Notes on Relativistic Quantum Field Theories

– For supplements to LEPS Theoretical Base Set – Year 2017

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0 Notations

■ metric of 4-vector space:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{("Bjorken-Drell metric")} \tag{1}$$

The square of a 4-vector $\underline{p} = (p^0, \boldsymbol{p}) = (p^0, p^1, p^2, p^3)$ is

$$\underline{p}^2 = \underline{p}^T g\, \underline{p} = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (p^0)^2 - \boldsymbol{p}^2\,,$$

where a superscript T denotes the transpose. An underline on a quantity (e.g. $\underline{p}$) indicates a 4-vector but it is frequently suppressed. When components of a 4-vector are given, they are contravariant components by default. The contravariant components of a 4-vector $\underline{p}$ are denoted as p^μ with upper suffixes. To make sure that components are contravariant, we often denote a 4-vector itself as p^μ. For given contravariant components, covariant components are obtained as

$$p_\mu = g_{\mu\nu} p^\nu\,,$$

where $g_{\mu\nu}$ is the $(\mu\nu)$ element of g. Thus, for the $\underline{p}$ given in the above, we write $p_\mu = (p^0, -\boldsymbol{p})$. The square of a 4-vector $\underline{p}$ is now also written as

$$\underline{p}^2 = p_\mu p^\mu\,,$$

where same suffixes μ in lower and upper positions must be summed over $\mu \in [0, 3]$.

■ Energy-momentum of a particle of the mass m is written as

$$p^\mu \stackrel{\leftarrow}{\equiv} \begin{pmatrix} E/c \\ \boldsymbol{p} \end{pmatrix} = mc \begin{pmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{pmatrix} = mu^\mu\,, \tag{2}$$

where $\gamma \equiv 1/\sqrt{1-\boldsymbol{\beta}^2}$ denotes the Lorentz factor of motion of the particle, $\boldsymbol{\beta}$ stands for the velocity nomalized by one of the light, c, and u^μ is the four-velocity.

■ Natural unit ($c = \hbar = 1$) is employed throughout our discussions. E.g. for the energy-momentum given in the above, we write

$$E = \sqrt{\boldsymbol{p}^2 + m^2} \quad \text{and} \quad \underline{p}^2 = m^2\,.$$

■ Nomalization of states

$$\text{Energy-momentum eigenstate:} \quad <p\,|\,p'>= 2E\delta^3(\boldsymbol{p}-\boldsymbol{p}'),\ \ E=\sqrt{m^2+\boldsymbol{p}^2}$$
$$\text{Completeness:} \quad \int \frac{d^3\boldsymbol{p}}{2E}\,|p><p|\ = 1\,. \tag{3}$$

Many authors including ones of Refs. [1] and [2] put a factor $(2\pi)^3$ in the front of δ function and in the denominator of the integration volume element. It is just a matter of convention. Note that

$$Dim[\,|p>] = [E]^{-1} \tag{4}$$

■ Levi-Civita symbols
These are totally antisymmetric tensor densities.

3 dimensional

$$\epsilon_{ijk} = \begin{cases} \pm 1 & (i,j,k) = \begin{matrix}\text{even}\\ \text{odd}\end{matrix} \text{ permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For the purpose of applying the Einstein's contraction rule to 3 vector suffices, we may use symbols like ϵ^{ijk}, ${\epsilon_{ij}}^{k}$ and so on. All these symbols are equivalent with ϵ_{ijk}. The vertical positions of indexes have no meaning.

4 dimensional

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & (\mu,\nu,\rho,\sigma,) = \text{even parmutation of } (0,1,2,3) \\ -1 & (\mu,\nu,\rho,\sigma,) = \text{odd parmutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Symbols with covariant suffices are defined with the metric tensor in such a way like ${\epsilon_{\mu}}^{\nu\rho\sigma} = g_{\mu\tau}\epsilon^{\tau\nu\rho\sigma}$. Therefore, we have $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$.

1 Introduction to Relativistic Quantum Field Theory

1.1 Starting from Quantum Mechanics

1.1.1 Quantum Mechanics

$$\text{Schrödinger eq.} \quad i\partial_t \Psi(t,\boldsymbol{x}) = H\Psi(t,\boldsymbol{x})\,, \quad H = -\frac{\boldsymbol{\partial}^2}{2m} + V(\boldsymbol{x}) = H^\dagger \tag{7}$$

$$\text{Eigenstates of } H \quad H\varphi_l(\boldsymbol{x}) = \epsilon_l \varphi_l(\boldsymbol{x})\,, \quad \{\varphi_l(\boldsymbol{x})\}\text{: orthogonal, complete} \tag{8}$$

$$\text{Solution of the Schrödinger eq.} \quad \Psi(t,\boldsymbol{x}) = \sum_l \Psi_l e^{-i\epsilon_l t}\varphi_l(\boldsymbol{x}) \tag{9}$$

Bras and kets

$$\Psi(t,\boldsymbol{x}) = <\boldsymbol{x}|\,\Psi(t)>\,, \qquad \varphi_l(\boldsymbol{x}) = <\boldsymbol{x}|\,\epsilon_l>\,, \tag{10}$$

$$i\partial_t\,|\Psi(t)> = \hat{H}\,|\Psi(t)>\,, \qquad \hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{x}) = \hat{H}^\dagger \qquad <7>$$

$$\hat{H}\,|\epsilon_l> = \epsilon_l\,|\epsilon_l>\,, \quad <\epsilon_l\,|\epsilon_{l'}> = 0 \text{ if } l \neq l'\,, \quad \sum_l |\epsilon_l><\epsilon_l| = 1 \qquad <8>$$

$$|\Psi(t)> = \sum_l <\epsilon_l|\,\Psi(0)>\,e^{-i\epsilon_l t}\,|\epsilon_l> \qquad <9>$$

1st quantization

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{11}$$

Eigen states

$$\begin{array}{l} \hat{\boldsymbol{x}}\,|\boldsymbol{x}> = \boldsymbol{x}\,|\boldsymbol{x}> \\ \hat{\boldsymbol{p}}\,|\boldsymbol{p}> = \boldsymbol{p}\,|\boldsymbol{p}> \end{array} \tag{12}$$

Conventional ("half relativistic") normalization

$$<\boldsymbol{x}|\,\boldsymbol{x}'> = \delta^3(\boldsymbol{x}-\boldsymbol{x}')\,, \qquad \int |\boldsymbol{x}>\,d^3\boldsymbol{x}\,<\boldsymbol{x}| = 1 \tag{13}$$

$$<\boldsymbol{p}|\,\boldsymbol{p}'> = 2E\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \qquad \int |\boldsymbol{p}>\,\frac{d^3\boldsymbol{p}}{2E}\,<\boldsymbol{p}| = 1 \tag{14}$$

Coordinate representation of the momentum operator

$$<\boldsymbol{x}|\,\hat{\boldsymbol{p}} = \frac{1}{i}\boldsymbol{\partial}\,<\boldsymbol{x}| \tag{15}$$

and of the momentum eigenstate

$$<\boldsymbol{x}|\,\boldsymbol{p}>=\sqrt{\frac{2E}{(2\pi)^3}}\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{16}$$

Requirement for the normalization factor is understood by employing the first equation of (14) in the $l.h.s$ of $<\boldsymbol{p}|\,\boldsymbol{p}'> = \int <\boldsymbol{p}|\,\boldsymbol{x}>\, d^3\boldsymbol{x} <\boldsymbol{x}\,|\boldsymbol{p}'>$ and remembering a formula

$$\int d^3\boldsymbol{x}\, e^{\pm i\boldsymbol{p}\cdot\boldsymbol{x}} = (2\pi)^3\delta^3(\boldsymbol{p})$$

1.1.2 N-body quantum mechanics

Schrödinger eq.

$$\begin{aligned} &i\partial_t\Psi^{(N)}(t;\boldsymbol{x}_1,\ldots\boldsymbol{x}_N) = H^{(N)}\Psi^{(N)}(t;\boldsymbol{x}_1,\ldots\boldsymbol{x}_N)\,,\\ &H^{(N)} = \sum_i^N H_i\,,\quad H_i = -\frac{\boldsymbol{\partial}^2}{2m}+V(\boldsymbol{x}_i)\qquad \text{(for simplicity)} \end{aligned} \tag{17}$$

$$H_i\varphi_l^{(i)}(\boldsymbol{x}) = \epsilon_l\varphi_l^{(i)}(\boldsymbol{x})\,,\qquad i=1,\ldots,N \tag{18}$$

$$\begin{aligned} &\Psi^{(N)}(t;\boldsymbol{x}_1,\ldots\boldsymbol{x}_N) =\\ &\quad\sum\nolimits_{l_1,\ldots,l_N}\Psi^{(N)}(l_1,\ldots,l_N)e^{-iE^{(N)}t}\left\{\varphi_{l_1}^{(1)}(\boldsymbol{x}_1)\cdots\varphi_{l_N}^{(N)}(\boldsymbol{x}_N)\right\}_P\,, \end{aligned} \tag{19}$$

where $E^{(N)} = \epsilon_{l_1}+\cdots+\epsilon_{l_N}$ and $\{\cdots\}_P$ denotes symmetrization for systems of identical bosons and antisymmetrization for identical fermions. Thus $\Psi^{(N)}$ is defined in the direct product of N Hilbert space.

■ Theorem of bosonic creation and annihilation operators
If an operator $\hat{a}$ and its hermite conjugate $\hat{a}^\dagger$ satisfy

$$[\hat{a},\hat{a}^\dagger] = 1\,, \tag{20}$$

then

1. Eigenvalues of an operator $\hat{N}\equiv\hat{a}^\dagger\hat{a}$ is nonnegative integers $\{0,1,\ldots,\infty\}$ and we can call it number operator.

