Inverse eigenvalue problem

Computing (the recurrence relation coefficients of) the OPs $a_j(z)$ can be done by solving the following IEP.

Definition (IEP)

Given: $Z = \mathrm{diag}(z_i)$ – points, $\mathbf{w} = (w_i)$ – weights
Find: Unitary Q and upper Hessenberg H such that

$$Q^H \mathbf{w} = \|\mathbf{w}\| \mathbf{e}_1, \quad Q^H Z Q = H.$$

Connection between Q, H and the OPs

Consider the following recurrence relation for b_i:

$$b_0 = \frac{1}{\|\mathbf{w}\|}, \quad z[b_0, b_1, \ldots, b_{N-1}] = [b_0, b_1, \ldots, b_{N-1}]H. \qquad (2)$$

H Hessenberg $\Rightarrow$ derive b_1 from 1st col, b_2 from 2nd, ...

$$Q^H\mathbf{w} = \|\mathbf{w}\|\mathbf{e}_1 \quad \Rightarrow \quad Q\mathbf{e}_1 = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \mathrm{diag}(\mathbf{w})[b_0(z_i)]$$

$$ZQ = QH \quad \Rightarrow \quad Q\mathbf{e}_k = \mathrm{diag}(\mathbf{w})[b_{k-1}(z_i)],\ k = 1, 2, \ldots, N.$$

Since $Q^HQ = I$, we have that:
(1) b_i are OPs wrt (1), (2) $a_i = b_i$ and thus (3) $WA = Q$.

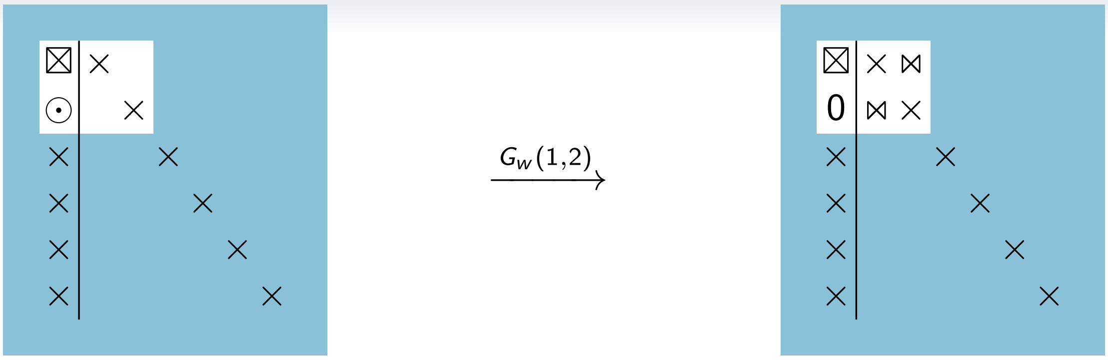
Algorithm for solving IEP

$$
\left[\begin{array}{c|cccc}
w_1 & z_1 & & & \\
w_2 & & z_2 & & \\
\vdots & & & \ddots & \\
w_N & & & & z_N
\end{array}\right]
$$

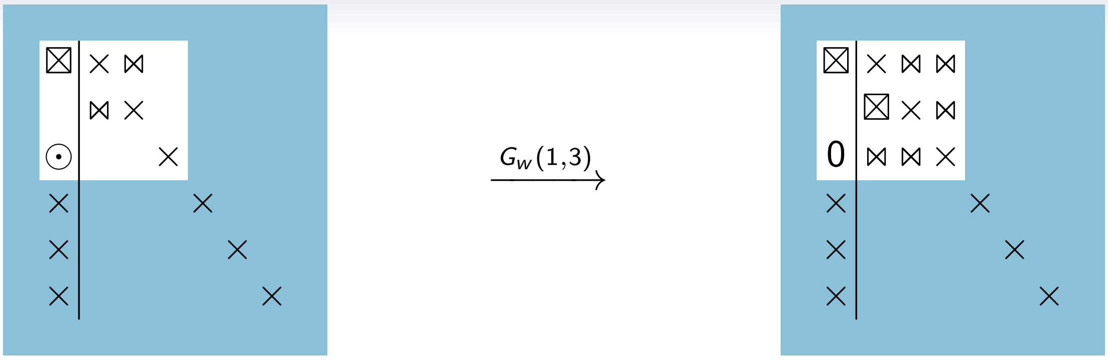
$\rightarrow$ sequence of unitary similarity transformations using Givens rotations $\rightarrow$

$$
\left[\begin{array}{c|ccccc}
\|\mathbf{w}\| & h_{11} \ldots & & & \ldots & h_{1,N} \\
 & h_{21} & & & & \\
 & & h_{32} & & & \\
 & & & \ddots & & \vdots \\
 & & & & h_{N,N-1} & h_{N,N}
\end{array}\right]
$$

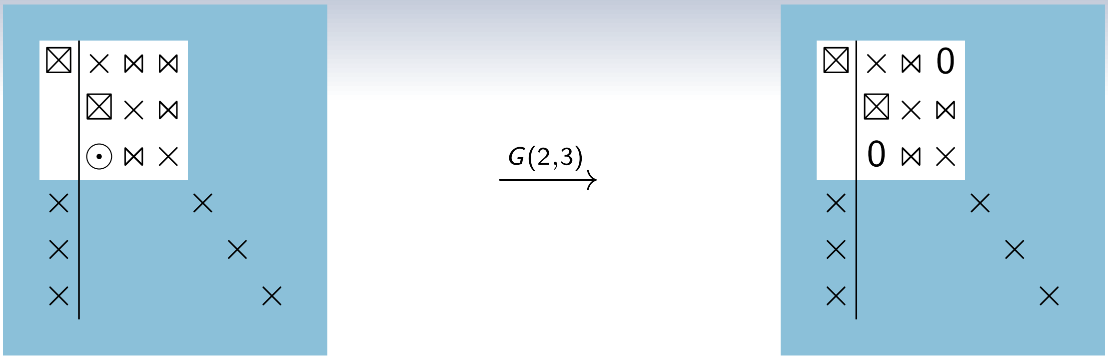
2 points



3 points (1)



