IRina ①

PS2

Q1. CE consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and HH $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

- given prices $\{p_t, w_t, r_t\}_0^{\infty}$, the allocation of repr. HH $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves HH's problem

$$\max_{\{c_t, i_t, x_{t+1}, k_t, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}$$

$$\text{s.t. } \sum_{t=0}^{\infty} p_t(c_t + i_t) \le \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi_0$$

$$x_{t+1} = (1-\delta)x_t + i_t$$

$$0 \le k_t \le x_t$$

$$0 \le l_t \le 1 \qquad \forall t \ge 0$$

$$c_t, x_{t+1} \ge 0$$

x_0 given

- given prices $\{p_t, w_t, l_t\}_{t=0}^{\infty}$, the allocation of repr firms $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves firm's problem:

$$\pi = \max_{\{y_t, l_t, r_t\}_{t=0}^{\infty}} \sum p_t (y_t - r_t k_t - w_t l_t)$$

$$\text{s.t. } y_t = F(k_t, l_t) \quad \forall t \ge 0 \qquad F(k_t, l_t) = z k^{\alpha} l^{1-\alpha}$$

$$y_t, k_t, l_t \ge 0$$

- markets clear, $\forall t$

$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

d) Rewrite the problem:

assume $\tau = 0$

$l_t = 1 \quad \forall t$

u is strictly $\uparrow$ & concave

$k_t = x_t \quad \forall t$

$$\max_{\{c_t, k_{t+1}, l_t\}_{0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

$$\text{s.t.} \sum_{t=0}^{\infty} p_t \big(c_t + \underbrace{k_{t+1} - (1-\delta) k_t}_{i_t} \big) = \sum_{t=0}^{\infty} p_t \left(r_t k_t + w_t l_t \right)$$

$c_t, k_{t+1} \geq 0 \quad \forall t$

k_0 given

FOCs:

wrt c_t: $\beta^t \frac{1-\sigma}{1-\sigma} c_t^{-\sigma} = \mu p_t$ 　　 wrt l_t: $-\beta^t \frac{\chi}{\cancel{1+\eta}} \cdot \cancel{(1+\eta)} \, l_t^{\eta} = 0$

$\beta^{t+1} c_{t+1}^{-\sigma} = \mu p_{t+1}$

wrt k_{t+1}: $\mu p_t = \mu (1 - \delta + r_{t+1}) p_{t+1}$

$$\frac{\beta c_{t+1}^{-\sigma}}{c_t^{-\sigma}} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1} - \delta} \quad \forall t$$

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = \frac{1}{1 + r_{t+1} - \delta}$$

Firm's FOC: (same as in PS1)

$-r_t p_t + p_t F_k(k_t, l_t) = 0$ 　　 $-w_t p_t + p_t F_l(k_t, l_t) = 0$

$r_t = F_k(k_t, l_t) = \alpha z k_t^{\alpha - 1} \underbrace{l_t^{1-\alpha}}_{1}$ 　　 $w_t = (1-\alpha) z k_t^{\alpha} \underbrace{l_t^{-\alpha}}_{=1}$

Substitute market clearing conditions & using firm's FOCs:

$$C_t = F(k_t, l_t) - [k_{t+1} - (1-\delta)k_t]$$

$$\frac{\beta\left(z k_{t+1}^{\alpha} l_{t+1}^{1-\alpha} - k_{t+2} + (1-\delta)k_{t+1}\right)^{-\sigma}}{\left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta)k_t\right)^{-\sigma}} = \frac{1}{1+r_{t+1}-\delta}$$

$l_t = 1$, $l_{t+1} = 1$

$$r_{t+1} = \left(\alpha z k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}\right), \quad l_{t+1} = 1$$

at steady state:

