Quantum Information and Computing

Many-body localization in random spin systems

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Outline

Introduction

- The aim of this project is to characterize the Many Body Localization (MBL) transition of a spin system in a random field.
- When an MBL transition occurs the system will move from an ergodic phase towards a non-ergodic one and thus fail to reach thermal equilibrium thus the long time behaviour will strongly depend on the initial conditions.
- The model we are gonna study is the Heseinberg chain (the XXX version) with quenched disorder in the z magnetic field direction.

The model

- We chose to use the z diagonal representation of the $\frac{1}{2}$ spin and we apply the magnetic field in the z direction.
- Following [1], we focused our attention on the spin sector for which the total spin operator S^z has eigenvalue equals to 0. Indeed since our Hamiltonian H commutes with the total spin operator we can block diagonalize the Hamiltonian where each block refers to states with a given total spin.
- Due to the above choice we restricted the analysis to even system sizes.

The model

If we use the usual z spin operator at each site i, σ_i^z, and the raising and lowering operators $\sigma_i^{\pm}$ the model Hamiltonian reads:

$$H = \frac{1}{4}\sum_{i=1}^{N}\left[2\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) + \sigma_i^z\sigma_{i+1}^z\right] + \frac{1}{2}\sum_{i=1}^{N} h_i\sigma_i^z$$

The fields $\{h_i\}$ are uniform random variables between $-h$ and h where h itself is a noise strength parameter. The fields are an example of *quenched disorder* and the strength h will control (as we will see) the MBL transition.

The matrix elements

- With (even) N number of particles in our system the number of states of the $S^z = 0$ sector is $\binom{N}{N/2}$
- The z parts of the Hamiltonian are diagonal in our basis so the only non trivial part is given by matrix element such as:

$$\langle t_1 \dots t_N | \, \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ \, | s_1 \dots s_N \rangle$$

- The above matrix elements are non zero if and only if $s_i \neq s_{i+1}$ and $s_i = -t_i \wedge s_{i+1} = -t_{i+1}$; in other words the action of $\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+$ results in two neighbour differently aligned spins flipped.

Energy spacing

- Throughout the text the index α will denote a disorder (i.e. fields) realization,
- The first quantity (from [1]) we will analyze in order to characterize the MBL transition is related to the energy spacing between eigenstates of the Hamiltonian.
- If $\delta_\alpha^{(n)} = |E_\alpha^{(n+1)} - E_\alpha^{(n)}|$ is the energy spacing between the eigenstate n and its adjacent $n+1$ we will compute the average of

$$r_\alpha^{(n)} = \min(\delta_\alpha^{(n)}, \delta_\alpha^{(n+1)}) / \max(\delta_\alpha^{(n)}, \delta_\alpha^{(n+1)})$$

over states and realizations.

Local magnetization

- The second quantity of interest is the local magnetization of each energy eigenstate:

$$m_{i\alpha}^{(n)} = \langle n | \sigma_i^z | n \rangle \tag{1}$$

- Given the above magnetization we will compute its difference between adjacent states, $|m_{i\alpha}^{(n+1)} - m_{i\alpha}^{(n)}|$ (averagining over disorder and sites).
- If the system thermalizes the aforementioned difference should disappear as the system size grows: by exploiting this property we can identify MBL and delocalized regions (at varying h).

Spin relaxation

- As a dynamical analysis, we consider the relation of an initially inhomogeneous spin density (which is the first non zero Fourier mode of the spin operator):

$$M_1 = \sum_{j=1}^{N} \sigma_j^z e^{i\frac{2\pi j}{L}} \tag{2}$$

- The initial condition we consider is characterized by the following density matrix ($\epsilon \ll 1$ is just a modulation parameter and Z a normalization factor):

$$\rho_0 = \frac{1}{Z}\left(1 + \epsilon M_1^\dagger\right) \tag{3}$$

Spin relaxation

- The initial and infinite time expected spin density are given by:

$$\langle M_1 \rangle_0 = \sum_n \langle n| \rho_0 M_1 |n\rangle = \frac{\epsilon}{Z} \sum_n \langle n| M_1^\dagger M_1 |n\rangle$$

$$\langle M_1 \rangle_\infty = \frac{\epsilon}{Z} \sum_n \langle n| M_1^\dagger |n\rangle \langle n| M_1 |n\rangle$$

