An Introduction to Relativistic Quantum Field Theory

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Contents

Notations

■ metric of 4-vector space:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(”Bjorken-Drell metric”)}$$

The square of a 4-vector $\underline{p} = (p^0, \boldsymbol{p}) = (p^0, p^1, p^2, p^3)$ is

$$\underline{p}^2 = \underline{p}^T g\, \underline{p} = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (p^0)^2 - \boldsymbol{p}^2 ,$$

where a superscript T denotes the transpose. An underline on a quantity (e.g. $\underline{p}$) indicates a 4-vector but it is frequently suppressed. When components of a 4-vector are given, they are contravariant components by default. The contravariant components of a 4-vector $\underline{p}$ are denoted as p^μ with upper suffixes. To make sure that components are contravariant, we often denote a 4-vector itself as p^μ. For given contravariant components, covariant components are obtained as

$$p_\mu = g_{\mu\nu} p^\nu ,$$

where $g_{\mu\nu}$ is the $(\mu\nu)$ element of g. Thus, for the $\underline{p}$ given in the above, we write $p_\mu = (p^0, -\boldsymbol{p})$. The square of a 4-vector $\underline{p}$ is now also written as

$$\underline{p}^2 = p_\mu p^\mu ,$$

where same suffixes μ in lower and upper positions must be summed over $\mu \in [0, 3]$.

■ Energy-momentum of a particle of the mass m is written as

$$p^\mu \leftrightarrows \begin{pmatrix} E/c \\ \boldsymbol{p} \end{pmatrix} = mc \begin{pmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{pmatrix} = m u^\mu ,$$

where $\gamma \equiv 1/\sqrt{1-\boldsymbol{\beta}^2}$ denotes the Lorentz factor of motion of the particle, $\boldsymbol{\beta}$ stands for the velocity nomalized by one of the light, c, and u^μ is the four-velocity.

■ Natural unit ($c = \hbar = 1$) is employed throughout our discussions. E.g. for the energy-momentum given in the above, we write

$$E = \sqrt{\boldsymbol{p}^2 + m^2} \quad \text{and} \quad \underline{p}^2 = m^2 .$$

■ Nomalization of states

Energy-momentum eigenstate: $<p\,|\,p'>= 2E\delta^3(\boldsymbol{p}-\boldsymbol{p}'),\ E=\sqrt{m^2+\boldsymbol{p}^2}$

Completeness: $\displaystyle\int \frac{d^3\boldsymbol{p}}{2E}\,|p><p|\ = 1\,.$

Many authors including ones of Refs. [2] and [3] put a factor $(2\pi)^3$ in the front of δ function and in the denominator of the integration volume element. It is just a matter of convention. Note that

$$Dim[\,|p>] = [E]^{-1}$$

■ Levi-Civita symbols

These are totally antisymmetric tensor densities.

3 dimensional

$$\epsilon_{ijk} = \begin{cases} {}^{+}_{-}\,1 \quad (i,j,k) = {}^{\text{even}}_{\text{odd}} \text{ permutation of } (1,2,3) \\ 0 \quad \text{otherwise} \end{cases}$$

For the purpose of applying the Einstein's contraction rule to 3 vector suffices, we may use symbols like ϵ^{ijk}, ${\epsilon_{ij}}^{k}$ and so on. All these symbols are equivalent with ϵ_{ijk}. The vertical positions of indexes have no meaning.

4 dimensional

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 \quad (\mu,\nu,\rho,\sigma,) = \text{even parmutation of } (0,1,2,3) \\ -1 \quad (\mu,\nu,\rho,\sigma,) = \text{odd parmutation of } (0,1,2,3) \\ 0 \quad \text{otherwise} \end{cases}$$

Symbols with covariant suffices are defined with the metric tensor in such a way like ${\epsilon_{\mu}}^{\nu\rho\sigma} = g_{\mu\tau}\epsilon^{\tau\nu\rho\sigma}$. Therefore, we have $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$. This notation follows [5].

Part I
Introduction

1 Starting from Quantum Mechanics

1.1 Basic matters

Observables are hermitian operators acting on a space of state vectors. Complete set of observables is a set of observables that commute with one another and for which there exists only one simultaneous eigenvector when the normalization is fixed. Set of all simultaneous eigenvectors of a complete set spans a Hilbert space [1]of state vectors associated with a quantum mechanical system under consideration. Therefore, the set constitutes a complete bases of the Hilbert space. Different choices of a particular set of complete bases, namely, that of a complete set of observables define different representations of the Hilbert space. Suppose $\{\hat{A}_1, \hat{A}_2, \dots\}$ is a complete set and $|a_1 a_2 \cdots>$ is a simultaneous eigenvector associated with a set of eigen values $\{a_1, a_2, \cdots\}$. Then an arbitrary state vector $|\Psi>$ can be expressed as

$$|\Psi>= \int [da_1]\,[da_2]\cdots <a_1 a_2 \cdots \,|\, \Psi> \; |a_1 a_2 \cdots> \tag{1}$$

where $\int [da_i]$ denotes sum or integration over a variable a_i with a measure chosen so that $\int [da_i]\; |a_i><a_i|$ is the projection of a space spanned by $|a_i>$. Factor $<a_1 a_2 \cdots \,|\, \Psi>$ in Eq. (1) is the wave function in the representation defined by the complete set. When we define another representation by a complete set $\{B_1, B_2, \dots\}$ and its simultaneous eigenvectors $|b_1 b_2 \cdots>$, it follows from Eq. (1) that the wave function in the new representation is given as

$$<b_1 b_2 \cdots \,|\, \Psi>= \int [da_1]\,[da_2]\cdots <a_1 a_2 \cdots \,|\, \Psi><b_1 b_2 \cdots \,|\, a_1 a_2 \cdots> \tag{2}$$

In quantum mechanics, canonically conjugate variables $\hat{A}$ and $\hat{B}$ satisfy a commutation relation instead of corresponding Poisson bracket in classical mechanics.

For a system composed of one particle, cartesian components $(\hat{x}_1, \hat{x}_2, \hat{x}_3) \equiv \hat{\boldsymbol{x}}$ of particle position are observables.

[1]When an observable of the set has continuous eigenvalues, the space is not a Hilbert space in a mathematically rigorous sense. However, we may follow a physics convention to use the term including such cases.

Eigenvector of a physical quantity $\hat{A}$ $(=\hat{A}^\dagger)$ associated with an eigenvalue a $(=a^*)$ is called eigenstate $|a>$; $\hat{A}\,|a>=a\,|a>$. Physical quantities

To a system composed of one particle, there associate coordinates of the position $\hat{\boldsymbol{x}}$ that are physical quantities.

$$\text{Schrödinger eq.}\quad i\partial_t\Psi(t,\boldsymbol{x})=H\Psi(t,\boldsymbol{x})\,,\quad H=-\frac{\boldsymbol{\partial}^2}{2m}+V(\boldsymbol{x})=H^\dagger \tag{3}$$

$$\text{Eigenstates of } H\quad H\varphi_l(\boldsymbol{x})=\epsilon_l\varphi_l(\boldsymbol{x})\,,\quad \{\varphi_l(\boldsymbol{x})\}\text{: orthogonal, complete} \tag{4}$$

$$\text{Solution of the Schrödinger eq.}\ \Psi(t,\boldsymbol{x})=\sum_l \Psi_l e^{-i\epsilon_l t}\varphi_l(\boldsymbol{x}) \tag{5}$$

Bras and kets

$$\Psi(t,\boldsymbol{x})=<\boldsymbol{x}|\,\Psi(t)>\,,\qquad \varphi_l(\boldsymbol{x})=<\boldsymbol{x}|\,\epsilon_l>\,, \tag{6}$$

$$i\partial_t\,|\Psi(t)>=\hat{H}\,|\Psi(t)>\,,\qquad \hat{H}=\frac{\hat{\boldsymbol{p}}^2}{2m}+V(\boldsymbol{x})=\hat{H}^\dagger \qquad <3>$$

$$\hat{H}\,|\epsilon_l>=\epsilon_l\,|\epsilon_l>\,,\quad <\epsilon_l\,|\epsilon_{l'}>=0 \text{ if } l\neq l'\,,\quad \sum_l |\epsilon_l><\epsilon_l|\,=1 \qquad <4>$$

$$|\Psi(t)>=\sum_l <\epsilon_l|\,\Psi(0)>e^{-i\epsilon_l t}\,|\epsilon_l> \qquad <5>$$

1st quantization

$$[\hat{x}_i,\hat{p}_j]=i\delta_{ij} \tag{7}$$

Eigen states

$$\begin{aligned}\hat{\boldsymbol{x}}\,|\boldsymbol{x}>&=\boldsymbol{x}\,|\boldsymbol{x}>\\ \hat{\boldsymbol{p}}\,|\boldsymbol{p}>&=\boldsymbol{p}\,|\boldsymbol{p}>\end{aligned} \tag{8}$$

Conventional ("half relativistic") normalization

$$<\boldsymbol{x}|\,\boldsymbol{x}'>=\delta^3(\boldsymbol{x}-\boldsymbol{x}')\,,\qquad \int |\boldsymbol{x}>d^3\boldsymbol{x}<\boldsymbol{x}|\,=1 \tag{9}$$

$$<\boldsymbol{p}|\,\boldsymbol{p}'>=2E\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,,\qquad \int |\boldsymbol{p}>\frac{d^3\boldsymbol{p}}{2E}<\boldsymbol{p}|\,=1 \tag{10}$$

Coordinate representation of the momentum operator

$$<\boldsymbol{x}|\,\hat{\boldsymbol{p}}=\frac{1}{i}\boldsymbol{\partial}<\boldsymbol{x}| \tag{11}$$

and of the momentum eigenstate

$$<\boldsymbol{x}|\,\boldsymbol{p}>=\sqrt{\frac{2E}{(2\pi)^3}}\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{12}$$

