PS2 IRina 1

Q1. CE consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and HH $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

- given prices $\{p_t, w_t, r_t\}_0^{\infty}$, the allocation of repr. HH $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves HH's problem

$$\max_{\{c_t, l_t, x_{t+1}, k_t, l_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \qquad u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}$$

s.t. $\sum_{t=0}^{\infty} p_t(c_t + i_t) \le \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi_0$

$x_{t+1} = (1-\delta)x_t + i_t$

$0 \le k_t \le x_t$

$0 \le l_t \le 1 \qquad \forall t \ge 0$

$c_t, x_{t+1} \ge 0$

x_0 given

- given prices $\{p_t, w_t, l_t\}_{t=0}^{\infty}$, the allocation of repr firms $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves firm's problem:

$$\pi = \max_{\{y_t, l_t, r_t\}_{t=0}^{\infty}} \sum_{\infty} p_t (y_t - r_t k_t - w_t l_t)$$

s.t. $y_t = F(k_t, l_t) \quad \forall t \ge 0 \qquad F(k_t, l_t) = z k_t^{\alpha} l_t^{1-\alpha}$

$y_t, k_t, l_t \ge 0$

- markets clear, $\forall t$

$y_t = c_t + i_t$

$l_t^d = l_t^s$

$k_t^d = k_t^s$

②

a) Rewrite HH problem;
assume $\pi = 0$

u is strictly ↑ & concave

$k_t = x_t \quad \forall t$

$$\max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

s.t. $$\sum_{t=0}^{\infty} p_t \left(c_t + k_{t+1} - (1-\delta) k_t \right) = \sum_{t=0}^{\infty} p_t \left(r_t k_t + w_t l_t \right)$$

$c_t, k_{t+1} \geq 0 \quad \forall t$

k_0 given

FOCs:

wrt c_t: $$\beta^t \frac{1-\sigma}{1-\sigma} c_t^{-\sigma} = \mu p_t$$

$$\beta^{t+1} c_{t+1}^{-\sigma} = \mu p_{t+1}$$

wrt l_t: $$-\beta^t \frac{\chi}{1+\eta} (1+\eta) l_t^{\eta} + \mu p_t w_t = 0$$

wrt k_{t+1}: $$\mu p_t = \mu (1-\delta + r_{t+1}) p_{t+1}$$

$$\frac{\beta c_{t+1}^{-\sigma}}{c_t^{-\sigma}} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}-\delta} \quad \forall t$$

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = \frac{1}{1+r_{t+1}-\delta}$$

$$\beta^t l_t^{\eta} \chi = \mu p_t w_t$$

$$\beta^{t+1} l_{t+1}^{\eta} \chi = \mu p_{t+1} w_{t+1}$$

$$\beta^{t+1} l_{t+1}^{\eta} \chi = \frac{\beta^t l_t^{\eta} \chi}{p_t w_t} \cdot w_{t+1} p_{t+1}$$

$$\frac{p_t w_t \cdot \beta l_{t+1}^{\eta} \chi}{l_t^{\eta} \chi} = w_{t+1} \cdot p_{t+1}$$

$$\beta \left(\frac{l_{t+1}}{l_t} \right)^{\eta} = \frac{w_{t+1} p_{t+1}}{w_t p_t}$$

$$\boxed{\beta \left(\frac{l_{t+1}}{l_t} \right)^{\eta} = \frac{w_{t+1}}{w_t} \cdot \frac{1}{1+r_{t+1}-\delta}}$$

