GAMMA-RAY ANGULAR CORRELATIONS FROM ALIGNED NUCLEI PRODUCED BY NUCLEAR REACTIONS[1]

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ABSTRACT

Two general procedures for the measurement and analysis of angular correlations of gamma radiations from nuclear reactions are described which have wide applications in nuclear spectroscopy for the determination of spins and gamma-ray multipolarities. Cases can be studied by these methods when the reaction proceeds through a compound state too complex to allow the usual analysis to be made, for example where several levels overlap or where direct interaction is dominant. The basis of these procedures is to exploit the simplifications brought about by making the reacting system axially symmetric. A sharp gamma-ray-emitting state formed in such a system can be regarded as aligned and described in terms of a relatively small number of population parameters for the magnetic substates. In the first procedure, a state Y^* is prepared by a nuclear reaction $X(h_1h_2)Y^*$ in which h_2 is unobserved. The state Y^* has axial symmetry about the beam axis. From coincidence angular correlation measurements of two cascade gamma rays from Y^*, the unknown population parameters for Y^* together with the nuclear spins and gamma-ray multipolarities can be determined. In the second procedure, h_2 is measured in a small counter at 0° or 180° relative to the incident beam. It is then shown that the quantum numbers of the magnetic substates of Y^* which can be populated do not exceed the sum of the spins of X, h_1, and h_2. In cases where the sum of the spins does not exceed $\frac{1}{2}$, the angular correlation of the gamma rays from the aligned state depends only upon the properties of the states in the residual nucleus. Theoretical expressions for angular correlations from aligned states are given, together with a method whereby existing extensive tables of coefficients can be used to calculate them. The results of two recent experiments are discussed as examples.

I. INTRODUCTION

The determination of the spins and parities of nuclear states as well as of a number of associated nuclear parameters has rested very largely on angular distribution measurements of nuclear reactions. This is a consequence of the intimate connection which the concepts of spin and parity have with the rotation and reflection of co-ordinate systems. The general theory of such processes exploits the invariance properties of the system through the use of the statistical tensors of Fano and Racah (1959), as has been described in a number of well-known papers. The formalism and, to a large degree, the notation of the survey of Devons and Goldfarb (1957) will be used in the present paper.

The interpretation of the simplest angular distribution, namely the angular distribution of an outgoing gamma radiation with respect to a bombarding beam and without observation of its polarization, is usually attended by ambiguity. The reason for this is that such a simple experiment does not provide very much information. More specifically, the results of a typical experiment can be represented by the formula

$$W(\theta) = a_0P_0(\cos\theta)+a_2P_2(\cos\theta)+a_4P_4(\cos\theta) \tag{1}$$

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where $W(\theta)$ is the intensity of the reaction observed at the angle θ relative to the incident beam and the a_k are the coefficients of the corresponding Legendre polynomials, $P_k(\cos\theta)$. It frequently happens that the angular distribution coefficients are a function of several unquantized parameters, for example, channel spin mixtures and orbital mixtures for the incoming particles and a multipole mixture for the outgoing gamma ray. Thus, it may be possible to adjust the parameters to fit the measured a_2/a_0 and a_4/a_0 for several choices of the spin of the state. In such a case, the spin could not be determined from the measurements.

It has been pointed out by Ferguson and Rutledge (1957) and by Biedenharn (1960) that substantially more information can be obtained from a triple correlation such as the reaction $(p\gamma\gamma)$. Assuming the axis of quantization to be defined by the incoming particle beam, the intensity of the two gamma rays in coincidence will be a function of the polar angles θ_1 and θ_2 for each radiation and the relative azimuthal angle ϕ between them. In this case, the intensity is given by the series

$$W(\theta_1, \theta_2, \phi) = \sum_{k_1 k_2 \kappa} a^{\kappa}_{k_1 k_2} X^{\kappa}_{k_1 k_2}(\theta_1, \theta_2, \phi) \tag{2}$$

where the $X^{\kappa}_{k_1 k_2}(\theta_1, \theta_2, \phi)$ are functions whose angular dependence has the form $P^{\kappa}_{k_1}(\cos\theta_1)P^{\kappa}_{k_2}(\cos\theta_2)\cos\kappa\phi$. k_1 and k_2 are limited by the multipolarities of the two gamma rays and are even if the nuclear states have sharp spin and parity. κ is positive or zero, may be even or odd, and does not exceed the smaller of k_1 and k_2. With the latter restricted to 4, corresponding to quadrupole radiation, the series (2) has 19 terms (Ferguson and Rutledge (1957) hereinafter designed FR). Provided the compound state has sharp spin and parity, then usually no more than five unquantized parameters comprising channel spin and orbital and multipole mixing will occur. Over a sufficiently varied set of points, the angle functions of series (2) are linearly independent, so that the coefficients $a^{\kappa}_{k_1 k_2}$ can all be determined, and it may be seen that the amount of available information, i.e. the number of measurable parameters, generously exceeds the number of unknowns. No experimental use has yet been made of this general approach. A number of investigators, Hoogenboom (1958), Litherland *et al.* (1959), and Broude *et al.* (1959) for example, have extracted a limited amount of information by the use of geometrical configurations, or "geometries", with one detector fixed and the other moving in a horizontal plane.

These considerations are predicated on the assumption that all of the states concerned have definite spins and parities. While this will be generally true for the lower states of the cascade, it will be obtained for the state into which an incident particle is captured only if this is a sharp, well-isolated resonance. The lighter nuclei under neutron, proton, or alpha-particle bombardment provide abundant instances where this requirement is satisfied. However, for deuteron bombardment and in the middle-weight and heavier nuclei, such isolated resonant structure is rare, the reaction ordinarily being dominated by stripping or direct interaction type of behavior for which the compound state has no definite spin. Here the description in terms of statistical tensors contains so many arbitrary parameters that, despite the large amount of information available, the analysis of the system must be regarded as impractical at present. It is the object of the present paper to describe some types of experiment wherein the representation of the system is greatly simplified so as to permit the results to be interpreted. The essence of the idea is to consider a state of sharp spin and parity, which is one of the later stages of a transmutation process, prepared in a way such that there are certain strong limitations on the populations of the magnetic substates. The subsequent decay of this state can then be treated as if it were aligned. The cases we will be concerned with are those in which the angular distribution of a gamma ray emitted by the state is measured and those in which the correlation of two gamma rays in cascade from the state is measured. In the latter, the amount of information that can be extracted from the correlation is the same as for the ($p\gamma\gamma$) triple correlation discussed above and will usually be well in excess of the number of arbitrary parameters describing the system. This method has been used by Warburton and Rose (1958) to establish the spin of the 6.89-Mev state of C^{14}. It is conjectured by these workers that the contrast between results for different spin choices will generally be much less marked than it was in the C^{14} study and they conclude that the method is probably of limited use. This is a question which has not been studied and does not appear to be amenable to a theoretical scrutiny. However, in the use made of triple angular correlations to date, clear-cut results have generally been found. We are proposing here that the method has indeed wide applicability and is one of the most powerful methods available for the determination of spins and multipole mixtures.

The ideas involved here can be grouped into two parts which we will call, for convenience, method I and method II. Method I is that in which a state of definite spin and parity is formed by a transmutation process involving incident and outgoing particles and proceeding through a compound state which may not have sharp spin and parity. The incident beam defines the axis of quantization and the outgoing particles are not observed. It can then be shown that the magnetic substates of the system are uncorrelated with each other and that they are symmetrically populated. The angular correlations of the radiations from this state will then be governed by a or $a-\frac{1}{2}$ polarization parameters according to whether the spin a is integral or half-integral. The measurement of the correlations of two cascade gamma rays following this will thus normally provide sufficient information to determine these parameters as well as multipole mixtures of the two radiations.

The theoretical expressions for such correlations are given in Section II of this paper together with the modifications necessary to allow for the finite size of the gamma-ray detectors. In Sections II and III a procedure is given for obtaining the theoretical correlation coefficients from the table of coefficients prepared by FR. In Section IV an experiment is discussed which provides a good illustration of method I. This is the experiment on the angular distributions and correlations of the gamma rays from the 4.24-Mev state in Mg^{24} discussed recently by Batchelor *et al.* (1960).

Method II consists of forming a state in a way similar to that of method I, but with the difference that the outgoing particle is detected in a counter located at either 0° or (near) 180° relative to the incident beam. Then, since the orbital magnetic quantum numbers for the incoming and outgoing particles are zero, the populations of the magnetic substates of the state under consideration will be limited by the spins of the target nucleus and of the incident and emergent particles.

In Section V theoretical expressions are given for the angular distribution of the gamma rays with respect to the axis defined by the incident and emergent particles. These expressions include the first order corrections for the finite size of the counter detecting the emergent particle. In Section VI the experimental results from the reaction $Mg^{26}(\alpha,n\gamma)Si^{29}$ are discussed. In this example the observation of the neutrons at 0° selects the magnetic substates $\pm\frac{1}{2}$ of the gamma-emitting state. Consequently there is no arbitrary parameter associated with the populations and the angular distribution coefficients depend only upon the properties of the residual nucleus Si^{29}. Theoretical expressions for such angular correlations are given and a method described whereby existing extensive tables of coefficients can be used to calculate them.

II. METHOD I

A state which is formed as the final one in a nuclear reaction involving the capture of an unpolarized particle incident along the z-axis followed by the emission of one or more unobserved radiations in cascade is clearly symmetric about the z-axis. The state is then aligned, this alignment being indistinguishable from the alignment achieved by methods other than nuclear reactions as summarized, for example, by Blin-Stoyle and Grace (1957). For an axially symmetric state with angular momenta a and a' only the tensor parameters $\rho_{k0}(aa')$ are different from zero as is readily seen from the requirement of invariance under rotation through an arbitrary angle about the z-axis. This treatment of correlations resulting from nuclear reactions is specifically considered by Biedenharn (1960). If, further, the state considered has definite parity, and if no polarization is present in the incident particles and target, then only the tensor parameters having even k will be non-zero. This again follows readily by considering reflection of the system in the origin, equivalent, in this case, to a proper rotation through 180° about the y-axis. We will henceforth make the assumption that the state has both spin and parity definite.

Using the standard relation connecting a statistical tensor and density matrix:

$$(3) \qquad \langle a\alpha|\rho|a\alpha'\rangle = \sum_{k_a\kappa_a} (-)^{a-\alpha'}(a\alpha,a-\alpha'|k_a\kappa_a)\rho_{k_a\kappa_a}(aa),$$

we find that the restriction $\kappa = 0$ implies that $\alpha' = \alpha$. The notation of Devons and Goldfarb is used here where α designates a magnetic quantum number of the state a; in general Greek letters designate the magnetic substates of states for which the corresponding Latin letter designates the spin magnitude.

$\langle a\alpha|\rho|a\alpha'\rangle$ is an element of the density matrix and $(a\alpha,a-\alpha'|k_a\kappa_a)$ is a Clebsch–Gordan vector addition coefficient. It is readily found that evenness of k_a implies symmetry between the population of the positive and negative magnetic substates, that is

$$\langle a-\alpha|\rho|a-\alpha\rangle = \langle a\alpha|\rho|a\alpha\rangle.$$

For convenience we will abbreviate the notation for these elements and introduce $P(\alpha) = \langle a\alpha|\rho|a\alpha\rangle$ to indicate the population of the substate α. Due to the symmetry between positive and negative values only the positive values and zero need be considered, but care must be taken not to ignore the contributions arising from negative values which normally occur in the summations.

The number of independent parameters required to describe the state a with the present limitations is $a+1$ or $a+\frac{1}{2}$ depending on whether a is integral or half-integral. The angular correlations are homogeneous in these parameters so that one parameter can be identified as a normalization factor which need not be measured. The essential number of parameters is consequently a or $a-\frac{1}{2}$. If the state decays with the emission of two gamma rays in cascade then two more unquantized parameters comprising the multipole mixing of the transitions must be expected. As a specific example, the correlation between two cascade gamma rays from a state of spin 5 will entail seven parameters. As we have seen, the measurement of such a correlation can yield 18 parameters so that ample information for the determination of the unknowns is generally available. We term the procedure based on these considerations method I. It consists of a bombardment experiment in which the incident beam direction is identified with the z-axis, the disintegration particles from the compound state are not observed, and the coincident correlations are measured relative to the z-axis of two cascade radiations, generally gamma rays, from a state below the compound state. The process is illustrated in Fig. 1. x and y are the spins of the target nucleus and compound nucleus

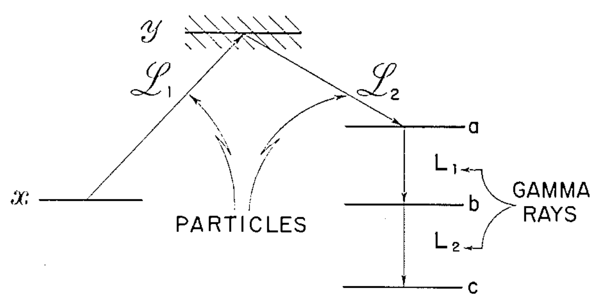

FIG. 1. Schematic energy level diagram to illustrate the quantum numbers used. The analysis of the paper is concerned with the gamma-ray cascade $aL_1\,b\,L_2\,c$. The quantum numbers $x\;y\;\mathscr{L}_1\mathscr{L}_2$ belong to the details of the formation of the state 'a' and are not involved directly in the analysis. x is the spin of the target nucleus, y is the spin of the compound state formed by capture of the particle with total angular momentum $\mathscr{L}_1$. $\mathscr{L}_2$ is the total angular momentum of the outgoing particle which in method I is not observed and in method II is observed in a counter located on the beam axis. The cross hatching on the state y indicates that it need not have sharp spin and parity. All of the details of the formation of the state 'a' are contained in the populations of its magnetic substates, $P(\alpha)$, which are treated as unknowns to be found from the angular correlations of the gamma rays.

and $\mathscr{L}_1$ and $\mathscr{L}_2$ the total angular momenta of incoming and outgoing particles respectively. a, b, and c are the spins of the subsequent states between which the radiations with multipolarities, L_1 and L_2, are observed. x, y, $\mathscr{L}_1$, and $\mathscr{L}_2$ and any interfering mixtures contained in them are immaterial to the analysis, which proceeds entirely from parameters describing the state a.