Requirement for the normalization factor is understood by employing the first equation of (10) in the $l.h.s$ of $<\boldsymbol{p}|\,\boldsymbol{p}'>=\int <\boldsymbol{p}|\,\boldsymbol{x}>d^3\boldsymbol{x}<\boldsymbol{x}\,|\boldsymbol{p}'>$ and remembering a formula

$$\int d^3\boldsymbol{x}\, e^{\pm i\boldsymbol{p}\cdot\boldsymbol{x}} = (2\pi)^3\delta^3(\boldsymbol{p})$$

1.2 N-body quantum mechanics

Schrödinger eq.

$$\begin{aligned} &i\partial_t\Psi^{(N)}(t;\boldsymbol{x}_1,\dots\boldsymbol{x}_N) = H^{(N)}\Psi^{(N)}(t;\boldsymbol{x}_1,\dots\boldsymbol{x}_N)\,, \\ &H^{(N)} = \sum_i^N H_i\,,\quad H_i = -\frac{\boldsymbol{\partial}^2}{2m} + V(\boldsymbol{x}_i) \quad \text{(for simplicity)} \end{aligned} \tag{13}$$

$$H_i\varphi_l^{(i)}(\boldsymbol{x}) = \epsilon_l\varphi_l^{(i)}(\boldsymbol{x})\,,\qquad i=1,\dots,N \tag{14}$$

$$\begin{aligned} &\Psi^{(N)}(t;\boldsymbol{x}_1,\dots\boldsymbol{x}_N) = \\ &\quad \sum\nolimits_{l_1,\dots,l_N}\Psi^{(N)}(l_1,\dots,l_N)e^{-iE^{(N)}t}\left\{\varphi_{l_1}^{(1)}(\boldsymbol{x}_1)\cdots\varphi_{l_N}^{(N)}(\boldsymbol{x}_N)\right\}_P\,, \end{aligned} \tag{15}$$

where $E^{(N)} = \epsilon_{l_1}+\cdots+\epsilon_{l_N}$ and $\{\cdots\}_P$ denotes symmetrization for systems of identical bosons and antisymmetrization for identical fermions. Thus $\Psi^{(N)}$ is defined in the direct product of N Hilbert space.

■ **Theorem of bosonic creation and annihilation operators** [4]
If an operator $\hat{a}$ and its hermite conjugate $\hat{a}^\dagger$ satisfy

$$[\hat{a},\hat{a}^\dagger] = 1\,, \tag{16}$$

then

1. Eigenvalues of an operator $\hat{N}\equiv\hat{a}^\dagger\hat{a}$ is nonnegative integers $\{0,1,\dots,\infty\}$ and we can call it number operator.

2. Vaccum state $|0>$ with respect to the dynamical freedom described by $\hat{a}$ and $\hat{a}^\dagger$ can be defined as the eigenstate of $\hat{N}$ belonging to its eigenvalue 0.

3. If we normalize the vacuum state by $<0|\,0>=1$, then the eigenstate of $\hat{N}$ belonging to an eigenvalue n is given below and it constitutes an orthonormal set.

$$|n> = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0>\,,\quad <n|\,m> = \delta_{nm} \tag{17}$$

Proof

We will make use of useful relationships

$$\begin{aligned} [ab,c] &= a\,[b,c] + [a,c]\,b \\ \text{and}\quad [a,bc] &= [a,b]\,c + b\,[a,c] \end{aligned}$$

which hold regardless the type of operators a, b and c. For a bosonic operator $\hat{a}$ satisfying Eq. (16), we have

$$\left[\hat{a}, \hat{a}^{\dagger n}\right] = n\hat{a}^{\dagger n-1}$$

so that

$$\left[\hat{a}^{\dagger}\hat{a}, \hat{a}^{\dagger n}\right] = n\hat{a}^{\dagger n} \tag{18}$$

and

$$(\hat{a}^{\dagger}\hat{a})(\hat{a}^{\dagger})^{n}\,|0> = \left[\hat{a}^{\dagger}\hat{a}, \hat{a}^{\dagger n}\right]\,|0> = n\hat{a}^{\dagger n}\,|0> \ .$$

This proves that $|n>$ in Eq. (17) is an eigenstate of $\hat{N}$ belonging to an eigenvalue n:

$$\hat{N}\,|n> = n\,|n>$$

Next, we compute

$$\begin{aligned} \hat{a}^{m}\hat{a}^{\dagger n}\,|0> &= \hat{a}^{m-1}\left[\hat{a}, \hat{a}^{\dagger n}\right]\,|0> \\ &= n\hat{a}^{m-1}\hat{a}^{\dagger n-1}\,|0> \\ &= n\hat{a}^{m-2}\left[\hat{a}, \hat{a}^{\dagger n-1}\right]\,|0> \\ &= n(n-1)\hat{a}^{m-2}\hat{a}^{\dagger n-2}\,|0> \\ &= \ldots \\ &= \begin{cases} n(n-1)\cdots(n-m+1)\hat{a}^{\dagger n-m}\,|0> \quad (m<n) \\ n!\,|0> \quad (m=n) \\ n!\hat{a}^{m-n}\,|0> \quad (m>n) \end{cases} \end{aligned}$$

and we obtain

$$<0|\,\hat{a}^{m}\hat{a}^{\dagger n}\,|0> = n!\delta_{n,m}$$

that shows $|n>$ are orthonormal.

■ **Theorem of fermionic creation and annihilation operators** [4]

If an operator $\hat{c}$ and its hermite conjugate $\hat{c}^{\dagger}$ satisfy

$$\{\hat{c}, \hat{c}^{\dagger}\} = 1 \quad \text{and} \quad \{\hat{c}, \hat{c}\} = 0\,, \tag{19}$$

then

1. Eigenvalues of an operator $\hat{N} \equiv \hat{c}^\dagger \hat{c}$ is 0 or 1 and we can call it number operator.

2. Vaccum state $|0>$ with respect to the dynamical freedom described by $\hat{c}$ and $\hat{c}^\dagger$ can be defined as the eigenstate of $\hat{N}$ belonging to its eigenvalue 0.

3. If we normalize the vacuum state by $<0|\,0>= 1$, then the eigenstates of $\hat{N}$ are $|0>$ and $|1>= c^\dagger\,|0>$.

Suppose now we have a set of $\hat{a}_l$ and $\hat{a}_l^\dagger$ for $l = 1, 2, \ldots, \infty$ corresponding to energy eigenvalues $\epsilon_1, \epsilon_2, \ldots$. We assume that we can have a set of operators such that each pair of $\hat{a}_l$ and $\hat{a}_l^\dagger$ satisfies the condition of creation-annihilation operators mentioned above and they are independent for different suffices:

$$\begin{aligned} &[\hat{a}_l, \hat{a}_m^\dagger] = \delta_{lm} \\ &[\hat{a}_l, \hat{a}_m] = [\hat{a}_l^\dagger, \hat{a}_m^\dagger] = 0 \end{aligned} \tag{20}$$

Having with these operators, we define

$$\hat{\varphi}(\boldsymbol{x}) = \sum_l \hat{a}_l \varphi_l(\boldsymbol{x})\,, \tag{21}$$

where $\varphi_l(\boldsymbol{x})$ is eigenvector of $H = -\boldsymbol{\partial}^2/2m + V(\boldsymbol{x})$ belonging to the lth eigenvalue ϵ_l. Assign $\hat{a}_l^\dagger$ as operator to create a particle in the lth energy eigenstate:

$$|\epsilon_l>= \hat{a}_l^\dagger\,|0> \tag{22}$$

Then we find

$$\begin{aligned} <0|\,\hat{\varphi}(\boldsymbol{x})\,|\epsilon_l> &= \sum_{l'} \varphi_{l'}(\boldsymbol{x}) <0|\,\hat{a}_{l'}\hat{a}_l^\dagger\,|0> \\ &= \sum_{l'} \varphi_{l'}(\boldsymbol{x}) <0|\,[\hat{a}_{l'}, \hat{a}_l^\dagger]\,|0> \\ &= \varphi_l(\boldsymbol{x})\,, \end{aligned} \tag{23}$$

where we have used relationships $\hat{a}\,|0>=<0|\,\hat{a}^\dagger = 0$. Comparing this result with the second equation in Eq. (6), we may write

$$<\boldsymbol{x}|\ =<0|\,\hat{\varphi}(\boldsymbol{x}) \tag{24}$$

If we denote by $\hat{a}_{\boldsymbol{x}}^\dagger$ the creation operator that creates a a particle at a position $\boldsymbol{x}$, we can write

$$\hat{a}_{\boldsymbol{x}}^\dagger = \hat{\varphi}^\dagger(\boldsymbol{x}) \tag{25}$$

The operator $\hat{\varphi}(\boldsymbol{x})$ defined in Eq. (21) is a primitive form of field operators discussed later.

N particle states can be constructed as

$$|\epsilon_{l_1}, \dots, \epsilon_{l_N} >= \hat{a}^\dagger_{l_1} \dots \hat{a}^\dagger_{l_N} |0> \tag{26}$$

and

$$<\boldsymbol{x}_1, \dots \boldsymbol{x}_N| \ =<0| \, \hat{\varphi}(\boldsymbol{x}_1) \dots \hat{\varphi}(\boldsymbol{x}_N) \tag{27}$$

We read

$$\Psi^{(N)}(t; \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) =<\boldsymbol{x}_1, \dots \boldsymbol{x}_N| \, \Psi^{(N)}(t)> \tag{28}$$

and

$$|\Psi^{(N)}(t)>= \sum_{l_1,\dots,l_N} \Psi^{(N)}(l_1, \dots, l_N) e^{-iE^{(N)}t} |\epsilon_{l_1}, \dots, \epsilon_{l_N} > \tag{29}$$

When N particles are identical bosons, these expressions in Eq. (26) and (27) are redundant because all operators in them are commuting with each other and the order of variables in these bra and ket have no meaning. Suppose we have n_1 particle in state of energy ϵ_1, n_2 in ϵ_2 and so on, we may write the *l.h.s.* of Eq. (26) as $|n_1, n_2, \dots >$. This notation is commonly used in condensed matter and nuclear physics.