Firm's FOC: (same as in PS1)

$$-r_t p_t + p_t F_k (k_t, l_t) = 0$$

$$r_t = F_k (k_t, l_t) = \alpha z k_t^{\alpha-1} l_t^{1-\alpha}$$

$$-w_t p_t + p_t F_l (k_t, l_t) = 0$$

$$w_t = (1-\alpha) z k_t^{\alpha} l_t^{-\alpha}$$

Substitute market clearing conditions & using firm's FOCs:

$$C_t = F(k_t, l_t) - [k_{t+1} - (1-\delta)k_t]$$

$$\frac{\beta\left(z k_{t+1}^{\alpha} l_{t+1}^{1-\alpha} - k_{t+2} + (1-\delta)k_{t+1}\right)^{-\sigma}}{\left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta)k_t\right)^{-\sigma}} = \frac{1}{1 + r_{t+1} - \delta}$$

$$\Downarrow$$

$$\left(\alpha z k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}\right)$$

at steady state:

$$C_t = C^* = C_{t+1}$$
$$k_{t+1} = k = k_{t+2}$$
$$l_t = l_{t+1} = l^*$$
$$r_t = r_{t+1} = r^*$$
$$w_t = w^* = w_{t+1}$$

$$\frac{1}{1+r-\delta} = \beta$$

$$1 + r - \delta = \frac{1}{\beta}$$

$$\boxed{r^* = \frac{1}{\beta} - 1 + \delta}$$

$$\frac{1}{\beta} - 1 + \delta = \alpha z k^{*\alpha-1} l^{*1-\alpha}$$

$$k^{*\alpha-1} = \frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}} \quad \Big| \; {}^{\frac{1}{\alpha-1}}$$

$$\boxed{k^* = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{\frac{1}{\alpha-1}}}$$

$$c^* = z k^{*\alpha} l^{*(1-\alpha)} - k^* + (1-\delta)k^* =$$

$$= z k^{*\alpha} l^{*(1-\alpha)} + k^*(1-\delta-1) = z k^{*\alpha} l^{*(1-\alpha)} - k^*\delta$$

$$= z \cdot \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*(1-\alpha)}}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{\frac{1}{\alpha-1}}$$

$$y^* = c^* + k^* - (1-\delta)k^* =$$

$$= c^* + k^*(1 - 1 + \delta) = c^* + k^*\delta =$$

$$= z\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{\alpha/\alpha-1} - \delta\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{1/\alpha-1} +$$

$$+ \delta\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{1/\alpha-1} = \boxed{z\left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{\alpha/\alpha-1}}$$

$$w^* = (1-\alpha) z k^{*\alpha} l^{*-\alpha} = (1-\alpha) z \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z l^{*1-\alpha}}\right)^{\alpha/\alpha-1} \cdot l^{*-\alpha} =$$

$$= (1-\alpha) z \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z}\right)^{\alpha/\alpha-1} \left(l^{*\alpha-1}\right)^{\frac{\alpha}{\alpha-1}} \cdot l^{*-\alpha} \; \boxed{= (1-\alpha) z \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha z}\right)^{\frac{\alpha}{\alpha-1}}}$$

$\boxed{\text{assume } l_t = 1}$

3)

$$w(k_0) = \max_{\{c_t, l_t, k_t\}_0^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}$$

s.t. $F(k_t, l_t) = z k_t^{\alpha} l_t^{1-\alpha}$

$F(k_t, l_t) = c_t + k_{t+1} - (1-\delta) k_t, \ \forall t$

$c_t \geq 0$

$k_t \geq 0$

$0 \leq l_t \leq 1 \quad \forall t$

$k_0 \leq \bar{k}_0$

$\bar{k}_0$ given

all info is contained in past capital

l_t choice is static, so we can find its solution given k_t, k_{t+1}

$$c_t = F(k_t, l_t) + (1-\delta) k_t - k_{t+1}$$

$$w(k_0) = \max_{k', l} \{ u(\underbrace{z k^{\alpha} + (1-\delta) k - k'}_{c}, l) \}$$

$0 \leq k' \leq z k^{\alpha}$

k^0 given

the idea is to use in BE,

$\tilde{c}(k, k') = F(k, \tilde{l}(k, k') - k' + (1-\delta) k)$,

where $\tilde{l}(k, k'')$ is from static problem.