- Comapring the two quantities we can try to understand the amount of 'information' about the initial conditions is kept on the long time behaviour i.e. we can characterize the thermalization of the system. In order to do this we will study:

$$f_\alpha^{(n)} = 1 - \frac{\langle n| M_1^\dagger |n\rangle \langle n| M_1 |n\rangle}{\langle n| M_1^\dagger M_1 |n\rangle}$$

Spin correlation

- Following [2] the last quantity we will study is:

$$C(t) = \frac{1}{L}\sum_{j=1}^{L} \langle\psi(0)|\, \sigma_j^z(0)\sigma_j^z(t)\, |\psi(0)\rangle$$

  where $\sigma_j^z(0) = \sigma_j^z$ and $\sigma_j^z(t) = e^{iHt}\sigma_j^z e^{-iHt}$.
- The state $|\psi(0)\rangle$ represent an alternating spin state such as $|- + \cdots - +\rangle$.
- The long time behaviour of $|C(t)|$ accounts for the amount of 'memory' from initial condition retained.

Results

- For the magnetic field strengths h we choose the ones of [1] i.e. $h \in [0.6, 1.0, 2.0, 2.7, 3.6, 5.0, 8.0]$.
- The system sizes probed were $L = 8, 10, 12, 14$ (only up to $L = 12$ for $C(t)$).
- We used, respectively, 10^4 disorder realizations for $L = 8, 10$, 10^3 for $L = 12$ and 50 for $L = 14$.

Energy spacing

- By plotting the average $r_\alpha^{(n)}$ we observe two limiting behaviours at small and high field.
- At small field strength the average value is around 0.53 which is typical of GOE energy spacing statistics.
- As the strength highers this average drops towards a value ≈ 0.39 which is typical of Poisson spacings.

