n_r} v_{n_r} - U_{n_r}\left(u_{n_r} + v_{n_r}\right) \end{aligned}\right\} \begin{aligned} \varepsilon_{n_r} f^+_{n_r} &= E_{n_r} f^-_{n_r} \\ \varepsilon_{n_r} f^-_{n_r} &\simeq \left[E_{n_r} + 2U_{n_r}\right] f^+_{n_r} \end{aligned} \longrightarrow \varepsilon^2_{n_r} = E_{n_r}\left[E_{n_r} + 2U_{n_r}\right]$$

$$\Rightarrow u^2_{n_r} - v^2_{n_r} = 1 = f^+_{n_r} f^-_{n_r} = \frac{E_{n_r}}{\varepsilon_{n_r}}\left(f^-_{n_r}\right)^2 = 1 \Longrightarrow \frac{\left(f^-_{n_r}\right)^2}{\varepsilon^2_{n_r}} = \frac{1}{E_{n_r}\varepsilon_{n_r}}$$

Then:

$$v^2_{n_r} = \frac{1}{4}\left(\frac{E_{n_r} - \varepsilon_{n_r}}{\varepsilon_{n_r}}\right)^2 \left(f^-_{n_r}\right)^2 = \frac{1}{4} \frac{\left(E_{n_r} - \varepsilon_{n_r}\right)^2}{E_{n_r}\varepsilon_{n_r}}$$

$$\chi(n) \simeq \frac{1}{8} \sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{n_r} \frac{\left(E_{n_r} - \varepsilon_{n_r}\right)^3}{E_{n_r}\varepsilon_{n_r}} = \frac{1}{8} \sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{n_r} \frac{\left[E_{n_r} - \sqrt{E_{n_r}\left(E_{n_r} + 2U_{n_r}\right)}\right]^3}{E_{n_r}\sqrt{E_{n_r}\left(E_{n_r} + 2U_{n_r}\right)}}$$

$$= \frac{1}{8} \sum_m \int_{-\infty}^{\infty} \frac{dk}{2\pi} E_{n_r}(k) \frac{\left[1 - \sqrt{1 + \frac{2U_{n_r}(k)}{E_{n_r}(k)}}\right]^3}{\sqrt{1 + \frac{2U_{n_r}(k)}{E_{n_r}(k)}}}$$

(2)

* $E_{n_r m}(k) = \dfrac{\hbar^2 k^2}{2m} + \hbar\omega_\perp (2n_r + m)$ with $C_{n_r m} = \dfrac{3}{2^{2n_r+m}} \begin{pmatrix} 2n_r+m \\ n_r \end{pmatrix}$; $F(j,\sigma) = \sigma^{j+1} e^{\sigma}\, \Gamma(-j,\sigma)$

$U_{n_r m}(k) = g_{1D}\, n_{1D}\, C_{n_r m}\, F\left[2n_r+m, \dfrac{k^2 \ell_\perp^2}{2}\right]$

* Dimensionless form: $q = k\ell_\perp$; unit of energy: $\hbar\omega_\perp$

 * $\dfrac{g_{1D} n_{1D}}{\hbar\omega_\perp} = \dfrac{4\pi\hbar^2 a}{m \cdot 2\pi \ell_\perp^2} \dfrac{n_{1D}}{\hbar\omega_\perp} = 2 n_{1D} a \;\longrightarrow\; U_{n_r m}(q) = 2n_{1D} a\, C_{n_r m}\, F\left[2n_r+m, q^2/2\right]$

 * $E_{n_r m}(q) = \dfrac{q^2}{2} + 2n_r + m$

$$\chi(n) = \left(\frac{\hbar\omega_\perp}{\ell_\perp}\right) \frac{1}{8} \sum_m \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n_r} E_{n_r m}(q) \frac{\left[1 - \sqrt{1 + \frac{2U_{n_r m}(q)}{E_{n_r m}(q)}}\right]^3}{\left[1 + \frac{2U_{n_r m}(q)}{E_{n_r m}(q)}\right]^{1/2}} = \frac{\hbar\omega_\perp}{\ell_\perp} \sum_{n_r m} B_{n_r m}$$