2. Vaccum state $|0>$ with respect to the dynamical freedom described by $\hat{a}$ and $\hat{a}^\dagger$ can be defined as the eigenstate of $\hat{N}$ belonging to its eigenvalue 0.

3. If we normalize the vacuum state by $<0|\,0>=1$, then the eigenstate of $\hat{N}$ belonging to the eigenvalue n is given by

$$|n>=\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0>\,,\quad <n|\,m>=\delta_{nm} \tag{21}$$

■ **Theorem of fermionic creation and annihilation operators**
If an operator $\hat{c}$ and its hermite conjugate $\hat{c}^\dagger$ satisfy

$$\{\hat{c}, \hat{c}^\dagger\} = 1 \quad \text{and} \quad \{\hat{c}, \hat{c}\} = 0\,, \tag{22}$$

where $\{\dots\}$ is anti-commutator, then

1. Eigenvalues of an operator $\hat{N} \equiv \hat{c}^\dagger \hat{c}$ is 0 or 1 and we can call it number operator.
2. Vaccum state $|0>$ with respect to the dynamical freedom described by $\hat{c}$ and $\hat{c}^\dagger$ can be defined as the eigenstate of $\hat{N}$ belonging to its eigenvalue 0.
3. If we normalize the vacuum state by $<0|\,0>= 1$, then the eigenstates of $\hat{N}$ are $|0>$ and $|1>= c^\dagger\,|0>$.

Suppose now we have a set of $\hat{a}_l$ and $\hat{a}_l^\dagger$ for $l = 1, 2, \dots, \infty$ corresponding to energy eigenvalues $\epsilon_1, \epsilon_2, \dots$. We assume that we can have a set of operators such that each pair of $\hat{a}_l$ and $\hat{a}_l^\dagger$ satisfies the condition of creation-annihilation operators mentioned above and they are independent for different suffices:

$$\begin{array}{l} [\hat{a}_l, \hat{a}_m^\dagger] = \delta_{lm} \\ [\hat{a}_l, \hat{a}_m] = [\hat{a}_l^\dagger, \hat{a}_m^\dagger] = 0 \end{array} \tag{23}$$

Having with these operators, we define

$$\hat{\varphi}(\boldsymbol{x}) = \sum_l \hat{a}_l \varphi_l(\boldsymbol{x})\,, \tag{24}$$

where $\varphi_l(\boldsymbol{x})$ is eigenvector of $H = -\boldsymbol{\partial}^2/2m + V(\boldsymbol{x})$ belonging to the lth eigenvalue ϵ_l. Assign $\hat{a}_l^\dagger$ as operator to create a particle in the lth energy eigenstate:

$$|\epsilon_l>= \hat{a}_l^\dagger\,|0> \tag{25}$$

Then we find

$$\begin{aligned} <0|\,\hat{\varphi}(\boldsymbol{x})\,|\epsilon_l> &= \sum_{l'} \varphi_{l'}(\boldsymbol{x})\,<0|\,\hat{a}_{l'}\hat{a}_l^\dagger\,|0> \\ &= \sum_{l'} \varphi_{l'}(\boldsymbol{x})\,<0|\,[\hat{a}_{l'}, \hat{a}_l^\dagger]\,|0> \\ &= \varphi_l(\boldsymbol{x})\,, \end{aligned} \tag{26}$$

where we have used relationships $\hat{a}\,|0>=<0|\,\hat{a}^\dagger = 0$. Comparing this result with the second equation in Eq. (10), we may write

$$<\boldsymbol{x}| \; =<0|\,\hat{\varphi}(\boldsymbol{x}) \tag{27}$$

If we denote by $\hat{a}^\dagger_{\boldsymbol{x}}$ the creation operator that creates a a particle at a position $\boldsymbol{x}$, we can write

$$\hat{a}^\dagger_{\boldsymbol{x}} = \hat{\varphi}^\dagger(\boldsymbol{x}) \tag{28}$$

The operator $\hat{\varphi}(\boldsymbol{x})$ defined in Eq. (24) is a primitive form of field operators discussed later.

N particle states can be constructed as

$$|\epsilon_{l_1}, \ldots, \epsilon_{l_N}> = \hat{a}^\dagger_{l_1} \ldots \hat{a}^\dagger_{l_N} \, |0> \tag{29}$$

and

$$<\boldsymbol{x}_1, \ldots \boldsymbol{x}_N| \;=<0| \, \hat{\varphi}(\boldsymbol{x}_1) \ldots \hat{\varphi}(\boldsymbol{x}_N) \tag{30}$$

We read

$$\Psi^{(N)}(t; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) =<\boldsymbol{x}_1, \ldots \boldsymbol{x}_N| \, \Psi^{(N)}(t)> \tag{31}$$

and

$$|\Psi^{(N)}(t)>= \sum_{l_1,\ldots,l_N} \Psi^{(N)}(l_1, \ldots, l_N) e^{-iE^{(N)}t} \, |\epsilon_{l_1}, \ldots, \epsilon_{l_N}> \tag{32}$$

When N particles are identical bosons, these expressions in Eq. (29) and (30) are redundant because all operators in them are commuting with each other and the order of variables in these bra and ket have no meaning. Suppose we have n_1 particle in state of energy ϵ_1, n_2 in ϵ_2 and so on, we may write the *l.h.s.* of Eq. (29) as $|n_1, n_2, \ldots>$. This notation is commonly used in condensed matter and nuclear physics.

Basis $|\boldsymbol{x}_1, \ldots \boldsymbol{x}_N>$ or $|\epsilon_{l_1}, \ldots, \epsilon_{l_N}>$ span N-particle Hilbert space. Set of all N-particle Hilbert space spanned by $|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots>$ is called the Fock space. A state vector $|\Psi>$ of the Fock space can be expanded as

$$|\Psi>= \sum_N \int \prod_i^N d^3\boldsymbol{x}_i \, |\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N><\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N| \, \Psi> \tag{33}$$

1.1.3 Schrödinger field theory

In place of Eq. (24), if we define

$$\hat{\Psi}(t, \boldsymbol{x}) = \sum_l \hat{a}_l e^{-i\epsilon_l t} \varphi_l(\boldsymbol{x}) \,, \tag{34}$$

this operator satisfies the single particle Schrödinger equation:

$$i\partial_t \hat{\Psi}(t, \boldsymbol{x}) = H \hat{\Psi}(t, \boldsymbol{x}) \tag{$\widehat{7}$}$$

We already know that any number of Schrödinger particles can be generated by $\hat{\varphi}(\boldsymbol{x}) = \hat{\Psi}(0, \boldsymbol{x})$. Comparing Eq. (34) with Eq. (9), we observe $\hat{\Psi}(t, \boldsymbol{x})$ is obtained by replacing c-number amplitude Ψ_l in $\Psi(t,\boldsymbol{x})$ by annihilation operator $\hat{a}_l$. The field operator $\hat{\Psi}$ satisfies the following equaltime commutation relations:

$$\begin{aligned}
&[\hat{\Psi}(t,\boldsymbol{x}),\hat{\Psi}^\dagger(t,\boldsymbol{x'})] = \delta^3(\boldsymbol{x}-\boldsymbol{x'}) \\
&[\hat{\Psi}(t,\boldsymbol{x}),\hat{\Psi}(t,\boldsymbol{x'})] = [\hat{\Psi}^\dagger(t,\boldsymbol{x}),\hat{\Psi}^\dagger(t,\boldsymbol{x'})] = 0
\end{aligned} \tag{35}$$

If we require Eq. (35), then Eq. (23) follows. Schrödinger equation $(\widehat{7})$ follows from a Lagrangian density

$$\mathcal{L} = i\Psi^*\partial_t\Psi + \frac{1}{2m}\Psi^*\boldsymbol{\partial}^2\Psi - V\Psi^*\Psi \tag{36}$$

Here we note $dim[\mathcal{L}] = E^4$ and $dim[\Psi] = E^{3/2}$. Canonical momentum field is given as

$$\Pi \overset{\leftarrow}{=} \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\Psi^* \,, \tag{37}$$

which has the same dimension as Ψ. The set (35) is equivalent with

$$\begin{aligned}
&[\hat{\Psi}(t,\boldsymbol{x}),\hat{\Pi}(t,\boldsymbol{x'})] = \delta^3(\boldsymbol{x}-\boldsymbol{x'}) \\
&[\hat{\Psi}(t,\boldsymbol{x}),\hat{\Psi}(t,\boldsymbol{x'})] = [\hat{\Pi}(t,\boldsymbol{x}),\hat{\Pi}(t,\boldsymbol{x'})] = 0
\end{aligned} \tag{38}$$

This is the canonical commutation relation. We may reverse our argument starting from Eq. (36), writing down the "field equation" $(\widehat{7})$ and requiring the equaltime canonical commutation relation (38). This procedure is called the 2nd quantizaton.

1.2 Canonical quantizaton

$\mathcal{L}$agrangian density ($dim = E^4$) is given as a functional of field $\varphi(x)$ and its space time derivatives

$$\mathcal{L} = \mathcal{L}[\varphi(x), \partial^\mu\varphi(x)] \tag{39}$$

Euler-Lagrange Eq.

$$\frac{\partial\mathcal{L}}{\partial\varphi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))} = 0 \tag{40}$$

Canonical momentum field

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}(x)} \tag{41}$$

where $\dot{\varphi}(x) = \partial_0\varphi(x) = \partial^0\varphi(x)$.
$\mathcal{H}$amiltonian density

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \ . \tag{42}$$

By solving Eq. (41) for $\dot{\varphi}(x)$, $\mathcal{H}$ is a function solely of $\pi(x), \varphi(x)$ and $\partial_i \varphi(x)$. In the classical field theory, temporal developments of $\varphi(x)$ and $\pi(x)$ are given by Hamiltonian $H = \int \mathrm{d}^3\boldsymbol{x}\, \mathcal{H}$ through the canonical equation of motion

$$\dot{\varphi}(x) = -i[\varphi(x), H] \;, \qquad \dot{\pi}(x) = -i[\pi(x), H] \tag{43}$$

where $-i[\cdots]$ is the Poisson braket. The Euler equation (40) and the canonical equation (43) are equivalent.

