Irina Shchemelinina

PS 1

(1) CE (AD) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and HH $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

(HH capital stock)

- given prices $\{p_t, w_t, r_t\}_0^{\infty}$, the allocation of representative HH, $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves HH's problem:

$$\max_{\{c_t, i_t, x_{t+1}, k_t, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t (c_t + i_t) \le \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi_0 \quad (*)$$

$$x_{t+1} = (1-\delta) x_t + i_t$$

$$0 \le k_t \le x_t$$

$$0 \le l_t \le 1 \qquad \forall t \ge 0$$

$$c_t, x_{t+1} \ge 0$$

x_0 given

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of repr. firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves firm's problem:

$$\pi = \max_{\{y_t, l_t, k_t\}_{t=0}^{\infty}} \sum p_t (y_t - r_t k_t - w_t l_t)$$

$$\text{s.t. } y_t = F(k_t, l_t) \quad \forall t \ge 0$$

$$y_t, k_t, l_t \ge 0$$

- markets clear in $\forall t$:

$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

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HH capital stock

$\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

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(*)

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$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

(2) SPP - Sequential formation

$$v(\bar{k}_0) = \max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $F(k_t, l_t) = c_t + k_{t+1} - (1-\delta) k_t \quad \forall t$

$c_t \geq 0$

$k_t \geq 0$ $\qquad c_t = F(k_t, l_t) - k_{t+1} + (1-\delta) k_t$

$0 \leq l_t \leq 1 \quad \forall t$

$k_0 \leq \bar{k}_0$

$l_t = 1 \quad \forall t$

$\delta = 1$

also, assuming u is contin. differentiable, strictly ↑ strictly concave & bounded; F is contin. differentiable, strictly ↑, strictly concave and HOD1, then define

$f(k_t) = F(k_t, 1) + (1-\delta) k_t \; ; \forall t$

output

$c_t = f(k_t) - k_{t+1}$ $\qquad f(k) = F(k, 1) + (1-\delta) k \quad \forall k$

(3)
$$v(\bar{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(\underbrace{f(k_t) - k_{t+1}}_{c_t})$$

s.t.

$0 \leq k_{t+1} \leq f(k_t) \quad \forall t$ $\qquad$ ignore constraints assuming INADA conditions hold

$c_t \geq 0$

$k_0 = \bar{k}_0 > 0$, given

FOC: wrt k_{t+1}

$$-\beta^t u'(f(k_t) - k_{t+1}) + \beta^{t+1} u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1}) = 0 \quad \forall t$$

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1}) \quad \forall t$$

Impose transversality condition:

$$\lim_{t\to\infty} \beta^t u'(f(k_t)-k_{t+1}) f'(k_t) k_t = 0$$

From Th 12, Krueger, under above assumptions, allocation $\{k_{t+1}\}_{t=0}^{\infty}$ satisfying Euler equations and TVC solves sequential SPP for given k_0.

In a steady state equilibrium, which is social optimum, $c_t = c^*$, $k_{t+1} = k^* = k_0$ and

$$u'(f(k^*)-k^*) = \beta u'(f(k^*)-k^*) f'(k^*)$$

$$\beta f'(k^*) = 1$$

$$f'(k^*) = \underbrace{1+\rho}_{1/\beta} \qquad \beta = \frac{1}{1+\rho}$$

$$f'(k) = F_k(k,1) + 1 - \delta = 1 \Rightarrow \underbrace{F_k(k^*,1) = 1+\rho}$$

$$f(k^*) - k^* = c^* \qquad l^* = 1$$

From CE definition, we can rewrite HH problem $(*)$ & assuming $\pi = 0$, $l_t = 1$ & u is strictly $\uparrow$, $k_t = x_t$

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } \sum_0^{\infty} p_t (c_t + \underbrace{k_{t+1} - (1-\hat{\delta}) k_t}_{i_t}) = \sum_{t=0}^{\infty} p_t (r_t k_t + w_t)$$

$$c_t, k_{t+1} \geq 0 \quad \forall t$$

k_0 given.

