heavy-particle emission, the trend of the result is clear. At energies of 20–30 Mev where the compound nucleus picture is most valid the nucleus appears blackest and shows an absorption about equal to the geometrical cross section of 1.00 barn. Extrapolating the flat portion of the curves to 370 Mev shows agreement with the total absorption cross section at that energy estimated by Belmont and Miller,[1] while the 685-mb value at 90 Mev compares favorably with the 730-mb inelastic cross section for 90-Mev neutrons on copper measured by Hadley and York.[28]

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# Gamma-Ray Angular Distribution in Coulomb Excitation*

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In view of discrepancies in the literature regarding signs in formulas for the angular distribution of gamma rays emitted in Coulomb excitation, this distribution is worked out for the special case of $0\rightarrow 2$ transitions. The calculation is quantum mechanical and neglects higher than first-order effects in the Coulomb energy. The signs and forms obtained are confirmed by a semiclassical calculation.

Considerable interest has recently been attached to the angular distribution of gamma rays from nuclear states excited by the Coulomb field of impinging particles. The angular correlation in question is between the direction of the incident particle and that of the gamma emitted in de-excitation. The original work of Alder and Winther[1] pointed out the possibilities of the subject and gave correctly the relative signs of the contributions to the coefficients of the Legendre functions in their formula. In their numerical application however, some of these signs appear to have been incorrectly used. This applies to the sign of the term multiplying $P_4$ as a whole in their formula as well. In the work of Biedenharn and Class[2] some relative signs in the term multiplying $P_2$ are incorrect, agreeing with Alder and Winther's arithmetic rather than their formula. Since the general formalism used by Biedenharn and Class requires minute attention regarding conventions of notation it appears desirable to outline a short calculation of these coefficients, which while not covering the most general case suffices for ascertaining all factors in the general formula with the exception of providing the general interpretation of the nuclear quadrupole matrix element. It thus determines the signs of the terms giving the coefficients $a_2(\xi)$ in the quantum generalization of the Alder-Winther formula. In this the interaction between the field of the incident particle and the multipole moment of the nucleus, and that between the nucleus and the radiation field are treated as perturbations; the Coulomb term of the interaction between nucleus and incident particle is treated exactly, however. The particular calculation has assumed excitation of a nucleus originally in a $J=0$ state to a $J=2$ state, and the subsequent $\gamma$ decay of the nucleus back to a $J=0$ state. The two steps involved in the process are thus, first, the Coulomb excitation to one of these $J=2$ nuclear sublevels, and, secondly, the radiative transition from the sublevel to the ground state. Each sublevel gives rise to a characteristic distribution of radiation; the anisotropy of the $\gamma$ distribution is due to the various sublevels being excited with different amplitudes.

The intensity of the $\gamma$ emission is proportional to the square of the absolute value of the matrix element of the interaction energy for this process. This matrix element in turn is proportional to the sum over sublevels of the product of the amplitude for excitation of a given sublevel and the amplitude of the radiation emitted in a given direction in the de-excitation of this level. Here it is assumed that the sublevels are part of a degenerate level. Since the polarization of the gamma ray is not observed, the intensities arising from the two possible polarizations of the radiation are added. The angular correlation being the only quantity of interest here, only the dependence of the excitation amplitude on the magnetic quantum number $\mu$, and the dependence of the radiation amplitude upon $\mu$ and the direction of emission

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[1] K. Alder and A. Winther, Phys. Rev. **91**, 1578 (1953).
[2] L. C. Biedenharn and C. M. Class, Phys. Rev. **98**, 691 (1955).

enter the present considerations. Therefore constant factors, and factors depending upon the radial nuclear integrals are dropped so as not to complicate the presentation.

Considerable simplification obtains if the axis of quantization is chosen to be the direction of the incident particle. In this case the various terms of the scattered wave contain as factors $Y_L^\mu$, $\mu$ being the magnetic quantum number associated with a given sublevel, and $L$ referring to a particular partial scattered wave. Therefore, if the intensity is summed over all angles of scattering of the incident particle, no interference terms from different sublevels or from different angular momenta in the scattered wave can occur. It then suffices to determine the probability of scattering into a state of given $L$ exciting the sublevel $\mu$, multiplying this by the intensity of radiation of given polarization, and summing this over $L$, $\mu$, and the two transverse directions of polarization.