$C_t = C^* = C_{t+1}$

$k_{t+1} = k = k_{t+2}$, $r_t = r_{t+1} = r^*$

$l_t = 1 = l^*$, $w_t = w^* = w_{t+1}$

$$\frac{1}{1+r-\delta} = \beta$$

$$1+r-\delta = \frac{1}{\beta}$$

$$\boxed{r^* = \frac{1}{\beta} - 1 + \delta}$$

$$\frac{1}{\beta} - 1 + \delta = \alpha z k^{*\alpha-1}$$

$$k^{*\alpha-1} = \frac{\frac{1}{\beta}-1+\delta}{\alpha z} \quad \Big| \ ^{\frac{1}{\alpha-1}}$$

$$\boxed{k^* = \left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}}}$$

$$c^* = zk^{*\alpha} - k^* + (1-\delta)k^* =$$

$$= zk^{*\alpha} + k^*(1-\delta-1) = zk^{*\alpha} - k^*\delta$$

$$\boxed{= z\cdot\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}}}$$

$$y^* = c^* + k^* - (1-\delta)k^* =$$

$$= c^* + k^*(1-1+\delta) = c^* + k^*\delta =$$

$$= z\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\alpha/\alpha-1} - \delta\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{1/\alpha-1} +$$

$$+ \delta\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{1/\alpha-1} = \boxed{z\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\alpha/\alpha-1}}$$

$$\boxed{w^* = (1-\alpha)zk^{*\alpha} = (1-\alpha)z\left(\frac{\frac{1}{\beta}-1+\delta}{\alpha z}\right)^{\alpha/\alpha-1}}$$

(4)

(3)

$$v(\bar{k}_0) = \max_{\{c_t, l_t, k_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}$$

s.t. $F(k_t, l_t) = z k_t^\alpha l_t^{1-\alpha}$

$F(k_t, l_t) = c_t + k_{t+1} - (1-\delta) k_t \quad \forall t$

$c_t \geq 0$

$k_t \geq 0$

$0 \leq l_t \leq 1 \quad \forall t$

$k_0 \leq \bar{k}_0$

$\bar{k}_0$ given

all info is contained in past capital

l_t choice is static, so we can find its solution given k_t, k_{t+1}

$$c_t = F(k_t, l_t) + (1-\delta) k_t - k_{t+1}$$

$$v(k_0) = \max_{k', l} \{ u(\underbrace{z k^\alpha + (1-\delta) k - k'}_{c}, l) \}$$

$0 \leq k' \leq z k^\alpha$

k^0 given

The idea is to use in BE,

$\tilde{c}(k, k') = F(k, \tilde{l}(k, k')) - k' + (1-\delta) k)$,

where $\tilde{l}(k, k'')$ is from static problem.

BE:

$$V(k) = \max_{\substack{k' \\ 0 \leq k' \leq z k^\alpha}} \{ F(k, \tilde{c}) + (1-\delta) k - k' + \beta V(k') \}$$

k^0 given

Static problem:

$$\max_{c_t, l_t} \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

s.t. $c_t = z k_t^\alpha l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t$

$l_t, c_t \geq 0$

$\partial c_t: \frac{1}{1-\sigma} \cdot (1-\sigma) c_t^{-\sigma} = \lambda$

$\partial l_t: -\frac{\chi}{1+\eta} (1+\eta) l_t^\eta + \lambda z k_t^\alpha l_t^{-\alpha} (1-\alpha) = 0$

$c_t^{-\sigma} = \lambda$

$\chi l_t^\eta = (1-\alpha) \lambda z k_t^\alpha l_t^{-\alpha}$

$\chi l_t^\eta = (1-\alpha) c_t^{-\sigma} z k_t^\alpha l_t^{-\alpha}$

$$\tilde{l}_t = \left[\frac{(1-\alpha) c_t^{-\sigma} z k_t^\alpha}{\chi} \right]^{\frac{1}{\eta+\alpha}} \Leftarrow (\tilde{l}_t)^{\eta+\alpha} = \frac{(1-\alpha) c_t^{-\sigma} z k_t^\alpha}{\chi} \Leftarrow \frac{l_t^\eta}{l_t^{-\alpha}} = \frac{(1-\alpha) c_t^{-\sigma} z k_t^\alpha}{\chi}$$