Energy spacing

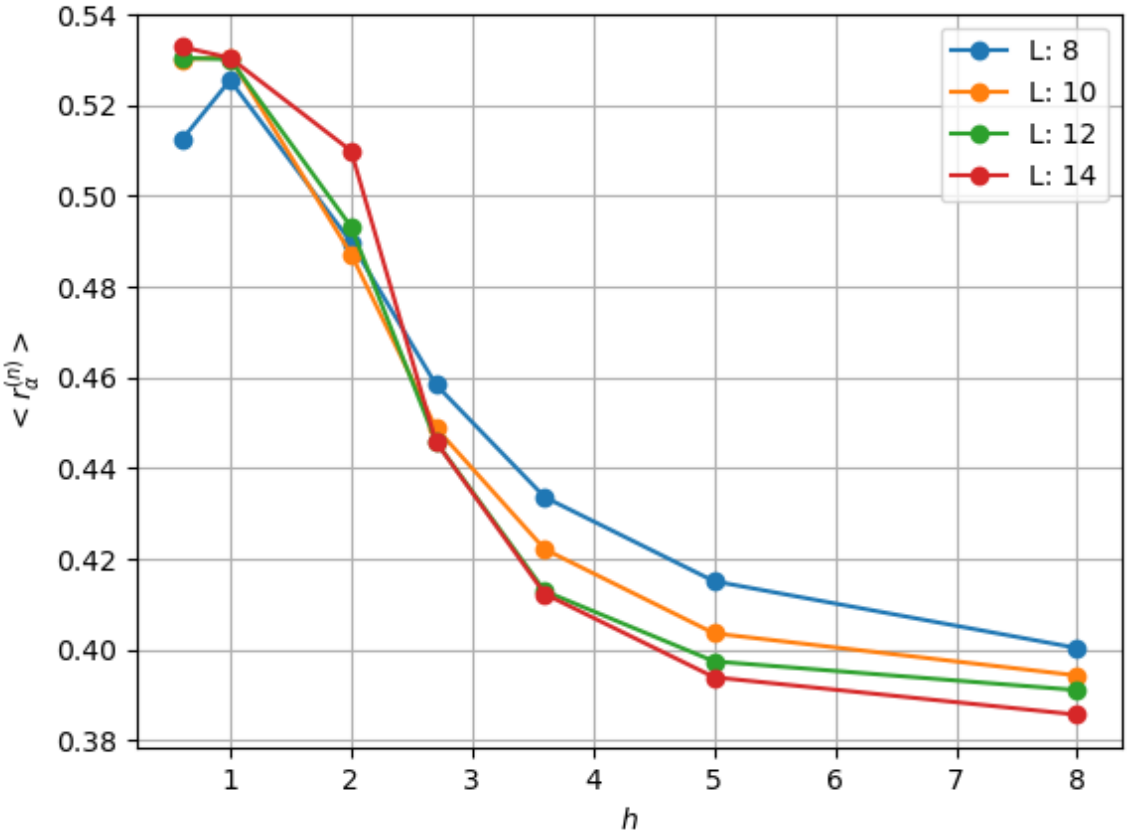

Figure: Ratio between adjacent energy gaps.

Local magnetization

- When looking at the local magnetization a more definite crossover between ergodic and non-ergodic phases appears: the adjacent eigenstates difference is basically constant for any system size at high magnetic field while is decaying at small ones. This last behaviour is typical of thermal eigenstates since a small energy difference (and hence a small temperature difference) would result in similar values for σ_i^z.