FOC: wrt c_t: $\beta^t u'(c_t) = \mu p_t$ wrt k_{t+1}: $\mu p_t = \mu [1-\hat{\delta} + r_{t+1}] p_{t+1}$

$$\beta^{t+1} u'(c_{t+1}) = \mu p_{t+1}$$

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}-\delta} \quad \forall t$$

From firm's FOC: $r_t = F_k(k_t, 1)$

$f'(k_t) = F(k_t, 1) + (1-\delta)k_t$, so $r_t = f'(k_t) - (1-\delta)$

same as in SPP

$$\begin{cases} c_t = f(k_t) - k_{t+1} \to \text{market clearing condition from goods market} \\ F_k(k_t, 1) = f'(k_t) \\ l_t = 1 \end{cases}$$

Substitute market clearing goods condition & firm's FOC above $\Rightarrow$

$$\frac{\beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = \frac{1}{(1+r_{t+1}-\delta)}$$

$$\frac{(1+r_{t+1}-\delta)\,\beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = 1$$

$$\frac{f'(k_{t+1}) \cdot \beta u'(f(k_{t+1}) - k_{t+2})}{u'(f(k_t) - k_{t+1})} = 1$$

this is the same Euler equation as in SPP earlier

needs transversality condition

$$\lim_{t\to\infty} \underbrace{\lambda_t}_{=p_t} k_{t+1} = 0$$

It can be shown that allocation $\{k_{t+1}\}_{t=0}^{\infty}$ are Pareto optimal and so SPP's allocations coincide with equil.'s allocations.

(4) BE:

$$v(k) = \max_{0\le k'\le f(k)} \{u(f(k)-k') + \beta v(k')\}$$

SP dynamic problem:

$$w(\bar{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t)-k_{t+1}) \Rightarrow$$

s.t. $0 \le k_{t+1} \le f(k_t)\ \forall t$

k_0 given

$$\Rightarrow \max_{0\le k_1\le f(k_0);\ k_0 \text{ given}} u(\overbrace{f(k_0)-k_1}^{c_0}) + \beta\Big[\max_{\substack{\{k_{t+1}\}\\ 0\le k_{t+1}\le f(k_t)}} \sum_{t=1}^{\infty} \beta^{t-1}\cdot u(f(k_t)-k_{t+1})\Big] \Rightarrow$$

$$w(\bar{k}_0) = \max_{\substack{0\le k_1\le f(k_0)\\ k_0 \text{ given}}} \{u(f(k_0)-k_1) + \beta w(k_1)\}$$

$w(\bar{k}_0) = v(\bar{k}_0)$ under Principle of Optimality.

(5) $u(c) = \log c$

$f(k,l) = z k^{\alpha} l^{1-\alpha}$

$l = 1$

$\delta = 1$

$c_t = f(k_t) - k_{t+1}$

$f(k) = F(k,1) = z k^{\alpha}$

Functional equation:

$$v(k) = \max_{0\le k'\le zk^{\alpha}} \{\log(zk^{\alpha}-k') + \beta v(k')\}$$

By guess & verify: log utility + $\delta = 1$

$v(k) = A + B\ln(k) \Rightarrow$ substitute the guess for

$$\max_{0\le k'\le f(k)} \{\log(zk^{\alpha}-k') + \beta(A + B\ln(k'))\}$$

FOC: $\frac{1}{zk^{\alpha}-k'} = \frac{\beta B}{k'} \Rightarrow k' = \frac{\beta B z k^{\alpha}}{1+\beta B}$

Evaluate @ k' optimal

$$\log\left(\underbrace{Zk^\alpha - \frac{\beta B Z k^\alpha}{1+\beta B}}_{\frac{Zk^\alpha(1+\beta B-\beta B)}{1+\beta B}}\right)+\beta A+\beta B\log\left(\frac{\beta Z B k^\alpha}{1+\beta B}\right)$$

$$= \log(Zk^\alpha) - \log(1+\beta B)+\beta A+\beta B \alpha\log(Zk)+\beta B\log\left(\frac{\beta B}{1+\beta B}\right)$$

$$=\alpha\log(Zk) - \log(1+\beta B)+\beta A+$$

$$+\beta B\alpha\log(Zk)+\beta B\log\left(\frac{\beta B}{1+\beta B}\right)=$$

$$=\alpha\log Z+\underline{\alpha\log k}-\log(1+\beta B)+\beta A+\beta B\alpha\log Z+\underline{\beta B\alpha\log k}$$

$$+\beta B\log\left(\frac{\beta B}{1+\beta B}\right)$$

Check:

$$A+B\ln(k) = \underbrace{\alpha\log Z-\log(1+\beta B)+\beta A+\beta B\alpha\log Z+\beta B\log\left(\frac{\beta B}{1+\beta B}\right)}_{A}+$$

$$+\log k(\underbrace{\alpha+\beta B\alpha}_{B})$$

$$B=\alpha+\beta B\alpha$$

$$B-\beta B\alpha=\alpha$$

$$B(1-\beta\alpha)=\alpha$$

$$B=\frac{\alpha}{1-\beta\alpha}$$

$$A=\alpha\log Z-\log\left(1+\beta\cdot\frac{\alpha}{1-\beta\alpha}\right)+\beta A+\beta\alpha\log(Z)\cdot\frac{\alpha}{1-\beta\alpha}+\beta\cdot\frac{\alpha}{1-\beta\alpha}\log\left(\frac{\frac{\beta\alpha}{1-\beta\alpha}}{1+\beta\cdot\frac{\alpha}{1-\beta\alpha}}\right)$$

$$A(1-\beta)=\alpha\log Z\cdot\left(1+\frac{\beta\alpha}{1-\beta\alpha}\right)+\frac{\beta\alpha}{1-\beta\alpha}\log\left[\frac{\beta\alpha}{1-\beta\alpha}\times\frac{1-\beta\alpha}{1-\beta\alpha+\beta\alpha}\right]-\log\left[1+\frac{\beta\alpha}{1-\beta\alpha}\right]$$

$$A=\frac{1}{1-\beta}\left[\alpha\log Z\cdot\left[\frac{1}{1-\beta\alpha}\right]+\frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha)-\log\left(\frac{1}{1-\beta\alpha}\right)\right]$$

$$A=\frac{1}{1-\beta}\cdot\left[\frac{\alpha}{1-\beta\alpha}\cdot\left(\beta\log Z+\beta\log(\alpha\beta)\right)-\log\left(\frac{1}{1-\beta\alpha}\right)\right]$$

policy fn: $k' = g(k) = \frac{\beta B z k^{\alpha}}{1+\beta B} = \frac{\beta \cdot \frac{\alpha}{1-\beta\alpha} \cdot z k^{\alpha}}{1+\beta \cdot \alpha/(1-\beta\alpha)} =$

$= \frac{\beta\alpha z k^{\alpha}}{1-\beta\alpha} : \left(1 + \frac{\beta\alpha}{1-\beta\alpha}\right) = \frac{\beta\alpha z k^{\alpha}}{\cancel{1-\beta\alpha}} \times \frac{\cancel{1-\beta\alpha}}{1} =$

$$= \underline{\alpha\beta z k^{\alpha}}$$

$k_1 = \alpha\beta z k_0^{\alpha}$
$k_2 = \alpha\beta z k_1^{\alpha}$
$k_3 = \alpha\beta z k_2^{\alpha}$
$\vdots$

(6) $c_t = c^* = c_{t+1}$
$k_{t+1} = k^*$
From SPP steady state solution:

$\beta \cdot f'(k^*) = 1$ $\qquad f(k^*) = z k^{*\alpha} + (1-\delta)k$, δ "1"

$\beta \cdot \alpha z k^{*\alpha-1} = 1 \quad |^{\frac{1}{\alpha-1}}$ $\qquad f'(k^*) = \alpha z k^{*\alpha-1}$

$\alpha z k_*^{\alpha-1} = \frac{1}{\beta}$

$\underbrace{k^* = \left(\frac{1}{\beta\alpha z}\right)^{\frac{1}{\alpha-1}}}$

$c^* = f(k^*) - k^* =$
$= z k^{*\alpha} - k^* =$
$= z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1} - \left(\frac{1}{\beta\alpha z}\right)^{1/\alpha-1}$

$r^* = f'(k^*) = \alpha z k^{*\alpha-1} = \alpha z \left[\left(\frac{1}{\beta\alpha z}\right)^{\frac{1}{\alpha-1}}\right]^{\alpha-1} = \frac{\alpha z}{\beta\alpha z} = \frac{1}{\beta}$

$y^* = F(k^*, 1) = z k^{*\alpha} = z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1}$

$w^* = F_e(k^*, 1) = (1-\alpha) z k^{*\alpha} = (1-\alpha) z \cdot \left(\frac{1}{\beta\alpha z}\right)^{\alpha/\alpha-1}$