Basis $|\boldsymbol{x}_1, \dots \boldsymbol{x}_N >$ or $|\epsilon_{l_1}, \dots, \epsilon_{l_N} >$ span N-particle Hilbert space. Set of all N-particle Hilbert space spanned by $|\boldsymbol{x}_1, \boldsymbol{x}_2, \dots >$ is called the Fock space. A state vector $|\Psi>$ of the Fock space can be expanded as

$$|\Psi>= \sum_N \int \prod_i^N d^3\boldsymbol{x}_i \, |\boldsymbol{x}_1, \dots, \boldsymbol{x}_N><\boldsymbol{x}_1, \dots, \boldsymbol{x}_N| \, \Psi> \tag{30}$$

1.3 Schrödinger field theory

In place of Eq. (21), if we define

$$\hat{\Psi}(t, \boldsymbol{x}) = \sum_l \hat{a}_l e^{-i\epsilon_l t} \varphi_l(\boldsymbol{x}) \,, \tag{31}$$

this operator satisfies the single particle Schrödinger equation:

$$i\partial_t \hat{\Psi}(t, \boldsymbol{x}) = H \hat{\Psi}(t, \boldsymbol{x}) \qquad (\widehat{3})$$

We already know that any number of Schrödinger particles can be generated by $\hat{\varphi}(\boldsymbol{x}) = \hat{\Psi}(0, \boldsymbol{x})$. Comparing Eq. (31) with Eq. (5), we observe $\hat{\Psi}(t, \boldsymbol{x})$ is obtained by replacing c-number amplitude Ψ_l in $\Psi(t, \boldsymbol{x})$ by annihilation operator $\hat{a}_l$. The field operator $\hat{\Psi}$ satisfies the following equaltime commutation relations:

$$\begin{aligned} &[\hat{\Psi}(t, \boldsymbol{x}), \hat{\Psi}^\dagger(t, \boldsymbol{x'})] = \delta^3(\boldsymbol{x} - \boldsymbol{x'}) \\ &[\hat{\Psi}(t, \boldsymbol{x}), \hat{\Psi}(t, \boldsymbol{x'})] = [\hat{\Psi}^\dagger(t, \boldsymbol{x}), \hat{\Psi}^\dagger(t, \boldsymbol{x'})] = 0 \end{aligned} \tag{32}$$

If we require Eq. (32), then Eq. (20) follows. Schrödinger equation $\widehat{3}$ follows from a Lagrangian density

$$\mathcal{L} = i\Psi^* \partial_t \Psi + \frac{1}{2m}\Psi^* \boldsymbol{\partial}^2 \Psi - V\Psi^*\Psi \tag{33}$$

Here we note $dim[\mathcal{L}] = E^4$ and $dim[\Psi] = E^{3/2}$. Canonical momentum field is given as

$$\Pi \stackrel{\leftarrow}{=} \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^* \,, \tag{34}$$

which has the same dimension as Ψ. The set (32) is equivalent with

$$\begin{aligned} &[\hat{\Psi}(t, \boldsymbol{x}), \hat{\Pi}(t, \boldsymbol{x'})] = \delta^3(\boldsymbol{x} - \boldsymbol{x'}) \\ &[\hat{\Psi}(t, \boldsymbol{x}), \hat{\Psi}(t, \boldsymbol{x'})] = [\hat{\Pi}(t, \boldsymbol{x}), \hat{\Pi}(t, \boldsymbol{x'})] = 0 \end{aligned} \tag{35}$$

This is the canonical commutation relation. We may reverse our argument starting from Eq. (33), writing down the "field equation" $\widehat{3}$ and requiring the equaltime canonical commutation relation (35). This procedure is called the 2nd quantizaton.

2 Canonical quantizaton

Lagrangian density ($dim = E^4$) is given as a functional of field $\varphi(x)$ and its space time derivatives

$$\mathcal{L} = \mathcal{L}[\varphi(x), \partial^\mu \varphi(x)] \tag{36}$$

Euler-Lagrange Eq.

$$\frac{\partial \mathcal{L}}{\partial \varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))} = 0 \tag{37}$$

Canonical momentum field

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \tag{38}$$

where $\dot{\varphi}(x) = \partial_0 \varphi(x) = \partial^0 \varphi(x)$.
$\mathcal{H}$amiltonian density

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \tag{39}$$

By solving Eq. (38) for $\dot{\varphi}(x)$, $\mathcal{H}$ is a function solely of $\pi(x), \varphi(x)$ and $\partial_i \varphi(x)$. In the classical field theory, temporal developments of $\varphi(x)$ and $\pi(x)$ are given by Hamiltonian $H = \int d^3\boldsymbol{x}\ \mathcal{H}$ through the canonical equation of motion

$$\dot{\varphi}(x) = -i[\varphi(x), H] \,, \qquad \dot{\pi}(x) = -i[\pi(x), H] \tag{40}$$

where $-i[\cdots]$ is the Poisson braket. The Euler equation (37) and the canonical equation (40) are equivalent in the classical level.

Following the canonical quantization method, fields $\varphi(x)$ and $\pi(x')$ are postulated to satisfy equal time commutation relations at a time $x^0 = x'^0 = t_0$ (which is called the time of quantization):

$$\begin{gathered}[\hat{\varphi}(t_0,\boldsymbol{x}),\hat{\pi}(t_0,\boldsymbol{x'})] = i\delta^3(\boldsymbol{x}-\boldsymbol{x'})\ , \\ [\hat{\varphi}(t_0,\boldsymbol{x}),\hat{\varphi}(t_0,\boldsymbol{x'})] = 0\ , \quad [\hat{\pi}(t_0,\boldsymbol{x}),\hat{\pi}(t_0,\boldsymbol{x'})] = 0\end{gathered} \tag{41}$$

These relations define a set of field operators at time $x^0 = t_0$. If the Hamiltonian $\hat{H}$ is provided, time evolution of field operators as Heisenberg operators are given as

$$i\partial_t\,\hat{\varphi}_H(x) = [\hat{\varphi}_H(x),\hat{H}]\ , \qquad i\partial_t\,\hat{\pi}_H(x) = [\hat{\pi}_H(x),\hat{H}]\,, \tag{42}$$

where $[\cdots]$ denotes commutation relation. Formal solutions to these equations are written as

$$\begin{aligned}\hat{\varphi}_H(x) &= e^{i\hat{H}(t-t_0)}\hat{\varphi}(t_0,\boldsymbol{x})e^{-i\hat{H}(t-t_0)} \\ \hat{\pi}_H(x) &= e^{i\hat{H}(t-t_0)}\hat{\pi}(t_0,\boldsymbol{x})e^{-i\hat{H}(t-t_0)}\end{aligned} \tag{43}$$

Substituting operators defined in Eq. (41) into Eq. (43), we obtain a set of Heisenberg field operators at arbitrary time. They satisfy equal time commutation relations at arbitrary time t:

$$\begin{gathered}[\hat{\varphi}_H(t,\boldsymbol{x}),\hat{\pi}_H(t,\boldsymbol{x'})] = i\delta^3(\boldsymbol{x}-\boldsymbol{x'})\ , \\ [\hat{\varphi}_H(t,\boldsymbol{x}),\hat{\varphi}_H(t,\boldsymbol{x'})] = 0\ , \quad [\hat{\pi}_H(t,\boldsymbol{x}),\hat{\pi}_H(t,\boldsymbol{x'})] = 0\end{gathered} \tag{44}$$

In a situation that the Hamiltonian can be constructed by Eq. (39) in the classical level, we may obtain $\hat{H}$ by replacing fields by these Heisenberg operator fields under certain care on the order of noncommuting operators. The time t in $\hat{H} = \int d^3\boldsymbol{x}\mathcal{H}(t,\boldsymbol{x})$ can be chosen arbitrary since $i\partial_t\hat{H} = [\hat{H},\hat{H}] = 0$. It is then shown [7] that the set of the Heisenberg equations (42) is equivalent with the set of Euler equation (37) for the Heisenberg field $\hat{\varphi}_H(x)$ and the definition of $\hat{\pi}_H(x)$ given by Eq.(38). [2] Namely, they are equivalent in the quantum level. The validity of Eq. (44) shows that once the equal time commutation relation is set, it is valid for arbitrary time [3] , provided

[2]In the derivation in [7], a formula

$$[F(A,B,\dots),\pi] = (\partial F/\partial A)[A,\pi] + (\partial F/\partial B)[B,\pi] + \dots$$

is employed.

[3]However, one may not equate time derivatives of the both hand sides since it is valid at fixed times.

Hamiltonian exists. In this sense, the quantization time is arbitral when the canonical quantization method works in usual manner.

In summary, field equation (37) can be read as one for the Heisenberg field operator of which the operator nature is set by the equal time commutation relation (41) at time t_0 that can be chosen arbitrary.