The expression for the correlation can be obtained immediately from one given by Devons and Goldfarb (1957) for two cascade radiations from an unpolarized state of spin a. We will assume as they do that the cascade proceeds through the states b and c with the emission of radiation of multipolarity 2^{L_1} and 2^{L_2}. The required expression is obtained from equation (11.33) of Devons and Goldfarb (1957) by removing the assumption that the initial state is unpolarized. It is

$$(4)\qquad W \sim \sum \rho_{k0}(aa)\, \langle c||L_2||b\rangle\, \langle c||L_2'||b\rangle^*\, \langle b||L_1||a\rangle\, \langle b||L_1'||a\rangle^*$$

$$\times \overset{*}{\epsilon}_{k_1\kappa_1}(L_1L_1')\overset{*}{\epsilon}_{k_2\kappa_2}(L_2L_2')\hat{a}^2\hat{k}_1\hat{k}_2 \begin{Bmatrix} b & L_1 & a \\ b & L_1' & a \\ k_2 & k_1 & k \end{Bmatrix} \hat{b}^2\hat{k}_2 \begin{Bmatrix} c & L_2 & b \\ c & L_2' & b \\ 0 & k_2 & k_2 \end{Bmatrix}$$

$$\times (k_2\kappa_2, k_1\kappa_1|k0)\ (00\,k_20|k_20)\ \hat{c}.$$

The summation in equation (4) is over $k\, k_1k_2\kappa_1\kappa_2L_1L_1'L_2L_2'$. The basic change required to remove the condition of a unpolarized is to replace the factor $\delta_{k_a0}/\hat{a}$ of the original equation by $\rho_{k0}(aa)$, in which k is assumed even as discussed above. The other δ-functions have been removed, putting in the explicit parameters demanded by them. From the Clebsch–Gordan coefficient $(k_2\kappa_2,k_1\kappa_1|k0)$ we must have $\kappa_1 = -\kappa_2 = \kappa$. The coefficient $(00k_20|k_20) = 1$. The circumflex over any index or quantum number, e.g. a, indicates the function

$$\hat{a} = (2a+1)^{1/2}.$$

The angular dependence of the correlation is contained specifically in the factors $\epsilon_{k_1\kappa_1}(L_1L_1')$ and $\epsilon_{k_2\kappa_2}(L_2L_2')$. These can both be written as

$$(5)\qquad \epsilon_{k\kappa}(L\,L') = c_{k0}(L\,L')\, D^k_{\kappa 0}(\mathscr{R})$$

$$= \frac{\hat{L}\,\hat{L}'(-)^{L'-1}}{\sqrt{2\pi}\,\hat{k}}(L1, L'-1|k0)\ .\ Y_k^{\kappa *}(\theta,\phi)$$

where $c_{k0}(L\,L')$ is a radiation parameter for gamma rays as described by Devons and Goldfarb, $D^k_{\kappa 0}(\mathscr{R})$ is an element of the rotation matrix for the rotation $\mathscr{R} = (\phi,\theta,0)$ which brings the z-axis into the direction of the counter. The second line of equation (5) is obtained by substituting explicit expressions for these quantities. $Y_k^\kappa(\theta,\phi)$ is the spherical harmonic of the polar angles θ and ϕ.

The reduced matrix elements, $\langle b||L_1||a\rangle$, etc. are real. The usual multipole amplitude ratios are defined by*

*We write the reduced matrix elements in the normal order. The consistent use of this convention will avoid the confusion of sign (Ofer 1959) which arises from the convention of Biedenharn and Rose (1953). The penalty for using the normal order is the appearance of factors of the type $(-)^{L-L'}$ in the formulae. We feel, however, that this is a smaller difficulty from the experimentalists' viewpoint than the one of having the sign of a mixing ratio depend on whether the transition involved is the first or last one in the correlation.

$$\delta_1 = \frac{\langle b||L_1+1||a\rangle}{\langle b||L_1||a\rangle}, \tag{6a}$$

$$\delta_2 = \frac{\langle c||L_2+1||b\rangle}{\langle c||L_2||b\rangle}. \tag{6b}$$

Finally the second $9-j$ symbol of the formula can be reduced to a Racah coefficient through

$$\begin{Bmatrix} c\, L_2\, b \\ c\, L_2'\, b \\ 0\, k_2\;\; k_2 \end{Bmatrix} = \frac{(-)^{c+k_2-L_2-b}}{\hat{c}\,\hat{k}_2} W(bL_2bL_2', ck_2).$$

Dropping constant factors which affect only the normalization of the expression we have

$$\begin{aligned} W \sim \sum\ & \rho_{k0}(aa)\, \delta_1^{p_1}\, \delta_2^{p_2} (-)^{f_1}\, \hat{L}_1\hat{L}_1'\hat{L}_2\hat{L}_2' \\ & \times (L_1 1, L_1'-1|k_1 0)\ (L_2 1, L_2'-1|k_2 0)\ (k_1-\kappa, k_2\kappa|k0) \\ & \times \begin{Bmatrix} b\ \ L_1\, a \\ b\ \ L_1'\, a \\ k_2\, k_1\ k \end{Bmatrix} W(bL_2bL_2', ck_2)\ Y_{k_1}^{\kappa}\ (\theta_1\phi_1)\ Y_{k_2}^{-\kappa}(\theta_2\phi_2). \end{aligned} \tag{7}$$

The summation is over $k\, k_1 k_2 \kappa L_1 L_1' L_2 L_2'$. $f_1 = c-b+L_1'-L_2+L_2'+k_2$. k_1 and k_2 are even as a consequence of the states b and c having definite parity. We thus have

$$(k_1-\kappa, k_2\kappa|k0) = (k_1\kappa, k_2-\kappa|k0).$$

The summation over κ extends over positive and negative values which can be grouped in pairs

$$\begin{aligned} (k_1-\kappa, k_2\kappa|k0)\ & Y_{k_1}^{\kappa}\ Y_{k_2}^{-\kappa} + (k_1\kappa, k_2-\kappa|k0)\ Y_{k_1}^{-\kappa}\ Y_{k_2}^{\kappa} \\ & = (k_1-\kappa, k_2\kappa|k0)\ [Y_{k_1}^{\kappa}\ Y_{k_2}^{-\kappa} + Y_{k_1}^{\kappa *}\ Y_{k_2}^{-\kappa *}] \\ & = (k_1-\kappa, k_2\kappa|k0)\ 2\ \mathrm{Re}\ [Y_{k_1}^{\kappa}\ (\theta_1,\phi_1)\ Y_{k_2}^{-\kappa}\ (\theta_2,\phi_2)] \end{aligned}$$

revealing that W is real, which is known, of course, as a general result. For explicit evaluation we write the spherical harmonics in terms of the associated Legendre polynomials (Condon and Shortley 1951)

$$Y_k^{\kappa}(\theta,\phi) = (-)^{(\kappa+|\kappa|)/2} \frac{\hat{k}}{\sqrt{4\pi}} \left[\frac{(k-|\kappa|)!}{(k+|\kappa|)!}\right]^{1/2} P_k^{|\kappa|}\ (\cos\theta) e^{i\kappa\phi}$$

and

$$\begin{aligned} & \mathrm{Re}\ [Y_{k_1}^{\kappa}(\theta_1,\phi_1)\ Y_{k_2}^{-\kappa}\ (\theta_2,\phi_2)] \\ & = \frac{(-)^{|\kappa|}}{4\pi} 2^{1-\delta_{\kappa 0}}\, \hat{k}_1\hat{k}_2 \left[\frac{(k_1-|\kappa|)!\ (k_2-|\kappa|)!}{(k_1+|\kappa|)!\ (k_2+|\kappa|)!}\right]^{1/2} P_{k_1}^{|\kappa|}\ (\cos\theta_1)\ P_{k_2}^{|\kappa|}\ (\cos\theta_2) \cos\kappa\,\phi. \\ & = \frac{(-)^{|\kappa|}}{4\pi} 2^{1-\delta_{\kappa 0}}\, X_{k_1k_2}^{\kappa}\ (\theta_1,\theta_2,\phi). \end{aligned} \tag{8}$$

The terms representing interfering multipoles are duplicated in the summation of equation (7). It is consequently convenient to contract this formula and write

$$(9)\quad W \sim \sum{}' \, 2^{n} \rho_{k0}(aa)\, \delta_1^{p_1} \delta_2^{p_2} (-)^{f_2} \hat{L}_1 \hat{L}_1' \hat{L}_2 \hat{L}_2' \times (L_1 1, L_1' - 1 | k_1 0)(L_2 1, L_2' - 1 | k_2 0)\ (k_1 - \kappa, k_2 \kappa | k 0) \times \begin{Bmatrix} b & L_1 & a \\ b & L_1' & a \\ k_2 & k_1 & k \end{Bmatrix} W\,(bL_2\, bL_2', ck_2)\, X^{\kappa}_{k_1 k_2}\ (\theta_1, \theta_2, \phi).$$

The summation here is over $k\, k_1 k_2 \kappa L_1 L_1' L_2 L_2'$. The prime on the $\sum$ indicates the restriction $L_1 \leqslant L_1'$, $L_2 \leqslant L_2'$, and $\kappa \geqslant 0$. The duplication is accounted for by the factor 2^n. n is equal to the number of pairs $(L_1 L_1')$ and $(L_2 L_2')$ in which the interfering multipoles are different, plus zero for $\kappa = 0$ and plus one for $\kappa > 0$. $f_2 = c - b + L_1' - L_2 + L_2' + k_2 + |\kappa|$.

Equation (9) has the form of equation (2) so that the explicit expression for the coefficients $a^{\kappa}_{k_1 k_2}$ can be written down. The indices $k_1 k_2$, which characterize the angle functions $X^{\kappa}_{k_1 k_2}(\theta_1, \theta_2, \phi)$, are even and κ is integral and not greater than the smaller of k_1 and k_2. If $L_1, L_1', L_2, L_2' \leqslant 2$, then $(k_1 k_2 \kappa)$ may in general have the values (000), (020), (040), (200), (220), (221), (222), (240), (241), (242), (400), (420), (421), (422), (440), (441), (442), (443), (444). These 19 sets of indices define 19 corresponding angle functions which are mutually orthogonal in the intervals $0 \leqslant \theta_1 \leqslant \pi$, $0 \leqslant \theta_2 \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ and are thus linearly independent.

Extensive tabulations of the coefficients occurring in equation (9) have been made for which a convenient bibliography is given by Gibbs and Way (1958, 1959). The more extensive tabulations are by Simon *et al.* (1954) and Smith and Stevenson (1957). Composite coefficients in which several of the factors are combined and which facilitate numerical work considerably have been tabulated by Sharp *et al.* (1953), Ferentz and Rosenzweig (1955), Rose (1958), and Wapstra *et al.* (1959). The tabulation of FR is calculated specifically for $(p\gamma\gamma)$ reactions, but can be conveniently used in the present application with some limitations. Its use will be discussed in detail.

The variables in equation (9) that are generally to be found from experiment are the various $\rho_{k0}(aa)$, δ_1, and δ_2. The population parameters, $P(\alpha)$, constitute an alternate set of variables to the $\rho_{k0}(aa)$ and are related linearly to them. The choice between these two is largely a matter of convenience and is discussed further in Section IV. In some instances there may be knowledge of the details of the formation of the state of spin a which imposes a limit on k. In such cases it is convenient to use the $\rho_{k0}(aa)$ variables so that the restriction can be explicitly imposed. The $P(\alpha)$ have the important restriction that they must be positive. The formulation in terms of these parameters allows this restriction to be easily imposed.

The functions defined by Biedenharn (1960) and given by

$$P_{kk_1k_2}\ (\Omega_0 \Omega_1 \Omega_2) = (4\pi)^{3/2} \frac{i^{\,k+k_1-k_2}}{\hat{k}\,\hat{k}_1\,\hat{k}_2^2} \sum_{\kappa,\kappa_1,\kappa_2} (k\kappa, k_1\kappa_1 | k_2\kappa_2)(-)^{\kappa} Y_k^{\kappa}\ (\Omega_0)\ Y_{k_1}^{\kappa_1}\ (\Omega_1)\ Y_{k_1}^{-\kappa_2}(\Omega_2),$$

where $\Omega_i = (\theta_i,\phi_i)$ represent the polar angles of the initial, first, and second radiations, play a role in triple correlation theory analogous to that of the Legendre polynomials in double correlation theory. Specializing these functions for $\theta_0 = \phi_0 = 0$, and introducing the population parameters $P(\alpha)$, the correlation function becomes

$$(10)\quad W \sim \sum P(\alpha)\, \delta_1^{p_1}\delta_2^{p_2}(-)^{f_3}\hat{k}\, \hat{k}_1\hat{k}_2\, (a\alpha,a-\alpha|k0)$$
$$\times G_{k\, k_1k_2}(L_1L_1'\, b\, a)\, F_{k_2}(L_2L_2'\, b\, c)\, P_{k\, k_1k_2}(\theta_1,\theta_2,\phi).$$

The summation is over $\alpha k\, k_1k_2L_1L_1'L_2L_2'$. $f_3 = a-L_2+L_2'+(k-k_1-k_2)/2-\alpha$. $G_{k\, k_1k_2}(L_1L_1'\, b\, a)$ is the coefficient defined and tabulated by Rose (1958) and $F_{k_2}(L_2L_2'\, b\, c)$ is that tabulated by Ferentz and Rosenzweig (1955). The coefficients $Z_1(L_2bL_2'b;k_2c)$ defined and tabulated by Sharp *et al.* (1953) are related to the $F_{k_2}(L_2L_2'bc)$ coefficients by

$$Z_1(Lb\, L'b;\, ck) = \hat{b}\, (-)^{L+L'+b-c}\, \mathrm{Re}\, i^{(L'-\pi'-L+\pi+3k)}\, F_k(L\, L'b\, c)$$

where π and π' are 0 for electric and 1 for magnetic radiations.