Following the canonical quantization method, fields $\varphi(x)$ and $\pi(x')$ at a time $x^0 = x'^0 = t_0$ (which is called the time of quantization) are postulated to satisfy an equal time commutation relation

$$\begin{aligned} [\varphi(x), \pi(x')] &= i\delta^3(\boldsymbol{x} - \boldsymbol{x'}) \;, \\ [\varphi(x), \varphi(x')] = 0 \;, &\quad [\pi(x), \pi(x')] = 0 \end{aligned} \tag{44}$$

Assuming the existence of the Hamiltonian and using this equal time commutation relation, it is shown that the Euler equation (40) and the Heisenberg equation (43) are equivalent in the level of quantum theory. [4] Here, we shold read $[\cdots]$ in Eq. (43) as commutation relation. In that case, using EOM of fields, it is shown that the commutation relation (44) holds at arbitrary time once it is set.[4] In this sense, the quantization time is arbitral when the canonical quantization method works in usual manner. Formal solution of $\varphi(x)$ for the Heisenberg Eq. (43) is written as

$$\varphi(x) = e^{iH(t-t_0)} \varphi(\boldsymbol{x}, t_0) e^{-iH(t-t_0)} \tag{45}$$

1.3 Noethers Theorem

For each degree of freedom of continuous transformation of fields against which the action remains invariant, there exist a conserved current $J^\mu(x)$. Particularly, let a infinitesimal transformation with s parameters $\delta\alpha_\lambda$ be written as

$$\begin{aligned} & x^\mu \mapsto x^{\mu'} = x^\mu + \delta x^\mu \;, \quad \delta x^\mu = \textstyle\sum_{\lambda=1}^s X^\mu_\lambda \delta\alpha_\lambda \\ & \varphi(x) \mapsto \varphi'(x') = \varphi(x) + \delta\varphi(x) \;, \quad \delta\varphi(x) = \textstyle\sum_{\lambda=1}^s \Phi_\lambda(x) \delta\alpha_\lambda \end{aligned} \tag{46}$$

This transformation may be associated with a space-time transformation specified by X^μ_λ as shown in the first line in Eq. (46). When the transformation is related only with internal degree of freedom of the field , one may set $X^\mu_\lambda = 0$. When the action $\mathcal{A}$ is invariant under this transformation, s currents

$$J^\mu_\lambda(x) \quad \overset{\leftarrow}{=} \quad -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \left(\Phi_\lambda(x) - \partial_\nu \varphi(x) X^\nu_\lambda\right) - \mathcal{L}(x) X^\mu_\lambda \tag{47}$$

are all conserved ones. Namely,

$$\partial_\mu J^\mu_\lambda(x) = 0\,, \quad \lambda = 1, \ldots, s \tag{48}$$

hold and corresponding charges

$$Q_\lambda = \int d^3\boldsymbol{x} J^0_\lambda(x) \tag{49}$$

are conserved. Eq. (47) defines the Noether currents and quantities Q_λ in Eq. (49) are called Noether charges.

1.3.1 Energy-Momentum tensor

Our action must be invariant under space-time translations. The infinitesimal transformation

$$x^\mu \mapsto x^{\mu'} = x^\mu + \delta a^\mu \tag{50}$$

has 4 continuous parameters δa^μ and in our notation $X^\mu_\nu = \delta^\mu_\nu$. The field is invariant

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x) \tag{51}$$

and $\Phi_\nu = 0$. The Noether current (47) reads

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x) - \mathcal{L}(x) g^\mu_\nu \tag{52}$$

1.3.2 Angular Momentum tensor

Consider an infinitesimal spatial rotation

$$x^\mu \mapsto x^{\mu'} = x^\mu + \delta\omega^\mu{}_\nu x^\nu \tag{53}$$

This transformation has 6 continuous parameters $\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$. δx^μ in Eq. (46) is given as

$$\begin{aligned}
\delta x^\mu &= \delta\omega^\mu{}_\nu x^\nu \\
&= g^\mu_\xi g_{\nu\eta} \delta\omega^{\xi\eta} x^\nu \\
&= \frac{1}{2}(g^\mu_\xi g_{\nu\eta} - g^\mu_\eta g_{\nu\xi}) x^\nu \delta\omega^{\xi\eta} \\
&= \frac{1}{2}(a_{\xi\eta})^\mu{}_\nu x^\nu \delta\omega^{\xi\eta}
\end{aligned} \tag{54}$$

where

$$(a_{\xi\eta})^\mu{}_\nu = g^\mu_\xi g_{\nu\eta} - g^\mu_\eta g_{\nu\xi} \tag{55}$$

We have

$$X^{\mu}{}_{\xi\eta} = \frac{1}{2}(a_{\xi\eta})^{\mu}{}_{\nu}x^{\nu} \tag{56}$$

The field is transformed as

$$\varphi_{\alpha}(x) \mapsto \varphi'_{\alpha}(x') = \delta\varphi_{\alpha}(x)\,, \quad \delta\varphi_{\alpha}(x) = \frac{1}{2}(S_{\xi\eta})_{\alpha}^{\ \beta}\varphi_{\beta}(x)\delta\omega^{\xi\eta} \tag{57}$$

where

$$(S_{\mu\nu})_{\alpha}^{\ \beta} = \begin{cases} 0 \quad \cdots \text{scalar} \\ (a_{\mu\nu})_{\alpha}^{\ \beta} \quad \cdots \text{vector} \\ \frac{1}{4}[\gamma_{\mu},\gamma_{\nu}]_{\alpha}^{\ \beta} \quad \cdots \text{Dirac spinor} \end{cases} \tag{58}$$

We have

$$\Phi_{\alpha\xi\eta} = \frac{1}{2}(S_{\xi\eta})_{\alpha}^{\ \beta}\varphi_{\beta}(x) \tag{59}$$

The Noether current

$$\begin{aligned} M^{\mu}_{\xi\eta} &\overset{\leftarrow}{\equiv} 2J^{\mu}_{\ \xi\eta} \\ &= -2\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\left(\Phi_{\alpha\xi\eta}(x) - \partial_{\nu}\varphi_{\alpha}(x)X^{\nu}{}_{\xi\eta}\right) - 2\mathcal{L}(x)X^{\mu}_{\ \xi\eta} \\ &= \left(\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\partial_{\nu}\varphi_{\alpha}(x) - \mathcal{L}g^{\mu}_{\nu}\right)\cdot 2X^{\nu}{}_{\xi\eta} - 2\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\Phi_{\alpha\xi\eta}(x) \\ &= (a_{\xi\eta})^{\nu}{}_{\rho}x^{\rho}\cdot T^{\mu}_{\nu} - \frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}(S_{\xi\eta})_{\alpha}^{\ \beta}\varphi_{\beta}(x) \qquad (60) \\ &\overset{\rightarrow}{\equiv} L^{\mu}_{(\xi\eta)} + S^{\mu}_{(\xi\eta)} \qquad (61) \end{aligned}$$

The last equation defines orbital and spin parts of the current. The total angular momentum vector is defined as

$$J^{k} = \frac{1}{2}\epsilon^{kij}M^{ij}\,, \quad M_{\xi\eta} = \int d^{3}\boldsymbol{x}M^{0}_{\xi\eta} \tag{62}$$

1.3.3 Electric Charge

For a complex field, gauge transformation of the first kind is defined as

$$\varphi(x) \mapsto \varphi'_{\alpha}(x) = e^{ie\theta}\varphi(x)\,, \quad \varphi^{*}(x) \mapsto \varphi^{*\prime}(x) = e^{-ie\theta}\varphi^{*}(x) \tag{63}$$

This transformation consists a U(1). Considering infinitesimal θ, we read in Eq. (46) that

$$X^{\mu} = 0\,, \quad \Phi = ie\varphi\,, \quad \Phi^{*} = -ie\varphi^{*}\,, \tag{64}$$

Invariance of the $\mathcal{L}$agragian density leads

$$\partial_\mu J^\mu = 0 \tag{65}$$

for

$$J^\mu = ie\left(\varphi^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\varphi\right) \tag{66}$$

The corresponding charge is $Q = \int d^3\boldsymbol{x} J^0$.

1.3.4 Internal Global Symmetries

We consider a internal global SU(n) symmetry. The field $\varphi_a(x)$ is subject to a transformation

$$\varphi_a(x) \mapsto \varphi'_a(x) = (e^{i\alpha_i G_i})_a^{\ b}\varphi_b(x) \tag{67}$$

where $\alpha_i, (i = 1, \ldots, n^2 - 1)$ are continuous real parameters, G_i are matrix representations of generators. Considering infinitesimal α_i, we have in Eq. (46) that

$$\Phi_{ai} = i(G_i)_a^{\ b}\varphi_b\,, \tag{68}$$

The Noether current is given by

$$J_i^\mu \quad = \quad -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}(G_i)_a^{\ b}\varphi_b \tag{69}$$

and the corresponding $n^2 - 1$ charges are

$$C_i = -i\int d^3\boldsymbol{x}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a}(G_i)_a^{\ b}\varphi_b \tag{70}$$

Among $n^2 - 1$ generators G_i, only $n - 1$ commutes to each other. Accordingly, $n - 1$ charges among $n^2 - 1$ can be diagonalized at the same time.