3 Noether's Theorem

For each degree of freedom of continuous transformation of fields against which the action remains invariant, there exist a conserved current $J^\mu(x)$. Particularly, let a infinitesimal transformation with s parameters $\delta\alpha_\lambda$ be written as

$$\begin{array}{l} x^\mu \mapsto x^{\mu'} = x^\mu + \delta x^\mu\,, \quad \delta x^\mu = \sum_{\lambda=1}^{s} X^\mu_\lambda \delta\alpha_\lambda \\ \varphi(x) \mapsto \varphi'(x') = \varphi(x) + \delta\varphi(x)\,, \quad \delta\varphi(x) = \sum_{\lambda=1}^{s} \Phi_\lambda(x)\delta\alpha_\lambda \end{array} \tag{45}$$

This transformation may be associated with a space-time transformation specified by X^μ_λ as shown in the first line in Eq. (45). When the transformation is related only with internal degree of freedom of the field , one may set $X^\mu_\lambda = 0$. When the action $\mathcal{A}$ is invariant under this transformation, s currents

$$J^\mu_\lambda(x) \quad \stackrel{\leftarrow}{=} \quad -\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\left(\Phi_\lambda(x) - \partial_\nu\varphi(x) X^\nu_\lambda\right) - \mathcal{L}(x) X^\mu_\lambda \tag{46}$$

are all conserved ones. Namely,

$$\partial_\mu J^\mu_\lambda(x) = 0\,, \quad \lambda = 1, \ldots, s \tag{47}$$

hold and corresponding charges

$$Q_\lambda = \int d^3\boldsymbol{x} J^0_\lambda(x) \tag{48}$$

are conserved. Eq. (46) defines the Noether currents and quantities Q_λ in Eq. (48) are called Noether charges.

3.0.1 Energy-Momentum tensor

Our action must be invariant under space-time translations. The infinitesimal transformation

$$x^\mu \mapsto x^{\mu'} = x^\mu + \delta a^\mu \tag{49}$$

has 4 continuous parameters δa^μ and in our notation $X^\mu_\nu = \delta^\mu_\nu$. The field is invariant

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x) \tag{50}$$

and $\Phi_\nu = 0$. The Noether current (46) reads

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi(x) - \mathcal{L}(x) g^\mu_\nu \tag{51}$$

3.0.2 Angular Momentum tensor

Consider an infinitesimal spatial rotation

$$x^\mu \mapsto x^{\mu'} = x^\mu + \delta\omega^\mu{}_\nu x^\nu \tag{52}$$

This transformation has 6 continuous parameters $\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$. δx^μ in Eq. (45) is given as

$$\begin{aligned} \delta x^\mu &= \delta\omega^\mu{}_\nu x^\nu \\ &= g^\mu_\xi g_{\nu\eta} \delta\omega^{\xi\eta} x^\nu \\ &= \frac{1}{2}(g^\mu_\xi g_{\nu\eta} - g^\mu_\eta g_{\nu\xi}) x^\nu \delta\omega^{\xi\eta} \\ &= \frac{1}{2}(a_{\xi\eta})^\mu{}_\nu x^\nu \delta\omega^{\xi\eta} \end{aligned} \tag{53}$$

where

$$(a_{\xi\eta})^\mu{}_\nu = g^\mu_\xi g_{\nu\eta} - g^\mu_\eta g_{\nu\xi} \tag{54}$$

We have

$$X^\mu{}_{\xi\eta} = \frac{1}{2}(a_{\xi\eta})^\mu{}_\nu x^\nu \tag{55}$$

The field is transformed as

$$\varphi_\alpha(x) \mapsto \varphi'_\alpha(x') = \delta\varphi_\alpha(x)\,, \quad \delta\varphi_\alpha(x) = \frac{1}{2}(S_{\xi\eta})_\alpha{}^\beta \varphi_\beta(x) \delta\omega^{\xi\eta} \tag{56}$$

where

$$(S_{\mu\nu})_\alpha{}^\beta = \begin{cases} 0 & \cdots \text{scalar} \\ (a_{\mu\nu})_\alpha{}^\beta & \cdots \text{vector} \\ \frac{1}{4}[\gamma_\mu, \gamma_\nu]_\alpha{}^\beta & \cdots \text{Dirac spinor} \end{cases} \tag{57}$$

We have

$$\Phi_{\alpha\xi\eta} = \frac{1}{2}(S_{\xi\eta})_\alpha{}^\beta \varphi_\beta(x) \tag{58}$$

The Noether current

$$\begin{aligned}
M^{\mu}_{\xi\eta} &\stackrel{\leftarrow}{=} 2J^{\mu}_{\xi\eta} \\
&= -2\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\left(\Phi_{\alpha\xi\eta}(x) - \partial_\nu\varphi_\alpha(x)X^{\nu}_{\xi\eta}\right) - 2\mathcal{L}(x)X^{\mu}_{\xi\eta} \\
&= \left(\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\partial_\nu\varphi_\alpha(x) - \mathcal{L}g^{\mu}_{\nu}\right)\cdot 2X^{\nu}{}_{\xi\eta} - 2\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\Phi_{\alpha\xi\eta}(x) \\
&= (a_{\xi\eta})^{\nu}{}_{\rho}x^{\rho}\cdot T^{\mu}_{\nu} - \frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}(S_{\xi\eta})_{\alpha}^{\ \beta}\varphi_\beta(x) \qquad (59) \\
&\stackrel{\rightarrow}{=} L^{\mu}_{(\xi\eta)} + S^{\mu}_{(\xi\eta)} \qquad (60)
\end{aligned}$$

The last equation defines orbital and spin parts of the current. The total angular momentum vector is defined as

$$J^k = \frac{1}{2}\epsilon^{kij}M^{ij}\,, \quad M_{\xi\eta} = \int d^3\boldsymbol{x}M^0_{\xi\eta} \tag{61}$$

3.0.3 Electric Charge

For a complex field, gauge transformation of the first kind is defined as

$$\varphi(x)\mapsto\varphi'_\alpha(x) = e^{ie\theta}\varphi(x)\,, \quad \varphi^*(x)\mapsto\varphi^{*\prime}(x) = e^{-ie\theta}\varphi^*(x) \tag{62}$$

This transformation consists a U(1). Considering infinitesimal θ, we read in Eq. (45) that

$$X^\mu = 0\,, \quad \Phi = ie\varphi\,, \quad \Phi^* = -ie\varphi^*\,, \tag{63}$$

Invariance of the $\mathcal{L}$agragian density leads

$$\partial_\mu J^\mu = 0 \tag{64}$$

for

$$J^\mu = ie\left(\varphi^*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varphi\right) \tag{65}$$

The corresponding charge is $Q = \int d^3\boldsymbol{x}J^0$.

3.0.4 Internal Global Symmetries

We consider a internal global SU(n) symmetry. The field $\varphi_a(x)$ is subject to a transformation

$$\varphi_a(x)\mapsto\varphi'_a(x) = (e^{i\alpha_i G_i})_a^{\ b}\varphi_b(x) \tag{66}$$

where $\alpha_i, (i = 1, \ldots, n^2 - 1)$ are continuous real parameters, G_i are matrix representations of generators. Considering infinitesimal α_i, we have in Eq. (45) that

$$\Phi_{ai} = i(G_i)_a^{\ b}\varphi_b \,, \tag{67}$$

The Noether current is given by

$$J_i^\mu \quad = \quad -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}(G_i)_a^{\ b}\varphi_b \tag{68}$$

and the corresponding $n^2 - 1$ charges are

$$C_i = -i\int d^3\boldsymbol{x}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_a}(G_i)_a^{\ b}\varphi_b \tag{69}$$

Among n^2-1 generators G_i, only $n-1$ commutes to each other. Accordingly, $n-1$ charges among n^2-1 can be diagonalized at the same time.

■ SU(2)
The conserved (internal) vector is the isospin $\boldsymbol{I} = (C_1, C_2, C_3)$.

$$\varphi_a \in \square \quad : \quad G_i = \frac{1}{2}\sigma_i \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{70}$$

$$\varphi_a \in \square\square \quad : \quad G_i = t_i \,, \quad (t_i)_{jk} = -i\epsilon_{ijk} \tag{71}$$

■ SU(3)

$$\varphi_a \in \square : \qquad G_i = \tfrac{1}{2}\lambda_i$$

$$\begin{gathered}
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{72}$$

$$\varphi_a \in \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} : \quad G_i = T_i \,, \quad (T_a)_{bc} = -if_{abc}(\text{structure const.}) \tag{73}$$

$$\begin{gathered} T_\pm = G_1 \pm iG_2 \,, \quad V_\pm = G_4 \pm iG_5 \,, \quad U_\pm = G_6 \pm iG_7 \,, \\ T_3 = G_3 \,, \quad Y = \tfrac{2}{\sqrt{3}}G_8 \end{gathered} \tag{74}$$

4 Interacting Fields (1/2)

4.0.1 The Interaction Picture

● Schrödinger Picture
For a given Hamiltonian H,

$$\begin{cases} i\partial_t \,|\Psi\!>_S = H\,|\Psi\!>_S \\ i\partial_t \mathcal{O}_S = 0 \end{cases} \tag{75}$$

It is assumed that $\mathcal{O}_S$ does not depend on t explicitly. Formal solution to the Schrödinger equation is written as

$$|\Psi\!>_S = e^{-iHt}\,|\Psi_0\!>_S \tag{76}$$

Expectation value of $\mathcal{O}_S$ in a state $|\Psi\!>_S$ is given as

$${}_S\!<\!\Psi|\,\mathcal{O}_S\,|\Psi\!>_S \tag{77}$$

● Heisenberg Picture
Related to the Schrödinger picture by

$$\begin{cases} |\Psi\!>_H = e^{iHt}\,|\Psi\!>_S \\ \mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt} \end{cases} \tag{78}$$

so that

$${}_H\!<\!\Psi|\,\mathcal{O}_H\,|\Psi\!>_H = {}_S\!<\!\Psi|\,\mathcal{O}_S\,|\Psi\!>_S \tag{79}$$