(5)

(4) $l_{ss} = 0.4$

$$l_{ss} = \left[\frac{(1-\alpha)c^{*-\sigma} z k^{*\alpha}}{\chi}\right]^{\frac{1}{\eta+\alpha}}$$

$\alpha = 1/3$
$z = 1$
$\sigma = 2$
$\eta = 1.$

$$0.4 = \left[\frac{(1-\frac{1}{3})c^{*-2} \cdot 1 \cdot k^{*1/3}}{\chi}\right]^{\frac{1}{1+1/3}}$$

let $\beta = 0.96$
$\delta = 0.9$

$$k^* = \left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

$$c^* = 1 \cdot \left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1/3}{1/3-1}} - 1\left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

Code is attached.

$\chi = 0.0817$ based on code

(5), (6) code and figures attached.

(7) Below is based on paper "Improved & Generalised Upper bounds on the Complexity of PO" by B. Scherrer 2018.

State space X is finite; A, action space, at each stage there is a reward fn $r(a)$; $\pi: X \to A$, policy map. max the expected discounted sum of rewards from $\forall i$ state. $\beta \in (0,1)$

$v_* = \max_\pi v_\pi$ optimal value; $V_\pi = r_\pi + \beta V_\pi$ is BE with

V_π is a fixed point of operator $T_\pi: v \to r_\pi + \beta v$;

$v_* = \max_\pi \{r_\pi + \beta v_*\} = \max_\pi T_\pi v_*$, where v_* is a fixed point of operator $T: v \to \max_\pi T_\pi v$.

WTS, the sequence $\|v_* - v_{\pi_k}\|_\infty$ built by Howard's PI is contraction with coeff. β.