■ SU(2)
The conserved (internal) vector is the isospin $\boldsymbol{I} = (C_1, C_2, C_3)$.

$$\varphi_a \in \square \quad : \qquad G_i = \frac{1}{2}\sigma_i \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{71}$$

$$\varphi_a \in \square\!\square \quad : \qquad G_i = t_i\,, \quad (t_i)_{jk} = -i\epsilon_{ijk} \tag{72}$$

■ SU(3)

$$\varphi_a \in \square : \qquad G_i = \frac{1}{2}\lambda_i$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{73}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\varphi_a \in \ydiagram{2,1} : \quad G_i = T_i\,, \quad (T_a)_{bc} = -if_{abc}(\text{structure const.}) \tag{74}$$

$$\begin{gathered} T_\pm = G_1 \pm iG_2\,, \quad V_\pm = G_4 \pm iG_5\,, \quad U_\pm = G_6 \pm iG_7\,, \\ T_3 = G_3\,, \quad Y = \frac{2}{\sqrt{3}}G_8 \end{gathered} \tag{75}$$

1.4 Interacting Fields (1/2)

1.4.1 The Interaction Picture

● Schrödinger Picture

For a given Hamiltonian H,

$$\begin{cases} i\partial_t \,|\Psi\!>_S = H\,|\Psi\!>_S \\ i\partial_t \mathcal{O}_S = 0 \end{cases} \tag{76}$$

No explicit t dependence in $\mathcal{O}_S$ is assumed. Formal solution to the Schrödinger equation is written as

$$|\Psi\!>_S = e^{-iHt}\,|\Psi_0\!>_S \tag{77}$$

Expectation value of $\mathcal{O}_S$ in a state $|\Psi\!>_S$ is given as

$${}_S\!<\!\Psi|\,\mathcal{O}_S\,|\Psi\!>_S \tag{78}$$

● Heisenberg Picture

Related to the Schrödinger picture by

$$\begin{cases} |\Psi\!>_H = e^{iHt}\,|\Psi\!>_S \\ \mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt} \end{cases} \tag{79}$$

so that

$$ {}_H\!<\Psi|\,\mathcal{O}_H\,|\Psi>_H = {}_S\!<\Psi|\,\mathcal{O}_S\,|\Psi>_S \tag{80} $$

They evolve in time as

$$ \begin{cases} i\partial_t\,|\Psi>_H = 0 \\ i\partial_t\mathcal{O}_H = [\mathcal{O}_H, H] \end{cases} \tag{81} $$

● The Interaction Picture

$$ H = H_0 + H_{int} \tag{82} $$

Related to the Schrödinger picture by

$$ \begin{cases} |\Psi>_I = e^{iH_0 t}\,|\Psi>_S \\ \mathcal{O}_I = e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t} \end{cases} \tag{83} $$

so that

$$ {}_I\!<\Psi|\,\mathcal{O}_I\,|\Psi>_I = {}_S\!<\Psi|\,\mathcal{O}_S\,|\Psi>_S \tag{84} $$

The interaction Hamiltonian in this picture is time dependent;

$$ H_I \overset{\leftarrow}{=} (H_{int})_I = e^{iH_0 t} H_{int} e^{-iH_0 t} \tag{85} $$

$|\Psi>_I$ and $\mathcal{O}_I$ evolve in time as

$$ \begin{cases} i\partial_t\,|\Psi>_I = e^{iH_0 t}(-H_0 + H)\,|\Psi>_S \\ \qquad\qquad = e^{iH_0 t} H_{int} e^{-iH_0 t} e^{iH_0 t}\,|\Psi>_S \\ \qquad\qquad = H_I\,|\Psi>_I \\ i\partial_t\mathcal{O}_I = [\mathcal{O}_I, H_0] \end{cases} \tag{86} $$

1.4.2 Dyson's Formula

Formal solution of Eq. (86) can not be written in the form like one in Eq. (77) since H_I is time dependent. Writing the solution as

$$ \begin{aligned} &|\Psi(t)>_I = U(t,t_0)\,|\Psi(t_0)>_I \\ &U(t,t) = 1 \quad \text{and} \quad U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1)\,, \end{aligned} \tag{87} $$

the time evolution unitary operator $U(t,t_0)$ is given as

$$ U(t,t_0) = Texp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)\,, \tag{88} $$

where T stands for time ordered product

$$T[\mathcal{O}_2(t')\mathcal{O}_1(t)] = \theta(t'-t)\mathcal{O}_2(t')\mathcal{O}_1(t) + \theta(t-t')\mathcal{O}_1(t)\mathcal{O}_2(t')\,. \tag{89}$$

In fact, for $t > t_0$

$$\begin{aligned} i\partial_t U(t,t_0) &= T\left[H_I(t) exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)\right] \\ &= H_I(t) T exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right) \\ &= H_I(t)U(t,t_0) \end{aligned} \tag{90}$$

and the conditions for $U(t,t_0)$ in Eq. (87) is obviously satisfied. We may write Eq. (88) in a form of series as

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} H_I(t')dt' \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\; T[H_I(t')H_I(t'')] + \cdots \end{aligned} \tag{91}$$

However,

$$\begin{aligned} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\; T[\mathcal{O}(t')\mathcal{O}(t'')] &= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\; \mathcal{O}(t')\mathcal{O}(t'') + \int_{t_0}^{t} dt' \int_{t'}^{t} dt''\; \mathcal{O}(t'')\mathcal{O}(t') \\ &= 2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\; \mathcal{O}(t')\mathcal{O}(t'') \end{aligned}$$

and

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} H_I(t')dt' \\ &\quad + (-i)^2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + \cdots \end{aligned} \tag{92}$$

1.4.3 The S-matrix

Dealing with the scattering, we ***assume*** that initial and final states are eigenstates of H_0. To be more concrete, let $\mathcal{O}_0$ be a complete set of observables that includes H_0 and no one of them depends on t in an explicit manner. In the interaction picture, observables in $\mathcal{O}_0$ are constant as operators. [See Eq. (86).] What we have assumed is that the initial and final states are eigenstates of $\mathcal{O}_0$.

The scattering process takes place as follows. At $t_1 \to -\infty$, the system is in the initial state $|i\!>$, the system evolves in time by $U(t_2,t_1)$ under the effect of H_I, then, at $t_2 \to \infty$, the system turns in to the scattered state. The amplitude to find a final state $|f\!>$ in the scattered state is

$$\lim_{\substack{t_1\to-\infty\\ t_2\to\infty}} <\!f|\,U(t_2,t_1)\,|i\!> \;\stackrel{\to}{=}\; <\!f|\,S\,|i\!> \tag{93}$$

This is the definition of the S-matrix. Obviously, S is unitary. Substituting Eq. (91) or (92) in Eq. (93), we may consider the operator S as given in the form of a perturbation series. When the expression of Eq. (91) is used, it reads

$$\begin{aligned} S-1 &= -i\int_{-\infty}^{\infty} H_I(t')dt' \\ &\quad + \frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt'dt''\; T[H_I(t')H_I(t'')] + \cdots \end{aligned} \tag{94}$$

1.5 Scalar Fields

Irreducible representations of the Loerentz group solely determines forms of free field equations. Irreducible representations are classified by spins of particles.

1.5.1 Real Scalar Free Field

Spinless and neutral.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\underline{\partial}\varphi(\underline{x})\right)^2 - \frac{1}{2}m^2\varphi^2(\underline{x}) \\ &= \frac{1}{2}\partial_\mu\varphi(\underline{x})\cdot\partial^\mu\varphi(\underline{x}) - \frac{1}{2}m^2\varphi^2(\underline{x}) \end{aligned} \tag{95}$$

where a dot in the last equation indicates the former derivative acting only on the first φ. We frequently omit this dot. Under a Lorentz transformation $\underline{x} \mapsto \underline{x}' = L\underline{x}$, the field $\varphi(\underline{x})$ transforms as

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x)\,. \tag{96}$$

Here and hereafter, we omitt underlines on Lorentz vectors.

$$\begin{gathered} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi \\ \pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\varphi(x)} = \dot\varphi \end{gathered} \tag{97}$$

Euler-Lagrange equation

$$\left(\Box + m^2\right)\varphi(x) = 0 \qquad \text{Klein-Gordon} \tag{98}$$

where $\Box \stackrel{\leftarrow}{=} \partial^2 = \partial_\mu \partial^\mu = \partial_0^2 - \boldsymbol{\partial}^2$.
Classical solution:

$$\varphi_{cl}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}2k^0} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^*(\boldsymbol{k})e^{ik\cdot x}\right] \tag{99}$$

where $k^0 = +\sqrt{\boldsymbol{k}^2 + m^2}$. Canonical momentum field reads,

$$\pi_{cl}(x) = \partial_0 \varphi_{cl}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}2k^0} \left[-ik^0 a(\boldsymbol{k})e^{-ik\cdot x} + ik^0 a^*(\boldsymbol{k})e^{ik\cdot x}\right] \tag{100}$$

Canonical quantization

$$\begin{aligned} [\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] &= i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \\ [\varphi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{y})] = 0\,, &\quad [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0\,. \end{aligned} \tag{101}$$

This is equivalent with requiring

$$\begin{aligned} [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] &= 2k^0\delta^3(\boldsymbol{k}-\boldsymbol{k}')\ , \\ \left[a(\boldsymbol{k}), a(\boldsymbol{k}')\right] = 0\ , &\quad \left[a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')\right] = 0 \end{aligned} \tag{102}$$

and writing

$$\varphi(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}2k^0} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right] \tag{103}$$

$$\pi(x) = \frac{-i}{2} \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} \left[a(\boldsymbol{k})e^{-ik\cdot x} - a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right]\ , \tag{104}$$

They have the same form as Eqs. (99) and (100) but now the coefficients $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ are quantum operators.