Local magnetization

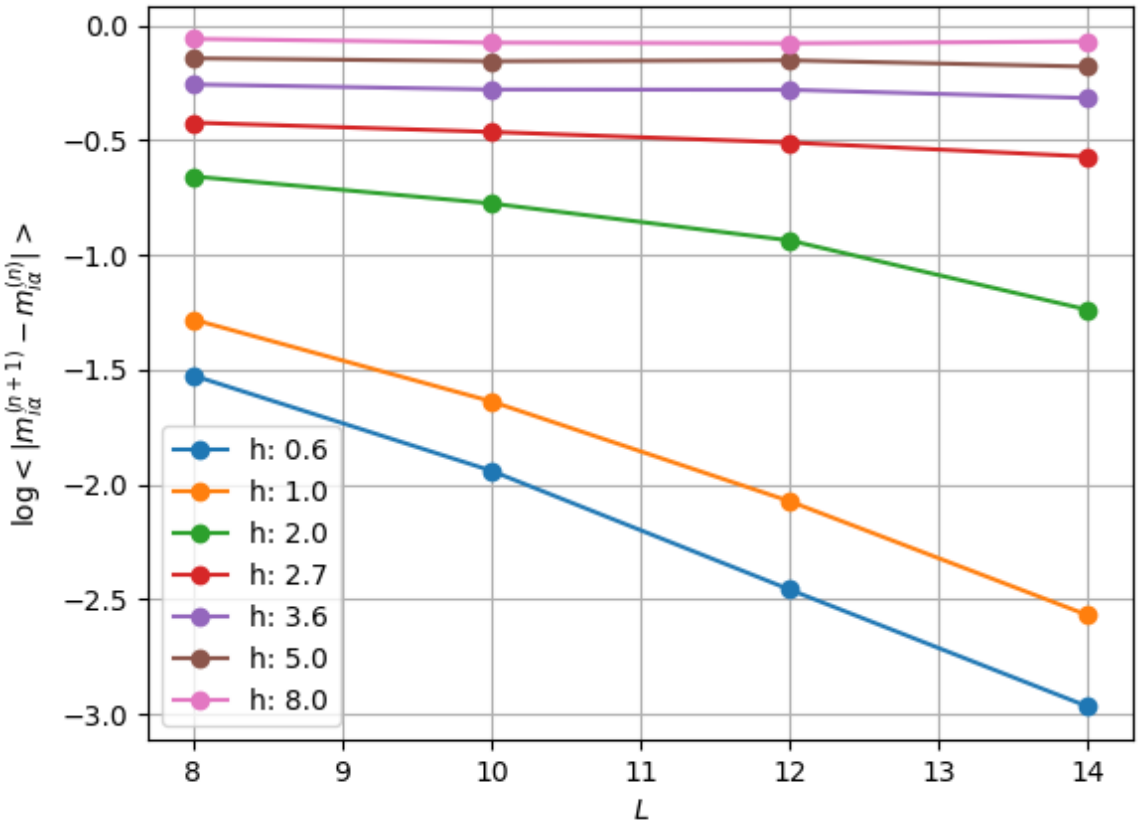

Figure: Log difference between magnetization at adjacent eigenstates.

Spin relaxation and correlation

- Being related to the memory of initial conditions, the average $f_\alpha^{(n)}$, a, moves from 1 to 0 as we increase h (we can think that if the system is localized then we have $\langle M_1 \rangle_\infty \approx \langle M_1 \rangle_0$ hence $\langle f_\alpha^{(n)} \rangle \approx 0$).
- In Figure 4 we show the norm of $C(t)$ for $L = 8, 10, 12$ measured for 50 log spaced timesteps. We observe that as h increases the asymptotic value of $|C(t)|$ highers, meaning that the correlation with initial conditions is stronger. For $L = 12$ we observe a more definite decay at long times of the correlation function; larger systems studied in [2] confirm this behaviour.

Spin relaxation

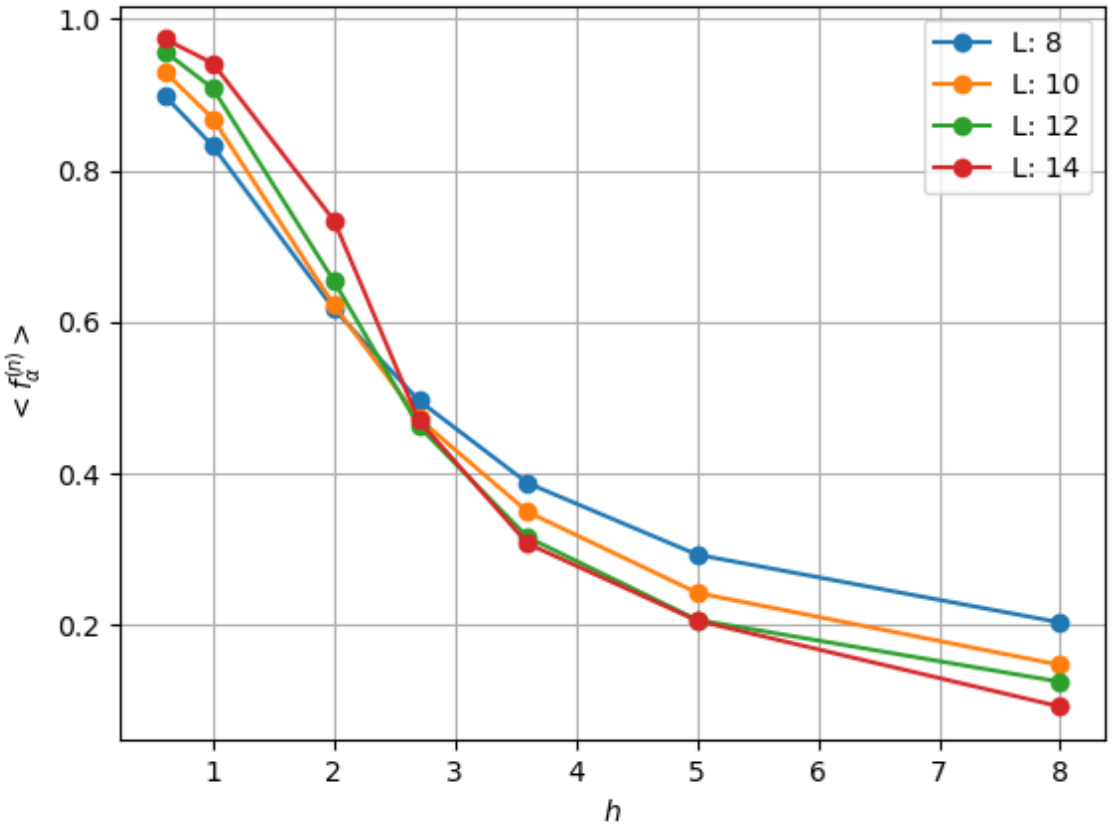

Figure: Comparison of initial state and asymptotic one.