BE:

$$V(k) = \max_{\substack{k' \\ 0 \leq k' \leq z k^{\alpha}}} \{ u[F(k, \tilde{c}) + (1-\delta) k - k'] + \beta V(k') \}$$

k^0 given

Static problem:

$$\max_{c_t, l_t} \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

s.t. $c_t = z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t$

$l_t, c_t \geq 0$

$\partial c_t: \frac{1}{1-\sigma} \cdot (1-\sigma) c_t^{-\sigma} = \lambda$

$\partial l_t: -\frac{\chi}{1+\eta} (1+\eta) l_t^{\eta} + \lambda z k_t^{\alpha} l_t^{-\alpha} (1-\alpha) = 0$

$c_t^{-\sigma} = \lambda$

$\chi l_t^{\eta} = (1-\alpha) \lambda z k_t^{\alpha} l_t^{-\alpha}$

$\chi l_t^{\eta} = (1-\alpha) c_t^{-\sigma} z k_t^{\alpha} l_t^{-\alpha}$

$$\tilde{l}_t = \left[\frac{(1-\alpha) c_t^{-\sigma} z k_t^{\alpha}}{\chi} \right]^{\frac{1}{\eta+\alpha}} \Leftarrow (l_t)^{\eta+\alpha} = \frac{(1-\alpha) c_t^{-\sigma} z k_t^{\alpha}}{\chi} \Leftarrow \frac{l_t^{\eta}}{l_t^{-\alpha}} = \frac{(1-\alpha) c_t^{-\sigma} z k_t^{\alpha}}{\chi}$$

$l_{ss} = 0.4$

$$l_{ss} = \left[\frac{(1-\alpha)c^{*-\sigma} z k^{*\alpha}}{\chi}\right]^{\frac{1}{\eta+\alpha}}$$

$\alpha = 1/3$
$z = 1$
$\sigma = 2$
$\eta = 1$

$$0.4 = \left[\frac{(1-\frac{1}{3})c^{*-2} \cdot 1 \cdot k^{*1/3}}{\chi}\right]^{\frac{1}{1+1/3}}$$

$$k^* = \left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

det $\beta = 0.96$
$\delta = 0.9$

$$c^* = 1 \cdot \left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1/3}{1/3-1}} - 1\left(\frac{\frac{1}{0.96} - 1 + 1}{1/3 \cdot 1}\right)^{\frac{1}{1/3-1}}$$

For time and iterations see attached reported details (next page)

Code is attached.

$\chi = 0.0817$ based on code

(5), (6) code and figures attached.

(7) Below is based on paper "Improved & Generalised upper bounds on the complexity of PO" by B. Scherrer 2018.

State space X is finite; A, action space, at each stage there is a reward fn $r(a)$; $\pi: X \to A$, policy map. max the expected discounted sum of rewards from $\forall i$ state. $\beta \in (0,1)$

$v_* = \max_\pi v_\pi$ optimal value; $V_\pi = r_\pi + \beta V_\pi$ is BE with V_π is a fixed point of operator $T_\pi : v \to r_\pi + \beta v$;

$v_* = \max_\pi \{r_\pi + \beta v_*\} = \max_\pi T_\pi v_*$, where v_* is a fixed point of operator $T: v \to \max_\pi T_\pi v$.

WTS, the sequence $\|v_* - v_{\pi_k}\|_\infty$ built by Howard's PI is contraction with coeff. β.