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Addendum: Proof of (101) $\Leftrightarrow$ (102)

Proof of the necessity of Eq. (102) is straightforward, We show the sufficiency of Eq. (101) in the following. We may write Eqs. (103) and (104) as

$$\begin{aligned} \varphi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} Q_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \\ \pi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} P_{\boldsymbol{k}}(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \end{aligned} \tag{105}$$

with

$$Q_{\boldsymbol{k}}(t) = \frac{1}{2k^0}\left[a(\boldsymbol{k})e^{-ik^0 t} + a^\dagger(-\boldsymbol{k})e^{ik^0 t}\right] \tag{106}$$

$$P_{\boldsymbol{k}}(t) = \frac{i}{2}\left[a^\dagger(\boldsymbol{k})e^{ik^0t} - a(-\boldsymbol{k})e^{-ik^0t}\right] \tag{107}$$

Relationships $Q^\dagger_{\boldsymbol{k}} = Q_{-\boldsymbol{k}}$ and $P^\dagger_{\boldsymbol{k}} = P_{-\boldsymbol{k}}$ ensure that φ and π are real. From the linear independence of Fourier components, we have

$$\begin{aligned} 0 = [\varphi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{y})] \iff [Q_{\boldsymbol{k}}(t), Q_{\boldsymbol{k}'}(t)] = 0\,, \\ 0 = [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] \iff [P_{\boldsymbol{k}}(t), P_{\boldsymbol{k}'}(t)] = 0\,, \end{aligned} \tag{108}$$

and

$$\begin{aligned} i\delta^3(\boldsymbol{x}-\boldsymbol{y}) &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3} i\delta^3(\boldsymbol{k}-\boldsymbol{k}')e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k}'\boldsymbol{y}} \\ &= [\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] \\ &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3} [Q_{\boldsymbol{k}}, P_{\boldsymbol{k}'}]e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{k}'\cdot\boldsymbol{y}} \\ &\iff \\ [Q_{\boldsymbol{k}}, P_{\boldsymbol{k}'}] &= i\delta^3(\boldsymbol{k}-\boldsymbol{k}') \end{aligned} \tag{109}$$

Eqs. (106) and (107) reads,

$$a(\boldsymbol{k}) = \left(k^0 Q_{\boldsymbol{k}}(t) + iP^\dagger_{\boldsymbol{k}}(t)\right)e^{ik^0t}\,, \tag{110}$$

$$a^\dagger(\boldsymbol{k}) = \left(k^0 Q^\dagger_{\boldsymbol{k}}(t) - iP_{\boldsymbol{k}}(t)\right)e^{-ik^0t}\,, \tag{111}$$

and Eq. (102) follows from Eqs. (108) and (109).

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Notice that $dim[\varphi] = E^1$ as can be seen from Eq. (95) and $dim[\pi] = E^2$. Thus they are physical quantities quite different from those in the case of the Schrödinger field theory. Nevertheless, operators a and $a^\dagger$ satisfy conditions to be annihilation and creation operators. We define the vacuum state by

$$a(\boldsymbol{k})\,|0\!>= 0 \tag{112}$$

Hamiltonian density in the classical level is obtained from Eq. (42) as

$$\begin{aligned} \mathcal{H} &= \pi(x)\dot{\varphi}(x) - \frac{1}{2}\left\{(\dot{\varphi}(x))^2 - (\boldsymbol{\partial}\varphi(x))^2\right\} + \frac{1}{2}m^2\varphi^2(x) \\ &= \frac{1}{2}\left\{\pi^2(x) + (\boldsymbol{\partial}\varphi(x))^2 + m^2\varphi^2(x)\right\} \end{aligned} \tag{113}$$

To get the quantum level Hamiltonian, we substitute Eqs (103) and (104) into Eq. (113). However, since $\mathcal{H}$ is quadratic in operators, we need to specify the ordering of them and write

$$\hat{H} = \int d^3\boldsymbol{x}\hat{\mathcal{H}}(x)\,, \quad \hat{\mathcal{H}}(x) = :\frac{1}{2}\left\{\pi^2(x) + (\boldsymbol{\partial}\varphi(x))^2 + m^2\varphi^2(x)\right\}:\,, \tag{114}$$

where :...: denotes the normal product. As the result, we obtain an expression

$$\hat{H} = \int \frac{d^3\boldsymbol{k}}{2k^0} k^0 a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{115}$$

This form indicates that our Hamiltonian is diagonalized. To see this let us define a single particle state $|\boldsymbol{p}> = a^\dagger(\boldsymbol{p})\,|0>$. [1] We see from Eq. (102) that $\hat{H}\,|\boldsymbol{p}> = E\,|\boldsymbol{p}>$ with $E = \sqrt{\boldsymbol{p}^2 + m^2}$. Thus, our field acquires the particle interpretation through the canonical quantization.

The total number operator is defined as

$$\hat{N} = \int \frac{d^3\boldsymbol{k}}{2k^0} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{116}$$

It holds that

$$\begin{aligned}
&\hat{N} a(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}'}{2k^{0\prime}} \left\{ a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}') - 2k^0 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \right\} a(\boldsymbol{k}') = a(\boldsymbol{k})(\hat{N} - 1)\,, \\
&\hat{N} a^\dagger(\boldsymbol{k}) = a^\dagger(\boldsymbol{k})(\hat{N} + 1)
\end{aligned} \tag{117}$$

Hamiltonian in Eq. (115) coinsides with one derived (with a care on the operator orderings) from Eq. (52) as $\int d^3\boldsymbol{x} T^{00}(x)$. Other components of $T^{0\mu}$ give an expression for the momentum as

$$\hat{\boldsymbol{P}} = \int \frac{d^3\boldsymbol{k}}{2k^0} \boldsymbol{k} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{118}$$

We certainly have $\hat{\boldsymbol{P}}\,|\boldsymbol{p}> = \boldsymbol{p}\,|\boldsymbol{p}>$. We omit here to write expressions for angular momenta corresponding to Eq. (62).

1.5.2 Complex Scalar Free Field

Spinless and charged.

$$\mathcal{L} = \partial_\mu \varphi^\dagger(x) \cdot \partial^\mu \varphi(x) - m^2 \varphi^\dagger(x) \varphi(x) \tag{119}$$

$$\left(\Box + m^2\right) \varphi(x) = 0\,, \qquad \left(\Box + m^2\right) \varphi^\dagger(x) = 0\,, \tag{120}$$

$$\begin{aligned}
\varphi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k}) e^{-ik\cdot x} + b^\dagger(\boldsymbol{k}) e^{ik\cdot x} \right] \\
\varphi^\dagger(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[b(\boldsymbol{k}) e^{-ik\cdot x} + a^\dagger(\boldsymbol{k}) e^{ik\cdot x} \right]
\end{aligned} \tag{121}$$

[1] This state is normalized as

$$<p|\,p'> = <0|\,[a(p), a^\dagger(p')]\,|0> = 2k^0 \delta^3(\boldsymbol{p} - \boldsymbol{p}')$$

Canonical quantization

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})] = [b(\boldsymbol{k}), b^\dagger(\boldsymbol{k'})] = \delta^3(\boldsymbol{k} - \boldsymbol{k'}) \,, \qquad \text{other commutators} = 0 \tag{122}$$

Four momentum

$$\hat{P}^\mu = \int \frac{d^3\boldsymbol{k}}{2k^0} k^\mu \left(a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + b^\dagger(\boldsymbol{k}) b(\boldsymbol{k}) \right) \tag{123}$$

$\hat{P}^0$ is the Hamiltonian. Vacuum state is defined as $a(\boldsymbol{p})\,|0>= b(\boldsymbol{p})\,|0>= 0$. There are two kinds of single particle states $a^\dagger(\boldsymbol{p})\,|0>$ and $b^\dagger(\boldsymbol{p})\,|0>$ both an eigenstate of $\hat{P}^\mu$. So far, a and b are just particles independent of each other and having the common mass m. Let us examine the electric charge given from Eq. (66). We find

$$\hat{Q} = e \int \frac{d^3\boldsymbol{k}}{2k^0} \left(a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) - b^\dagger(\boldsymbol{k}) b(\boldsymbol{k}) \right) \tag{124}$$

Thus they carry opposite electric charges.

We consider now inversion symmetries $U = P, C, T$. The vacuum is invariant under these inversions.

- Space inversion: $(\boldsymbol{x}, t) \mapsto (-\boldsymbol{x}, t)$

  $$Pa(\boldsymbol{k})P^{-1} = \pm a(-\boldsymbol{k}) \,, \quad Pb(\boldsymbol{k})P^{-1} = \pm b(-\boldsymbol{k}) \,, \tag{125}$$

  $+$ for scalar and $-$ for pseudoscalar.

- Charge conjugation

  $$Ca(\boldsymbol{k})C^{-1} = \pm b(\boldsymbol{k}) \,, \quad Cb(\boldsymbol{k})C^{-1} = \pm a(\boldsymbol{k}) \,, \tag{126}$$

- Time reversal: $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}, -t)$

  $$Ta(\boldsymbol{k})T^{-1} = \pm a(-\boldsymbol{k}) \,, \quad Cb(\boldsymbol{k})C^{-1} = \pm b(-\boldsymbol{k}) \,, \tag{127}$$

  and T is antilinear so that

  $$T\varphi(\boldsymbol{x}, t)T^{-1} = \pm \varphi(\boldsymbol{x}, -t) \tag{128}$$

  The sign is fixed through the invariance of interactions with other kind of fields which have definite signatures under the T transformation. Usually, $+$ for scalar and $-$ for pseudoscalar.