Spin correlation

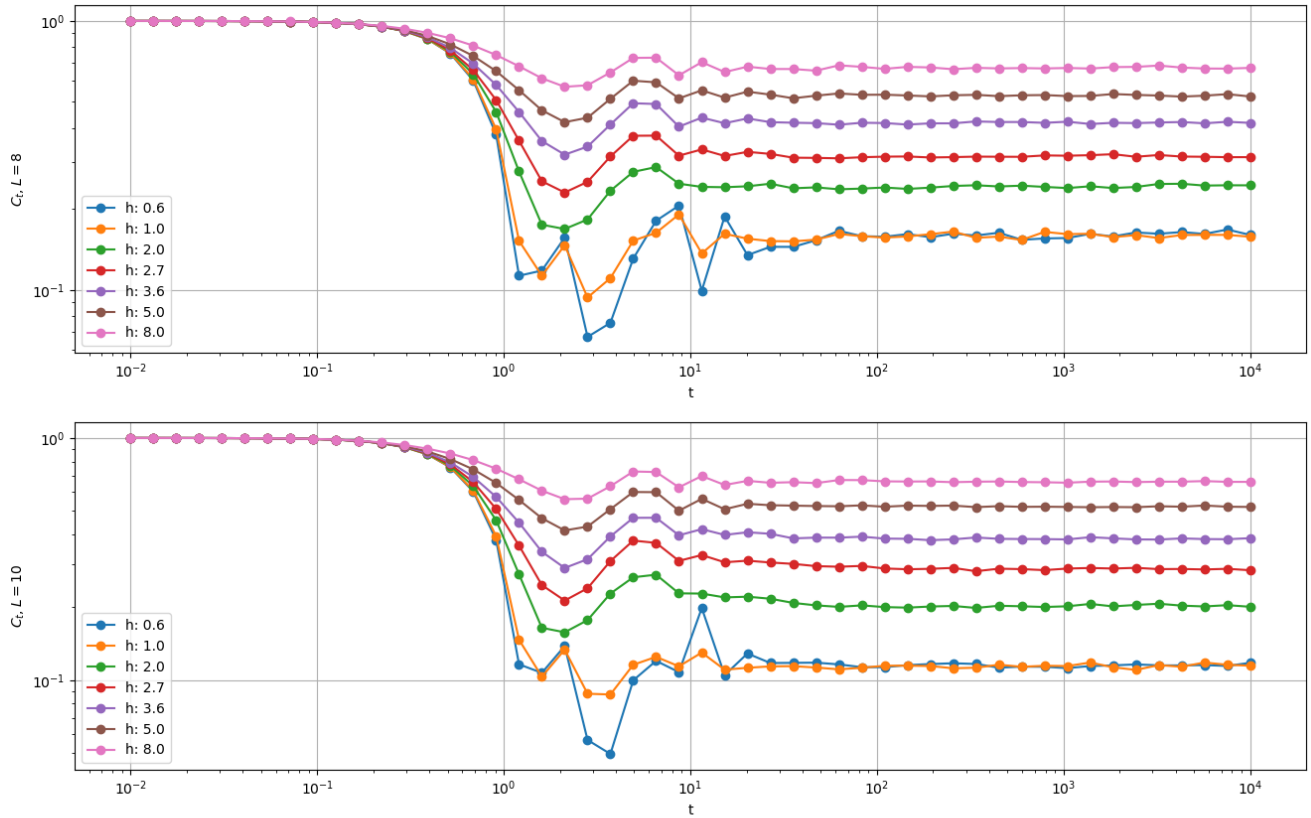

Figure: Evolution of spin correlation norm $|C(t)|$ for $L = 8, 10$.