This excitation probability is determined by solving the time-independent Schrödinger equation for the system of the incident particle and the nucleus. It is expressed in terms of the familiar radial integrals $I(L,L')$ for quadrupole excitation. The angular distribution of the gamma radiation emitted in the transition from a particular $J=2$ sublevel to ground is computed next. These probabilities are then combined, and the resulting angular distribution expressed in terms of the usual $a_2$ and $a_4$ coefficients. No account is taken of the change in angular distribution due to higher order processes.

### LIST OF NOTATIONS

$\mathbf{r}$, $\mathbf{r}_p$ = coordinates of bombarding particle and nuclear proton relative to the center of mass of the target nucleus.

$m$ = reduced mass of system of bombarding particle and nucleus.

$k=mv/\hbar$ is $2\pi$ times the wave number associated with the relative motion.

$Ze$ = charge of target nucleus.

$\eta=Ze^2/\hbar v$.

$F_L(kr)$ = regular solution of the differential equation for $r$ times the radial wave function in a Coulomb field.

$b_s^{L,\mu}$ = coefficients describing the combination of two spherical harmonics, defined in Eq. (11).

$\sigma_L$ = Coulomb phase shift $=\arg\Gamma(L+1+i\eta)$.

The total Hamiltonian for the system of bombarding particle plus nucleus may be written as

$$H=H^0+H^N+H', \tag{1}$$

in which

$$H^0=-(\hbar^2/2m)\Delta+Ze^2/r \tag{2}$$

is the Hamiltonian for the relative motion of the scattered particle and the nucleus. For simplicity, the bombarding particle is chosen to be singly charged. If this is not the case, the only effect is to change the definition of the constant $A$ in Eq. (10), and also the quantity $\eta$. The nuclear Hamiltonian $H^N$ satisfies

$$\begin{aligned}(H^N-E_0)v_0&=0,\\ (H^N-E_f)w_\mu&=0,\end{aligned} \tag{3}$$

in which the wave function $v_0$ refers to the ground state of the nucleus, assumed to have $J=0$, and $w_\mu$ is the wave function for the $\mu$th sublevel of the $J=2$ excited state. These wave functions are taken to be of the form

$$\begin{aligned}w_\mu(\mathbf{r}_p)&=R_f(r_p)Y_2^\mu(\theta_p,\varphi_p),\\ v_0(\mathbf{r}_p)&=R_i(r_p)Y_0^0(\theta_p,\varphi_p),\end{aligned} \tag{4}$$

the $Y_l^\mu$ being spherical harmonics normalized so that

$$\int|Y_l^\mu|^2d\Omega=1. \tag{5}$$

A single nuclear proton is considered here in the interests of simplicity. The generalization to many nuclear protons can be made by employing sums of single-proton contributions and theorems regarding forms of matrix elements for tensor quantities developed in Wigner's book on applications of group theory.[3] The well-known expression for the transition quadrupole moment would thus be reproduced and the discussion of this matter is omitted here since the primary interest is in the form of the factor multiplying the quadrupole moment. The phases of the spherical harmonics, which enter through the coefficients $b_s^{L,\mu}$ in Eq. (11), are chosen as in the book by van der Waerden.[4] The interaction energy $H'$ is given by

$$H'=Ze^2/|\mathbf{r}-\mathbf{r}_p|-Ze^2/r. \tag{6}$$

The part of Eq. (6) contributing to quadrupole excitation is

$$(4\pi/5)Ze^2Y_2^{\mu*}(\theta,\varphi)Y_2^\mu(\theta_p,\varphi_p). \tag{7}$$

An approximate wave function for the system is

$$\Psi(\mathbf{r},\mathbf{r}_p)=v_0(\mathbf{r}_p)\psi^c(\mathbf{r})+\textstyle\sum_\mu w_\mu(\mathbf{r}_p)\psi^\mu(\mathbf{r}). \tag{8}$$

