

(1)

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PS1

(1) CE (AD) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$

k_t^d Net capital stock, $y_t \geq 0$, and l_t^d

- given $\{c_t, i_t, x_t+1, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of representative agent $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$, solves firm's problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $\{x_t+1, k_t^s, l_t^s, y_t\}_{t=0}^{\infty}$

$$\sum_{t=0}^{\infty} p_t (c_t + i_t) = \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi_t \quad (\star)$$

$$x_{t+1} = (1-\delta)x_t + i_t$$

$$0 \leq r_t \leq x_t$$

$$0 \leq l_t \leq 1 \quad \pi_t \geq 0$$

$$c_t, x_{t+1} \geq 0$$

To find

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of rep. firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$, solves firm's problem.

$$\mathcal{L} = \max \sum_{t=0}^{\infty} p_t (y_t - r_t k_t - w_t l_t)$$

$$\text{s.t. } y_t = F(k_t, l_t) \quad \forall t \geq 0$$

$$y_t, k_t, l_t \geq 0$$

- markets clear in $\forall t$:

$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

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PS1

① CE (AD) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and KRE
 w/c capital stock

$\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ s.t.

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, no allocation of
 representative agent $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$

solves agent's problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$

$$\text{s.t. } \sum_{t=0}^{\infty} p_t (c_t + i_t) \leq \sum_{t=0}^{\infty} p_t (r_t k_t + w_t l_t) + \pi_0 \quad (*)$$

$$x_{t+1} = (1-\delta)x_t + i_t$$

$$0 \leq k_t \leq x_t$$

$$0 \leq l_t \leq 1 \quad \pi_t \geq 0$$

$$c_t, x_{t+1} \geq 0$$

to given

- given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of
 repor. firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$, solves firm's problem.

$$\mathcal{L} = \max \sum_{t=0}^{\infty} p_t (y_t - r_t k_t - w_t l_t)$$

$$\text{s.t. } y_t = F(k_t, l_t) \quad t \geq 0$$

$$y_t, k_t, l_t \geq 0$$

- markets clear in $t \geq 0$:

$$y_t = c_t + i_t$$

$$l_t^d = l_t^s$$

$$k_t^d = k_t^s$$

(2) SPP - Sequential formulation

$$w(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$\exists c_t, k_t, k_{t+1}$

s.t. $F(c_t, k_t) = c_t + k_{t+1} - (1-\delta)k_t \quad \forall t$

$$c_t \geq 0$$

$$k_t \geq 0$$

$$0 \leq k_t \leq 1$$

$$k_0 \geq k_0$$

$$c_t = F(k_t, k_{t+1}) - k_{t+1} + (1-\delta)k_t$$

$$k_t = 1 \quad \forall t$$

$$s=1$$

Also, assuming u is contin. differentiable, strictly ↑, strictly concave & bounded; F is contin. differentiable, strictly ↑, strictly concave and $\nabla F(1, 1) \neq 0$, then define

$$f(k_t) = F(k_t, 1) + (1-\delta)k_t \quad \forall t$$

output

$$c_t = f(k_t) - k_{t+1}$$

$$f(k) = F(k, 1) + (1-\delta)k$$

$$\forall k$$

(3) $w(k_0) = \max \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$

s.t.

$$0 \leq k_{t+1} \leq f(k_t) \quad \forall t$$

$$c_t \geq 0$$

$$k_0 = k_0 > 0, \text{ given}$$

ignore constraints
assuming INADA conditions
held

FOC: s.t. k_{t+1}

$$-\beta^t u'(f(k_t) - k_{t+1}) + \beta^{t+1} u'(f(k_{t+1}) - k_{t+2}) = 0 \quad \forall t$$

$$\bullet f'(k_{t+1}) = 0 \quad \forall t$$

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) \cdot f'(k_{t+1})$$

Impose transversality condition:

✓

$$\lim_{t \rightarrow \infty} \beta^t u'(f(k_t) - r_{t+1}) f'(k_t) k_t = 0$$

From Th 12, Krugman, under above assumptions,
 allocation $\{k_t\}_{t=0}^{\infty}$ satisfying Euler equations
 and TVC solves sequential SPP. for given k_0 .