Spin correlation

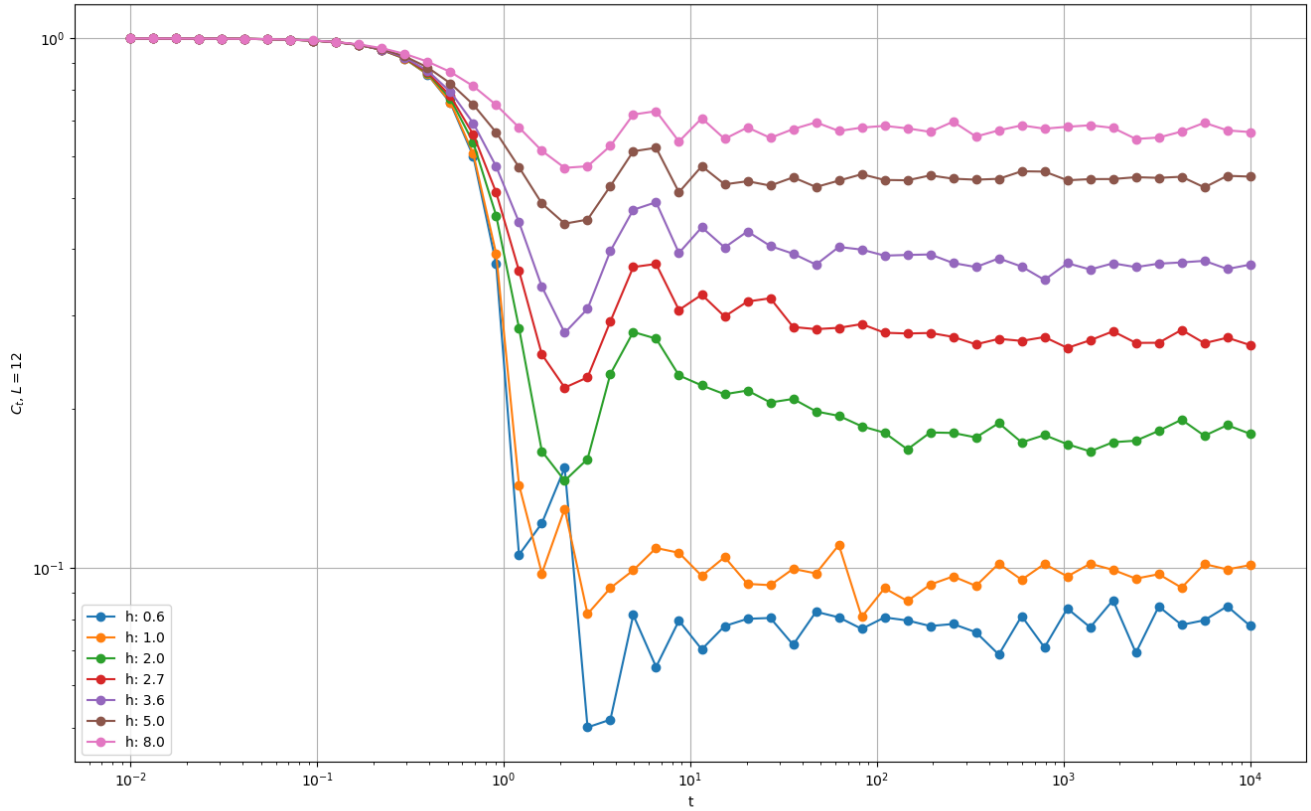

Figure: Evolution of spin correlation norm $|C(t)|$ $L = 12$.