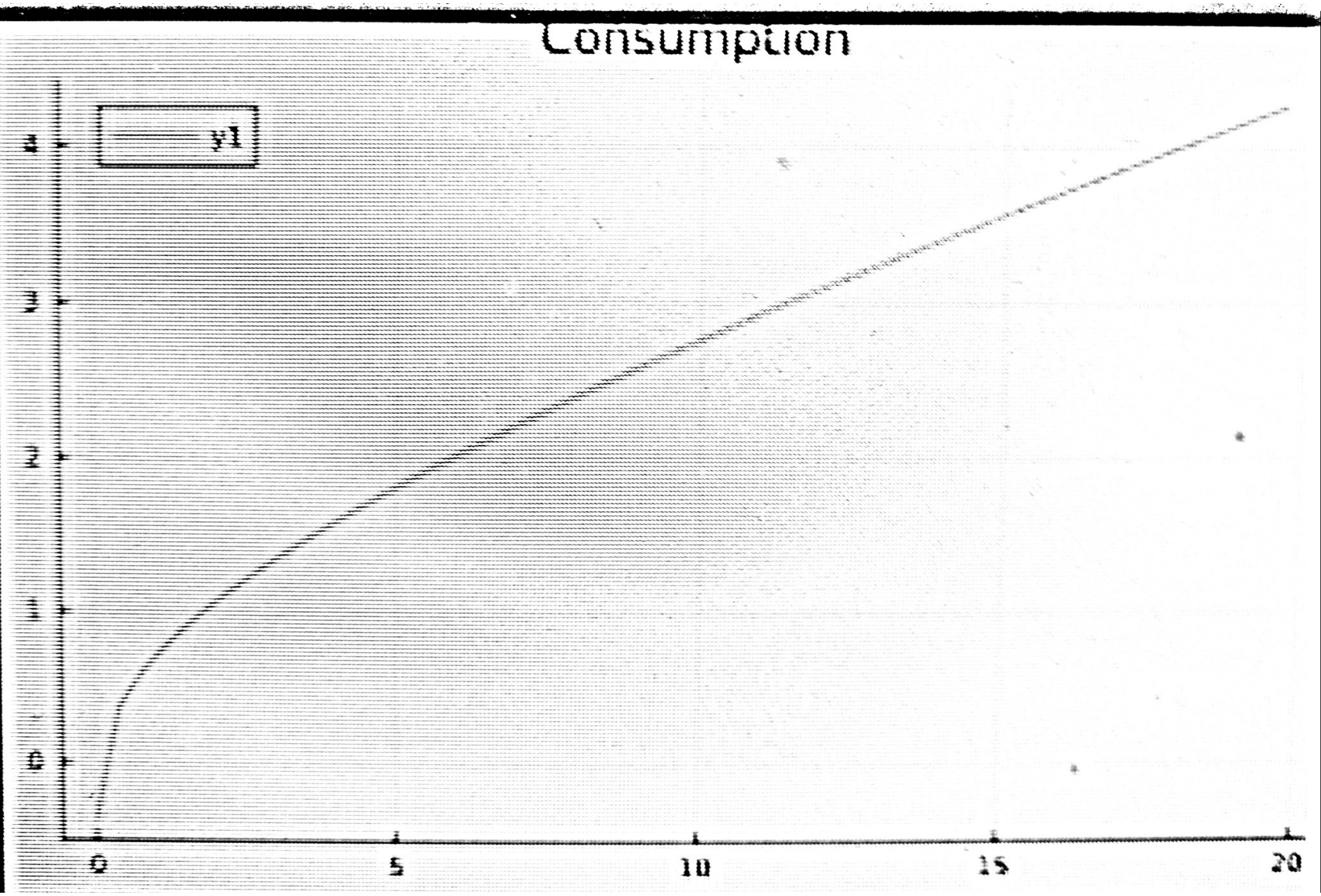
$\forall k \geq 0,$

$$v_{\pi *} - v_{\pi k} = T_{\pi *} v_{\pi *} - T_{\pi *} v_{\pi k-1} + T_{\pi *} v_{\pi k-1} -$$
$$- T_{\pi k} v_{\pi k-1} + T_{\pi k} v_{\pi k-1} - T_{\pi k} v_{\pi k} \leq$$
$$\leq \beta (v_{\pi *} - v_{\pi k-1}) + \beta (v_{\pi k-1} - v_{\pi k}) \leq$$
$$\beta (v_{\pi *} - v_{\pi k-1})$$