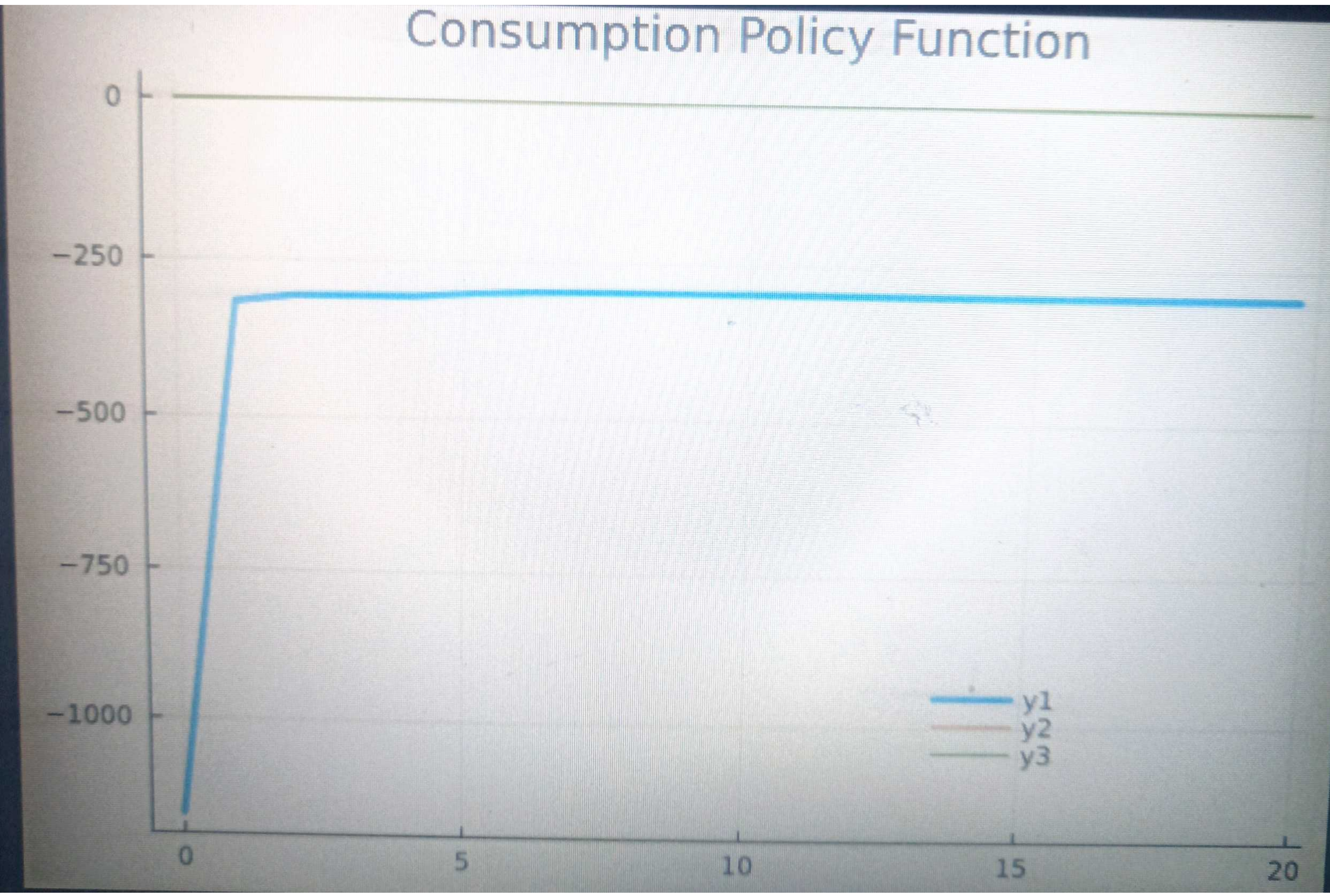
$$v_{\pi *} - v_{\pi k} = T_{\pi *} v_{\pi *} - T_{\pi *} v_{\pi k-1} + T_{\pi *} v_{\pi k-1} -$$
$$- T_{\pi k} v_{\pi k-1} + T_{\pi k} v_{\pi k-1} - T_{\pi k} v_{\pi k} \le$$
$$\le \beta (v_{\pi *} - v_{\pi k-1}) + \beta (v_{\pi k-1} - v_{\pi k}) \le$$
$$\beta (v_{\pi *} - v_{\pi k-1})$$