Inverse Participation Ratio

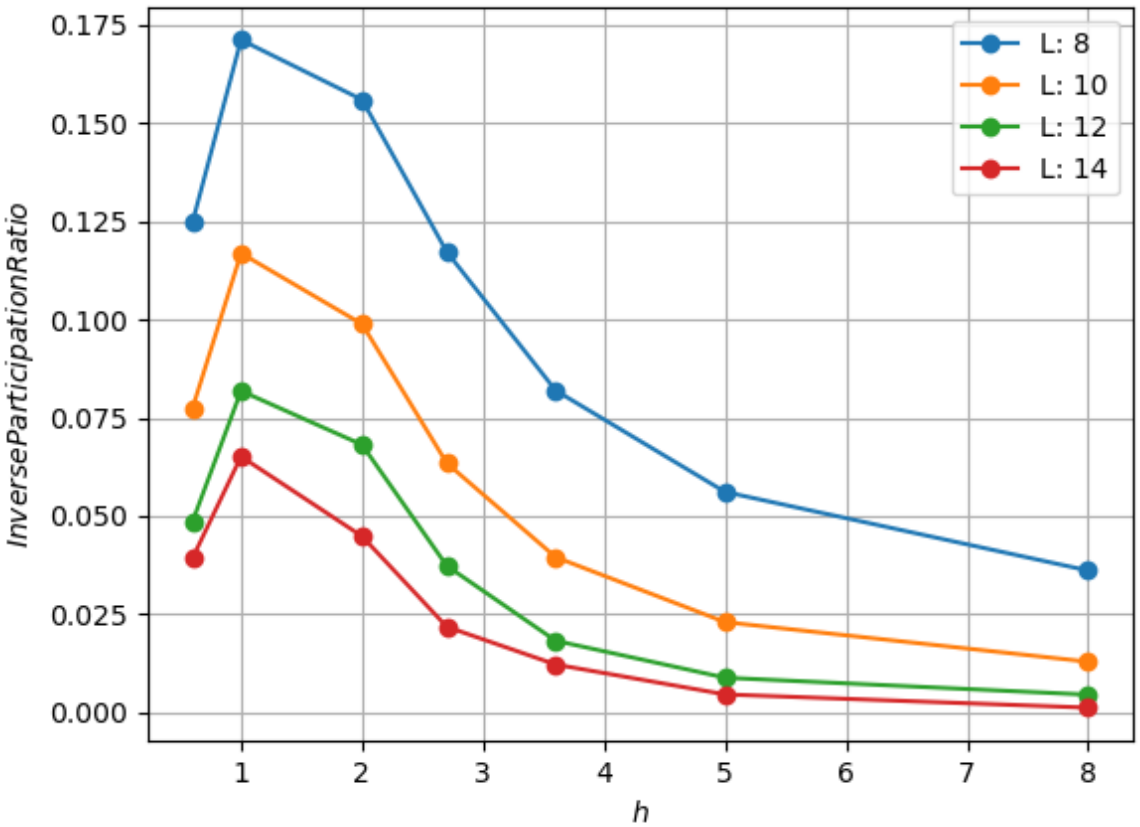

Figure: Inverse Participation Ratio for state used in time evolution, normalized to the number of states of the spin sector.

Conclusion

- The literature provides an estimate of $h_c \approx 3.5 \pm 1.0$ (see [1]). We may state the by looking at the four analyzed quantities we collected enough evidence for the occurrence of such a phase transition after $h \approx 2.5$; we can observe this by looking at the intersection between the different curves in figures 1 and 3 or the change of behaviour of the magnetization in 2.

Code self-commentary

- Since we focused only on the $S^z = 0$ spin sector the basis used in the construction of the Hamiltonian was dealt with in the implementation using a 'dictionary' kind approach mapping usual states into sector ones.
- As said in the main text the only non trivial part in the construction of the Hamiltonian was the XY interaction part.
- All the XY matrix elements are generated by the values $\langle t| \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ |s\rangle$; given that s can be viewed as a bit string, the only t state for which the element is non zero is $t = (1 - \delta_{t_i, s_i})\,(t \oplus \mathrm{mask}(i, i+1))$ where $\oplus$ is the XOR operator and $\mathrm{mask}(i, j) = 2^i + 2^j$ when viewed as integer.

Bibliography

- Pal A., Huse D.A. "Many-body localization phase transition" PHYSICAL REVIEW B 82, 174411 2010
- "Extended slow dynamical regime close to the many-body localization transition" David J. Luitzv D.J. Laflorencie N.. Alet F. (arXiv)