I attached Matlab code,
Julia code is in progress

(7) $\alpha = 1/3$
$Z = 1$
$\delta = 1$
$\beta \cong 0,96$

shock to $K^* \to 0.8\, K^*$
shock to $\alpha \to 1.052$

$$K^* = \left(\frac{1}{\beta \cdot 1/3 \cdot 1}\right)^{\frac{1}{1/3 - 1}}$$

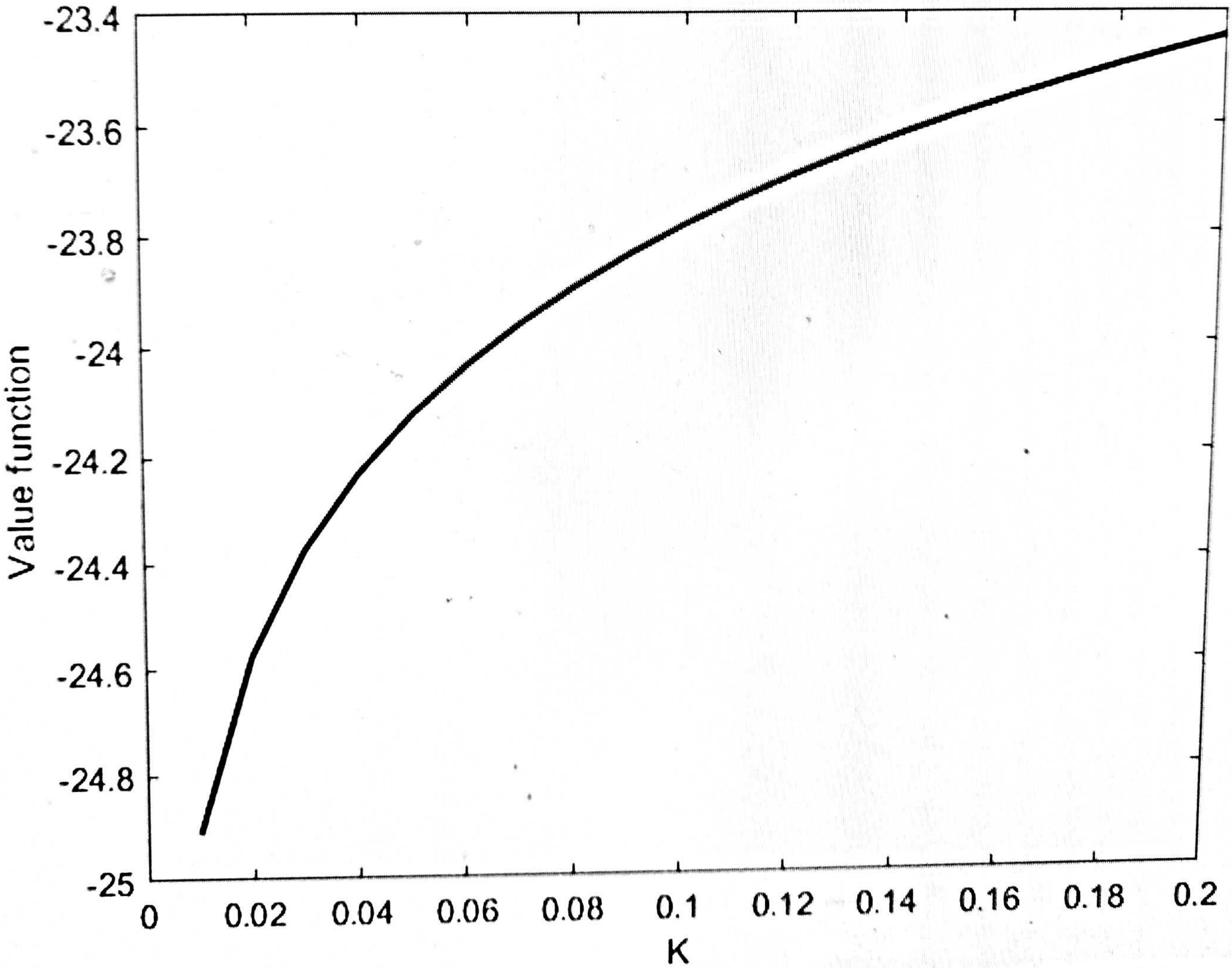
$$C^* = \left(\frac{1}{\beta \cdot 1/3 \cdot 1}\right)^{1/3/(1/3-1)} - \left(\frac{1}{1/3\,\beta}\right)^{1/(1/3-1)}$$