1.5.3 Internal Symmetry

Complex scalar field with internal symmetry

$$\mathcal{L} = \sum_a \left[\partial_\mu \varphi_a^\dagger(x) \cdot \partial^\mu \varphi_a(x) - m^2 \varphi_a^\dagger(x)\varphi_a(x)\right] \tag{129}$$

We consider scalar fields $\varphi_a(x)$ transforms under a global $SU(n)$ as in Eq. (67). We assume $\{\varphi_a(x)\}$ composes an irreducible representation ν and write them in a combined form $\varphi = (\varphi_1, \ldots, \varphi_s)$ where s is the multiplicity of the representation ν. We may write the $\mathcal{L}$agrangian exactly in the same form as Eq. (119) understanding our φ is now multicomponent. We may then write the Noether current and charge in Eqs (69) and (70) interms creation and annihilation operators. States $\{a_j^\dagger\,|0\!>\}$ composes a multiplet of ν. (To be continued.)

1.5.4 Scalar Yukawa Theory - A Toy Model -

Let us consider a toy model composed of scalar fields:

$$\mathcal{L} \;=\; \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2}(\partial_\mu \phi)^2 - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g\psi^*\psi\phi \tag{130}$$

ψ is a mock nucleon complex scalar field with the mass M and ϕ is a mock pion real scalar field with the mass m. The first parts of $\mathcal{L}$ except for the last term compose free $\mathcal{L}$agrangians $\mathcal{L}_N$ and $\mathcal{L}_\pi$ for nucleon and pion fields, respectively. The last term is the interaction $\mathcal{L}$agrangian in which nucleons and pions interact each other with a coupling constant g. The coupling g has the dimension of energy and the dimensionless parameter is g/E, where E is the energy scale of the process of interest. This means that the interaction term $\mathcal{L}_{int} = -g\psi^*\psi\phi$ is relevant at low energies. The relativistic nature gets important at $E \gg M, m$ and we may choose $g \ll M, m$ so that the perturbation series (92) converges.

Each fields are quantized through equal-time commutation relations at time, say, 0. One may write down field equations but they are not solvable due to the presence of $\mathcal{L}_{int}$. Remember they describe time evolution of field operators in the Heisenberg picture. In the interaction picture, however, we may let fields obey free field equations. In practice, conjugate fields are given by

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} = \dot{\phi}\,, \quad \pi_\psi = \frac{\partial \mathcal{L}}{\partial(\dot{\psi})} = \dot{\psi}^*\,, \quad \pi_{\psi^*} = \frac{\partial \mathcal{L}}{\partial(\dot{\psi}^*)} = \dot{\psi}\,, \tag{131}$$

and classical Hamiltonian density is written as

$$\begin{aligned}\mathcal{H} &= \pi_\phi \dot{\phi} + \pi_\psi \dot{\psi} + \pi_{\psi^*} \dot{\psi}^* - \mathcal{L} \\ &= \frac{1}{2}\left\{\pi_\phi^2 + (\boldsymbol{\partial}\phi)^2 + m^2\phi^2\right\} + \left\{\pi_\psi \pi_{\psi^*} + \boldsymbol{\partial}\psi^* \cdot \boldsymbol{\partial}\psi + M^2 \psi^*\psi\right\} \\ &\quad + g\psi^*\psi\phi \end{aligned} \tag{132}$$

The first two terms are $\mathcal{H}_\pi$ and $\mathcal{H}_N$ corresponding to $\mathcal{L}_\pi$ and $\mathcal{L}_N$, respectively, and we assign them as the free $\mathcal{H}$amiltonian $\mathcal{H}_0 = \mathcal{H}_\pi + \mathcal{H}_N$. Writing $\mathcal{H}_{int} = g\psi^*\psi\phi$, we have a decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$, which corresponds to Eq. (82). After substituting quantized fields (at the fixed time) into these expressions (and taking the normal ordering), we obtain $\mathcal{H}$amiltonian operators. The time evolutions of fields in the interaction picture are given from Eq. (86) as

$$\begin{aligned} i\partial_t \phi_I &= [\phi_I, H_0] = [\phi_I, H_\pi]\,, \\ i\partial_t \psi_I &= [\psi_I, H_0] = [\psi_I, H_N]\,, \\ i\partial_t \psi_I^\dagger &= [\psi_I^\dagger, H_0] = [\psi_I^\dagger, H_N]\,, \end{aligned} \tag{133}$$

where we added an suffix I to indicate quantities in the interaction picture. We already know that Eqs. (133) are equivalent to free field equations all given as the Klein-Gordon equation with masses of each fields. Therefore, the fields ϕ_I, ψ_I and $\psi_I^\dagger$ have Fourier expanded forms as in Eqs. (103) and (121). To be specific, we write

$$\begin{aligned} \phi_I(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right] \\ \psi_I(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \left[b(\boldsymbol{p})e^{-ip\cdot x} + c^\dagger(\boldsymbol{p})e^{ip\cdot x}\right] \\ \psi_I^\dagger(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \left[c(\boldsymbol{p})e^{-ip\cdot x} + b^\dagger(\boldsymbol{p})e^{ip\cdot x}\right] \end{aligned} \tag{134}$$

Operators a, b and c and their conjugates satisfy commutation relations given in Eqs. (102) and (122) and they establishes the particle interpretations. Parts of the free $\mathcal{H}$amiltonian $\mathcal{H}_\pi$ and $\mathcal{H}_N$ are now described by these operators as in Eq. (115) and the time component of Eq. (123). We may define number operator $\hat{N}_\pi$ as in Eq. (116) and ones for nucleons as

$$\hat{N}_N = \int \frac{d^3\boldsymbol{p}}{2p^0} b^\dagger(\boldsymbol{p})b(\boldsymbol{p})\,, \quad \hat{N}_{\bar{N}} = \int \frac{d^3\boldsymbol{p}}{2p^0} c^\dagger(\boldsymbol{p})c(\boldsymbol{p})\,, \tag{135}$$

where we have assigned $b^\dagger$ and $c^\dagger$ as creation operators for a nucleon (N) and an anti-nucleon ($\bar{N}$), respectively. In the interaction picture, $\mathcal{L}_{int}$ term in Eq.(130) [or $\mathcal{H}_{int}$ term in Eq. (132)] contains creation and annihilation operators in each field and they may change number of particles. In practice, the number operators we have just defined do not commute with H_I, therefore neither with H and do not conserve. Though numbers of particles are not conserved, the total electric charge would be conserved since the $\mathcal{L}$agrangian (130) is invariant under constant phase change of the field ψ and its conjugate for ψ^*. According to the Noether's theorem and Eq. (124), the charge

$$Q = e \int \frac{d^3\boldsymbol{p}}{2p^0} \left(b^\dagger(\boldsymbol{p}) b(\boldsymbol{p}) - c^\dagger(\boldsymbol{p}) c(\boldsymbol{p}) \right) \tag{136}$$

commutes with H and conserved.

Keeping the formula (92) in mind, let us examine particular expression of H_I:

$$\begin{aligned}
H_I(t) &= g \int d^3\boldsymbol{x} {:}\psi_I^\dagger(x)\psi_I(x)\phi_I(x){:} \\
&= \frac{g}{\sqrt{(2\pi)^9}} \int \frac{d^3\boldsymbol{p}}{2p^0} \frac{d^3\boldsymbol{p}'}{2p^{0\prime}} \frac{d^3\boldsymbol{k}}{2k^0} \int d^3\boldsymbol{x} : [e^{-ipx} c(\boldsymbol{p}) + e^{ipx} b^\dagger(\boldsymbol{p})] \\
&\qquad [e^{-ip'x} b(\boldsymbol{p}') + e^{ip'x} c^\dagger(\boldsymbol{p}')][e^{-ikx} a(\boldsymbol{k}) + e^{ikx} a^\dagger(\boldsymbol{k})] : \\
&= \frac{g}{\sqrt{(2\pi)^3}} \int \frac{d^3\boldsymbol{p}}{2p^0} \frac{d^3\boldsymbol{p}'}{2p^{0\prime}} \frac{d^3\boldsymbol{k}}{2k^0} \\
&\quad [c(\boldsymbol{p}) b(\boldsymbol{p}') a(\boldsymbol{k}) \delta^3(\boldsymbol{p} + \boldsymbol{p}' + \boldsymbol{k}) e^{-i(p+p'+k_+)^0 t} \\
&\quad + b^\dagger(\boldsymbol{p}) c^\dagger(\boldsymbol{p}') a^\dagger(\boldsymbol{k}) \delta^3(\boldsymbol{p} + \boldsymbol{p}' + \boldsymbol{k}) e^{i(p+p'+k_+)^0 t} \\
&\quad + b^\dagger(\boldsymbol{p}) c^\dagger(\boldsymbol{p}') a(\boldsymbol{k}) \delta^3(\boldsymbol{p} + \boldsymbol{p}' - \boldsymbol{k}) e^{i(p+p'-k_+)^0 t} \\
&\quad + c^\dagger(\boldsymbol{p}) c(\boldsymbol{p}') a(\boldsymbol{k}) \delta^3(\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{k}) e^{i(p-p'-k_-)^0 t} \\
&\quad + b^\dagger(\boldsymbol{p}) b(\boldsymbol{p}') a(\boldsymbol{k}) \delta^3(\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{k}) e^{i(p-p'-k_-)^0 t} \\
&\quad + a^\dagger(\boldsymbol{k}) c(\boldsymbol{p}) b(\boldsymbol{p}') \delta^3(\boldsymbol{p} + \boldsymbol{p}' - \boldsymbol{k}) e^{-i(p+p'-k_-)^0 t} \\
&\quad + a^\dagger(\boldsymbol{k}) c^\dagger(\boldsymbol{p}') c(\boldsymbol{p}) \delta^3(\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{k}) e^{-i(p-p'-k_-)^0 t} \\
&\quad + a^\dagger(\boldsymbol{k}) b^\dagger(\boldsymbol{p}) b(\boldsymbol{p}') \delta^3(-\boldsymbol{p} + \boldsymbol{p}' - \boldsymbol{k}) e^{i(p-p'+k_-)^0 t}]
\end{aligned} \tag{137}$$