The formulae developed so far have assumed ideal counters of negligible size. The modifications required for counters of finite size have been considered by Rose (1958) and Ferguson (1961). For counters with axial symmetry about a line through the target the correlation function from equation (9) is

$$(11)\quad W \sim \sum 2^n\, \rho_{k0}(aa)\, \delta_1^{p_1}\, \delta_2^{p_2}(-)^{f_2}\, \hat{L}_1\hat{L}_1'\hat{L}_2\hat{L}_2'$$
$$\times (L_11,L_1'-1|k_10)\, (L_21,L_2'-1|k_20)\, (k_1-\kappa,k_2\kappa|k0)$$
$$\times \begin{Bmatrix} b & L_1 & a \\ b & L_1' & a \\ k_2 & k_1 & k \end{Bmatrix} W(bL_2\, bL_2',\, ck_2)\, Q_{k_1}\, Q_{k_2}\, X_{k_1k_2}^{\kappa}\, (\theta_1,\, \theta_2,\, \phi)$$

where Q_{k_1} and Q_{k_2} are attenuation factors for the counters that detect the gamma rays 1 and 2 and are defined by

$$Q_k = \frac{J_k}{J_0},$$

$$J_k = \int_0^{\pi} \epsilon(\xi)\, P_k(\cos\xi)\, \sin\xi\, d\xi,$$

where $\epsilon(\xi)$ is the efficiency of the counter for a gamma ray propagating at an angle ξ to the axis of the counter. Tabulations of Q_k are given by Gove and Rutledge (1958), Rutledge (1959), and Rose (1953). In equation (11) the polar angles θ_1, θ_2, and ϕ define the position of the axes of the counters. The summation in equation (11) is over $k\, k_1k_2\kappa L_1L_1'L_2L_2'$ with $L_1 \leqslant L_1'$, $L_2 \leqslant L_2'$, and $\kappa \geqslant 0$.

The angular resolution corrections are applied to equation (10) in exactly the same way, that is, by the introduction of the two factors Q_{k_1} and Q_{k_2}. It may be of interest that the corrections required for counters not having axial symmetry can be applied in principle. They involve, however, substantially more complication, which can normally be avoided through the use of cylindrical counters.

The table of FR has been found to facilitate considerably the calculation of theoretical results. These have been prepared specifically for $(p\gamma\gamma)$ reactions, where p signifies here a particle of arbitrary spin, using the channel spin formalism and assuming that the compound state as well as the succeeding ones have sharp spin and parity. It is readily shown that the capture of a particle from a channel spin state, s, through orbital momentum l into a state of spin a gives rise to populations

$$P(\alpha) \propto (s\alpha,\ l0|a\alpha)^2.$$

Different channel spins combine incoherently, so that a mixture of channel spins having intensities $T(s)$ gives rise to populations

$$P(\alpha) \propto \sum_s T(s)\,(s\alpha,\ l0|a\alpha)^2.$$

By choosing $l = a$ for even–even nuclei or $l = a\pm\frac{1}{2}$ for even–odd nuclei, then any distribution of the populations, $P(\alpha)$, can be represented by an equivalent set of $T(s)$. Thus the tables of FR can be used by substituting for $P(\alpha)$ the variable $T(s)$. It should be stressed that this variable is completely fictitious and bears no relation to the actual formation of the state a as envisaged in the present application. The tabulation is limited to $l \leqslant 3$, which limits its use to $\alpha \leqslant 3$. While it appears to be true that cases for $a > 3$ could be accommodated if knowledge of the formation provided assurance that substates $\alpha > 3$ were not populated, the loss of generality makes this approach unattractive.

The triple correlation function obtained from the tables can be written

$$(12)\quad W = \sum{}'(-)^{f_4} D^{\kappa}_{k_1k_2}\ (sabcll'L_1L_1'L_2L_2')\ T(s)\ \delta_1^{p_1}\delta_2^{p_2}Q_{k_1}Q_{k_2}X^{\kappa}_{k_1k_2}(\theta_1,\theta_2,\phi)$$

the summation being over $k_1k_2\kappa sL_1L_1'L_2L_2'$. $f_4 = L_2+L_2'+k_1$. $D^{\kappa}_{k_1k_2}\ (sabcll'L_1L_1'L_2L_2')$ is the coefficient tabulated by FR. Introducing the definition of $D^{\kappa}_{k_1k_2}\ (sabcll'L_1L_1'L_2L_2')$, equation (12) can be written

$$(13)\quad W = \hat{b}^2\sum{}'\,2^n(-)^{f_5}\,Z(lal'a,\,sk)\ \hat{L}_1\hat{L}_1'\hat{L}_2\hat{L}_2'\ (k_1-\kappa,k_2\kappa|k0)$$
$$\times(L_11,L_1'-1|k_10)\ (L_21,L_2'-1|k_20)\begin{Bmatrix} b & L_1 & a\\ b & L_1' & a\\ k_2 & k_1 & k\end{Bmatrix} W(bL_2\,bL_2',\,ck_2)$$
$$\times T(s)\ \delta_1^{p_1}\delta_2^{p_2}\ Q_{k_1}Q_{k_2}\,X^{\kappa}_{k_1k_2}(\theta_1,\theta_2,\phi).$$

The summation is over $k\ k_1k_2\kappa sL_1L_1'L_2L_2'$.

$$f_5 = s+a+b+c+L_1'+L_2+L_2'+k_2+\kappa+(l-l'-k)/2$$

$Z(lal'a,\,sk)$ is the coefficient defined by Blatt and Biedenharn (1952). Comparing equation (13) with equation (11), it is evident that they are identical if

$$\rho_{k0}(aa) = \frac{1}{\hat{a}^2}\sum_s\ (-)^{f_5-f_2}\,Z(lal'a,\,sk)\ T(s).$$

Only one value of l is required. Recognizing also that k, k_1, and k_2 are even, we obtain

(14) $$\rho_{k0}(aa) = \frac{1}{\hat{a}^2} \sum_s (-)^{s+a-k/2} Z(lala, sk)\, T(s).$$

By substituting this expression for $\rho_{k0}(aa)$ in equation (3) we obtain for $P(\alpha)$

(15) $$P(\alpha) = \sum_s (s\alpha, a-\alpha|l0)^2\, T(s).$$

Further discussion of the use of these results is given in Section III.

III. APPLICATIONS OF METHOD I

We will discuss in this section practical aspects of the application of the ideas of Section II. Section IV describes as a specific example, the reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$ (Batchelor *et al.* 1960) and also some possible future applications.

The compound state formed by the capture of a particle with an energy of several million electron volts or more must be expected to have mixed spins and parities. The state formed after the emission of a particle is very likely to be a bound state and to have sharp spin and parity, thus satisfying the requirements for the application of method I. Reactions of the type $(pp'\gamma\gamma)$, $(\alpha p\gamma\gamma)$, $(dp\gamma\gamma)$, $(dn\gamma\gamma)$, $(d\alpha\gamma\gamma)$, $(He^3\, p\gamma\gamma)$, etc. generally have these characteristics. Details of the formation of the first gamma-emitting state are immaterial to the analysis, so that the reaction may proceed via direct interaction, a compound nucleus, or any combination of these without affecting the results. The processes $(pp'\gamma\gamma)$ and $(\alpha\alpha'\gamma\gamma)$ involve compound state excitations of lower energy than generally obtained for processes induced by deuterons, He^3, and particles of high internal energy. Few competing reactions will be present, resulting in simple gamma spectra, so that these inelastic scattering processes offer a particularly suitable field for the application of the method. Cases can be found where the radiation from the state a of Fig. 1 is a particle instead of a gamma ray, and the particle–gamma correlation can be treated by this approach provided the state a has the necessary sharp spin and parity. Gamma-ray transitions are the only possible ones among the bound states, and since interest is usually focussed on the lower states, gamma-ray studies will play the principal role in applications.

The correlation function of method I is described by equation (2) in which the coefficients $a^{\kappa}_{k_1k_2}$ are functions of the population parameters, $P(\alpha)$, and the multipole amplitude ratios δ_1 and δ_2. It is, in principle, possible to fit a set of experimental points using equation (2) and then from the coefficients, $a^{\kappa}_{k_1k_2}$, so determined, find the population parameters and multipole mixtures consistent with them. If some spins are unknown then a number of sets of possible spin assignments must be examined and it can be expected that generally, for one set only, parameters can be found that are consistent with all of the coefficients. Such a procedure is analogous to the use normally made of equation (1) in analyzing angular distributions. However, a practical difficulty in this procedure is that it is necessary to make measurements sufficient in number and in the ranges of the angles to determine all of the coefficients. Under fairly general conditions there are 19 coefficients, and since it is usually desirable to measure a number of points rather greater than the number of parameters, a considerable number of measurements is evidently needed.

Since the information content of equation (2) is great, it is sufficient in most cases to obtain by experiment only a part of it. This can be done by fixing two of the angles $(\theta_1, \theta_2, \phi)$ at either 0°, 90°, or 180° and varying the third one. This choice of angles causes the functions $X^{\kappa}_{k_1 k_2}(\theta_1, \theta_2, \phi)$ to vanish for κ odd. The correlation as a function of the variable angle will then have the form of equation (1) and will provide information in the form of the ratios a_2/a_0 and a_4/a_0. It can be written

$$W^i(\theta) = \sum_k a_k^i\,(abc)\,P_k(\cos\theta)$$

where $a_k^i\,(abc)$ is the coefficient of the Legendre polynomial of order k for the spin sequence $a \to b \to c$. The superscript i distinguishes between a number of possible arrangements obtained by fixing different pairs of angles at certain of the three indicated positions. Seven such arrangements are shown in Fig. 2.

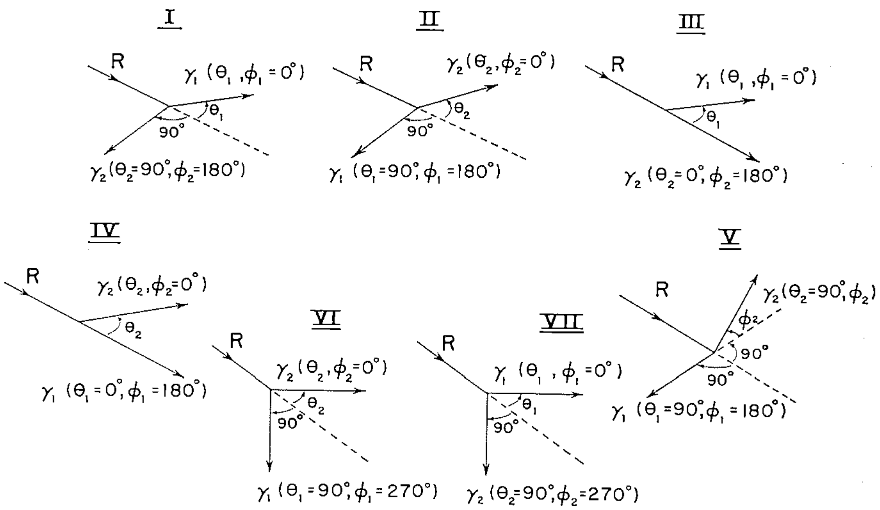

FIG. 2. Diagrams showing the locations and motions of the two counters for various special cases. The diagrams are stereographic projections of the counter directions in three dimensions. The incident beam R and one counter always lie in the horizontal plane. In cases I to IV the directions R, γ_1, and γ_2 are coplanar. In cases VI and VII one counter lies on the perpendicular to the horizontal plane and below it. In case V one counter, γ_2, moves in azimuth about the beam direction.

Here independent azimuthal angles have been assigned to each ray which are related to ϕ by $\phi_1 - \phi_2 = \phi$. The values of the three fixed angles are given for each case and the remaining unspecified one is variable. For cases I to IV the incident beam and the two gamma-ray directions are coplanar. For cases VI and VII the fixed counter is in a plane perpendicular to that containing the beam and the moving gamma-ray counter. For case V both the moving ray and the fixed one lie in a plane whose perpendicular is the beam direction. In this case it is clearly immaterial which gamma-ray counter is moved.

The series (2), with the restrictions described in Section II and with κ even, has 14 terms rather than the 19 terms obtained when the odd κ are present. Thus more than 14 independent quantities cannot be obtained from measurements made in these "geometries". Although the information obtained from the different cases is not always independent of the others, a sufficient number of these is available to establish the unknown parameters. It is advantageous to know the normalization of the cases relative to each other. Thus if a_k^i is a Legendre polynomial coefficient for a case, i, the a_0^i/a_0^0, a_2^i/a_0^0, and a_4^i/a_0^0, where a_0^0 is a particular coefficient, will be significant quantities rather than only a_2^i/a_0^i and a_4^i/a_0^i when the relative normalizations are unknown.

The direct angular distributions of the two individual gamma rays are also described in terms of the parameters used for the coincidence correlations and provide additional information. Indeed, these may be regarded as further geometries which supplement the others. Formulae for these distributions are easily obtained from equation (7) by integrating over all directions of the unobserved radiation. The only surviving terms will have $k_i = \kappa = 0$ where i is 1 or 2 depending on whether the first or second transition is unobserved.

Formulae for the coefficients for each case are readily obtained by specializing the angles as indicated. The associated Legendre polynomials $P_{k_1}^{\kappa}(\theta_1)$ and $P_{k_2}^{\kappa}(\theta_2)$, where κ is even, can each be expressed as a sum of the ordinary Legendre polynomials $P_k(\theta_1)$ and $P_k(\theta_2)$. In terms of the tabulation of Ferguson and Rutledge the reduction can be written

$$(16) \qquad u_k^i(abcsL_1L_1'L_2L_2') = \sum_{k_1k_2\kappa} Q_{k_1}\, Q_{k_2}\, \alpha_{k\ k_1k_2}^{i\ \kappa}\, D_{k_1k_2}^{\kappa}\ (sabcll'L_1L_1'L_2L_2').$$

The complete Legendre polynomial coefficients are obtained from

$$(17) \qquad a_k^i(abc) = \sum{}' u_k^i(abcsL_1L_1'L_2L_2')\, T(s)\, \delta_1^{p_1}\, \delta_2^{p_2}.$$

The summation is over $sll'L_1L_1'L_2L_2'$ with $l \leqslant l'$, $L_1 \leqslant L_1'$, $L_2 \leqslant L_2'$. The factor 2^n required in the contracted sum (as in, for example, equation (9)) has been included in the tabulated coefficients.