$$r^* = \frac{1}{\beta}$$

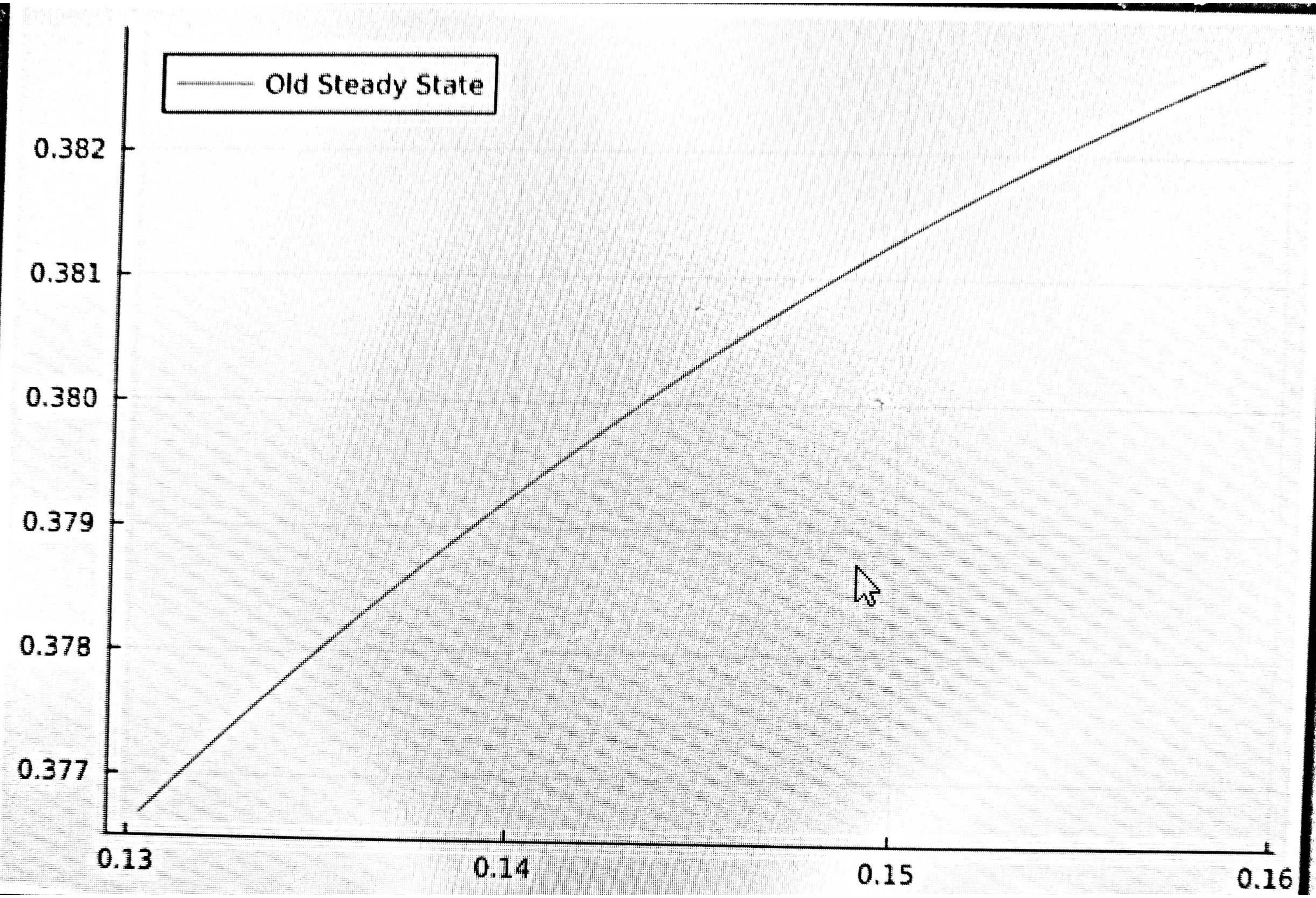
$$y^* = \left(\frac{1}{\beta \cdot 1/3}\right)^{1/3/(1/3-1)}$$