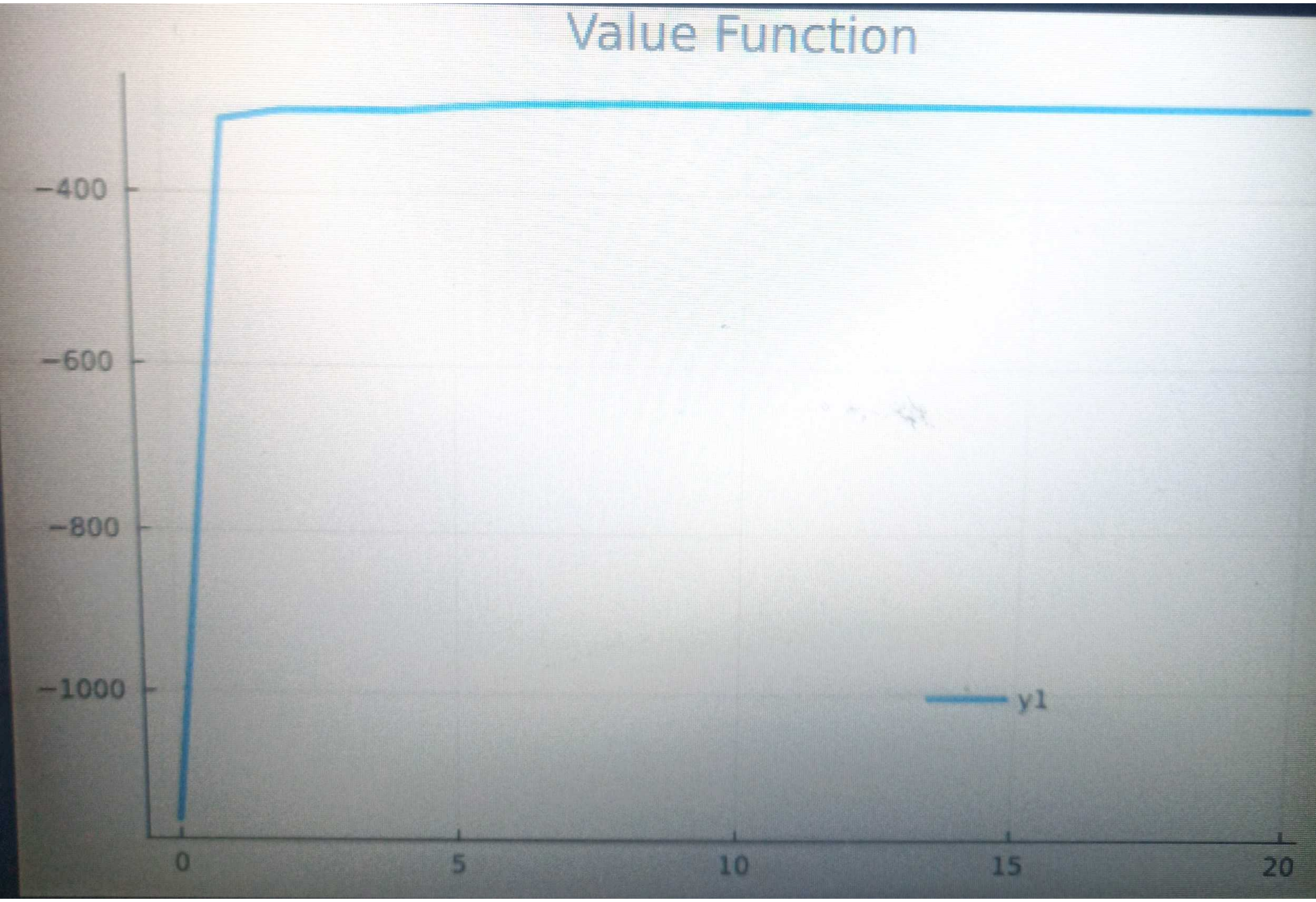
$T_{\pi *} v_{\pi k-1} \le T_{\pi k} v_{\pi k-1}$