They evolve in time as

$$\begin{cases} i\partial_t \,|\Psi\!>_H = 0 \\ i\partial_t \mathcal{O}_H = [\mathcal{O}_H, H] \end{cases} \tag{80}$$

● The Interaction Picture

$$H = H_0 + H_{int} \tag{81}$$

Related to the Schrödinger picture by

$$\begin{cases} |\Psi\!>_I = e^{iH_0t}\,|\Psi\!>_S \\ \mathcal{O}_I = e^{iH_0t}\mathcal{O}_S e^{-iH_0t} \end{cases} \tag{82}$$

so that

$${}_I\!<\!\Psi|\,\mathcal{O}_I\,|\Psi\!>_I = {}_S\!<\!\Psi|\,\mathcal{O}_S\,|\Psi\!>_S \tag{83}$$

The interaction Hamiltonian in this picture is time dependent;

$$H_I \stackrel{\leftarrow}{=} (H_{int})_I = e^{iH_0 t} H_{int} e^{-iH_0 t} \tag{84}$$

$|\Psi>_I$ and $\mathcal{O}_I$ evolve in time as

$$\begin{cases} i\partial_t \,|\Psi>_I = e^{iH_0 t}(-H_0 + H)\,|\Psi>_S \\ \qquad\qquad = e^{iH_0 t} H_{int} e^{-iH_0 t} e^{iH_0 t}\,|\Psi>_S \\ \qquad\qquad = H_I\,|\Psi>_I \\ i\partial_t \mathcal{O}_I = [\mathcal{O}_I, H_0] \end{cases} \tag{85}$$

4.0.2 Dyson's Formula

Formal solution of Eq. (85) can not be written in the form like one in Eq. (76) since H_I is time dependent. Writing the solution as

$$\begin{aligned} &|\Psi(t)>_I = U(t,t_0)\,|\Psi(t_0)>_I \\ &U(t,t) = 1 \quad \text{and} \quad U(t_3,t_2)U(t_2,t_1) = U(t_3,t_1)\,, \end{aligned} \tag{86}$$

the time evolution unitary operator $U(t, t_0)$ is given as

$$U(t,t_0) = Texp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)\,, \tag{87}$$

where T stands for time ordered product [5–7, 9, 10]

$$T[\mathcal{O}_2(t')\mathcal{O}_1(t)] = \theta(t'-t)\mathcal{O}_2(t')\mathcal{O}_1(t) + \theta(t-t')\mathcal{O}_1(t)\mathcal{O}_2(t')\,. \tag{88}$$

In fact, for $t > t_0$

$$\begin{aligned} i\partial_t U(t,t_0) &= T\left[H_I(t)exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)\right] \\ &= H_I(t)Texp\left(-i\int_{t_0}^{t} H_I(t')dt'\right) \\ &= H_I(t)U(t,t_0) \end{aligned} \tag{89}$$

and the conditions for $U(t, t_0)$ in Eq. (86) is obviously satisfied. We may write Eq. (87) in a form of series as

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} H_I(t')dt' \\ &\quad + \frac{(-i)^2}{2}\int_{t_0}^{t} dt'\int_{t_0}^{t} dt''\; T[H_I(t')H_I(t'')] + \cdots \end{aligned} \tag{90}$$

However,

$$\begin{aligned}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\, T[\mathcal{O}(t')\mathcal{O}(t'')] &= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \mathcal{O}(t')\mathcal{O}(t'') + \int_{t_0}^{t} dt' \int_{t'}^{t} dt''\, \mathcal{O}(t'')\mathcal{O}(t') \\ &= 2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \mathcal{O}(t')\mathcal{O}(t'')\end{aligned}$$

and

$$\begin{aligned}U(t,t_0) &= 1 - i\int_{t_0}^{t} H_I(t')dt' \\ &\quad +(-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + \cdots \end{aligned} \tag{91}$$

4.0.3 The S-matrix

Dealing with the scattering, we ***assume*** that initial and final states are eigenstates of H_0. To be more concrete, let $\mathcal{O}_0$ be a complete set of observables that includes H_0 and no one of them depends on t in an explicit manner. In the interaction picture, observables in $\mathcal{O}_0$ are constant as operators. [See Eq. (85).] What we have assumed is that the initial and final states are eigenstates of $\mathcal{O}_0$.

The scattering process takes place as follows. At $t_1 \to -\infty$, the system is in the initial state $|i\!>$, the system evolves in time by $U(t_2, t_1)$ under the effect of H_I, then, at $t_2 \to \infty$, the system turns in to the scattered state. The amplitude to find a final state $|f\!>$ in the scattered state is

$$\lim_{\substack{t_1 \to -\infty \\ t_2 \to \infty}} <\!f|\, U(t_2,t_1)\, |i\!> \; \overset{\rightarrow}{=} \; <\!f|\, S\, |i\!> \tag{92}$$

This is the definition of the S-matrix. Obviously, S is unitary. Substituting Eq. (90) or (91) in Eq. (92), we may consider the operator S as given in the form of a perturbation series. It reads

$$\begin{aligned}S &= 1 - i\int_{-\infty}^{\infty} H_I(t')dt' \\ &\quad + \frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt'dt''\, T[H_I(t')H_I(t'')] + \cdots \end{aligned} \tag{93}$$

In the following discussions, it will be licit in most cases to write $H_I(t) = \int d^4x \mathcal{H}_{int}(x)$. If we adopt this expression, the nth order S matrix reads [4]

$$S^{(n)} = \frac{(-i)^n}{n!}\int dx_1' \cdots dx_n'\, T[\mathcal{H}_{int}(x_1') \cdots \mathcal{H}_{int}(x_n')] \tag{94}$$

[4] $\mathcal{H}_{int}(x)$ denotes interaction Hamiltonian density and it should be transformed to the

5 Scalar Fields

Irreducible representations of the Loerentz group solely determines forms of free field equations. Irreducible representations are classified by spins of particles.

5.1 Real Scalar Free Field

Spinless and neutral.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\underline{\partial}\varphi(\underline{x})\right)^2 - \frac{1}{2}m^2\varphi^2(\underline{x}) \\ &= \frac{1}{2}\partial_\mu\varphi(\underline{x})\cdot\partial^\mu\varphi(\underline{x}) - \frac{1}{2}m^2\varphi^2(\underline{x}) \qquad (95)\end{aligned}$$

where a dot in the last equation indicates the former derivative acting only on the first φ. Under a Lorentz transformation $\underline{x} \mapsto \underline{x}' = L\underline{x}$, the field $\varphi(\underline{x})$ transforms as

$$\varphi(x) \mapsto \varphi'(x') = \varphi(x)\,. \qquad (96)$$

Here and hereafter, we omitt underlines on Lorentz vectors.

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} &= \partial^\mu\varphi \\ \pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}(x)} &= \dot{\varphi}\end{aligned} \qquad (97)$$

Euler-Lagrange equation

$$\left(\Box + m^2\right)\varphi(x) = 0 \qquad \text{Klein-Gordon} \qquad (98)$$

interaction picture according to Eq. (84) before adopted in the formula. However, in most cases in the following discussions, this transformation is achieved just by replacing fields in $\mathcal{H}_{int}(x)$ by ones in the interaction picture and we may use the same notation for the interaction Hamiltonian density in the interaction picture as one for the Heisenberg picture. An exception is a case when $\mathcal{H}_{int}(x)$ involves derivative couplings.

where $\Box \overset{\leftarrow}{=} \partial^2 = \partial_\mu \partial^\mu = \partial_0^2 - \boldsymbol{\partial}^2$.
Classical solution of the field equation (98) is written as [5]

$$\varphi_{cl}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^*(\boldsymbol{k})e^{ik\cdot x}\right] \tag{99}$$

where $k^0 = +\sqrt{\boldsymbol{k}^2 + m^2}$. Canonical conjugate field reads,

$$\pi_{cl}(x) = \dot{\varphi}_{cl}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[-ik^0 a(\boldsymbol{k})e^{-ik\cdot x} + ik^0 a^*(\boldsymbol{k})e^{ik\cdot x}\right] \tag{100}$$

Hamiltonian density in the classical level is evaluated from Eq. (39) as

$$\begin{aligned} \mathcal{H} &= \pi(x)\dot{\varphi}(x) - \frac{1}{2}\left\{(\dot{\varphi}(x))^2 - (\boldsymbol{\partial}\varphi(x))^2\right\} + \frac{1}{2}m^2\varphi^2(x) \\ &= \frac{1}{2}\left\{\pi^2(x) + (\boldsymbol{\partial}\varphi(x))^2 + m^2\varphi^2(x)\right\} \end{aligned} \tag{101}$$

Canonical quantization

$$\begin{gathered} [\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \\ [\varphi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{y})] = 0\,, \quad [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0\,. \end{gathered} \tag{102}$$

The fields $\varphi(x)$ and $\pi(x)$ are settled as operators at a time t. As Heisenberg operators, they obey the Heisenberg equation (42) with quantum Hamiltonian given by

$$\hat{H} = \int d^3\boldsymbol{x}\hat{\mathcal{H}}(x)\,, \quad \hat{\mathcal{H}}(x) = :\frac{1}{2}\left\{\pi^2(x) + (\boldsymbol{\partial}\varphi(x))^2 + m^2\varphi^2(x)\right\}:\,, \tag{103}$$

where $:\ldots:$ denotes the normal product. Since we have the Hamiltonian, the Heisenberg equation (42) is equivalent to the quantum level Euler equation (98) and its solution and the canonical conjugate field are written as

$$\varphi(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right]\,, \tag{104}$$