Case V of Fig. 2 is slightly different. A Fourier cosine series arises naturally from the specialization of θ_1 and θ_2 made in this case. Thus the expansion used here in place of equation (1) is

$$W(\phi) = \sum_k a_k \cos k\, \phi.$$

The coefficients $\alpha_{k\ k_1k_2}^{i\ \kappa}$ are tabulated by FR.* It is important to note the occurrence of the attenuation factors Q_{k_1} and Q_{k_2} in equation (16). This shows that the effect of finite counter solid angle is not to attenuate each term of the expansion by a simple factor Q_k. The attenuation factors of all orders

*Earlier copies covered only cases I to IV and contained some errors. In subsequent copies the values were extended to include all the cases of Fig. 2 and the errors were corrected.

A group of errors has been recently found in the main table of $D_{k_1k_2}^{\kappa}$ $(sabcll'L_{12}L_{12}'L_2L_2')$ arising from an incorrect Clebsch–Gordan coefficient, $(k\kappa, k_2\kappa_2|k_1\kappa_1) = (40,43|43)$, used as input. This has caused the coefficients having $k_1k_2\kappa = 443$, $a \geqslant 2$, $ll' = 22$, 13, and 33 to be incorrect. This will not affect the cases discussed here since for them κ is even.

of k_1 and k_2 occur in each coefficient and the effects are complicated and unpredictable. This point was first noticed and discussed by Hoogenboom (1958).

The evaluation of equation (16) with the aid of a desk calculator is not difficult, so that, within the limitations of FR, the necessary theoretical coefficients are readily obtained. They can also be obtained with greater range, but less convenience, from the tables of Rose (1958) and of Ferentz and Rosenzweig (1955).

In Tables I to IV are given the coefficients of the Legendre polynomial

TABLE I

Legendre polynomial coefficients, $u_k^i(abc\alpha L_1L_1'L_2L_2')$, of equation (18), for various cases and the spin sequence $abc = 0 \rightarrow 2 \rightarrow 0$, $\alpha = 0, L_2 = L_2' = 2$ for all entries. Cases VI, VII, γ_1, and γ_2 are isotropic due to $a = 0$

Case	k = 0	k = 2	k = 4
I	1.2908	−.8583	.6641
II	1.2908	−.8583	.6641
III	1.0000	.3049	.6641
IV	1.0000	.3049	.6641
V*	1.1620	.4362	.3632

*In this case the expansion $W(\phi) = \Sigma_k a_k \cos k\phi$ is assumed.

coefficients for the spin sequence $a \rightarrow 2 \rightarrow 0$, where a takes the values 0, 1, 2, and 3. All of the cases of Fig. 2 are included as well as the angular distribution of the first gamma ray with the second unobserved and of the second gamma ray with the first unobserved, which are readily obtained by a similar calculation. The latter are designated respectively as cases γ_1 and γ_2. The transformation of equation (15), from the channel spin, s, to the magnetic quantum number, α, has been included. These coefficients can be designated by $u_k^i(abc\alpha L_1L_1'L_2L_2')$. The Legendre polynomial coefficients for any case, i, excepting case V, will be written as

$$a_k^i(abc) = \sum{}' u_k^i(abc\alpha L_1L_1'L_2L_2')\, P(\alpha)\, \delta_1^{p_1}\, \delta_2^{p_2} \tag{18}$$

with the summation over $\alpha L_1L_1'L_2L_2'$ and with $L_1 \leqslant L_1'$, $L_2 \leqslant L_2'$. Case V as before assumes a Fourier cosine series and the coefficients belong to $\cos k\phi$ rather than $P_k(\cos\theta)$. The attenuation coefficients $Q_2 = .9239$ and $Q_4 = .7623$ have been used in computing the tables. These are appropriate to 5-in. diameter by 6-in. long NaI(Tl) scintillators situated with their front faces 6.2 inches from the target and for gamma rays of minimum absorption coefficient, i.e. having energies of about 5 Mev. For crystals of other sizes and for other gamma-ray energies, these attenuation factors can be approximated by suitably adjusting the distance between the counter and the target. In this way the tabulation is applicable to a fairly wide range of experimental conditions.

An interesting feature of the coefficients is that for pure quadrupole radiation ($L_1L_1' = 22$); there are three pairs of cases, I–II, III–IV, and VI–VII in which

TABLE II

Legendre polynomial coefficients, $u_k^i(abc\alpha L_1L_1'L_2L_2')$, of equation (18), for various cases and the spin sequence $abc = 120$. $L_2 = L_2' = 2$ for all entries

		L_1L_1'								
		11			12			22		
		k								
Case	α	0	2	4	0	2	4	0	2	4
I	0	0.4809	0.5436	0.0000	0.5249	0.9151	0.0000	0.8052	0.1906	−0.3622
	1	2.1990	0.0966	0.0000	0.9067	−3.7782	0.0000	2.1468	0.4590	−0.9660
II	0	0.9694	0.9772	−0.7244	0.2201	−0.5044	−1.0799	0.8052	0.1906	−0.3622
	1	1.7105	−0.3370	0.7244	1.2115	−2.3587	1.0799	2.1468	0.4590	−0.9660
III	0	1.4619	0.4796	0.0000	0.0000	2.7285	0.0000	0.5381	0.0807	0.3622
	1	1.5381	−1.1198	0.0000	0.0000	0.1346	0.0000	2.4619	0.3766	−1.6904
IV	0	0.9076	0.3095	0.7244	1.2395	0.4090	1.0799	0.5381	0.0807	0.3622
	1	2.0924	−0.9497	−0.7244	−1.2395	2.4540	−1.0799	2.4619	0.3766	−1.6904
VI	0	1.1230	0.0991	0.0000	−1.4596	0.0954	0.0000	0.8820	−0.2485	0.0000
	1	2.1971	−0.0991	0.0000	0.0281	−0.0954	0.0000	1.3913	0.2485	0.0000
VII	0	1.0571	−0.0326	0.0000	−0.5249	1.9649	0.0000	0.8820	−0.2485	0.0000
	1	2.2630	0.0326	0.0000	−0.9067	−1.9649	0.0000	1.3913	0.2485	0.0000
V*	0	0.6451	−0.4322	0.0000	−0.7302	0.7873	0.0000	1.5072	−0.2161	−0.7264
	1	2.2109	−0.0480	0.0000	1.3744	1.3600	0.0000	1.4089	0.1440	0.0000
γ_1	0	1.0000	−0.0924	0.0000	0.0000	1.2395	0.0000	1.0000	−0.4619	0.0000
	1	2.0000	0.0924	0.0000	0.0000	−1.2395	0.0000	2.0000	0.4619	0.0000
γ_2	0	1.0000	0.4619	0.0000	0.0000	0.0000	0.0000	1.0000	−0.4619	0.0000
	1	2.0000	−0.4619	0.0000	0.0000	0.0000	0.0000	2.0000	0.4619	0.0000

*In this case the expansion $W(\phi) = \Sigma_k a_k \cos k\phi$ is assumed.

TABLE III

Legendre polynomial coefficients, $u_k^i(abc\alpha L_1L_1'L_2L_2')$, of equation (18), for various cases and the spin sequence $abc = 220$. $L_2 = L_2' = 2$ for all entries

		L_1L_1'								
		11			12			22		
		k								
Case	α	0	2	4	0	2	4	0	2	4
I	0	1.6419	0.0565	0.0000	0.4686	−2.0398	0.0000	1.2059	−0.9483	0.2131
	1	1.4527	0.0565	0.0000	0.9892	−4.1223	0.0000	1.9876	−0.2613	0.8658
	2	2.4389	−1.1799	0.0000	0.1041	3.0383	0.0000	1.8409	0.7456	−0.1302
II	0	0.8971	−0.4780	1.2737	0.8012	−0.4909	1.1783	1.2059	−0.9483	0.2131
	1	2.1745	0.5282	−1.2958	1.3218	−2.5735	1.1783	1.9876	−0.2613	0.8658
	2	2.4619	−1.1172	0.0221	−0.5610	−0.0594	−2.3565	1.8409	0.7456	−0.1302
III	0	2.2012	1.0168	0.0000	0.0000	−2.9770	0.0000	0.4852	0.1242	0.1269
	1	1.1684	−0.4771	0.0000	0.0000	0.1469	0.0000	2.3564	−0.9864	−0.6866
	2	1.6305	0.5273	0.0000	0.0000	5.9540	0.0000	2.1584	0.5355	1.5084
IV	0	1.4619	0.4824	1.2737	−1.3525	−0.4463	−1.1783	0.4852	0.1242	0.1269
	1	2.4619	0.3300	−2.1007	−1.3524	2.6776	−1.1783	2.3564	−0.9864	−0.6866
	2	1.0761	0.2546	0.8270	2.7049	0.8925	2.3565	2.1584	0.5355	1.5084
VI	0	0.6410	0.9855	0.0663	0.5513	0.9372	0.0000	1.2089	−0.6064	−0.1319
	1	1.3636	0.1317	−0.0884	0.0306	−0.1041	0.0000	2.5272	−0.1109	0.1758
	2	2.4619	−1.1172	0.0221	−2.1438	−0.8332	0.0000	1.7830	0.7173	−0.0439
VII	0	0.6815	1.0168	0.0000	−0.4686	−1.1026	0.0000	1.2089	−0.6064	−0.1319
	1	1.3461	0.1631	0.0000	−0.9892	−2.1438	0.0000	2.5272	−0.1109	0.1758
	2	2.4389	−1.1800	0.0000	−0.1041	3.2464	0.0000	1.7830	0.7173	−0.0439
V*	0	0.8896	0.7203	0.0000	0.7967	0.7029	0.0000	1.0827	0.1487	0.5188
	1	1.3445	0.0800	0.0000	1.4996	1.4839	0.0000	2.5535	−0.1028	0.0000
	2	3.0060	0.0000	0.0000	−1.5935	0.1561	0.0000	1.4237	0.0057	0.0000
γ_1	0	1.0000	0.4619	0.0000	0.0000	−1.3525	0.0000	1.0000	−0.1414	−0.3734
	1	2.0000	0.4619	0.0000	0.0000	−1.3524	0.0000	2.0000	−0.1414	0.4978
	2	2.0000	− 0.9239	0.0000	0.0000	2.7049	0.0000	2.0000	0.2828	−0.1245
γ_2	0	1.0000	0.3300	0.8712	0.0000	0.0000	0.0000	1.0000	−0.1414	−0.3734
	1	2.0000	0.3300	−1.1616	0.0000	0.0000	0.0000	2.0000	−0.1414	0.4978
	2	2.0000	−0.6599	0.2904	0.0000	0.0000	0.0000	2.0000	0.2828	−0.1245

*In this case the expansion $W(\phi) = \Sigma_k a_k \cos k\phi$ is assumed.

TABLE IV

Legendre polynomial coefficients, $u_k^i(abc\alpha L_1L_1'L_2L_2')$, of equation (18), for various cases and the spin sequence $abc = 320$. $L_2 = L_2' = 2$ for all entries

		L_1L_1'								
		11			12			22		
		k								
Case	α	0	2	4	0	2	4	0	2	4
I	0	0.3806	0.3070	0.0000	−0.3781	−1.5918	0.0000	0.9285	1.1584	−0.0107
	1	1.5283	−0.9801	0.0000	−0.9081	−0.1471	0.0000	1.6050	0.0123	0.4392
	2	2.3812	−0.0533	0.0000	−0.0974	0.0974	0.0000	1.3776	0.6802	−1.5950
	3	2.4966	1.1533	0.0000	−0.9542	6.3169	0.0000	2.4100	−0.3547	0.8345
II	0	1.1336	0.9731	−1.1198	0.7626	1.1856	0.6613	0.9285	1.1584	−0.0107
	1	2.1150	0.7736	0.7737	1.1244	2.1819	−2.3146	1.6050	0.0123	0.4392
	2	2.0000	−0.3049	0.6810	−0.7403	−0.3619	1.1022	1.3776	0.6802	−1.5950
	3	1.5381	−1.0150	−0.3350	−3.4843	1.6698	0.5511	2.4100	−0.3547	0.8345
III	0	1.0923	0.2110	0.0000	0.0000	−5.2189	0.0000	1.7854	0.5649	1.5288
	1	2.6467	−2.0312	0.0000	0.0000	−1.9835	0.0000	2.4158	−0.2197	−1.2513
	2	3.0164	1.2805	0.0000	0.0000	0.0000	0.0000	0.4754	−0.1524	−0.2367
	3	0.2446	0.1130	0.0000	0.0000	2.5270	0.0000	2.3234	−1.4122	−0.3729
IV	0	0.6304	0.2231	0.4498	−2.0242	−1.4312	−1.7635	1.7854	0.5649	1.5288
	1	1.4457	0.6700	−1.5001	−3.0362	−1.8129	2.8656	2.4158	−0.2197	−1.2513
	2	2.0000	0.6097	1.6872	0.0000	0.0000	0.0000	0.4754	−0.1524	−0.2367
	3	2.9239	−1.9295	−0.6368	5.0604	−1.4312	−1.1022	2.3234	−1.4122	−0.3729
VI	0	1.2360	0.3877	−0.6368	1.2616	0.2457	1.1022	1.2358	−0.5978	1.4382
	1	2.4394	0.9322	0.2908	1.9118	−0.3690	−0.5511	2.3604	0.2228	−0.5268
	2	2.0000	−0.3049	0.6810	0.7403	0.3619	−1.1022	1.4789	0.7298	−1.7459
	3	1.5381	−1.0150	−0.3350	−1.5760	−0.2385	0.5511	2.4100	−0.3547	0.8345

TABLE IV (*Concluded*)

Legendre polynomial coefficients, $u_k^i(abc\alpha L_1L_1'L_2L_2')$, of equation (18), for various cases and the spin sequence $abc = 320$. $L_2 = L_2' = 2$ for all entries