In a steady state equilibrium, which is social optimum, $c_t = c^*$, $r_{t+1} = r^* = r_0$ and

$$u'(f(k^*) - r^*) = \beta u'(f(k^*) - r^*) f'(k^*)$$

$$\beta f'(k^*) = 1$$

$$f'(k^*) = 1 + \rho \quad \rho = \frac{1}{1+\beta}$$

$$f'(k) = F_R(k, 1) + 1 - \delta^{-1} \Rightarrow F_R(k^*, 1) = 1 + \rho$$

~~$$f(k^*) - r^* = c^*$$~~
$$r^* = 1$$

From CE definition, we can rewrite KMM problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

& assuming $\pi = 0$

$$\ell = 1$$

& u is strictly π
 $k_t = \alpha c_t$

$$\text{s.t. } \sum_{t=0}^{\infty} \beta^t (c_t + k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} \beta^t (r_t k_t + w_t)$$

$$c_t, k_{t+1} \geq 0$$

k_0 given.

$$\begin{aligned} \text{FOC: } & \beta^t u'(c_t) = \mu_t & \text{w.r.t } k_{t+1} \& \mu_t = \mu_0 (1-\delta + c_{t+1}) \rho \mu_t \\ & \text{w.r.t } c_t \quad \beta^{t+1} u'(c_{t+1}) = \mu_{t+1} \end{aligned}$$

(b)

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_t + \delta} \quad \text{as}$$

From firm's FOC: $r_t = f'_k(k_{t+1})$

$$f'_k(k_t) = f(k_{t+1}) + (1-\delta)k_t \quad \text{so } r_t = f'_k(k_t) - (1-\delta)$$

$$\left. \begin{array}{l} c_t = f(k_t) - r_t + 1 \rightarrow \text{market clearing condition} \\ \text{and } k_{t+1} = f'_k(k_t) \end{array} \right\} \text{from goods market}$$

$$\left. \begin{array}{l} r_t = 1 \\ \text{Substitute market clearing goods condition} \end{array} \right\} \text{from firm's FOC above} \Rightarrow$$

$$\frac{\beta u'(f(k_t) - r_{t+1})}{u'(f(k_t) - r_{t+1})} = \frac{1}{(1+r_t + \delta)^{-1}}$$

$$\frac{(1+r_t + \delta) \beta u'(f(k_{t+1}) - r_{t+2})}{u'(f(k_t) - r_{t+1})} = 1$$

$$\frac{u'(f(k_t) - r_{t+1})}{u'(f(k_{t+1}) - r_{t+2})} = 1$$

$$\frac{f'(k_{t+1}) \cdot \beta u'(f(k_{t+1}) - r_{t+2})}{u'(f(k_t) - r_{t+1})} = 1$$

needs transversality condition

$$\lim_{t \rightarrow \infty} \frac{\partial r_t}{\partial k_t} k_{t+1} = 0$$

this is same Euler equation as
in SPP earlier

It can be shown that allocation $\frac{\partial r_t}{\partial k_t}$
Parity optimal and so SPP's allocation
coincide with equl.'s allocation.

(4)

$$v(k) = \max_{0 \leq k' \leq f(k)} \{ u(f(k) - k') + \beta v(k') \}$$

SP dynamic problem:

$$w(\bar{k}_0) = \max_{\substack{0 \leq k_t+1 \leq f(k_t) \\ k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \Rightarrow$$

$$\text{s.t. } 0 \leq k_{t+1} \leq f(k_t) \quad \forall t \\ k_0 \text{ given}$$

$$\Rightarrow \max_{0 \leq k_1 \leq f(k_0); k_0 \text{ given}} u(f(k_0) - k_1) + \beta \left[\max_{0 \leq k_{t+1} \leq f(k_t)} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right]$$

$$w(\bar{k}_0) = \max_{0 \leq k_1 \leq f(k_0); k_0 \text{ given}} \{ u(f(k_0) - k_1) + \beta w(k_1) \}$$