[5] The following consideration lays behind:

$$\begin{aligned} \varphi(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int d^4k\delta(k^2 - m^2)\left[\theta(k^0) + \theta(-k^0)\right]\tilde{\varphi}(k)e^{-ik\cdot x} \\ &= \frac{1}{\sqrt{(2\pi)^3}} \int d^4k\delta(k^2 - m^2)\left[\theta(k^0)\tilde{\varphi}(k)e^{-ik\cdot x} + \theta(k^0)\tilde{\varphi}(-k)e^{ik\cdot x}\right] \\ &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[\tilde{\varphi}(k)e^{-ik\cdot x} + \tilde{\varphi}(-k)e^{ik\cdot x}\right] \\ &\overset{\rightarrow}{=} \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^*(\boldsymbol{k})e^{ik\cdot x}\right] \end{aligned}$$

$$\pi(x) = \frac{-i}{2}\int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}}\left[a(\boldsymbol{k})e^{-ik\cdot x} - a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right] , \tag{105}$$

They have the same form as Eqs. (99) and (100) but now the coefficients $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ are quantum operators defined by the equal-time commutation relations (102) at time $x^0 = t$. The requirements of Eq. (102) is equivalent with requiring

$$\begin{aligned} [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})] &= 2k^0\delta^3(\boldsymbol{k}-\boldsymbol{k'}) , \\ [a(\boldsymbol{k}), a(\boldsymbol{k'})] = 0 , &\quad \left[a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})\right] = 0 \end{aligned} \tag{106}$$

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Addendum: Proof of (102) $\Leftrightarrow$ (106)

Proof of the necessity of Eq. (106) is straightforward, We show the sufficiency of Eq. (102) in the following. We may write Eqs. (104) and (105) as

$$\begin{aligned} \varphi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} Q_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \\ \pi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} P_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} , \end{aligned} \tag{107}$$

with

$$Q_{\boldsymbol{k}}(t) = \frac{1}{2k^0}\left[a(\boldsymbol{k})e^{-ik^0t} + a^\dagger(-\boldsymbol{k})e^{ik^0t}\right] \tag{108}$$

$$P_{\boldsymbol{k}}(t) = \frac{i}{2}\left[a^\dagger(\boldsymbol{k})e^{ik^0t} - a(-\boldsymbol{k})e^{-ik^0t}\right] \tag{109}$$

Relationships $Q^\dagger_{\boldsymbol{k}} = Q_{-\boldsymbol{k}}$ and $P^\dagger_{\boldsymbol{k}} = P_{-\boldsymbol{k}}$ ensure that φ and π are real. From the linear independence of Fourier components, we have

$$\begin{aligned} 0 = [\varphi(t,\boldsymbol{x}), \varphi(t,\boldsymbol{y})] &\iff [Q_{\boldsymbol{k}}(t), Q_{\boldsymbol{k'}}(t)] = 0 , \\ 0 = [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] &\iff [P_{\boldsymbol{k}}(t), P_{\boldsymbol{k'}}(t)] = 0 , \end{aligned} \tag{110}$$

and

$$\begin{aligned} i\delta^3(\boldsymbol{x}-\boldsymbol{y}) &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3} i\delta^3(\boldsymbol{k}-\boldsymbol{k}')e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\boldsymbol{k}'\boldsymbol{y}} \\ &= [\varphi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] \\ &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3} [Q_{\boldsymbol{k}}, P_{\boldsymbol{k}'}]e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\boldsymbol{k}'\cdot\boldsymbol{y}} \\ &\iff \\ \left[Q_{\boldsymbol{k}}, P_{\boldsymbol{k}'}\right] &= i\delta^3(\boldsymbol{k}-\boldsymbol{k}') \end{aligned} \tag{111}$$

Eqs. (108) and (109) reads,

$$a(\boldsymbol{k}) = \left(k^0 Q_{\boldsymbol{k}}(t) + iP^\dagger_{\boldsymbol{k}}(t)\right)e^{ik^0t} , \tag{112}$$

$$a^\dagger(\boldsymbol{k}) = \left(k^0 Q^\dagger_{\boldsymbol{k}}(t) - iP_{\boldsymbol{k}}(t)\right) e^{-ik^0 t}\,, \tag{113}$$

and Eq. (106) follows from Eqs. (110) and (111).
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Notice that $dim[\varphi] = E^1$ as can be seen from Eq. (95) and $dim[\pi] = E^2$. Thus they are physical quantities quite different from those in the case of the Schrödinger field theory.

We are now making use of a fact that operators in Eq. (106) satisfiy the condition of the bosonic creation-annihilation operators, (20), for the case of continuous eigenvalues. The total number operator can be defined as

$$\hat{N} = \int \frac{d^3\boldsymbol{k}}{2k^0} a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{114}$$

It holds that

$$\begin{aligned} \hat{N} a(\boldsymbol{k}) &= \int \frac{d^3\boldsymbol{k}'}{2k^{0\prime}} \left\{ a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}') - 2k^0 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \right\} a(\boldsymbol{k}') \\ &= a(\boldsymbol{k})(\hat{N} - 1)\,, \\ \hat{N} a^\dagger(\boldsymbol{k}) &= a^\dagger(\boldsymbol{k})(\hat{N} + 1) \end{aligned} \tag{115}$$

The Hamiltonian (103) reads from Eqs. (104) and (105) as,

$$\hat{H} = \int \frac{d^3\boldsymbol{k}}{2k^0} k^0 a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) \tag{116}$$

□ Total momentum comes here [6]
We have

$$[\hat{P}^\mu, \hat{N}] = 0 \tag{117}$$

[6] Applying Noether's theorem to the invariance under space-time translations, we have an expression for the conserved energy-momentum vector as

$$P^\mu = \int T^{0\mu}(x) d^3\boldsymbol{x}$$

where the conserved Noether current is given as

$$T^\mu_\nu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))} \partial_\nu \varphi(x) - g^\mu_\nu \mathcal{L}$$

$$\hat{P}^\mu = \int \frac{d^3\boldsymbol{k}}{2k^0} k^\mu a^\dagger(\boldsymbol{k}) a(\boldsymbol{k})$$

and this relationship establishes particle interpretation. Namely a state $a^\dagger(\boldsymbol{k})\,|0>$ is interpreted as one particle eigenstate of the momentum associated with an eigenvalue $\boldsymbol{k}$.

$$\hat{\boldsymbol{P}}\hat{a}^\dagger(\boldsymbol{k})\,|0>= \boldsymbol{k}\hat{a}^\dagger(\boldsymbol{k})\,|0> \tag{118}$$

so that we may write

$$\hat{a}^\dagger(\boldsymbol{k})\,|0>= \,|\boldsymbol{k}> \tag{119}$$

State normalization

$$<\boldsymbol{p}|\,\boldsymbol{p}'>=<0|\,[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')]\,|0>= 2k^0\delta^3(\boldsymbol{p}-\boldsymbol{p}')$$

The Hamiltonian (116) is diagonalized by creation-annihilation operators:

$$[\hat{H}, \hat{a}^\dagger(\boldsymbol{k})] = k^0\hat{a}^\dagger(\boldsymbol{k})\,, \quad [\hat{H}, \hat{a}(\boldsymbol{k})] = -k^0\hat{a}(\boldsymbol{k}) \tag{120}$$

so that

$$\hat{H}\hat{a}^\dagger(\boldsymbol{k})\,|0>= k^0\hat{a}^\dagger(\boldsymbol{k})\,|0> \tag{121}$$

Wave function:
For

$$|\Psi^{(1)}> \overset{\leftarrow}{=} \int \frac{d^3\boldsymbol{k}}{2k^0}\Psi^{(1)}(\boldsymbol{k})a^\dagger(\boldsymbol{k})\,|0>\,,$$

$$\hat{N}\,|\Psi^{(1)}>= \,|\Psi^{(1)}>\,,$$

$$<\boldsymbol{k}|\,\Psi^{(1)}>= \Psi^{(1)}(\boldsymbol{k})$$

The state is normalized through a relationship

$$<\Psi^{(1)}|\,\Psi^{(1)}>= \int \frac{d^3\boldsymbol{k}}{2k^0}|\Psi^{(1)}(\boldsymbol{k})|^2$$

A state of n particles at momenta $\boldsymbol{k}_1$, $\boldsymbol{k}_2, \ldots \boldsymbol{k}_n$, is written as

$$|\boldsymbol{k}_1 \cdots \boldsymbol{k}_n>= \frac{1}{\sqrt{n!}}a^\dagger(\boldsymbol{k}_1)\cdots a^\dagger(\boldsymbol{k}_n)\,|0>$$

Since $a^\dagger(\boldsymbol{k}_1), \ldots a^\dagger(\boldsymbol{k}_n)$ commute among themselves, the state is symmetric under change of orders of momenta. We have [7]

$$
\begin{aligned}
\hat{N}\,|\boldsymbol{k}_1\cdots\boldsymbol{k}_n\!> &= \frac{1}{\sqrt{n!}}\int\frac{d^3\boldsymbol{k}}{2k^0}a^\dagger(\boldsymbol{k})\left[a(\boldsymbol{k}),a^\dagger(\boldsymbol{k}_1)\cdots a^\dagger(\boldsymbol{k}_n)\right]\,|0\!> \\
&= \frac{1}{\sqrt{n!}}\int\frac{d^3\boldsymbol{k}}{2k^0}a^\dagger(\boldsymbol{k})\left(\left[a(\boldsymbol{k}),a^\dagger(\boldsymbol{k}_1)\right]a^\dagger(\boldsymbol{k}_2)\cdots a^\dagger(\boldsymbol{k}_n)\right. \\
&\quad \left. +\,a^\dagger(\boldsymbol{k}_1)\left[a(\boldsymbol{k}),a^\dagger(\boldsymbol{k}_2)\cdots a^\dagger(\boldsymbol{k}_n)\right]\right)\,|0\!> \\
&= \cdots \\
&= n\,|\boldsymbol{k}_1\cdots\boldsymbol{k}_n\!>
\end{aligned}
$$