Here $\psi^c$ is the plane wave incident along the $Z$-axis and is expanded as

$$\psi^c=(4\pi)^{\frac{1}{2}}\textstyle\sum_L i^L(2L+1)^{\frac{1}{2}}Y_L^0\exp(i\sigma_L)F_L(kr)/(kr). \tag{9}$$

The coefficients $\psi^\mu$ in the expansion Eq. (8) represent the scattered wave. These are found to satisfy the approximate equation

$$(H^0+E_f-E)\psi^\mu\simeq A\psi^cY_2^{\mu*}/r^3, \tag{10}$$

where the coefficient $A$ contains the nuclear quadrupole moment. It is convenient to define coefficients $b_s^{L,\mu}$ for the spherical harmonics by

$$Y_2^\mu Y_L^0=\textstyle\sum_s b_s^{L,\mu}Y_{L+s}^\mu. \quad (s=\pm 2,0) \tag{11}$$

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[3] E. P. Wigner, *Gruppentheorie* (Friedrich Vieweg and Son, Braunschweig, 1931).

[4] B. L. van der Waerden, *Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932).

Using the Green's function for the operator $H_0+E_f-E$ describing scattering to solve Eq. (10), one finds that the coefficients $\psi^\mu(\mathbf{r})$ are asymptotically

$$\psi^\mu \sim \text{const} \sum_L i^L(1/r)\exp i(k'r-L\pi/2-\eta'\ln 2k'r+\sigma_L') \times\sum_s \exp(i\sigma_{L-s})I(L-s,L)\beta_s^{L-s,-\mu}Y_L^{-\mu}, \quad (13)$$

the quantities $k'$ and $\eta'$ referring to values appropriate to the outgoing wave:

$$\hbar^2k'^2/(2m)=E-E_f. \quad (14)$$

The integrals $I(L,\Lambda)$ are defined as

$$I(L,\Lambda)=\int_0^\infty (1/r^3)F_L(kr)F_\Lambda(k'r)dr. \quad (15)$$

The quantities $\beta_s^{L-s,\mu}$ are related to the coefficients $b_s^{L-s,\mu}$ by

$$\beta_s^{L-s,\mu}=-i^s(2L-2s+1)^{\frac{1}{2}}b_s^{L-s,\mu}. \quad (16)$$

The presence of the factor $Y_L^\mu$ in Eq. (13) insures the absence of interference between the contributions from different sublevels and different $L$ values, assuming that only the relative probability of exciting different sublevels enters. These probabilities are proportional to

$$|\sum_s \exp(i\sigma_{L-s})I(L-s,L)\beta_s^{L-s,\mu}|^2, \quad (17)$$

in view of the symmetry property

$$b_s^{L-s,\mu}=b_s^{L-s,-\mu}. \quad (18)$$

The interaction between the nucleus and the radiation field with vector potential $\mathbf{A}$ gives rise to a term in the Hamiltonian

$$(ie\hbar/mc)(\mathbf{A}\cdot\nabla). \quad (19)$$

Thus the intensity of the radiation emitted in the transition from the $\mu$th sublevel to the ground state will be proportional to the square of the matrix element

$$H'=\int w_\mu^*\pi_s e^{i\mathbf{k}\cdot\mathbf{r}}\nabla v_0 d\mathbf{r}, \quad (20)$$

$\pi_s$ being a unit vector in the direction of the electric field for polarization $s$. In this the radiation field has been expanded in plane waves, $\mathbf{k}$ denoting the wave vector of the emitted radiation. Considering the fact that $\Delta J=2$ in this transition, it is seen that no magnetic dipole radiation can occur, and thus the angular part of Eq. (20) gives rise to a term proportional to

$$A_{\mu,s}=\int Y_2^{\mu*}\left(\frac{\mathbf{r}}{r}\right)\left(\frac{\mathbf{r}}{r}\cdot\pi_s\right)P_1\left(\frac{\mathbf{k}\cdot\mathbf{r}}{kr}\right)d\Omega. \quad (21)$$