		L_1L_1'								
		11			12			22		
		k								
Case	α	0	2	4	0	2	4	0	2	4
VII	0	0.7648	−0.0772	0.0000	0.3781	−2.3479	0.0000	1.2358	−0.5978	1.4382
	1	1.5709	−1.0228	0.0000	0.9081	−1.9633	0.0000	2.3604	0.2228	−0.5268
	2	2.3811	−0.0533	0.0000	0.0973	−0.0973	0.0000	1.4789	0.7298	−1.7459
	3	2.4966	1.5133	0.0000	0.9542	4.4085	0.0000	2.4100	−0.3547	0.8345
V*	0	0.5229	−0.2881	0.0000	1.0207	−0.5671	0.0000	1.3870	−0.8644	−0.1816
	1	2.0534	−0.0320	0.0000	0.5729	−1.3621	0.0000	1.9097	−0.1440	0.0000
	2	2.3926	0.0000	0.0000	0.0000	−0.1460	0.0000	0.4711	−0.0100	0.0000
	3	1.9352	0.0000	0.0000	−2.6456	−1.4312	0.0000	2.9112	0.0000	0.0000
γ_1	0	1.0000	−0.3696	0.0000	0.0000	−2.0242	0.0000	1.0000	0.1320	0.6534
	1	2.0000	−0.5543	0.0000	0.0000	−3.0362	0.0000	2.0000	0.1980	0.2178
	2	2.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2.0000	0.0000	−1.5246
	3	2.0000	0.9239	0.0000	0.0000	5.0604	0.0000	2.0000	−0.3300	0.6534
γ_2	0	1.0000	0.5279	−0.4356	0.0000	0.0000	0.0000	1.0000	0.1320	0.6534
	1	2.0000	0.7919	−0.1452	0.0000	0.0000	0.0000	2.0000	0.1980	0.2178
	2	2.0000	0.0000	1.0164	0.0000	0.0000	0.0000	2.0000	0.0000	−1.5246
	3	2.0000	−1.3198	−0.4356	0.0000	0.0000	0.0000	2.0000	−0.3300	0.6534

*In this case the expansion $W(\phi) = \Sigma_k a_k \cos k\phi$ is assumed.

the coefficients are identical. Under these conditions the correlations represented by these three pairs will be identical independently of the populations, $P(\alpha)$. This is an instance of a more general symmetry present if $L_1 = L_1' = L_2 = L_2' = b$ and $c = 0$. From equation (9) it is then readily found that

$$W(\theta_1,\theta_2,\phi) = W(\theta_2,\theta_1,\phi). \tag{19}$$

The nuclear parameters $P(\alpha)$, δ_1, and δ_2 are related to the Legendre polynomial coefficients, a_k^i, by equations (17) or (18). The problem is to obtain the best fit to the measured coefficients using these equations and every a priori admissible spin sequence, $a \rightarrow b \rightarrow c$. The nuclear parameters occur non-linearly and are rather more numerous than can be conveniently handled by hand computation. In some instances, prior knowledge of the system fixes some of the parameters, at least approximately. It may be possible to select certain measurements from which the remaining parameters can be found. In this way a number of data equal to the number of variables may be fitted more or less exactly. If the spin assignments are correct, then the remainder of the data also will probably be fitted reasonably well by these parameters. They will then constitute one solution to the problem which the limited experience with the method to date indicates will be probably unique. The considerable amount of numerical calculations associated with the analysis makes an electronic computer an almost necessary adjunct to the work. While solutions can be found by trial and error methods aided by physical insight as remarked above, there is no convincing assurance that less obvious ones, perhaps equally good or better, do not exist. The enormous capacity of an electronic computer can reduce this uncertainty to a negligible point.

If an electronic computer is available then it is feasible to analyze the measured data by the method of least squares. If a number of geometries described above have been measured, these can be analyzed first in terms of Legendre polynomials and then the resulting coefficients, a_k^i, treated as data for a further least squares fit in terms of the population parameters and multipole mixtures.* This procedure has the advantage of reducing the amount of data to be handled. It should be recognized that statistical correlations are normally present between the coefficients found from angular distribution analysis which are usually small enough to be ignored, but in some instances,† could be significant. The fits can be made using as data either the coefficients, a_k^i, together with adjustable normalization constants, N_i, introduced into equation (17), or the ratios a_k^i/a_0^i. The former approach leads to a slightly simpler program and was used in the analysis of Batchelor *et al.* (1960). It entails evidently more unknowns, but this is not generally of consequence to a computer.

*Broude and Gove (1960 and to be published) have studied a considerable number of correlations by this method, and have found that the convergence of the non-linear least squares fitting procedure is often unsatisfactory due to the presence of secondary minima in the fit and singular points for the normal equations. They have adopted a procedure in which different geometries were normalized so as to agree approximately at the common points. Linear least squares fits were then made having the $P(\alpha)$ as fitted variables for a series of values for the multipole mixtures. The best of these were then used as a first approximation for further refinement by the general program.

†For example, if the correlation were measured only over a small range of angles.

IV. AN EXAMPLE OF METHOD I, $Mg^{24}(pp'\gamma\gamma)Mg^{24}$

There have been several examples of the experimental study of gamma–gamma correlations from nuclei aligned by radiative capture reactions (Hoogenboom 1958; Litherland *et al.* 1959; and Broude *et al.* 1959 for example). However, since most of these experiments involved isolated resonances in the compound nucleus, the populations of the substates of the compound nucleus were not treated as unknowns to be determined by experiment. In other cases it is not possible to be sure that an isolated resonance is being studied and, consequently, there is considerable uncertainty in the interpretation of the results. For an example, consider the $Mg^{24}(pp'\gamma\gamma)Mg^{24}$ reaction at bombarding energies of greater than 5 Mev. Seward (1959) has shown that there are a number of broad overlapping resonances for the excitation of the 1.37-Mev state in Mg^{24} in this region, which makes it unlikely that a particular point on the yield curve can be used for an analysis of the gamma-ray angular correlations in terms of a single resonance in the compound nucleus. It is in this type of situation that analysis of the gamma-ray angular correlations in terms of an incoherent sum over the angular correlations from the magnetic substates of an isolated state in the residual nucleus becomes most useful. The energy stability of the particles initiating the reaction must, however, be adequate to ensure that the population parameters remain constant during an experiment because the population parameters can vary as rapidly as the yield of the reaction. The population parameters determined by experiment can then in principle be used to shed further light on the formation of the state in the residual nucleus formed by the reaction and in this respect the analysis represents a convenient two-step method of analysis of the results. The analysis in terms of population parameters also can be used when a single resonance in the compound nucleus is apparently being studied because there is then no lingering uncertainty due to the possibility that the resonance is not completely isolated.

Situations where the initial state, a, is unpolarized, or very nearly so, are unfavorable for the application of this method. This is because the correlation then reduces to a double one described by equation (1) and provides only a limited amount of information rather than the considerable amount we have been assuming on the basis of equation (2). The degree of alignment depends on the details of the nuclear reaction and a consideration of these enables some general conclusions concerning favorable and unfavorable situations to be drawn.

If the state a is unpolarized, then only the tensor parameter $\rho_{00}(aa)$ is non-zero. In the opposite situation, in which the populations of the magnetic substates differ strongly (but, of course, remain symmetric between positive and negative values), the higher terms $\rho_{20}(aa)$, $\rho_{40}(aa)$ ... are non-zero. The cases in which these have magnitudes comparable to $\rho_{00}(aa)$ we can regard as particularly favorable ones. Selection rules which limit the order, k, of the tensor $\rho_{k0}(aa)$ are conveniently found from the Wigner $9-j$ coefficients

$$\begin{Bmatrix} x & \mathscr{L}_1 & y \\ x & \mathscr{L}_1' & y' \\ 0 & k & k \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} y & \mathscr{L}_2 & a \\ y' & \mathscr{L}_2' & a \\ k & 0 & k \end{Bmatrix}$$

which are contained as factors in the formal expression for the tensors in terms of the parameters of the process $x \xrightarrow{\mathscr{L}_1} y \xrightarrow{\mathscr{L}_2} a$. The "triangle conditions" state that the three quantum numbers occurring in any row or column of a non-zero coefficient must be capable of forming a closed triangle. The zero in the first coefficient is due to the initial state, x, being unpolarized, and in the second, is due to the fact that the radiation $\mathscr{L}_2$ is unobserved. It is evident that k is limited by the triangles $(\mathscr{L}_1\mathscr{L}_1'k)$, $(yy'k)$, and (aak). Remembering that k is even, we see from the first two that only the $k = 0$ term will be present if either $\mathscr{L}_1$ and $\mathscr{L}_1'$, or y and y' are 0 or $\frac{1}{2}$. The third condition provides nothing new, being implied in the definition of $\rho_{k\kappa}(aa)$. No limitation is imposed on k by $\mathscr{L}_2$ and $\mathscr{L}_2'$. Clearly high angular momentum of the incoming particle and high spin of the compound state are required for favorable cases.

In general, reactions which can be excited close to their thresholds will be favorable ones. In these the outgoing angular momentum, $\mathscr{L}_2$, will have small values because of the small outgoing energy, causing most of the angular momentum change between the states x and a to reside in $\mathscr{L}_1$. Thus, in conformity with the above requirement, $\mathscr{L}_1$ will be as large as is compatible with the various spins. Another and basically equivalent argument is to consider the magnetic substates of the state a. Since the incoming particle travels along the z-axis, the z component of its orbital momentum is zero. On the other hand, the outgoing orbital momentum, being unobserved, is analogous to an unpolarized spin and has all magnetic substates equally populated. The populated substates of the state a are derived from those of x, $\mathscr{L}_1$, and $\mathscr{L}_2$ and will be quite limited in the favorable cases considered now. This type of argument is the basis of method II and is discussed in detail in Section V. Consider, as an example, the reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$; if the outgoing protons are s-waves, then only the $\alpha = 0$ and the $\alpha = \pm 1$ substates are populated, the latter arising from the flip of the proton spin. Thus for $a \geqslant 2$ the nuclear states are strongly aligned. The yield of gamma radiation from the excited state of interest may be low just above the threshold. Higher bombarding energies will then be necessary where the alignment of the nuclear state may be reduced if higher outgoing angular momenta become strong. A compromise may then be involved between the degree of alignment and the yield of the reaction.

At a resonance in the compound nucleus it may be advantageous to describe the alignment in terms of the statistical tensors rather than the population parameters. For example, if a 4+ state in Mg^{24} is excited by the inelastic scattering of protons and the major contribution to the cross section comes from a 5/2+ resonance in the compound nucleus Al^{25}, then all the $2j+1$ substates of the 4+ state are populated and one is faced with the possibility of fitting an angular correlation with four population ratios. However, as mentioned in Section II, the alternative description of the alignment of the 4+ state in terms of statistical tensors is simpler since the major contribution to the angular correlation comes from the terms containing the statistical tensors ρ_{00}, ρ_{20}, ρ_{40}. The other possible tensors ρ_{60} and ρ_{80} are small or zero. If the resonance in Al^{25} is a pure $5/2+$ resonance, this implies that ρ_{60} and ρ_{80} are identically zero because the tensor ρ_{k0} is proportional to $Z(2\frac{5}{2}2\frac{5}{2}, \frac{1}{2}k)$ (cf. equation (14) of Section II) which arises from the $9-j$ symbol

$$\begin{Bmatrix} x & \mathscr{L}_1 & y \\ x & \mathscr{L}_1' & y' \\ 0 & k & k \end{Bmatrix} = \begin{Bmatrix} 1/2 & 2 & 5/2 \\ 1/2 & 2 & 5/2 \\ 0 & k & k \end{Bmatrix}.$$

Consequently, the populations $P(\alpha)$ of the $4+$ state in Mg^{24} are not independent but are related by equations that can be derived from equation (3) of Section II. In general, some knowledge of the mode of formation of the axially symmetrical state in the residual nucleus is of great value since then one has a first approximation to the otherwise unknown ratios of the populations or alignment tensors. Modifications to these ratios can then be made to take into account the perturbing effects of other resonances in the compound nucleus.

The reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$ has been studied by Batchelor *et al.* (1960), who exploited the ideas outlined above. The 4.24-Mev state in Mg^{24} was studied in the range of incident proton energies from 5.0 to 6.0 Mev. The two resonances shown in Fig. 3 were found and the angular distribution of

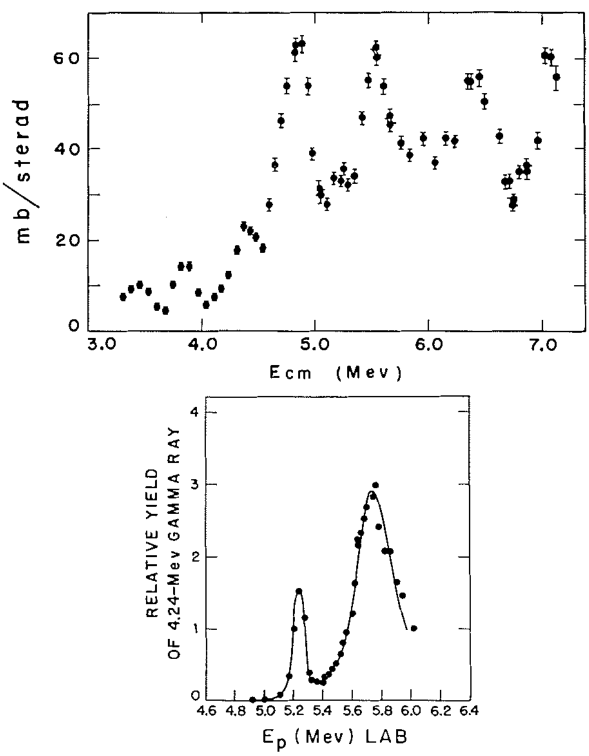

FIG. 3. Yield curve for the reactions $Mg^{24}(pp')Mg^{24*}$. The upper figure is taken from Seward (1959) and shows the yield of protons to the first excited state as a function of the center of mass energy. The lower one is taken from Batchelor *et al.* (1960) and shows the yield of 4.24-Mev gamma rays.

the gamma rays de-exciting the 4.24-Mev state in Mg^{24} are shown in Fig. 4. Since the angular distribution of the 4.24-Mev gamma ray at the 5.72-Mev resonance was stronger than that at the 5.24-Mev resonance, the 5.72-Mev resonance was chosen for study. The stronger angular distribution implies a