$$w^* = (1 - 1/3)\left(\frac{1}{1/3\,\beta}\right)^{1/3/(1/3-1)}$$

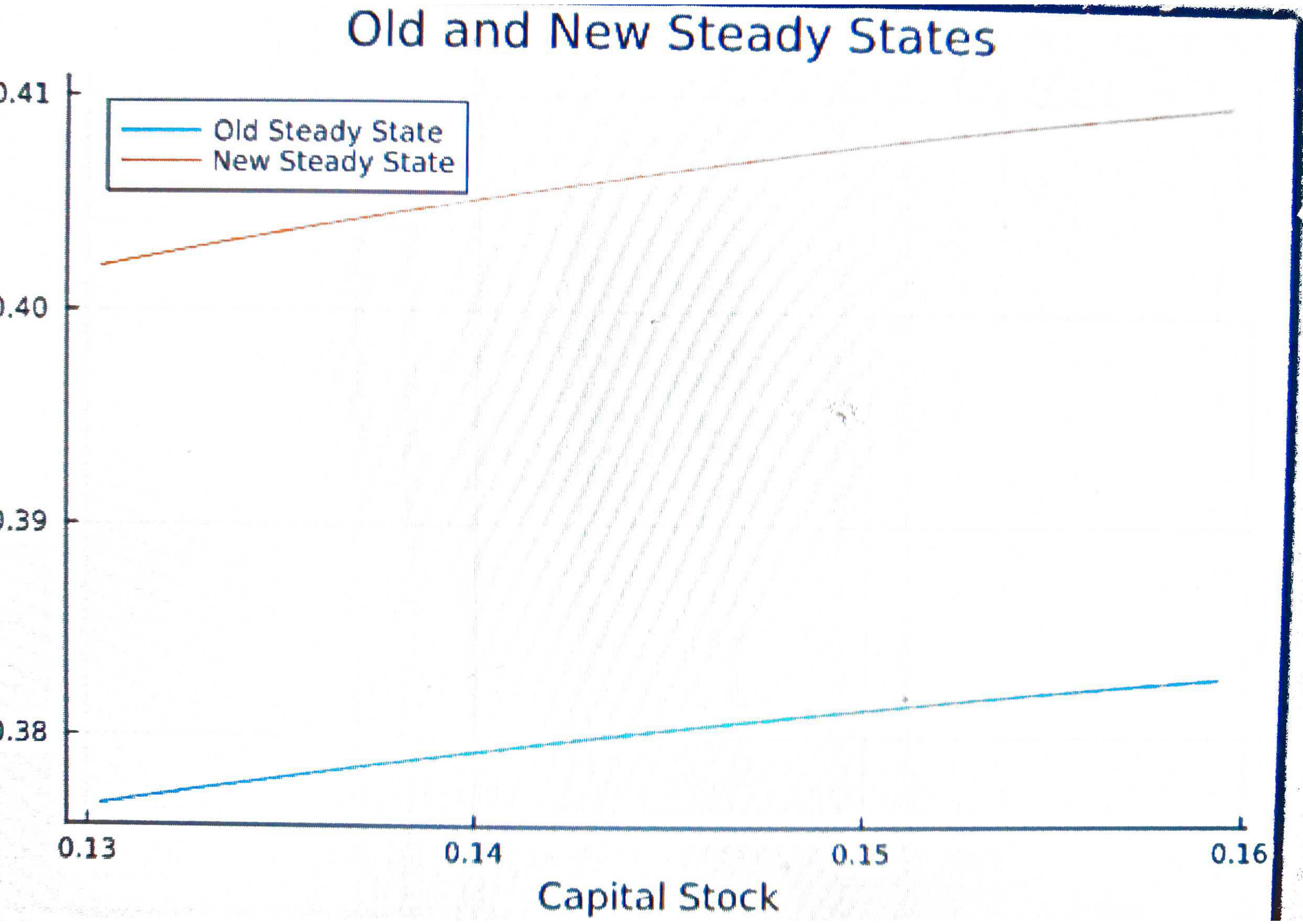
(8) Done



-23.4
-23.6
-23.8
-24
-24.2
-24.4
-24.6
-24.8
-25
Value function
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
K



Old Steady State
0.382
0.381
0.380
0.379
0.378
0.377
0.13
0.14
0.15
0.16


Old and New Steady States
Old Steady State
New Steady State
0.41
0.40
0.39
0.38
0.13
0.14
0.15
0.16
Capital Stock