Letting $\Theta$ and $\Phi$ be the colatitude and azimuth of the emitted $\gamma$ ray, the direction cosines of the two transverse polarization vectors may be taken as

$$\pi_a:\ (\cos\Theta\cos\Phi,\ \cos\Theta\sin\Phi,\ -\sin\Theta),$$
$$\pi_b:\ (-\sin\Phi,\ \cos\Phi,\ 0).$$

Substituting these vectors in Eq. (21) and performing the integrations leads to the following expression for the relative amplitudes $A_{\mu,s}$ of radiation from the various sublevels.

For polarization $a$ one finds

$$\begin{aligned}
A_{2,a}&=D\cdot(1/\sqrt{6})CSe^{-2i\Phi},\\
A_{1,a}&=D\cdot(-1/\sqrt{6})(C^2-S^2)e^{-i\Phi},\\
A_{0,a}&=-D\cdot CS,\\
A_{-1,a}&=D\cdot(1/\sqrt{6})(C^2-S^2)e^{i\Phi},\\
A_{-2,a}&=D\cdot(1/\sqrt{6})CSe^{2i\Phi}.
\end{aligned} \quad (22)$$

For polarization $b$ there obtains

$$\begin{aligned}
A_{2,b}&=D\cdot(-i/\sqrt{6})Se^{-2i\Phi},\\
A_{1,b}&=D\cdot(i/\sqrt{6})Ce^{-i\Phi},\\
A_{0,b}&=0,\\
A_{-1,b}&=D\cdot(i/\sqrt{6})Ce^{i\Phi},\\
A_{-2,b}&=D\cdot(i/\sqrt{6})Se^{2i\Phi}.
\end{aligned} \quad (23)$$

Here the abbreviations $C\equiv\cos\Theta$, $S\equiv\sin\Theta$ have been used. The value of the constant of proportionality $D$ is immaterial for the present discussion. Squaring these amplitudes and summing over the two polarization directions gives for the relative intensities

$$\begin{aligned}
\gamma^{\pm2}&=(2/15)-(2/21)P_2(\cos\Theta)-(4/105)P_4(\cos\Theta),\\
\gamma^{\pm1}&=(2/15)+(1/21)P_2(\cos\Theta)+(16/105)P_4(\cos\Theta),\\
\gamma^0&=(2/15)+(2/21)P_2(\cos\Theta)-(8/35)P_4(\cos\Theta).
\end{aligned} \quad (24)$$

For convenience the angular dependence has been written in terms of Legendre polynomials $P_2$ and $P_4$.

The combination of the probabilities for excitation as in Eq. (17) and the relative intensities as in Eq. (24) yields for the angular distribution of the $\gamma$ rays

$$W(\Theta)=\text{const}\cdot\sum_{L,\mu}\gamma^\mu|\sum_s \beta_s^{L-s,\mu}I(L-s,L)\times\exp(i\sigma_{L-s})|^2. \quad (25)$$

Employing Eq. (24) for $\gamma^\mu$ in the above, and using the known forms for $\beta_s^{L-s,\mu}$, it is found that $W(\Theta)$ may be written as

$$W(\Theta)=\text{const}[b_0+b_2P_2(\cos\Theta)+b_4P_4(\cos\Theta)] \quad (26)$$

with the following meaning of the symbols. The coefficient $b_0$ is given by

$$b_0=\frac{2}{15}\sum_L\left[|I(L-2,L)|^2\frac{3L(L-1)}{2(2L-1)}+|I(L,L)|^2\frac{L(2L+1)(L+1)}{(2L-1)(2L+3)}+|I(L+2,L)|^2\frac{3(L+2)(L+1)}{2(2L+3)}\right]. \quad (27)$$