Because $v_{\pi *} - v_{\pi k}$ is non-negative,

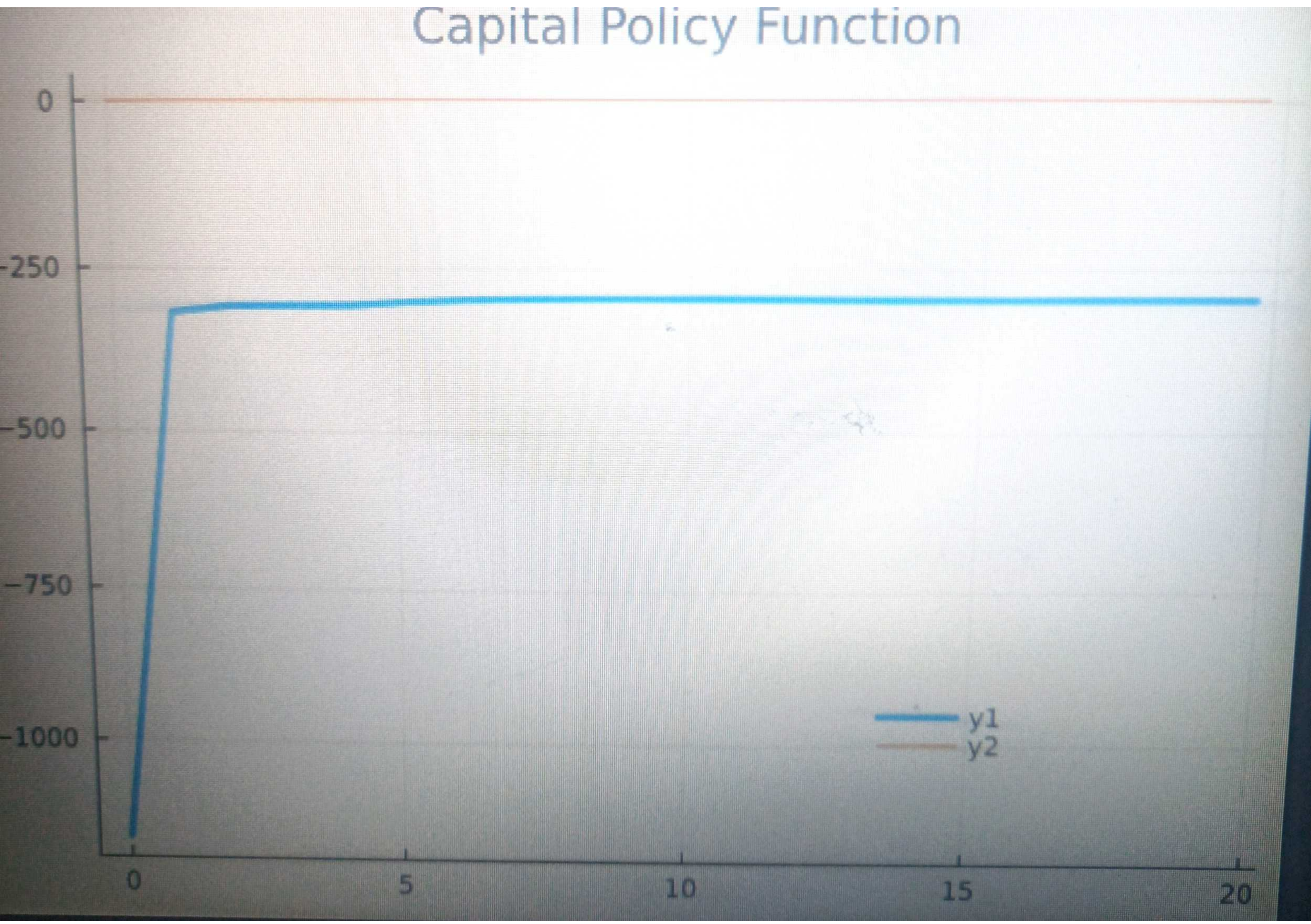
$\| v_{\pi *} - v_{\pi k} \|_\infty \le \beta \| v_{\pi *} - v_{\pi k-1} \|_\infty \Rightarrow$ contraction.

```
7;
   Prices: r = 0.9416666666666668; w = 1.4290287909594477;
 -------------------------

  0.162118 seconds (268.99 k allocations: 15.943 MiB, 92.95% compilation time)
ERROR: UndefVarError: gr not defined
Stacktrace:
 [1] top-level scope
   @ c:\Users\ROG\Desktop\PS02_VFI_plain_static.jl:110


 -------------------------
 Steady State variables
   k = 9.84910377210457; y = 2.1435431864391714; c = -7.705560585665399; l = 0.399859881014876
7;
   Prices: r = 0.9416666666666668; w = 1.4290287909594477;
 -------------------------

  0.119624 seconds (128.51 k allocations: 8.583 MiB, 90.48% compilation time)

julia> 
```