3 points (2)



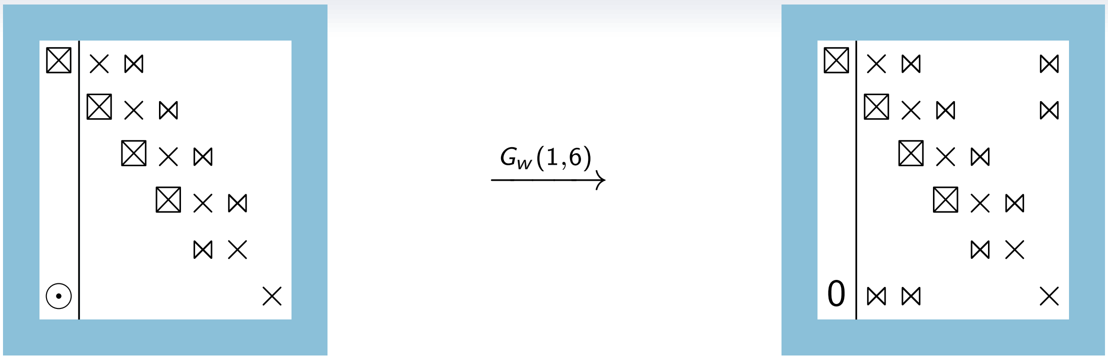
. . . skip some steps and jump to 6 points . . .

One-variable OP Algorithm

5 points (4)

$$
\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & \bowtie & \\
 & & & \boxtimes & \times & \bowtie & \\
 & & & \odot & \bowtie & \times & \\
\times & & & & & & \times
\end{array}
\xrightarrow{G(4,5)}
\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & 0 & \\
 & & & \boxtimes & \times & \bowtie & \\
 & & & 0 & \bowtie & \times & \\
\times & & & & & & \times
\end{array}
$$

6 points (1)



6 points (2)

$$\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \bowtie \\
 & \boxtimes & \times & \bowtie & & & \bowtie \\
 & & \boxtimes & \times & \bowtie & & \\
 & & & \boxtimes & \times & \bowtie & \\
 & & & & \bowtie & \times & \\
 & \odot & \bowtie & & & & \times
\end{array}\right]
\xrightarrow{G(2,6)}
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & 0 \\
 & \boxtimes & \times & \bowtie & & & \bowtie \\
 & & \boxtimes & \times & \bowtie & & \bowtie \\
 & & & \boxtimes & \times & \bowtie & \\
 & & & & \bowtie & \times & \\
 & 0 & \bowtie & \bowtie & & & \times
\end{array}\right]$$

6 points (3)

$$
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \bowtie \\
 & & \boxtimes & \times & \bowtie & & \bowtie \\
 & & & \boxtimes & \times & \bowtie & \\
 & & & & \bowtie & \times & \\
 & & \odot & \bowtie & & & \times
\end{array}\right]
\xrightarrow{G(3,6)}
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & 0 \\
 & & \boxtimes & \times & \bowtie & & \bowtie \\
 & & & \boxtimes & \times & \bowtie & \bowtie \\
 & & & & \bowtie & \times & \\
 & & 0 & \bowtie & \bowtie & & \times
\end{array}\right]
$$

6 points (4)

$$
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & & \bowtie \\
 & & & \boxtimes & \times & \bowtie & \bowtie \\
 & & & & \bowtie & \times & \\
 & & & \odot & \bowtie & & \times
\end{array}\right]
\xrightarrow{G(4,6)}
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & & 0 \\
 & & & \boxtimes & \times & \bowtie & \bowtie \\
 & & & & \bowtie & \times & \bowtie \\
 & & & 0 & \bowtie & \bowtie & \times
\end{array}\right]
$$

6 points (5)

$$
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & & \\
 & & & \boxtimes & \times & \bowtie & \bowtie \\
 & & & & \boxtimes & \times & \bowtie \\
 & & & & \odot & \bowtie & \times
\end{array}\right]
\xrightarrow{G(5,6)}
\left[\begin{array}{c|cccccc}
\boxtimes & \times & \bowtie & & & & \\
 & \boxtimes & \times & \bowtie & & & \\
 & & \boxtimes & \times & \bowtie & & \\
 & & & \boxtimes & \times & \bowtie & 0 \\
 & & & & \boxtimes & \times & \bowtie \\
 & & & & 0 & \bowtie & \times
\end{array}\right]
$$

Algorithm 1: Transformation of the initial matrix $D = [\mathbf{w}|Z]$ into a matrix $[Q^H\mathbf{w}|Q^H ZQ]$ having zeros below the pivot elements

begin
  for $i = 2 : N$ **do**
    for $j = 1 : i - 1$ **do**
- make element $d_{i,j}$ zero
  by Givens rot. G^H with the pivot element (j, j):
  $D = G^H D$
- $D = D \begin{pmatrix} 1 & \\ & G \end{pmatrix}$ (similarity transformation)

    end
  end
end

Computational complexity to solve the IEP

- in general: $\mathcal{O}(N^3)$ FLOPS
- w_i real, z_i real: $\mathcal{O}(N^2)$ FLOPS
- z_i on the complex unit circle: $\mathcal{O}(N^2)$ FLOPS using Schur parametrization, H is a unitary Hessenberg matrix