It is normalized as

$$
\begin{aligned}
&<\!\boldsymbol{k}_1\cdots\boldsymbol{k}_n\,|\boldsymbol{k}'_1\cdots\boldsymbol{k}'_n\!> \\
&= \frac{1}{n!}\sum_{i'_1=1}^{n}<\!0|\,a_2\cdots a_n a_{1'}\cdots[a_1,a^\dagger_{i'_1}]\cdots a^\dagger_{n'}\,|0\!> \\
&= \frac{1}{n!}\sum_{i'_1=1}^{n}\sum_{i'_2\neq i'_i}^{n}<\!0|\,a_3\cdots a_n a_{1'}\cdots\cancel{a_{i'_1}}\cdots\cancel{a_{i'_2}}\cdots a^\dagger_{n'}\,|0\!>\left[a_1,a^\dagger_{i'_1}\right]\left[a_2,a^\dagger_{i'_2}\right] \\
&= \cdots \\
&= \frac{1}{n!}\sum_{i'_1\ldots i'_n=perm(1\ldots n)}^{n!\ \text{terms}}\prod_{l}^{n}2k^0_l\delta^3(\boldsymbol{k}_l-\boldsymbol{k}_{i'_l})
\end{aligned}
\tag{122}
$$

We may construct a state of n scalar particles described by a wave function $\Psi^{(N)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ as

$$
|\Psi^{(N)}\!>=\int\prod_{i}^{N}\frac{d^3\boldsymbol{k}_i}{2k^0_i}\Psi^{(N)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)\,|\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\!>
\tag{123}
$$

Since n particle momentum satate is symmetric under exchange of momenta, we may presuppose the function $\Psi^{(N)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ is also symmetric. Then we have

$$
<\!\boldsymbol{k}_1\cdots\boldsymbol{k}_n|\,\Psi^{(N)}\!>=\Psi^{(N)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)
$$

[7] Useful relationships

$$
\left[a,a^\dagger_1\cdots a^\dagger_n\right]=\sum_{i=1}^{n}a^\dagger_1\cdots[a,a^\dagger_i]\cdots a^\dagger_n
$$

$$
\left[a_1\cdots a_n,a^\dagger\right]=\sum_{i=1}^{n}a_1\cdots[a_i,a^\dagger]\cdots a_n
$$

When the number of particles is fixed, normalization of the state is written as

$$\| \,|\Psi^{(N)}> \|^2 = \int \prod_i^N \frac{d^3\boldsymbol{k}_i}{2k_i^0} \|\Psi^{(N)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)\|^2 = 1$$

The most general state in the Fock space may be written as

$$|\Psi>= \sum_N |\Psi^{(N)}>$$

In this case, $\| \,|\Psi^{(N)}> \|^2$ gives the probability to find the system with n particles.

Particle states composed like this are these in the Heisenberg picture. If we choose $e^{-ik^0 t_0}\hat{a}(\boldsymbol{k})$ and $e^{ik^0 t_0}\hat{a}^\dagger(\boldsymbol{k})$ as initial values of Heisenberg operators $\hat{a}_H(\boldsymbol{k}, t)$ and $\hat{a}_H^\dagger(\boldsymbol{k}, t)$ respectively at $t = t_0$, we have from Eq. (43) that [8]

$$a_H(\boldsymbol{k}, t) = e^{-ik^0 t} a(\boldsymbol{k})\,, \quad a_H^\dagger(\boldsymbol{k}, t) = e^{ik^0 t} a^\dagger(\boldsymbol{k}) \tag{124}$$

We introduce state vectors in the Schrödinger picture by

$${}_S<\Psi(t)|\,\mathcal{O}\,|\Psi(t)>_S=<\Psi|\,\mathcal{O}_H(t)\,|\Psi> \tag{125}$$

State vector in the Schrödinger picture obeys

$$i\partial_t\,|\Psi(t)>_S= \hat{H}\,|\Psi(t)>_S \tag{126}$$

Propagator

$$\left(\Box + m^2\right) \Delta_F(x) = \delta^4(x) \tag{127}$$

[8]From Eq. (120), we have

$$Ha^\dagger(\boldsymbol{k}) = a^\dagger(H + k^0)\,, \quad H^2 a^\dagger(\boldsymbol{k}) = a^\dagger(H + k^0)^2\,, \ldots$$

so that

$$\begin{aligned} e^{iH(t-t_0)} a^\dagger(\boldsymbol{k}) &= \sum_n^\infty \frac{i^n}{n!}(t - t_0)^n H^n a^\dagger(\boldsymbol{k}) \\ &= a^\dagger(\boldsymbol{k}) \sum_n^\infty \frac{i^n}{n!}(t - t_0)^n (H + k^0)^n \\ &= a^\dagger(\boldsymbol{k}) e^{i(H+k^0)(t-t_0)} \end{aligned}$$

$$\begin{aligned}\Delta_F(q) &= i\int d^4x e^{iqx} <0|\, \mathrm{T}\varphi(x)\varphi(0)\,|0> \\ &= \frac{-1}{q^2-m^2+i\epsilon}\,, \end{aligned} \tag{128}$$

where ϵ is infinitesimal positive number. T-product

$$T[\varphi(x)\varphi(y)] = \theta(x^0-y^0)\varphi(x)\varphi(y) + \theta(y^0-x^0)\varphi(y)\varphi(x) \tag{129}$$

One will find

$$\left(\Box+m^2\right)T[\varphi(x)\varphi(0)] = -i\delta^4(x)$$
$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iqx}\Delta_F(q) \tag{130}$$

5.2 Complex Scalar Free Field

Spinless and charged.

$$\mathcal{L} = \partial_\mu\varphi^\dagger(x)\cdot\partial^\mu\varphi(x) - m^2\varphi^\dagger(x)\varphi(x) \tag{131}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi^\dagger\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\dagger)} = \partial^\mu\varphi \tag{132}$$

$$\left(\Box+m^2\right)\varphi(x) = 0\,, \qquad \left(\Box+m^2\right)\varphi^\dagger(x) = 0\,, \tag{133}$$

$$\pi^\dagger(x) = \frac{\partial\mathcal{L}}{\partial\dot\varphi(x)} = \dot\varphi^\dagger\,, \qquad \pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\varphi^\dagger(x)} = \dot\varphi \tag{134}$$

$$\begin{aligned}\mathcal{H} &= \pi^\dagger\dot\varphi + \pi\dot\varphi^\dagger - \mathcal{L} \\ &= \pi^\dagger\pi + (\partial_i\varphi^\dagger)(\partial_i\varphi) + m\varphi^\dagger\varphi \end{aligned} \tag{135}$$

$$\begin{aligned}\varphi(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}}\left[a(\boldsymbol{k})e^{-ik\cdot x} + b^\dagger(\boldsymbol{k})e^{ik\cdot x}\right] \\ \varphi^\dagger(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}}\left[b(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right]\end{aligned} \tag{136}$$

Canonical quantization

$$\begin{gathered}[a(\boldsymbol{k}), a^\dagger(\boldsymbol{k'})] = [b(\boldsymbol{k}), b^\dagger(\boldsymbol{k'})] = 2k^0\delta^3(\boldsymbol{k}-\boldsymbol{k'})\,, \\ \text{other commutators} = 0\end{gathered} \tag{137}$$

Four momentum

$$\hat{P}^\mu = \int \frac{d^3\boldsymbol{k}}{2k^0}k^\mu\left(a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) + b^\dagger(\boldsymbol{k})b(\boldsymbol{k})\right) \tag{138}$$

$\hat{P}^0$ is the Hamiltonian. Vacuum state is defined as $a(\boldsymbol{p})\,|0>= b(\boldsymbol{p})\,|0>= 0$. There are two kinds of single particle states $a^\dagger(\boldsymbol{p})\,|0>$ and $b^\dagger(\boldsymbol{p})\,|0>$ both an eigenstate of $\hat{P}^\mu$. So far, a and b are just particles independent of each other and having the common mass m. Let us examine the electric charge given from Eq. (65). We find

$$\hat{Q} = e\int \frac{d^3\boldsymbol{k}}{2k^0}\left(a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) - b^\dagger(\boldsymbol{k})b(\boldsymbol{k})\right) \tag{139}$$

Thus they carry opposite electric charges.

We consider now inversion symmetries $U = P, C, T$. The vacuum is invariant under these inversions.

- Space inversion: $(\boldsymbol{x}, t) \mapsto (-\boldsymbol{x}, t)$

  $$Pa(\boldsymbol{k})P^{-1} = \pm a(-\boldsymbol{k})\,, \quad Pb(\boldsymbol{k})P^{-1} = \pm b(-\boldsymbol{k})\,, \tag{140}$$

  $+$ for scalar and $-$ for pseudoscalar.

- Charge conjugation

  $$Ca(\boldsymbol{k})C^{-1} = \pm b(\boldsymbol{k})\,, \quad Cb(\boldsymbol{k})C^{-1} = \pm a(\boldsymbol{k})\,, \tag{141}$$

- Time reversal: $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}, -t)$

  $$Ta(\boldsymbol{k})T^{-1} = \pm a(-\boldsymbol{k})\,, \quad Cb(\boldsymbol{k})C^{-1} = \pm b(-\boldsymbol{k})\,, \tag{142}$$

  and T is antilinear so that

  $$T\varphi(\boldsymbol{x}, t)T^{-1} = \pm\varphi(\boldsymbol{x}, -t) \tag{143}$$

  The sign is fixed through the invariance of interactions with other kind of fields which have definite signatures under the T transformation. Usually, $+$ for scalar and $-$ for pseudoscalar.