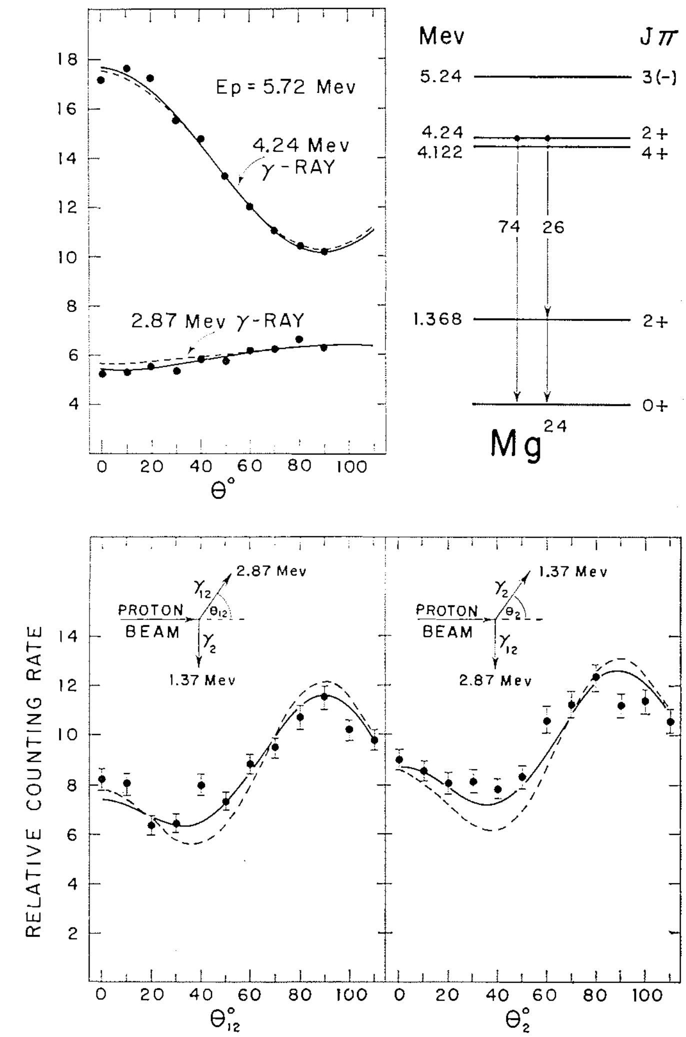

FIG. 4. Angular correlations for the 4.24 Mev–1.37 Mev ground-state cascade in the reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$. The correlations shown are: the angular distributions of the 4.24-Mev and 2.87-Mev components and the correlations for cases I and II of Fig. 2. The solid lines are theoretical fits for the spin assignments 2, 2, and 0 for the 4.24 Mev, 1.37 Mev, and ground states. The dotted lines show the fits obtained if the corrections for finite counter aperture are omitted.

greater degree of alignment of the 4.24-Mev state at the 5.72-Mev resonance and the measured a_2/a_0 coefficient of approximately 0.42 implies that the 5.72-Mev resonance probably has angular momentum 3/2+ and that the unobserved inelastic proton is predominantly *s*-wave. An isolated 3/2+ resonance in the reaction decaying to a 2+ state in Mg^{24} by the emission of an *s*-wave inelastic proton has a correlation coefficient $a_2/a_0 = +0.50$. The measured deviation from this value could be due to a *d*-wave admixture to the predominantly *s*-wave inelastic proton or it could be due to the 5.72-Mev resonance not being a single isolated resonance. However, as discussed above, the nature of the deviation is immaterial since the analysis of the angular correlation data is sufficiently general to take both possibilities into account.

The angular correlations taken at 5.72-Mev incident proton energy are shown in Fig. 4. Two geometrical arrangements of counters were used which correspond to the cases I and II of Fig. 2. Together with the angular distributions also shown in Fig. 4 these data were more than sufficient to determine the alignment parameters $T(s)$, discussed in Section III, and the multipole mixture of the 2.87-Mev radiation. The population parameters or the alignment parameters can be deduced from the $T(s)$ parameters with the help of equations (14) and (15) of Section II. These results are summarized in Table V and have been discussed by Batchelor *et al.* (1960). Table III gives

TABLE V

Comparison of measured and calculated angular distribution and angular correlation coefficients for the reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$ through the excited states at 4.24 Mev and 1.37 Mev. The spins of the 4.24 Mev, 1.37 Mev, and ground states are 2, 2, and 0 respectively. The incident proton energy was 5.72 Mev. γ_0 designates the angular distribution of the gamma-ray transition from the 4.24-Mev level to the ground state. In the lower part of the table are given the $E2/M1$ amplitude mixing ratio, δ_1, for the primary gamma ray and the ratios $P(1)/P(0)$ and $P(2)/P(0)$ of the population parameters of the 4.24-Mev state

Case	Parameter	Experiment	Theory
I	a_2/a_0	$-.373\pm.035$	$-.442$
	a_4/a_0	$.278\pm.047$	$.292$
II	a_2/a_0	$-.285\pm.07$	$-.401$
	a_4/a_0	$.169\pm.09$	$.318$
γ_0	a_2/a_0	$.398\pm.014$	$.393$
	a_4/a_0	$-.013\pm.017$	$-.002$
γ_1	a_2/a_0	$-.147\pm.015$	$-.119$
	a_4/a_0	$.010\pm.010$	0
	δ_1		23
	$P(1)/P(0)$		.77
	$P(2)/P(0)$		.09

the theoretical correlations for the sequence of spins $2 \rightarrow 2 \rightarrow 0$ and for attenuation parameters Q_k appropriate for 5-in. diameter by 6-in. long NaI(Tl) crystals situated with their front faces 6.2 inches from the target.

Since the two angular correlations shown in Fig. 4 are nearly identical, the considerations outlined in Section III imply that the 2.87-Mev radiation is predominantly quadrupole radiation. This is a consequence of equation (19) of the previous section. Similar results have recently been obtained by Broude and Gove (1960) for a number of *d*-shell nuclei, for example, the 5.22-Mev state of Mg^{24}. The solid curves are a least squares fit to the data using the parameters shown in Table V. The dotted curves are a similar fit except for the assumption that there is no attenuation of the correlations due to the finite size of the detectors. The fit shown in Fig. 4 was obtained using the Chalk River Datatron computer.[1] The parameters shown in Table V support the original conjecture that the 5.72-Mev resonance is predominantly a $3/2+$ resonance decaying by the emission of predominantly *s*-wave protons to a $2+$ state in Mg^{24}.

In the above discussion, aligned nuclear states were produced by reactions such as the $Mg^{24}(pp')Mg^{24*}$ reaction. The Mg^{24*} nuclei can be treated as aligned if the inelastic protons are not observed. Another very convenient method for producing aligned nuclear states is by Coulomb excitation. In this case the alignment parameters or populations of the substates can be calculated theoretically (Alder *et al.* 1956). The angular correlations of the cascade gamma rays from a state excited by Coulomb excitation are therefore determined by the spins and multipole mixtures of the final nuclear states. Alternatively the calculated alignment parameters can be regarded as a first approximation to the actual values, which are then determined more accurately by experiment. This is possible in principle because of the large number of experimentally determinable coefficients from a gamma-ray angular correlation measurement. From the experimental point of view angular correlations following Coulomb excitation of medium-weight and heavy nuclei should be as useful as the angular correlations following inelastic proton scattering in light nuclei.

V. METHOD II

If a nuclear state is formed by the absorption of an unpolarized particle in the direction of the quantization axis followed by the emission of a second particle which is detected along the axis, then the magnetic substates which can be populated do not exceed the sum of the spins of the target nucleus and the incident and emergent particles. This theorem arises basically from the fact that the orbital momenta contained in plane waves in the direction of the quantization axis have only zero projections on this axis. The system clearly has axial symmetry so that as discussed in Section II, the density matrix is diagonal in the magnetic quantum numbers, and further, is symmetric between positive and negative values. The state, which is assumed to have definite spin and parity, will then be described by a number of population parameters which will be limited by the spins of the target and the incident and emergent particles. The angular distribution of a gamma ray emitted from the state will be governed by these parameters and its multipole mixing ratio. The distribution will be of the form of equation (1) and the information obtainable will, in general, be the ratios a_2/a_0 and a_4/a_0. Thus, if no more than two arbitrary parameters describe the distribution, then the amount of information will be just sufficient or, in favorable cases, more than sufficient to determine them.

The determination of nuclear parameters based on these considerations we

[1]The computer program was written by one of the authors (A.J.F.).

term method II. It has evidently smaller applicability than method I but can be expected in suitable cases to yield multipole mixtures and spins. It is possible, in principle, to combine methods I and II by observing two gamma rays in coincidence with a particle detected by an axially symmetrical counter. This arrangement has the advantage of simplifying the gamma-ray spectrum because the energy selectivity of the particle counter can be used to ensure in many cases that only one state in the residual nucleus is studied at once. This point is discussed in more detail at the end of this section.

The mathematical formulation of the method is most simply carried out in terms of the density matrix. We shall assume that the state considered is the final one in the absorption and emission process described by

$$al_1s_1 \rightarrow b \rightarrow cl_2s_2.$$

Here, a, b, and c are the spins of the initial, intermediate, and final states and l_1, l_2, s_1, and s_2, are the orbital momenta and spins of the incident and emergent particles. We will use the channel spin coupling scheme:

$$\begin{aligned}\mathbf{S}_1 &= \mathbf{a}+\mathbf{s}_1,\\ \mathbf{b} &= \mathbf{S}_1+\mathbf{l}_1 = \mathbf{S}_2+\mathbf{l}_2,\\ \mathbf{S}_2 &= \mathbf{c}+\mathbf{s}_2.\end{aligned}$$

S_1 and S_2 are the incoming and outgoing channel spins. The density matrix for the intermediate state will be

$$\begin{aligned}\langle b\beta|\rho|b'\beta\rangle = \sum_{\alpha\sigma_1} & (a\alpha,s_1\sigma_1|S_1\beta)^2(S_1\beta,l_10|b\beta)\\ & \times(S_1\beta,\, l_1'0|b'\beta)\,\langle a\alpha|\rho|a\alpha\rangle\,\langle s_1\sigma_1|\rho|s_1\sigma_1\rangle\,\langle l_10|\rho|l_1'0\rangle\\ & \times\langle S_1l_1||V||b\rangle^*\,\langle S_1l_1'||V||b'\rangle.\end{aligned} \tag{20}$$

The states a and s_1 are assumed unpolarized so that

$$\langle a\alpha|\rho|a\alpha\rangle = \frac{1}{\hat{a}^2},$$

$$\langle s_1\sigma_1|\rho|s_1\sigma_1\rangle = \frac{1}{\hat{s}_1^2}.$$

For a plane wave

$$\langle l_10|\rho|l_1'0\rangle = \hat{l}_1\hat{l}_1'.$$

Interference between different channel spin states is removed by the orthogonality of the Clebsch–Gordan coefficients. The factors $\langle S_1l_1||V||b\rangle$ and $\langle S_1l_1'||V||b'\rangle$ are the reduced matrix elements of the perturbing Hamiltonian for absorption causing the transition.

The feature of significance in equation (20) for the present considerations is that for each term of the sum, $\beta = \alpha+\sigma_1$. Thus, the largest β results from the largest α and σ_1, which cannot, respectively, exceed a and s_1.

The density matrix for the final state can be derived from that of the intermediate state. It is

$$(21)\quad \langle c\gamma s_2\sigma_2 l_2 0|\rho|c\gamma s_2\sigma_2 l_2' 0\rangle = \langle c\gamma|\rho|c\gamma\rangle\,\langle s_2\sigma_2|\rho|s_2\sigma_2\rangle\,\langle l_2 0|\rho|l_2' 0\rangle$$
$$= \sum (S_2\beta, l_2 0|b\beta)\,(S_2'\beta, l_2' 0|b\beta)\,(c\gamma, s_2\sigma_2|S_2\beta)$$
$$\times (c\gamma, s_2\sigma_2|S_2'\beta)\,\langle b\beta|\rho|b'\beta\rangle\,\langle S_2 l_2||V||b\rangle\,\langle S_2' l_2'||V||b'\rangle^*.$$

The summation here is over $b\, b' S_2$ and S_2'. The first line of this equation expresses the fact that the density matrix for the final state is the direct product of the density matrices of the component spins and orbital momenta. The factors $\langle S_2 l_2||V||b\rangle$, $\langle S_2' l_2'||V||b'\rangle$ are the reduced matrix elements for the emission of the emergent particles. The magnetic quantum numbers of the outgoing orbital momentum, λ_2 and λ_2', different from zero are omitted from equation (21) because we are considering only particles travelling in the direction of the quantization axis. For counters of finite size, which are considered later, this assumption is not justified.

From equation (21) it is evident that $\beta = \gamma+\sigma_2$. Polarization of the outgoing particle is not detected so that all values of σ_2 will occur. Finally, the maximum value of γ that will occur will be

$$(22)\qquad \gamma_{\max} = \max(\alpha+\sigma_1+\sigma_2) = a+s_1+s_2.$$

This conclusion is independent of the presence of interfering spins b or orbitals l_1', l_2' or indeed interfering final spins c'. The density matrix is also clearly diagonal in γ, as was inferred from the more general considerations of axial symmetry.

The angular distribution of a gamma ray in a transition from a state whose population parameters are $P(\gamma) = \langle c\gamma|\rho|c\gamma\rangle$ to a state of spin d is given by

$$(23)\qquad W(\theta) = \sum_{\gamma kp} (-)^{f_6}\, P(\gamma)\delta^p(c\gamma, c-\gamma|k0)\, Z_1(LcL'c, dk)\, Q_k P_k(\cos\theta)$$

where L, L' are interfering multipolarities for the gamma ray, and δ is the multipole-mixing parameter as defined in equation (6). $f_6 = d+\gamma+L+L'+k/2$. $Z_1(LcL'c, dk)$ is the coefficient tabulated by Sharp *et al.* (1953). A factor 2 must be introduced into this formula for $L \neq L'$ corresponding to the 2^n of equation (9). As before $p = 0$, 1, or 2 for $(LL') = (11)$, (12), or (22) respectively. The quantities Q_k have been discussed in Section II. Equation (23) can be derived as a special case of equation (11).