(5)

$$u(c) = \log c$$

$$f(k, l) = \alpha k^\alpha l^{1-\alpha}$$

$$f(k) = F(k, 1) = \alpha k^\alpha$$

Functional equation:

$$w(\bar{k}_0) = v(\bar{k}_0) \text{ under principle of optimality.}$$

$$c_t = f(k_t) - k_{t+1}$$

$$v(k) = \max_{0 \leq k' \leq \bar{k}^\alpha} \{ \log(\bar{k}^\alpha - k') + \beta v(k') \beta \}$$

By guess & verify:

$$v(k) = A + B \ln(k) \rightarrow \text{log utility} + \delta = 1$$

max $\{ \log(\bar{k}^\alpha - k') + \beta (A + B \ln(k')) \}$ substitute the guess for $0 \leq k' \leq f(k)$

$$\text{FOC: } \frac{\partial}{\partial k'} \frac{\log(\bar{k}^\alpha - k') + \beta (A + B \ln(k'))}{k'} = \frac{\beta B}{1 + \beta B} \Rightarrow k' = \frac{\beta B \bar{k}^\alpha}{1 + \beta B}$$

Evaluerte @ κ' optimal

$$\log \left(ZK^d - \frac{\beta B ZK^d}{1+\beta B} \right) + \alpha A + \beta B \log \left(\frac{\beta Z B K^d}{1+\beta B} \right)$$

$$= \log(ZK^d) - \log(1+\beta B) + \alpha A + \beta B d \log \frac{Z}{1+\beta B} + \beta B \log \left(\frac{\beta B}{1+\beta B} \right)$$

$$= \alpha \log(ZK) - \log(1+\beta B) + \alpha A +$$

$$+ \beta B d \log(ZK) + \beta B \log \left(\frac{\beta B}{1+\beta B} \right) =$$

$$= \alpha \log Z + \frac{\alpha \log K}{B} - \log(1+\beta B) + \alpha A + \beta B d \log Z + \beta B d \log K$$

$$+ \beta B \log \left(\frac{\beta B}{1+\beta B} \right)$$

Check: $\frac{1}{1+\beta B}$

$$A + B \ln(\kappa) = \alpha \log Z - \log(1+\beta B) + \alpha A + \beta B d \log Z + \beta B \log \left(\frac{\beta B}{1+\beta B} \right) +$$

$$+ \log \left(\frac{d + \beta B d}{B} \right)$$

$$B = \alpha + \beta B \alpha$$

$$B - \beta B \alpha = \alpha$$

$$B(1-\beta\alpha) = \alpha$$

$$B = \frac{\alpha}{1-\beta\alpha}$$

$$A = \alpha \log Z - \log \left(1 + \beta \cdot \frac{\alpha}{1-\beta\alpha} \right) + \alpha A + \beta d \log(Z) \cdot \frac{\alpha}{1-\beta\alpha}$$

$$A(1-\beta) = \alpha \log Z \cdot \left(1 + \beta \frac{\alpha}{1-\beta\alpha} \right) + \beta d \log \left[\frac{\beta d}{1-\beta\alpha} \times \frac{1-\beta\alpha}{1-\beta\alpha + \beta d} \right] + \beta \cdot \frac{\alpha}{1-\beta\alpha} \log \left(\frac{\beta\alpha}{1-\beta\alpha} \right)$$

$$A = \frac{1}{1-\beta} \left[\alpha \log Z \cdot \left(\frac{1}{1-\beta\alpha} \right) + \frac{\beta\alpha}{1-\beta\alpha} \log(\beta\alpha) \cdot \log \left(\frac{1}{1-\beta\alpha} \right) \right]$$

$$A = \frac{1}{1-\beta} \cdot \frac{\alpha}{1-\beta\alpha} \cdot \left(\beta \log Z + \beta \log(\alpha\beta) \right) - \log \left(\frac{1}{1-\beta\alpha} \right)$$