The coefficient of the $P_2$ term, $b_2$, is

$$b_2=\frac{1}{21}\sum_L\Bigg[|I(L-2,L)|^2\frac{3L(L-1)(L-2)}{(2L-1)^2}$$
$$-|I(L,L)|^2\frac{(2L-3)(2L+5)L(L+1)(2L+1)}{(2L-1)^2(2L+3)^2}$$
$$+|I(L+2,L)|^2\frac{3(L+1)(L+2)(L+3)}{(2L+3)^2}$$
$$-I(L-2,L)I(L,L)\cos(\sigma_{L-2}-\sigma_L)$$
$$\times\frac{6L(L+1)(L-1)}{(2L-1)^2}-I(L+2,L)I(L,L)$$
$$\times\cos(\sigma_{L+2}-\sigma_L)\frac{6L(L+1)(L+2)}{(2L+3)^2}\Bigg]. \quad (28)$$

The $P_4$ term appears with the coefficient

$$b_4=-\frac{1}{35}\sum_L\Bigg[|I(L-2,L)|^2\frac{3L(L-1)(L-2)(L-3)}{(2L+1)(2L-1)^2}$$
$$+|I(L,L)|^2\frac{12L(L+1)(2L+1)(L-1)(L+2)}{(2L+3)^2(2L-1)^2}$$
$$+|I(L+2,L)|^2\frac{3(L+1)(L+2)(L+3)(L+4)}{(2L+3)^2(2L+1)}$$
$$-I(L-2,L)I(L,L)\cos(\sigma_{L-2}-\sigma_L)$$
$$\times\frac{20L(L+1)(L-1)(L-2)}{(2L-1)^2(2L+3)}-I(L+2,L)I(L,L)$$
$$\times\cos(\sigma_L-\sigma_{L+2})\frac{20L(L+1)(L+2)(L+3)}{(2L+3)^2(2L-1)}$$
$$+I(L-2,L)I(L+2,L)\cos(\sigma_{L-2}-\sigma_{L+2})$$
$$\times\frac{70L(L-1)(L+1)(L+2)}{(2L-1)(2L+1)(2L+3)}\Bigg]. \quad (29)$$

It is customary to compare this angular distribution with the angular correlation for the $\gamma-\gamma$ cascade passing through the same sequence of $J$ values. The Coulomb excitation distribution is written as

$$W(\Theta)=1+a_2B_2P_2(\cos\Theta)+a_4B_4P_4(\cos\Theta), \quad (30)$$

in which the coefficients $B_2$ and $B_4$ are defined by the $\gamma-\gamma$ correlation

$$W'(\Theta)=1+B_2P_2(\cos\Theta)+B_4P_4(\cos\Theta). \quad (31)$$

The coefficients $B_2$ and $B_4$ may be formed from the expressions for $\gamma^\mu(\Theta)$, since

$$W'(\Theta)\propto\sum_\mu\gamma^\mu(\Theta=0)\gamma^\mu(\Theta). \quad (32)$$

There results

$$B_2=5/14,\quad B_4=8/7, \quad (33)$$

and hence, comparing Eqs. (30) and (31),

$$a_2=(14/5)(b_2/b_0),\quad a_4=(7/8)(b_4/b_0), \quad (34)$$

giving the correlation coefficients in terms of Eqs. (27), (28), and (29).

The signs of the various terms in the coefficients $a_2$ and $a_4$ may be compared with those of the corresponding terms in the SCT discussion of Alder and Winther. This may be done by considering the asymptotic forms of the coefficients of the radial matrix elements in Eqs. (27), (28), and (29) as $L$ becomes very large; correspondingly, the asymptotic forms of the integrands in the SCT formulas are taken for $\epsilon$ large. If use is made of the correspondence established previously[5] between the radial matrix elements in the quantum and semiclassical pictures, it is found that the Alder-Winther formulas are reproduced exactly, in this limit, save for the sign in front of $a_4$.

In view of this disagreement as to the sign of $a_4$, the results of an independent semiclassical calculation seem to be of interest. This yields coefficients coinciding with those of Alder and Winther except as to the sign of $a_4$, thus checking the quantum-mechanical result obtained above, which is in fair agreement[6] with experiment.

The authors are indebted to Mr. G. Tice for checking some of the calculations.

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