The restriction $a+s_1+s_2 \leqslant 3/2$ or $\leqslant 1$ will lead to two population parameters $P(\gamma)$ corresponding to one unknown ratio. If, in this case, there is an additional unknown multipole mixture, and if the experiment yields the two observed quantities a_2/a_0 and a_4/a_0, then sufficient information is present to determine the unknowns. The above restriction is the one that is probably most generally applicable. If a multipole mixture is known to be absent, then the restriction can be relaxed. On the other hand, only the ratio a_2/a_0 may be obtainable from the experiment, in which case a closer restriction is needed. Clearly small spins of all the components of the reaction are required for the application of this method. In Section VI the application of these considerations to the reaction $Mg^{26}(\alpha n\gamma)Si^{29}$ is described.

The detector of the emergent particles can, in principle, detect either the forward-emitted or the backward-emitted particles. Only the forward position allows the counter to be precisely on the beam axis, but in some circumstances, it might be convenient to keep this direction unobstructed and place the counter in the backward position. While it might be expected that if the counter is in the backward position and as close to the incident beam as possible, then the requirements of the method will be approximately satisfied, the axial symmetry of the system is lost, and the estimation of corrections is considerably complicated. The use of a counter symmetric about the beam axis, i.e. a circular counter with a hole in it, through which the beam passes restores the axial symmetry of the system and allows formulae for counter size corrections to be developed.

For a cluster of identical counters located symmetrically about the beam axis at angular intervals $2\pi/N$, the efficiency matrix $\epsilon_{k\kappa}(cc)$ will be zero unless $\kappa = 0, N, 2N, \ldots$. For the typical cases in the use of method II where k does not exceed 4, the requirements of the method are realized by choosing $N = 5$. In other words, with a cluster of five counters arranged symmetrically about the beam axis to detect the reaction particles, equation (23) describes the angular distribution of a gamma ray emitted by the resulting state provided terms with $k > 4$ are absent. A single counter may be used in a similar way by measuring the correlation at five equally spaced azimuthal angles and averaging the correlations over these positions. The five measurements may be reduced to three using the symmetry valid for method II

$$W(\phi) = W(-\phi)$$

and starting the measurements at $\phi = 0$. A form of the multiple counter scheme which might have value in the combination of methods I and II mentioned early in this section is to set the polar angle of each counter relative to the beam axis equal to $\pi/2$ radians. $\epsilon_{k\kappa}(cc)$ will then vanish unless $\kappa = 0$, $2N, 4N, \ldots$.

Recalling that the correlation function is

$$W = \mathrm{Tr}(\rho\epsilon)$$

the effective density matrix involved in transitions from the state c in coincidence with the radiation l_2 will be

$$P(\gamma) = \sum_{l_2 l_2'\lambda_2\lambda_2'} \langle c\gamma|\rho|c\gamma\rangle \langle s_2\sigma_2|\rho|s_2\sigma_2\rangle \langle l_2\lambda_2|\rho|l_2'\lambda_2'\rangle\langle l_2'\lambda_2'|\epsilon|l_2\lambda_2\rangle. \tag{24}$$

For an ideal counter located on the beam axis

$$\langle l_2\lambda_2|\epsilon|l_2'\lambda_2'\rangle = \hat{l}_2\hat{l}_2'\delta_{\lambda_2 0}\delta_{\lambda_2' 0}.$$

The formula for a finite counter is readily found from the behavior of the efficiency matrix under co-ordinate axis rotations. The efficiency matrix for an ideal counter in an arbitrary position is

$$\langle l\lambda|\epsilon|l'\lambda'\rangle = D^{l}_{\lambda 0}(\mathscr{R})\,\langle l0|\epsilon|l'0\rangle D^{l'*}_{\lambda' 0}(\mathscr{R}) \tag{25}$$

where $\langle l0|\epsilon|l'0\rangle$ is the efficiency matrix element for an ideal counter on the

z-axis and $\mathscr{R}$ is the rotation bringing the z-axis into the arbitrary direction. The efficiency matrix for a finite counter is obtained by interpreting $\langle l\lambda|\epsilon|l'\lambda'\rangle$ as the efficiency for an element of solid angle in the direction defined by $\mathscr{R}$, and integrating this over all directions. Thus putting

$$\mathscr{R} = (\phi,\theta,0),$$

$$D^l_{\lambda 0}(\mathscr{R}) = \frac{\sqrt{4\pi}}{\hat{l}} Y_l^{\lambda *}(\theta,\phi).$$

$\langle l0|\epsilon|l'0\rangle$ will be a function of the angles θ and ϕ in a finite counter, so that we may write

$$\langle l0|\epsilon|l'0\rangle = \hat{l}\hat{l}'\epsilon(\theta,\phi).$$

Then

$$\langle l\lambda|\epsilon|l'\lambda'\rangle = 4\pi\int \epsilon(\theta,\phi)\, Y_l^{\lambda *}(\theta,\phi)\, Y_{l'}^{\lambda'}(\theta,\phi) \sin\theta\, d\theta\, d\phi.$$

If the counter is axially symmetric, $\epsilon(\theta,\phi)$ is independent of ϕ and the integration over azimuth introduces the factor $\delta_{\lambda\lambda'}$. The spherical harmonics reduce to associated Legendre polynomials and we get

$$(26)\quad \langle l\lambda|\epsilon|l'\lambda\rangle = \hat{l}\hat{l}'\left[\frac{(l-|\lambda|)!\,(l'-|\lambda|)!}{(l+|\lambda|)!\,(l'+|\lambda|)!}\right]^{1/2}\int \epsilon(\theta)P_l^{|\lambda|}(\theta)P_{l'}^{|\lambda|}(\theta)\sin\theta\, d\theta.$$

By writing $P_l^{|\lambda|}(\theta)P_{l'}^{|\lambda|}(\theta)$ as a sum over k of the function $P_k^{|\lambda|}(\theta)$, $\langle l\lambda|\epsilon|l'\lambda\rangle$ can be expressed in terms of the Q_k defined in Section II. For the present discussion we will use a more compact expression obtained by taking the leading terms of a Taylor expansion in terms of the half-angle, ξ, subtended by the counter. Using the approximation that for θ small

$$P_l(\cos\theta) \simeq \frac{(l+\lambda)!}{2^\lambda\lambda!(l-\lambda)!}(1-\cos^2\theta)^{\lambda/2}\left[1-\frac{(l-\lambda)(l+\lambda+1)}{2(\lambda+1)}(1-\cos\theta)\right]$$

and assuming that $\epsilon(\theta) = \text{const} = \epsilon$ for $0 \leqslant \theta \leqslant \xi$, and zero otherwise, we obtain

$$(27)\quad \langle l\lambda|\epsilon|l'\lambda\rangle \simeq \frac{\epsilon\xi^{2|\lambda|+2}\,\hat{l}\hat{l}'}{2^{2|\lambda|+1}(|\lambda|!)^2(|\lambda|+1)}\left[\frac{(l+|\lambda|)!(l'+|\lambda|)!}{(l-|\lambda|)!(l'-|\lambda|)!}\right]^{1/2}$$
$$\times\, [1-\xi^2\{(l-|\lambda|)(l+|\lambda|+1)+(l'-|\lambda|)(l'+|\lambda|+1)\}/4(|\lambda|+2)].$$

We can now obtain the effective density matrix for the state c by substituting equations (20), (21), and (27) in the expression (24), with the extension that we must include in equation (21) terms with $\lambda_2 \neq 0$. The effect of this will be to cause substates beyond $\gamma_{\max}$ to be populated; $\gamma_{\max}+1$ with a population proportional to ξ^4, $\gamma_{\max}+2$ with a population proportional to ξ^6, etc. In making the substitution noted above, it is convenient to introduce the coefficient defined by

$$(28)\quad A(\gamma l_2 l_2' \lambda_2) =$$
$$\sum \frac{\hat{l}_1 \hat{l}_1'}{\hat{a}^2 \hat{s}_1^2 \hat{s}_2^2} \frac{\hat{l}_2 \hat{l}_2'}{2^{2|\lambda_2|+1}(|\lambda_2|!)^2(|\lambda_2|+1)} \left[\frac{(l_2+|\lambda_2|)!(l_2'+|\lambda_2|)!}{(l_2-|\lambda_2|)!(l_2'-|\lambda_2|)!}\right]^{1/2}$$
$$\times (S_1\beta, l_1 0|b\beta)\ (S_1\beta, l_1' 0|b'\beta)$$
$$\times (S_2\Sigma_2, l_2\lambda_2|b\beta)\ (S_2'\Sigma_2, l_2'\lambda_2|b'\beta)\ (c\gamma, s_2\sigma_2|S_2\Sigma_2)(c\gamma, s_2\sigma_2|S_2'\Sigma_2)$$
$$\times \langle S_1 l_1||V||b\rangle^* \langle S_1 l_1'||V||b'\rangle \langle S_2 l_2||V||b\rangle \langle S_2' l_2'||V||b'\rangle^*$$

the summation being over $\beta, \Sigma_2, \sigma_2, b, b', l_1, l_1', S_1, S_2, S_2'$. Then to the order ξ^4 we can write for the population parameters

$$(29)\quad P(\gamma) = \xi^2 \epsilon \sum_{l_2\, l_2'} [A(\gamma l_2 l_2' 0) + \xi^2 \{A(\gamma l_2 l_2' 1)$$
$$+ A(\gamma l_2 l_2' -1) - (l_2(l_2+1) + l_2'(l_2'+1))\, A(\gamma l_2 l_2' 0)/8\}].$$

The factor ξ^2 outside the summation on the right-hand side of equation (29) reflects the fact that the response of the system is proportional to the approximate area of the counter $\pi\xi^2$. The first term within the square bracket, $A(\gamma l_2 l_2' 0)$, is that obtained with a particle counter of negligible size. The second term containing the factor ξ^2 is the leading correction term. In the summations of equations (28) and (29), as was noted also in the case of equation (7), interfering terms will be duplicated.

The complexity of the details of the formation of the state c is illustrated by equations (28) and (29), which are quoted here to show the form taken by the population parameters $P(\gamma)$. Cases can be found where b, l_1, l_2, and possibly less frequently S_1 and S_2, are sharp so that the $A(\gamma l_2 l_2' \lambda_2)$ can be written in terms of a small number of reduced matrix elements. However, such cases are amenable to a more general triple correlation analysis in which the emergent particle is observed at angles other than 0° or 180°, and this analysis will yield the large amount of information which is also a feature of method I. The terms of equation (28) with $\lambda_2 = 0$ are those obtained with an ideal counter and are zero for $\gamma > \gamma_{max}$. Those with $\lambda_2 \neq 0$ modify all of the populations, $P(\gamma)$, up to $\gamma = \gamma_{max} + \lambda_2$ in increasing orders of ξ^2. The magnitudes of the correction terms depend on a generally large number of reduced matrix elements and cannot be predicted. However, the coefficients $A(\gamma l_2 l_2' \lambda_2)$ can be expected to be of the same order of magnitude. Thus the corrections arising from λ_2 different from zero will be of the order of magnitude $\xi^{2|\lambda_2|}$. The practical application of these conclusions will be discussed in Section VI.

The combination of methods I and II, mentioned at the beginning of this section, in which two cascade gamma rays are observed in coincidence with an axially symmetrical particle counter, is necessary when the gamma-ray spectrum from a nuclear reaction becomes complicated, such as in the case of deuteron-induced reactions. This simplifies the gamma-ray pulse spectrum in a NaI(Tl) crystal to the point where a study of the individual gamma rays is possible. The theoretical formulae required here are identical with those for method I. The measurement of a triple coincidence between a particle and two gamma rays is normally highly inefficient. However, in deuteron-stripping reactions and some other direct interaction types of reaction, a large portion of the cross section for a particular particle group usually appears in a narrow cone in the forward direction. A counter situated at 0° to the beam and subtending about 30° at the target can consequently have an enhanced efficiency. In favorable cases, such as s-wave and p-wave stripping patterns with low Q, the effective geometrical efficiency can be as high as 50%. Also the alignment tensors ρ_{k0} of the resulting state in the residual nucleus are strongly limited by the stripping mechanism. In the case of an $l = 1$ transition only the tensors ρ_{00} and ρ_{20} are to a first approximation significant even though, for example, the tensors ρ_{40} and ρ_{60} can be allowed by the total angular momentum of the state in the residual nucleus. The $B^{10}(d,n)C^{11*}$ reaction leaving C^{11} in the 6.50-Mev state is a case in point. The total angular momentum of this state is probably 7/2.

VI. APPLICATIONS OF METHOD II

Section V contained a theoretical discussion of the selection of aligned nuclear states, from a nuclear reaction $X(h_1,h_2)Y^*$, by the observation of the particle h_2 in an axially symmetrical counter at 0° or 180° to the beam. It was pointed out in Section V that under these conditions the maximum value of the magnetic substate of the residual nucleus Y^* is limited to the maximum value of the sum of the spins of the particles X, h_1, and h_2. In the case where the maximum value of this sum is 0 or $\frac{1}{2}$ the angular correlation of the gamma rays de-exciting the nucleus Y^* is independent of the details of the formation. Consequently such reactions can be used when the yield of the reaction shows complex resonance structure.

In this section the reaction $Mg^{26}(He^4,n\gamma)Si^{29}$ will be discussed as an example of the use of method II. The $Mg^{26}(He^4,n\gamma)Si^{29}$ reaction was studied by Litherland and McCallum (1960) to provide an illustration of method II and in addition to measure magnetic dipole – electric quadrupole mixing amplitudes for the 1.28-Mev and 2.43-Mev gamma rays from Si^{29}. Equation (23) of Section V gives the full theoretical expression for the interpretation of such measurements.

Many other reactions which are as simple to interpret as the $Mg^{26}(He^4,n\gamma)Si^{29}$ reaction can be used to study the properties of the excited states of even–odd and odd–even nuclei. However, odd–odd nuclei cannot be studied in this way and reactions such as the $O^{16}(He^3,p\gamma)F^{18}$ and $Ne^{20}(d,He^4\gamma)F^{18}$ have to be used. In these cases the residual F^{18} states have their $\alpha = 0, \pm 1$ substates populated. Consequently, unless the details of the formation and decay of the compound state are specified, the observation of the proton or the alpha particle at 0° or 180° to the incident beam introduces an unknown parameter into equation (23). This parameter is the ratio of the populations of the $\alpha = \pm 1$ substates to the $\alpha = 0$ substate and it must be determined by experiment. The measurement of the angular correlation of the gamma rays from an excited state in an odd–odd nucleus is therefore unlikely to give an unambiguous answer to, for example, the $M1$–$E2$ amplitude mixture. If both a_2/a_0 and a_4/a_0 coefficients are observed, this situation is altered since there are then two unquantized parameters to determine and two measured quantities. Further information can be obtained by measuring the linear polarization of the gamma rays in coincidence with the particles and in favorable cases a linear polarization measurement also determines the parity change. Also if more than one gamma ray is emitted in coincidence with the particle then it may be possible to use the extra information obtained to determine the unknown population parameters.