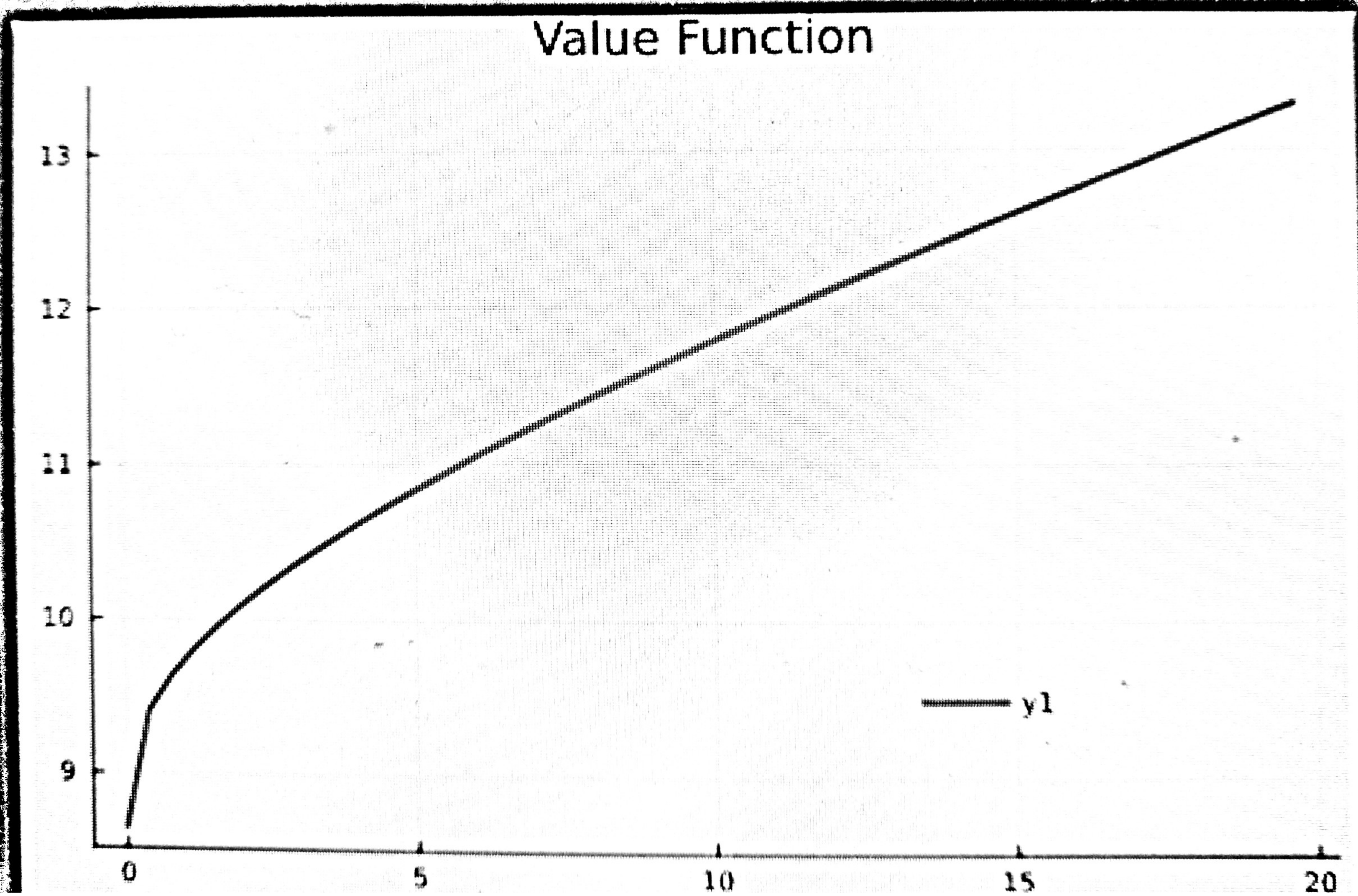
$T_{\pi *} v_{\pi k-1} \leq T_{\pi k} v_{\pi k-1}$

Because $v_{\pi *} - v_{\pi k}$ is non-negative,

$$\| v_{\pi *} - v_{\pi k} \|_\infty \leq \beta \| v_{\pi *} - v_{\pi k-1} \|_\infty \Rightarrow \text{contraction}.$$



Consumption
y1
4
3
2
1
0
0
5
10
15
20


Value Function
13
12
11
10
9
0
5
10
15
20
y1

$l_{ss} = 0.4$

$$l_{ss} = \left[\frac{(1-\alpha) c^{*^{-\sigma}} z k^{*\alpha}}{\chi}\right]^{\frac{1}{\eta+\alpha}}$$

$\alpha = 1/3$
$z = 1$
$\sigma = 2$
$\eta = 1.$

$$0.4 = \left[\frac{\left(1-\frac{1}{3}\right) c^{*^{-2}} \cdot 1 \cdot k^{*1/3}}{\chi}\right]^{\frac{1}{1+1/3}}$$

let $\beta = 0.96$
$\delta = 0.9$

$$k^* = \left(\frac{\frac{1}{0.96} - \cancel{1} + \cancel{1}}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

$$c^* = 1 \cdot \left(\frac{\frac{1}{0.96} - \cancel{1} + \cancel{1}}{1/3 \cdot 1}\right)^{\frac{1/3}{1/3-1}} - 1\left(\frac{\frac{1}{0.96} - \cancel{1} + \cancel{1}}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

Code is attached.

$\chi = 0.0817$ based on code

From screen:
VFI - Grid Search - n k=50
~~Iterations~~ = 431
Distance = 9.8932e-8
430.89 seconds

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