Consumption Policy Function
0
−250
−500
−750
−1000
0
5
10
15
20
y1
y2
y3



Value Function
-400
-600
-800
-1000
0
5
10
15
20
y1



Capital Policy Function
0
-250
-500
-750
-1000
0
5
10
15
20
y1
y2

```
tacktrace:
 [1] top-level scope
   @ c:\Users\ROG\Desktop\PS02_Howard PI.jl:101

plot (generic function with 4 methods)

ERROR: UndefVarError: k_grid not defined
Stacktrace:
 [1] top-level scope
   @ c:\Users\ROG\Desktop\PS02_Howard PI.jl:106

([4294.9160839261995, 4295.477161906606, 4472.873426832678, 4296.421542569186, 4297.2218260551
94, 4298.610151993237, 4301.701064979403, 4314.98800576327, 4261.423075184529, 4284.5529829298
44 … 4293.248538567495, 4293.274935093541, 4293.299239459529, 4293.321694691258, 4293.342507
3256985, 4293.3618540343505, 4293.379886851821, 4293.396737341022, 4293.412519939328, 4293.427
334667829], [1.0101739766261097, 1.0101739766261097, 0.5050919883130549, 1.0101739766261097, 1
.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261
097, 1.0101739766261097 … 1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101
739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097,
1.0101739766261097, 1.0101739766261097], [-1.001558256638005, -0.6412220571433522, -0.00286322
7613381758, -0.39937652375309485, -0.3026463735607683, -0.21310546465555102, -0.12847814062968
72, -0.047460020381939305, 0.030774377594652735, 0.10678447894075438 … 1.4945422488296805, 1
.555924599846671, 1.6170755419386333, 1.6780070068666975, 1.7387299714865496, 1.79925455977605
87, 1.859590131380262, 1.9197453587777984, 1.9797282947968755, 2.039546431908158])

julia> 
```

Ln 111, Col 1 (3115 selected)

```
Closest candidates are:
  iterate(::Union{LinRange, StepRangeLen}) at range.jl:872
  iterate(::Union{LinRange, StepRangeLen}, ::Integer) at range.jl:872
  iterate(::T) where T<:Union{Base.KeySet{<:Any, <:Dict}, Base.ValueIterator{<:Dict}} at dict.
jl:712
  ...
Stacktrace:
 [1] indexed_iterate(I::Function, i::Int64)
   @ Base .\tuple.jl:91
 [2] top-level scope
   @ Untitled-1:109

([4294.9160839261995, 4295.477161906606, 4472.873426832678, 4296.421542569186, 4297.2218260551
94, 4298.610151993237, 4301.701064979403, 4314.98800576327, 4261.423075184529, 4284.5529829298
44  …  4293.248538567495, 4293.274935093541, 4293.299239459529, 4293.321694691258, 4293.342507
3256985, 4293.3618540343505, 4293.379886851821, 4293.396737341022, 4293.412519939328, 4293.427
334667829], [1.0101739766261097, 1.0101739766261097, 0.5050919883130549, 1.0101739766261097, 1
.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261
097, 1.0101739766261097  …  1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101
739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097, 1.0101739766261097,
1.0101739766261097, 1.0101739766261097], [-1.001558256638005, -0.6412220571433522, -0.00286322
7613381758, -0.39937652375309485, -0.3026463735607683, -0.21310546465555102, -0.12847814062968
72, -0.047460020381939305, 0.030774377594652735, 0.10678447894075438  …  1.4945422488296805, 1
.555924599846671, 1.6170755419386333, 1.6780070068666975, 1.7387299714865496, 1.79925455977605
87, 1.859590131380262, 1.9197453587777984, 1.9797282947968755, 2.039546431908158])

julia> 
```

Ln 110, Col 1   Spaces: 4   UTF-8   CRLF   Julia   Main