where $k_\pm^0 = \sqrt{(\boldsymbol{p} \pm \boldsymbol{p}')^2 + m^2}$. At this stage, we may already have some insights about the interaction. Our $\mathcal{L}_{int}$ is a product of three field operators and each of them involve two terms with annihilation and creation operators. This is why there are $2^3 = 8$ terms in Eq. (137). Among them, the last 6 terms show possible processes. For instance, the third term corresponds a process in which a particle a (pion) disappears and particles b and c (nucleon and anti-nuclen) emerges. So this term corresponds to the pair creation of $N\bar{N}$ by π. One can confirm the momentum is conserved in the process. The energy is not conserved yet and there are exponential factors instead. Later, we will see these exponentials turn into delta functions corresponding to the energy conservation for each processes in the evaluation the scattering matrix. The first two terms in Eq. (137) will not contribute to scattering matrices since they violate the energy conservation. The second and higher order terms in Eq. (92) will be involved in the later discussion.

● Meson Decay

Consider a process $\pi \to N\bar{N}$. This process is involved in the lowest order term in Eq. (94) through the third term in Eq. (137). We write Initial and final states as

$$\begin{aligned} |i\!> &= a^\dagger(\boldsymbol{k})\,|0\!> \;\stackrel{\rightarrow}{=}\; |\pi(\boldsymbol{k})\!>, \\ |f\!> &= b^\dagger(\boldsymbol{p}_N)c^\dagger(\boldsymbol{p}_{\bar{N}})\,|0\!> \;\stackrel{\rightarrow}{=}\; |N(\boldsymbol{p}_N)\bar{N}(\boldsymbol{p}_{\bar{N}})\!> \end{aligned} \tag{138}$$

From Eq. (94), we read to the leading order in g that

$$\begin{aligned} <\!f|\,S\,|i\!> &= -ig\int d^4x <\!N(\boldsymbol{p}_N)\bar{N}(\boldsymbol{p}_{\bar{N}})|\,:\!\psi^\dagger(x)\psi(x)\phi(x)\!:|\pi(\boldsymbol{k})\!> \\ &= -ig\int d^4x <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})\!:\!\psi^\dagger(x)\psi(x)\phi(x)\!:\!a^\dagger(\boldsymbol{k})\,|0\!> \end{aligned} \tag{139}$$

Here we omitted indeces I on fields under understanding that we are in the interaction picture. In expanding fields as $\psi^\dagger \sim c+b^\dagger$, $\psi \sim b+c^\dagger$ and $\phi \sim a+a^\dagger$, we find only a term $\sim b^\dagger c^\dagger a$ contributes in Eq. (139). This corresponds to the third term in H_I in Eq. (137). We proceed from Eq. (139):

$$\begin{aligned} <\!f|\,S\,|i\!> &= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}'}{2k^{0\prime}}\int d^4x\; e^{i(p+p'-k')x} \\ &\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})[b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a(\boldsymbol{k}')]a^\dagger(\boldsymbol{k})\,|0\!> \\ &= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}'}{2k^{0\prime}}(2\pi)^4\delta^4(p+p'-k') \\ &\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})[b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a(\boldsymbol{k}')]a^\dagger(\boldsymbol{k})\,|0\!> \\ &= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}(2\pi)^4\delta^4(p+p'-k) \\ &\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')\,|0\!> \\ &= \frac{-ig}{\sqrt{(2\pi)^9}}(2\pi)^4\delta^4(p_N+p_{\bar{N}}-k) \end{aligned} \tag{140}$$

Reaction rate is defined in the same way as that for scatterting as

$$R_{fi} = \int d\Phi_f|<\!f|\,T\,|i\!>|^2, \tag{141}$$

where Φ_f and T are defined in Eqs. (**??**) and (**??**) respectively [2]. For decays of a particle with the mass M into n particles, differential decay width is defined

[2] A note on dimensions: $dim[d\Phi_f(<\!f|\,)^2] = 1/E^4$, $dim[T] = E^4$ and $dim[\,|i\!>]^2 = 1/E^{2k}$ for an initial state composed of k particles. In total, $dim[R] = E^{4-2k}$. $k=2$ and $dim[R] = E^0$ for scatterings. $k=1$ and $dim[R] = E^2$ for particle decays.

as

$$d\Gamma = \frac{(2\pi)^3}{2M} dR_{fi} = \frac{(2\pi)^3}{2M} \prod_i^n \frac{d^3\boldsymbol{p}_i}{2p_i^0} (2\pi)^4 \delta^4(P_f - P_i) | <f| \, T \, |i> |^2 \tag{142}$$

In particular, for two particle decays,

$$\begin{aligned}
d\Gamma_2 &= \frac{(2\pi)^3}{2M} \frac{d^3\boldsymbol{p}_1}{2p_1^0} d^4 p_2 \delta(p_2^2 - m_2^2)(2\pi)^4 \delta^4(P_f - P_i) | <f| \, T \, |i> |^2 \\
&= \frac{(2\pi)^7}{2M} \frac{P_1^{*2} dP_1^* d\Omega_1^*}{2p_1^{*0}} \delta(M^2 + m_1^2 - m_2^2 - 2Mp_1^{*0}) |T_{fi}|^2 \\
&= \frac{(2\pi)^9}{32\pi^2 M^2} P_1^* d\Omega_1^* |T_{fi}|^2
\end{aligned} \tag{143}$$

We note that $dim|T_{fi}|^2 = E^2$. The way of writing the coefficient in Eq. (143) is chosen so that it is clear that a factor $(2\pi)^9$ is absorbed in the state normalizations in notations of some authors including [1], [2] and [3].

Let's come back to our toy model. From Eq. (140), we have $T_{fi} = g/\sqrt{(2\pi)^9}$. Substituting this in Eq. (143), we obtain

$$\Gamma_2^{(1)} = \frac{g^2}{16\pi m^2} \lambda^{1/2}(m^2, M^2, M^2) \tag{144}$$

where a superscript (1) indicates the order of perturbation expansion in Eq. (94).

1.6 Interacting Fields (2/2)

To proceed beyond the second term in Eq. (94) requires some further preparations.

1.6.1 Propagator

Let's take a real scalar field.

$$\begin{aligned}
<0| \, \varphi(x)\varphi(y) \, |0> &= \int \frac{d^3\boldsymbol{k} d^3\boldsymbol{k}'}{(2\pi)^3 4k^0 k^{0\prime}} <0| \, a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}') \, |0> e^{-i(kx - k'y)} \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} e^{-ik(x-y)} \overset{\rightarrow}{=} D(x-y)
\end{aligned} \tag{145}$$

$$
\begin{aligned}
[\varphi(x), \varphi(y)] &= \int \frac{d^3\boldsymbol{k} d^3\boldsymbol{k}'}{(2\pi)^3 4k^0 k^{0\prime}} \Big([a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] e^{-i(kx - k'y)} \\
&\qquad\qquad + [a \dagger (\boldsymbol{k}), a(\boldsymbol{k}')] e^{i(kx - k'y)} \Big) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} \left(e^{-ik(x-y)} - e^{ik(x-y)} \right) \\
&= D(x-y) - D(y-x) \qquad (146)
\end{aligned}
$$

When $x - y$ is spacelike, there exists a Lorentz frame where $x^0 - y^0$. Then the *r.h.s.* of Eq. (146) vanishes. The whole expression is Lorentz invariant and it must vanish for all $(x-y)^2 < 0$. Nevertheless, $D(x-y)$ itself does not vanish even $x - y$ is spacelike.

The Feynman propagator is defined as

$$
\begin{aligned}
<0| T[\varphi(x)\varphi(y)] |0> &= \theta(x^0 - y^0) D(x-y) + \theta(y^0 - x^0) D(y-x) \qquad (147) \\
&= i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \qquad (148) \\
&\overset{\rightarrow}{\equiv} \Delta_F(x-y)
\end{aligned}
$$

Proof of Eq.(148) $\Leftrightarrow$ *Eq. (147)*

$$
\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2E_{\boldsymbol{p}}} \left(\frac{1}{p^0 - E_{\boldsymbol{p}} + i\epsilon} - \frac{1}{p^0 + E_{\boldsymbol{p}} - i\epsilon} \right), \qquad (149)
$$

$$
\begin{aligned}
\Delta_F(x-y) &= i \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&\quad \times \int \frac{dp^0}{2\pi} e^{-ip^0(x^0 - y^0)} \left(\frac{1}{p^0 - E_{\boldsymbol{p}} + i\epsilon} - \frac{1}{p^0 + E_{\boldsymbol{p}} - i\epsilon} \right) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{-1}{2\pi i} \Big(\theta(x^0 - y^0)(-2\pi i e^{-iE_{\boldsymbol{p}}(x^0 - y^0)}) \\
&\quad - \theta(y^0 - x^0)(+2\pi i e^{iE_{\boldsymbol{p}}(x^0 - y^0)}) \Big) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(\theta(x^0 - y^0) e^{-ip(x-y)} + \theta(y^0 - x^0) e^{ip(x-y)} \right) \\
&= \theta(x^0 - y^0) D(x-y) + \theta(y^0 - x^0) D(y-x) \\
&\qquad \blacksquare \qquad (150)
\end{aligned}
$$

