Numerical Linear Algebra2020-2021
Exercise session 5

Part 1: Theory

Least squares approximation by polynomials is defined as solving

$$p = \arg\min_{p \in \mathbb{P}_n} \|f - p\|_{2,\Omega} = \arg\min_{p \in \mathbb{P}_{n-1}} \left(\int_\Omega |f(\omega) - p(\omega)|^2 d\mu \right)^{\frac{1}{2}}$$

with $\mathbb{P}_{n-1}$ the space of polynomials with degree bounded by $n-1$, (Ω, μ) a measure space and f some given function. In this exercise session we focus on the discrete problem i.e. $(\Omega, \mu) = (I, w)$ with I some discrete set of size m and w a weighted counting measure:

$$p = \arg\min_{p \in \mathbb{P}_{n-1}} \|f(I) - p(I)\|_W = \left(\sum_{x_i \in I} w^2(x_i)((f(x_i) - p(x_i)))^2 \right)^{\frac{1}{2}}. \tag{1}$$

In fact, it is more natural to consider the equivalent differentiable form:

$$p = \arg\min_{p \in \mathbb{P}_{n-1}} \|f(I) - p(I)\|_W^2 = \left(\sum_{x_i \in I} w^2(x_i)(f(x_i) - p(x_i))^2 \right). \tag{2}$$

A you have seen, the typical approach is to make an *ansatz* based on a basis, that is we write p as

$$p = \sum_{j=1}^{n} a_n \psi_n$$

where $\{\psi_j\}_j$ is some basis for $\mathbb{P}_{n-1}$ with the convention that $\deg(\psi_j) = j - 1$. We assume $m \geq n - 1$
Using

$$W := \mathrm{diag}(w(\{x_i\}_{i \in I}))$$

find a matrix Ψ such that equations 1 & 2 are equivalent to the unweighted least squares problem

$$\mathbf{p} = \arg\min_{p \in \mathbb{P}_{n-1}} \|W(\Psi \mathbf{a} - \mathbf{f})\|_2^2$$

with $\mathbf{f} = [f(x_1), \ldots, f(x_m)]^T$ and $\mathbf{a} = [a_1, \ldots, a_m]^T$. Use the fact that expression 2 is differentiable to show that the solution to this minimization problem (for the case $I \subset \mathbb{R}$) is given by

$$(\Psi^T W^2 \Psi)\mathbf{a} = (\Psi^T W^2)\mathbf{f}.$$

This system is called the system of *normal equations*. Suppose now that we choose a system $\{\phi_j\}_j$ of polynomials that is orthonormal w.r.t. w i.e.

$$\langle \phi_{j_1}, \phi_{j_2} \rangle_W = \delta_{j_1 j_2}$$

Show that, with the corresponding Φ we have

$$(\Phi^T W^2 \Phi) = I_n.$$

If (as usual) the basis of polynomials is written in increasing degree, where ψ_k is of degree one higher than ψ_{k-1} it holds that

$$x\psi_k(x) \in \text{span}\{\psi_1, \dots, \psi_{k+1}\}$$

Show that this can be written as

$$\mathbf{x}\boldsymbol{\psi}(x) = \boldsymbol{\psi}(x)H + h_{m+1m}\psi_{m+1}(x)\mathbf{e}_m^T$$

with $\boldsymbol{\psi}(x) = [\psi_1(x), \dots, \psi(x_m)]$ What structure does H have? If the ψ's are replaced with the orthonormal ϕ, what is ϕ_{m+1}? Show that in this case we can write

$$X\Phi = \Phi H$$

with $X = \mathbf{diag}(\{x_i\}_i)$. Use this, together with the fact that $W\Phi$ is orthogonal to show that the diagonal matrix X and the matrix of recurrence coefficients H are unitarily similar. Show that this implies that H is in fact tridiagonal. What does this mean on the level of recurrence relations?

Part 2: An algorithm for the real, single variate case

You have seen that the determination of the recurrence coefficients can be done by applying a sequence of Givens rotations to the weights-extended diagonal node matrix i.e.

$$H = G_n \cdot \ldots \cdot G_1[\mathbf{w}, X]$$

Review slides 9-19 of the screencast on orthogonal polynomials. How do you find the sequence of orthogonal polynomials. Implement this. Test your algorithm for the Chebyshev nodes

$$x_i = \cos\left(\frac{2i-1}{2n}\pi\right), \quad i = 1, \dots, n.$$

and the associated weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

What do you notice? What does this mean?

For increasing n plot the computational time of this routine. For $n = 1000$, how many Givens rotations are called (use `Matlab`'s profiler)? Does this agree with your intuition?

Since at each fill-in deletion stage the matrix is almost tridiagonal, do we actually need to apply the Given rotations to entire rows/columns? Use this to implement a faster version for the real case. Plot its complexity and overlay it with the previous complexity plot.