(7)

$$\begin{aligned}
 \text{policy fn: } k' &= f(k) = \frac{\beta(1-\alpha)k^\alpha}{1+\beta(1-\alpha)} = \beta \cdot \frac{\frac{\alpha}{1-\alpha} k^\alpha}{1-\beta(1-\alpha)} = \\
 &= \frac{\beta \alpha k^\alpha}{1-\beta(1-\alpha)} : \left(1 + \frac{\beta \alpha k^\alpha}{1-\beta(1-\alpha)}\right) = \frac{\beta \alpha k^\alpha}{1-\beta \alpha k^\alpha} \times \frac{1-\beta \alpha k^\alpha}{1-\beta \alpha k^\alpha} = \\
 &= \underline{\alpha \beta \alpha k^\alpha}
 \end{aligned}$$

$$\begin{aligned}
 k_1 &= \alpha \beta \alpha k_0^\alpha \\
 k_2 &= \alpha \beta \alpha k_1^\alpha \\
 k_3 &= \alpha \beta \alpha k_2^\alpha \\
 &\vdots
 \end{aligned}$$

$$(6) C_t = C^* = C_t + \epsilon$$

$$k_{t+1} = k^*$$

From SPP steady state solution:

$$\beta \cdot f'(k^*) = 1$$

$$\beta \cdot \alpha z k^{*\alpha-1} = 1 \quad |^{1/\alpha}, \quad f'(k^*) = \alpha z k^{*\alpha-1}$$

$$f(k^*) = z k^{*\alpha} + (1-\alpha) k$$

$$\alpha z k^{*\alpha-1} = \frac{1}{\beta}$$

$$k^* = \underbrace{\left(\frac{1}{\beta \alpha z}\right)^{\frac{1}{\alpha-1}}}_{\alpha z k^{*\alpha-1}}$$

$$\begin{aligned}
 C^* &= f(k^*) - k^* = \\
 &= z k^{*\alpha} - k^* = \\
 &= z \cdot \left(\frac{1}{\beta \alpha z}\right)^{\frac{\alpha}{\alpha-1}} - \left(\frac{1}{\beta \alpha z}\right)^{\frac{1}{\alpha-1}}
 \end{aligned}$$

$$r^* = f'(k^*) = \alpha z k^{*\alpha-1} = \alpha z \left[\left(\frac{1}{\beta \alpha z}\right)^{\frac{1}{\alpha-1}}\right]^{\alpha-1} = \frac{\alpha z}{\beta \alpha z} = \frac{1}{\beta}$$

$$y^* = F(k^*, 1) = z k^{*\alpha} = z \cdot \left(\frac{1}{\beta \alpha z}\right)^{\frac{\alpha}{\alpha-1}}$$

$$w^* = Fe(k^*, \alpha) = (1-\alpha) z k^{*\alpha} = (1-\alpha) z \cdot \left(\frac{1}{\beta \alpha z}\right)^{\alpha-1}$$

I attached Matlab code,
Julia code is in progress

(7) $\delta = \frac{1}{\beta}$

$Z = 1$

$\gamma = 1$

$\beta = 0.96$

shock to $K^* \rightarrow 0.8 K^*$
shock to $\gamma \rightarrow 1.052$

$$K^* = \left(\frac{1}{\beta^{1/\beta-1}} \right)^{\frac{1}{1/\beta-1}}$$

$$C^* = \left(\frac{1}{\beta^{1/\beta-1}} \right)^{\frac{1}{1/\beta-1}} - \left(\frac{1}{1/\beta} \right)^{\frac{1}{1/\beta-1}}$$

$$r^* = \frac{1}{\beta}$$

$$y^* = \left(\frac{1}{\beta^{1/\beta-1}} \right)^{\frac{1}{1/\beta-1}}$$

$$w^* = \left(\frac{1}{\beta^{1/\beta-1}} \right)^{\frac{1}{1/\beta-1}}$$

(8) Done