$$B_{n_r m} = \frac{1}{4} \int_0^{\infty} \frac{dq}{2\pi} E_{n_r m}(q) \frac{\left[1 - \sqrt{1 + \frac{2U_{n_r m}(q)}{E_{n_r m}(q)}}\right]^3}{\left[1 + \frac{2U_{n_r m}(q)}{E_{n_r m}(q)}\right]^{1/2}} \underset{n_{1D} a \to 0}{\sim} \frac{1}{4} \int_0^{\infty} \frac{dq}{2\pi} E_{n_r m}(q) \left[\frac{-U_{n_r m}(q)}{E_{n_r m}(q)}\right]^3$$

$$= -\frac{1}{4} \int_0^{\infty} \frac{dq}{2\pi} \frac{U_{n_r m}(q)^3}{E_{n_r m}(q)^2} = \left[-\frac{1}{4} \int_0^{\infty} \frac{dq}{2\pi} \frac{\left[C_{n_r m} F(2n_r+m, q^2/2)\right]^3}{(q^2/2 + 2n_r + m)^2}\right] (2n_{1D} a)^3$$

Then: $$\boxed{\chi(n_{1D}) \simeq \left(\frac{\hbar\omega_\perp}{\ell_\perp}\right) (2n_{1D} a)^3 \left[-\frac{1}{4} \int_0^{\infty} \frac{dq}{2\pi} \frac{\left(C_{n_r m} F(2n_r+m, q^2/2)\right)^3}{(q^2/2 + 2n_r + m)^2}\right]}$$

• Let $c = 2n_{1D} a \;\rightarrow\; \dfrac{\Delta E_{LHY}}{L}(c) - \dfrac{1}{2} c \dfrac{d}{dc}\left(\dfrac{\Delta E_{LHY}}{L}\right) = \chi(c)$

For small $c \rightarrow \dfrac{\Delta E_{LHY}}{L}(c) \simeq \left(Ac^2 + Bc^3 + \cdots\right)\left(\dfrac{\hbar\omega_\perp}{\ell_\perp}\right) \Rightarrow$ $-\dfrac{B}{2}\left(\dfrac{\hbar\omega_\perp}{\ell_\perp}\right) c^3 \simeq \chi(c)$

- $$B = \sum_{n_r m} \frac{1}{2} \int_0^\infty \frac{dq}{2\pi} \frac{\left[C_{n_r m} F(2n_r \mp m, q^2/2)\right]^3}{(q^2/2 + 2n_r + m)^2} = \sum_{n_r m} B_{n_r m}$$

- $$B_{00} = \frac{1}{2} \int_0^\infty \frac{dq}{2\pi} \frac{\left[3F(0, q^2/2)\right]^3}{(q^2/2)^2} \simeq 2$$

- $$\sum_{(n_r, m) \neq (0,0)} B_{n_r m} \cong 0.256$$
  ↳ using Mathematica

- Then, I get $$\frac{\Delta E_{LHY}}{L} \underset{n_{1D}a \to 0}{\simeq} 2.256 \left(\frac{\hbar\omega_\perp}{e_1}\right)(2n_{10}a)^3 + A\,(2n_{10}a)^2 + \dots$$

- Using your notation: $$\left(\frac{\Delta E_{LHY}}{L}\right)_{3body} = \frac{1}{6} g_3^{(3)} n_{10}^3 \Rightarrow \quad g_3^{(3)} = \alpha \left(\frac{r_*}{\ell}\right)^3 \Rightarrow \quad r_* = 3a \Rightarrow g_3^{(3)} = \alpha \left(\frac{\hbar\omega_\perp}{e_1}\right) \cdot 3^3 a^3$$

  $$\left(\frac{\Delta E_{LHY}}{L}\right)_{3body} = \frac{1}{6} \cdot \alpha \left(\frac{\hbar\omega_\perp}{e_1}\right) \cdot \left(\frac{3}{2}\right)^3 (2n_{10}a)^3 = \left[\frac{9\alpha}{16}\right] \left(\frac{\hbar\omega_\perp}{e_1}\right)(2n_{10}a)^3$$

  I would then get $\alpha \simeq 4.01$ instead of 4.65

- Note: If I take only B_{00}, then I get $\alpha \simeq 3.57$ as in your paper.