Method II can be reliably used only if the effects of the finite size of the counter, detecting the heavy particle at 0° or 180° to the beam, are corrected for. These effects have been considered in a general way in Section V. It was shown that for a small counter of half-angle ξ situated at 0° or 180° to the beam there were extra terms introduced into equation (23). Again taking the $Mg^{26}(He^4,n\gamma)Si^{29}$ reaction as an example, the effect on the angular correlation of the gamma rays is to multiply the correlation from the $\alpha = \pm\frac{1}{2}$ substates by a factor $1+A\xi^2$ and to introduce a contribution from the $\alpha = \pm 3/2$ substate which was multiplied by a factor $B\xi^2$. The quantities A and B cannot be calculated unless the details of the reaction leading to the state c in the residual nucleus are specified (see equations (28) and (29)). Figure 5 shows the results of some calculations for a particular case which is relevant to the results of the experiment on the $Mg^{26}(He^4,n\gamma)Si^{29}$ discussed by Litherland and McCallum (1960). A 2+ resonance in Si^{30} is formed by the capture of d-wave alpha particles by Mg^{26}. This compound state decays by emitting $s_{1/2}$ or $d_{3/2}$ neutrons to a 5/2+ state in Si^{29} which in turn decays by emitting an electric quadrupole quantum to the 1/2+ ground state of Si^{29}. The neutrons are detected by an axially symmetrical counter of half-angle ξ. The effect of varying the half-angle ξ is shown for both the a_2/a_0 and a_4/a_0 coefficients. The approach to the theoretical values of $+4/7$ and $-4/7$ as ξ is reduced illustrates the necessity for a careful measurement of the variation of the a_2/a_0 and a_4/a_0 coefficients with the angle ξ.

When the angle ξ is small, it is possible to show for the case shown in Fig. 5 that the coefficients A and B defined above are given by equations (30) and (31).

$$A = -\frac{3}{4}\,\frac{2\sqrt{\frac{2}{7}}\,\delta+\frac{1}{7}\,\delta^2}{1+2\sqrt{\frac{2}{7}}\,\delta+\frac{2}{7}\,\delta^2} \tag{30}$$

$$B = \frac{3}{4}\,\frac{\frac{6}{7}\,\delta^2}{1+2\sqrt{\frac{2}{7}}\,\delta+\frac{2}{7}\,\delta^2} \tag{31}$$

These equations illustrate the important role played by the $s_{1/2}$–$d_{3/2}$ interference term δ in the correction for the finite solid angle of the neutron counter.

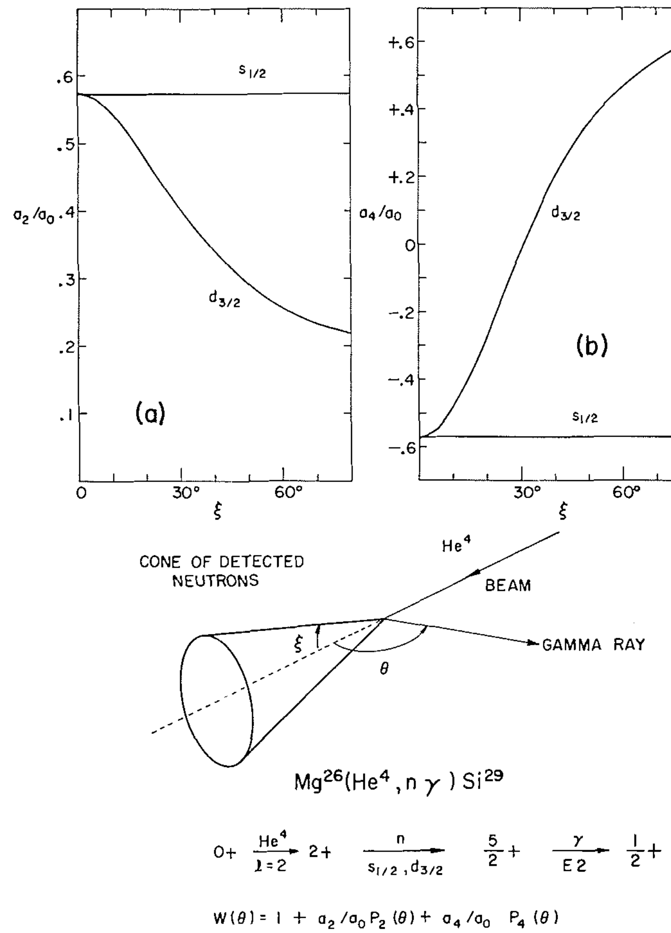

FIG. 5. Corrections to the angular distributions resulting from the finite aperture of the particle counter in the reaction $Mg^{26}(\alpha n\gamma)Si^{29}$. As the size of the counter increases, the corrections become increasingly dependent on the $d_{3/2}$:$s_{1/2}$ mixture of the outgoing neutrons. Graph (*a*) shows the behavior of the a_2/a_0 term, graph (*b*) the behavior of the a_4/a_0 term for both $s_{1/2}$ and $d_{3/2}$ neutrons as functions of the half-angle, ξ, subtended by the counter at the target.

The quantity δ is the amplitude of the $d_{3/2}$ neutron wave divided by the amplitude of the $s_{1/2}$ neutron wave. Equations (30) and (31) can be obtained from equations (28) and (29) with the coupling scheme for outgoing particles changed to the one used above or alternatively from expressions given by Sharp *et al.* (1953).

The correction for the finite solid angle of the gamma-ray counter is easier to apply in the case of method II than in the case of method I. It can be seen from equation (1) that the measured angular correlation coefficients are actually $(Q_2/Q_0)a_2/a_0$ and $(Q_4/Q_0)a_4/a_0$. Since the quantities Q_k can be calculated or obtained from tables (Gove and Rutledge 1958 and Rutledge 1959), the coefficients a_2/a_0 and a_4/a_0 can readily be obtained. Figure 6 shows

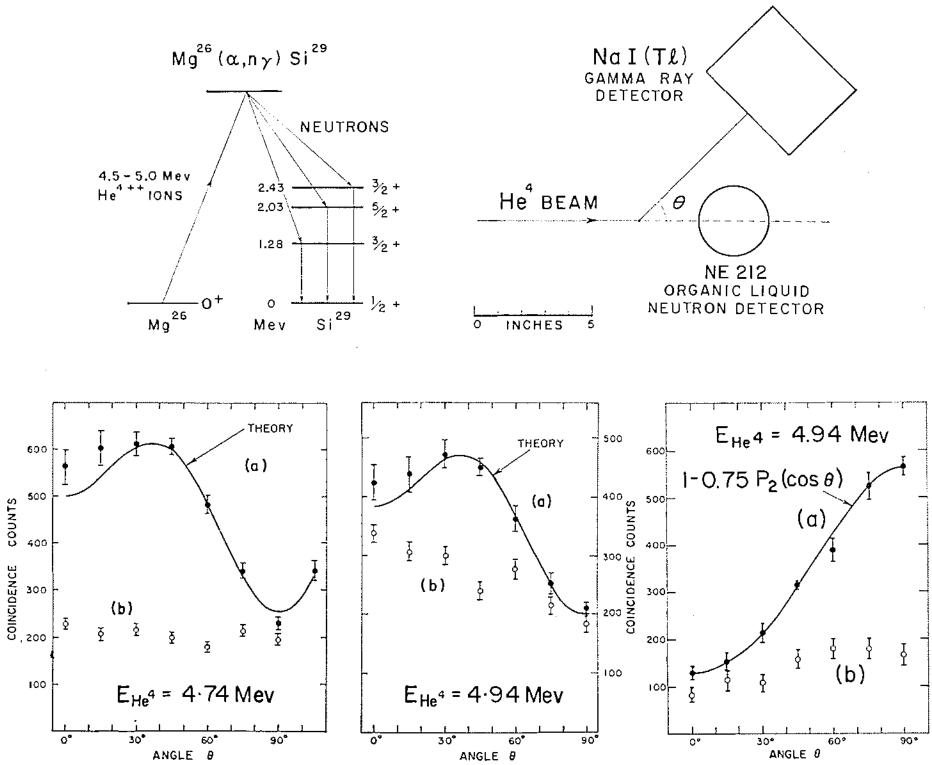

FIG. 6. Angular correlations obtained by method II. The left-hand and center lower graphs are the correlations of the 2.03-Mev gamma ray at two different bombarding energies. The solid line is the theoretical correlation for a $5/2+ \rightarrow 1/2+$ transition. The right-hand graph is the correlation of the 2.43-Mev gamma ray together with the theoretical correlation for a $3/2+ \rightarrow 1/2+$ transition. The open circles marked (*b*) in the lower parts of all three graphs are the correlations obtained with the neutron counter at 90° relative to the beam direction.

some experimental results obtained by Litherland and McCallum from the $Mg^{26}(He^4,n\gamma)Si^{29}$ reaction. The geometrical arrangement of the counters used is also shown. The angular correlation measurements were made with the neutron counter at 0° and at 90°. These two cases are labelled (*a*) and (*b*) in Fig. 6. The theoretical curves shown include only the effect of the finite solid angle of the gamma-ray counter. In Table VI the results of a least

TABLE VI

Measured and theoretical coefficients for the angular correlations shown in Fig. 6 analyzed according to method II. The reaction is $Mg^{26}(\alpha n\gamma)Si^{29}$ and the gamma rays studied are the 2.03-Mev and 2.43-Mev transitions to the ground state. In the case of the 2.43-Mev gamma ray, the $E2/M1$ amplitude ratio, δ, has two values, −0.26 and −1.10, which give agreement with the measured a_2/a_0 coefficient

Bombarding energy (Mev)	Gamma ray (Mev)	a_2/a_0		a_4/a_0	
		Measured	Theory	Measured	Theory
4.74	2.03	$+0.62\pm0.06$	0.53	-0.43 ± 0.06	−0.43
4.94	2.03	$+0.55\pm0.06$	0.53	-0.34 ± 0.06	−0.43
4.94	2.43	-0.75 ± 0.07	−0.75	—	0

squares analysis of the data are compared with theory. As discussed by Litherland and McCallum, the agreement between theory and experiment is good when only the correction for the finite size of the gamma-ray counters is made. The fact that no correction for neutron counter size is needed implies that the neutrons are predominantly *s*-wave, a conclusion which is supported by the measurement in one case of the angular distribution of the 2.03-Mev gamma rays with the neutrons unobserved. However, the measurement of the correlations with the neutron counter at 90° to the beam (marked (*b*) in Fig. 6) shows clearly the effect of the small admixture of *d*-wave neutrons to the predominantly *s*-wave neutrons.

The experiment reported by Litherland and McCallum was of a preliminary nature but serves to illustrate the simplifications possible by the use of method II.

VII. CONCLUSIONS

It is apparent that the method I, typified by the reaction $Mg^{24}(pp'\gamma\gamma)Mg^{24}$, has considerable advantages over method II, typified by the reaction $Mg^{26}(He^4,n\gamma)Si^{29}$. In method I, a measurement of the angular correlation of the gamma rays gives a possible total of 18 measurable ratios if both gamma rays contain a quadrupole component. These are the ratios $a^{\kappa}_{k_1k_2}/a^{0}_{00}$ for k_1, $k_2 = 0$, 2, or 4 and for $\kappa = 0$, 1, 2, 3, and 4. For an even–even nucleus there are at the most a unknown alignment parameters and two unknown multipole mixtures. As discussed in Section IV the number of unknown alignment parameters is usually less than a if a is large. In contrast the angular correlation measurements of method II yield at the most two measurable parameters a_2/a_0 and a_4/a_0 if the gamma radiation is limited to quadrupole radiation. As discussed in the previous subsection, these measured quantities are usually only just sufficient in favorable cases to permit the measurements to be interpreted.

The advantage of method II lies in the comparative simplicity of the gamma-ray pulse spectra obtained from the sodium iodide gamma-ray spectrometer. Since it is possible to use energy analysis on the pulses from the counter detecting the particles at 0° or 180° to the beam the number of gamma rays producing pulse spectra can often be drastically limited. This is in contrast with method I where the only simplification that is possible lies in the specification that gamma rays in coincidence are studied. To take an extreme example, it would be very difficult to study the gamma rays from the reaction $Mg^{26}(He^4,p\gamma)Al^{29}$ by method I but it would be relatively straightforward by method II since the gamma rays from the competing reaction $Mg^{26}(He^4,n\gamma)Si^{29}$ would be eliminated by the requirement that they be in coincidence with protons.

This discussion leads naturally to a synthesis of methods I and II. The weaknesses of method I can be removed by observing the outgoing heavy particles in an energy-sensitive detector in coincidence with the gamma rays. This detector would have to be of large solid angle to obtain an adequate triple coincidence counting rate but, provided it was axially symmetrical with respect to the incident beam, the analysis procedure of method I will still apply. As discussed in Section V, the strongly peaked deuteron-stripping angular distribution of the neutrons or protons contains a large part of the cross section leading to a particular excited state. Consequently a relatively small particle counter on the beam axis can be efficiently used to specify that a certain excited state has been formed.

In conclusion the types of the triple-angular correlation measurements discussed in this paper have been shown to be potentially powerful methods of determining nuclear spectroscopic data from nuclear reactions. The full exploitation of angular correlations of method I probably requires the use of an electronic computer, as in the example quoted in Section V, because the data obtained experimentally have to be analyzed by least squares to obtain several parameters. This analysis has then to be repeated for various values of the spins of the nuclear states involved. In contrast the data obtained by method II can be analyzed quite easily though as discussed above, the method is limited in its applicability.

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