1.6.2 T product

We decompose a real scalar field as

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x) \tag{151}$$

where $\varphi^{(+)}$ ($\varphi^{(-)}$) is the term which contain annihilation (creation) operator in Eq. (134). If $x^0 > y^0$,

$$\begin{aligned}
T[\varphi(x)\varphi(y)] &= (\varphi^{(+)}(x) + \varphi^{(-)}(x))(\varphi^{(+)}(y) + \varphi^{(-)}(y)) \\
&= \varphi^{(+)}(x)\varphi^{(+)}(y) + ([\varphi^{(+)}(x), \varphi^{(-)}(y)] + \varphi^{(-)}(y)\varphi^{(+)}(x)) \\
&\quad + \varphi^{(-)}(x)\varphi^{(+)}(y) + \varphi^{(-)}(x)\varphi^{(-)}(y) \\
&= {:}\varphi(x)\varphi(y){:} + D(x-y)
\end{aligned}$$

and if $y^0 > x^0$,

$$T[\varphi(x)\varphi(y)] = {:}\varphi(x)\varphi(y){:} + D(y-x)$$

Then, for arbitrary x^0 and y^0,

$$T[\varphi(x)\varphi(y)] = {:}\varphi(x)\varphi(y){:} + \Delta_F(x-y) \tag{152}$$

A similar evaluation shows for a complex scalar field that

$$T[\varphi(x)\varphi^\dagger(y)] = {:}\varphi(x)\varphi^\dagger(y){:} + \Delta_F(x-y) \tag{153}$$

•Wick's theorem

$$\begin{aligned}
& T[\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\ldots] \\
= \; & {:}\varphi_1\varphi_2\varphi_3\varphi_4\ldots{:} \\
+ \; & \sum_{k<l} \Delta_F(x_k - x_l){:}\varphi_1(x_1)\ldots \cancel{\varphi_k} \ldots \cancel{\varphi_l} \ldots{:} \\
+ \; & \sum_{k<l}\sum_{m<n} \Delta_F(x_k - x_l)\Delta_F(x_m - x_n){:}\ldots \cancel{\varphi_k} \ldots \cancel{\varphi_m} \ldots \cancel{\varphi_l} \ldots \cancel{\varphi_n} \ldots{:} \\
+ \; & \cdots
\end{aligned} \tag{154}$$

1.6.3 Scattering of Scalar Nucleons

Let's go back to the scalar Yukawa theory and consider nucleon scattering process $\psi\psi \to \psi\psi$.

$$\begin{aligned}
|i> &= b^\dagger(\boldsymbol{p}_a)b^\dagger(\boldsymbol{p}_b)\,|0> \stackrel{\rightarrow}{=} |N_a N_b> \\
|f> &= b^\dagger(\boldsymbol{p}_1)b^\dagger(\boldsymbol{p}_2)\,|0> \stackrel{\rightarrow}{=} |N_1 N_2>
\end{aligned} \tag{155}$$

The first contribution to $S-1$ in Eq. (94) arise from the second order term in H_I. It reads

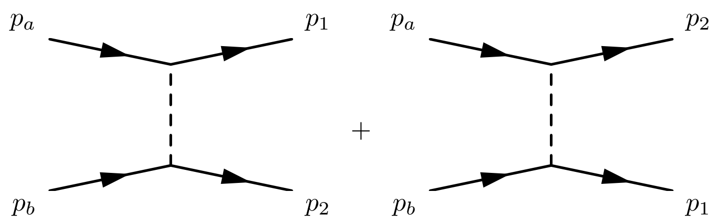
$$
\begin{aligned}
<f|\, S-1\, |i> &= \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 <f|\, T[\psi^\dagger(x_1)\psi(x_1)\phi(x_1) \\
&\qquad\qquad\qquad\qquad \times\psi^\dagger(x_2)\psi(x_2)\phi(x_2)]\,|i> \\
&= \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \Delta_F(x_1-x_2) \\
&\qquad <f|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):|i>
\end{aligned}
\tag{156}
$$

$$
\begin{aligned}
& <N_1N_2|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):|N_aN_b> \\
= & \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
& <0|\, b(\boldsymbol{p}_1)b(\boldsymbol{p}_2)b^\dagger(\boldsymbol{p}'_1)b^\dagger(\boldsymbol{p}'_2)b(\boldsymbol{p}'_a)b(\boldsymbol{p}'_b)b^\dagger(\boldsymbol{p}_a)b^\dagger(\boldsymbol{p}_b)\,|0> \\
= & \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
& <0|\, b(\boldsymbol{p}_1)\left\{2p_2^{0\prime}\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_2)+b^\dagger(\boldsymbol{p}'_2)b(\boldsymbol{p}_2)\right\} b^\dagger(\boldsymbol{p}'_1) \\
& b(\boldsymbol{p}'_a)\left\{2p_b^{0\prime}\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_b)+b^\dagger(\boldsymbol{p}_b)b(\boldsymbol{p}'_b)\right\} b^\dagger(\boldsymbol{p}_a)\,|0> \\
= & \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
& <0|\, \left\{4p_1^{0\prime}p_2^{0\prime}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_2)+4p_2^{0\prime}p_1^{0\prime}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_2)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_1)\right\} \\
& \left\{4p_a^{0\prime}p_b^{0\prime}\delta^3(\boldsymbol{p}'_a-\boldsymbol{p}_a)\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_b)+4p_a^{0\prime}p_b^{0\prime}\delta^3(\boldsymbol{p}'_a-\boldsymbol{p}_b)\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_a)\right\}\,|0> \\
= & \frac{1}{(2\pi)^6}\left\{e^{i(p_1x_1+p_2x_2)}+e^{i(p_2x_1+p_1x_2)}\right\}\left\{e^{-i(p_ax_1+p_bx_2)}+e^{-i(p_bx_1+p_ax_2)}\right\}
\end{aligned}
$$

Substituting this result and adopting expression (148),

$$
\begin{aligned}
<f|\,S-1\,|i> \quad &= \quad \frac{(-ig)^2}{2(2\pi)^6}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\int d^4x_1 d^4x_2 \Big\{ e^{i(p_1+k-p_a)x_1} e^{i(p_2-k-p_b)} \\
&\quad\quad + e^{i(p_2+k-p_a)x_1} e^{i(p_1-k-p_b)} + e^{i(p_1+k-p_b)x_1} e^{i(p_2-k-p_1)} \\
&\quad\quad + e^{i(p_2+k-p_b)x_1} e^{i(p_1-k-p_a)} \Big\} \\
&= \quad \frac{i(-ig)^2}{(2\pi)^6}\int \frac{(2\pi)^4 d^4k}{k^2-m^2+i\epsilon} \big\{ \delta^4(p_1+k-p_a)\delta^4(p_2-k-p_b) \\
&\quad\quad\quad\quad + \delta^4(p_2+k-p_a)\delta^4(p_1-k-p_b) \big\} \\
&= \quad \frac{i(-ig)^2}{(2\pi)^6} \left\{ \frac{1}{(p_1-p_a)^2-m^2} + \frac{1}{(p_2-p_a)^2-m^2} \right\} \\
&\quad\quad \times (2\pi)^4 \delta^4(p_1+p_2-p_a-p_b)
\end{aligned}
$$

$=$



(157)

1.7 Dirac Fields

Massive Spin 1/2.

1.7.1 Classical Free Field

$$\mathcal{L} = i\bar{\psi}(x)\partial\!\!\!/\psi(x) - m\bar{\psi}(x)\psi(x) \tag{158}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$.

$$\left(i\partial\!\!\!/ + m\right)\psi(x) = 0\,, \quad \bar{\psi}(x)\left(i \overleftarrow{\partial\!\!\!/} + m\right) = 0 \tag{159}$$

1.7.2 Quantized Free Field

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{s=\pm 1} \left[c(\boldsymbol{p},s)u(\boldsymbol{p},s)e^{-ipx} + d^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s)e^{ipx}\right] \tag{160}$$

$$\{c(\boldsymbol{p},s), c^\dagger(\boldsymbol{p}',s')\} = \{d(\boldsymbol{p},s), d^\dagger(\boldsymbol{p}',s')\} = 2p^0\delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}') \tag{161}$$

$$\begin{array}{ll} (p\!\!\!/ - m)u(\boldsymbol{p},s) = 0\,, & \bar{u}(\boldsymbol{p},s)(p\!\!\!/ - m) = 0\,, \\ (p\!\!\!/ + m)v(\boldsymbol{p},s) = 0\,, & \bar{v}(\boldsymbol{p},s)(p\!\!\!/ + m) = 0\,, \end{array} \tag{162}$$

Propagetor

$$\begin{aligned} S_F(q) &= i\int d^4x e^{iqx} <0|\, T[\psi(x)\bar{\psi}(0)]\,|0> \\ &= \frac{-1}{q\!\!\!/ - m + i\epsilon} = -\frac{q\!\!\!/ + m}{q^2 - m^2 + i\epsilon} \end{aligned} \tag{163}$$

1.7.3 Dirac Yukawa Theory

1.8 Massive Vector Fields

Spin 1.

$$\left(\Box + m^2\right) \varphi^\mu(x) = 0 \tag{164}$$

1.9 Local Gauge Symmetries

U(1), color SU(3), ...

1.9.1 Photon Field

Gauge bosons with unviolated symmetries. Electromagnetic field, gluons.

$$\Box A^\mu(x) = 0 \tag{165}$$

1.10 spin 3/2 Field

Δ

1.11 Renormalization

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