5.3 Internal Symmetry

Complex scalar field with internal symmetry

$$\mathcal{L} = \sum_a \left[\partial_\mu\varphi_a^\dagger(x)\cdot\partial^\mu\varphi_a(x) - m^2\varphi_a^\dagger(x)\varphi_a(x)\right] \tag{144}$$

We consider scalar fields $\varphi_a(x)$ transforms under a global $SU(n)$ as in Eq. (66). We assume $\{\varphi_a(x)\}$ composes an irreducible representation ν and write them in a combined form $\varphi = (\varphi_1, \ldots, \varphi_s)$ where s is the multiplicity of the representation ν. We may write the $\mathcal{L}$agrangian exactly in the same form as Eq. (131) understanding our φ is now multicomponent. We may then write the Noether current and charge in Eqs (68) and (69) interms creation and annihilation operators. States $\{a_j^\dagger\,|0>\}$ composes a multiplet of ν. (To be continued.)

5.4 Scalar Yukawa Theory - A Toy Model -

Let us consider a toy model composed of scalar fields:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2}(\partial_\mu \phi)^2 - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g\psi^* \psi \phi \tag{145}$$

ψ is a mock nucleon complex scalar field with the mass M and ϕ is a mock pion real scalar field with the mass m. The first parts of $\mathcal{L}$ except for the last term compose free $\mathcal{L}$agrangians $\mathcal{L}_N$ and $\mathcal{L}_\pi$ for nucleon and pion fields, respectively. The last term is the interaction $\mathcal{L}$agrangian in which nucleons and pions interact each other with a coupling constant g. The coupling g has the dimension of energy and the dimensionless parameter is g/E, where E is the energy scale of the process of interest. This means that the interaction term $\mathcal{L}_{int} = -g\psi^*\psi\phi$ is relevant at low energies. The relativistic nature gets important at $E \gg M, m$ and we may choose $g \ll M, m$ so that the perturbation series (90) converges.

Each fields are quantized through equal-time commutation relations at time, say, 0. One may write down field equations but they are not solvable due to the presence of $\mathcal{L}_{int}$. Remember they describe time evolution of field operators in the Heisenberg picture. In the interaction picture, however, we may let fields obey free field equations. In practice, conjugate fields are given by

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial(\dot\phi)} = \dot\phi\,, \quad \pi_\psi = \frac{\partial \mathcal{L}}{\partial(\dot\psi)} = \dot\psi^*\,, \quad \pi_{\psi^*} = \frac{\partial \mathcal{L}}{\partial(\dot\psi^*)} = \dot\psi\,, \tag{146}$$

and classical Hamiltonian density is written as

$$\begin{aligned}\mathcal{H} &= \pi_\phi\dot\phi + \pi_\psi\dot\psi + \pi_{\psi^*}\dot\psi^* - \mathcal{L} \\ &= \frac{1}{2}\left\{\pi_\phi^2 + (\boldsymbol{\partial}\phi)^2 + m^2\phi^2\right\} + \left\{\pi_\psi\pi_{\psi^*} + \boldsymbol{\partial}\psi^* \cdot \boldsymbol{\partial}\psi + M^2\psi^*\psi\right\} \\ &\quad + g\psi^*\psi\phi \end{aligned} \tag{147}$$

The first two terms are $\mathcal{H}_\pi$ and $\mathcal{H}_N$ corresponding to $\mathcal{L}_\pi$ and $\mathcal{L}_N$, respectively, and we assign them as the free $\mathcal{H}$amiltonian $\mathcal{H}_0 = \mathcal{H}_\pi + \mathcal{H}_N$. Writing $\mathcal{H}_{int} = g\psi^*\psi\phi$, we have a decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$, which corresponds to Eq. (81). After substituting quantized fields (at the fixed time) into these expressions (and taking the normal ordering), we obtain $\mathcal{H}$amiltonian operators. The time evolutions of fields in the interaction picture are given from Eq. (85) as

$$\begin{aligned} i\partial_t\phi_I &= [\phi_I, H_0] = [\phi_I, H_\pi]\,, \\ i\partial_t\psi_I &= [\psi_I, H_0] = [\psi_I, H_N]\,, \\ i\partial_t\psi_I^\dagger &= [\psi_I^\dagger, H_0] = [\psi_I^\dagger, H_N]\,, \end{aligned} \tag{148}$$

where we added an suffix I to indicate quantities in the interaction picture. We already know that Eqs. (148) are equivalent to free field equations all given as the Klein-Gordon equation with masses of each fields. Therefore, the fields ϕ_I, ψ_I and $\psi_I^\dagger$ have Fourier expanded forms as in Eqs. (104) and (136). To be specific, we write

$$
\begin{aligned}
\phi_I(x) &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} \left[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right] \\
\psi_I(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \left[b(\boldsymbol{p})e^{-ip\cdot x} + c^\dagger(\boldsymbol{p})e^{ip\cdot x}\right] \\
\psi_I^\dagger(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \left[c(\boldsymbol{p})e^{-ip\cdot x} + b^\dagger(\boldsymbol{p})e^{ip\cdot x}\right]
\end{aligned}
\tag{149}
$$

Operators a, b and c and their conjugates satisfy commutation relations given in Eqs. (106) and (137) and they establishes the particle interpretations. Parts of the free $\mathcal{H}$amiltonian $\mathcal{H}_\pi$ and $\mathcal{H}_N$ are now described by these operators as in Eq. (116) and the time component of Eq. (138). We may define number operator $\hat{N}_\pi$ as in Eq. (114) and ones for nucleons as

$$
\hat{N}_N = \int \frac{d^3\boldsymbol{p}}{2p^0} b^\dagger(\boldsymbol{p})b(\boldsymbol{p})\,, \quad \hat{N}_{\bar{N}} = \int \frac{d^3\boldsymbol{p}}{2p^0} c^\dagger(\boldsymbol{p})c(\boldsymbol{p})\,,
\tag{150}
$$

where we have assigned $b^\dagger$ and $c^\dagger$ as creation operators for a nucleon (N) and an anti-nucleon ($\bar{N}$), respectively. In the interaction picture, $\mathcal{L}_{int}$ term in Eq.(145) [or $\mathcal{H}_{int}$ term in Eq. (147)] contains creation and annihilation operators in each field and they may change number of particles. In practice, the number operators we have just defined do not commute with H_I, therefore neither with H and do not conserve. Though numbers of particles are not conserved, the total electric charge would be conserved since the $\mathcal{L}$agrangian (145) is invariant under constant phase change of the field ψ and its conjugate for ψ^*. According to the Noether's theorem and Eq. (139), the charge

$$
Q = e\int \frac{d^3\boldsymbol{p}}{2p^0} \left(b^\dagger(\boldsymbol{p})b(\boldsymbol{p}) - c^\dagger(\boldsymbol{p})c(\boldsymbol{p})\right)
\tag{151}
$$

commutes with H and conserved.

Keeping the formula (90) in mind, let us examine particular expression of H_I:

$$\begin{aligned}
H_I(t) &= g\int d^3\boldsymbol{x} {:}\psi_I^\dagger(x)\psi_I(x)\phi_I(x){:} \\
&= \frac{g}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}}{2k^0}\int d^3\boldsymbol{x} : [e^{-ipx}c(\boldsymbol{p})+e^{ipx}b^\dagger(\boldsymbol{p})] \\
&\qquad [e^{-ip'x}b(\boldsymbol{p}')+e^{ip'x}c^\dagger(\boldsymbol{p}')][e^{-ikx}a(\boldsymbol{k})+e^{ikx}a^\dagger(\boldsymbol{k})] : \\
&= \frac{g}{\sqrt{(2\pi)^3}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}}{2k^0} \\
&\quad [c(\boldsymbol{p})b(\boldsymbol{p}')a(\boldsymbol{k})\delta^3(\boldsymbol{p}+\boldsymbol{p}'+\boldsymbol{k})e^{-i(p+p'+k_+)^0t} \\
&\quad +b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a^\dagger(\boldsymbol{k})\delta^3(\boldsymbol{p}+\boldsymbol{p}'+\boldsymbol{k})e^{i(p+p'+k_+)^0t} \\
&\quad +b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a(\boldsymbol{k})\delta^3(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})e^{i(p+p'-k_+)^0t} \\
&\quad +c^\dagger(\boldsymbol{p})c(\boldsymbol{p}')a(\boldsymbol{k})\delta^3(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k})e^{i(p-p'-k_-)^0t} \\
&\quad +b^\dagger(\boldsymbol{p})b(\boldsymbol{p}')a(\boldsymbol{k})\delta^3(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k})e^{i(p-p'-k_-)^0t} \\
&\quad +a^\dagger(\boldsymbol{k})c(\boldsymbol{p})b(\boldsymbol{p}')\delta^3(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})e^{-i(p+p'-k_-)^0t} \\
&\quad +a^\dagger(\boldsymbol{k})c^\dagger(\boldsymbol{p}')c(\boldsymbol{p})\delta^3(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k})e^{-i(p-p'-k_-)^0t} \\
&\quad +a^\dagger(\boldsymbol{k})b^\dagger(\boldsymbol{p})b(\boldsymbol{p}')\delta^3(-\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})e^{i(p-p'+k_-)^0t}]
\end{aligned} \tag{152}$$

where $k^0_\pm = \sqrt{(\boldsymbol{p}\pm\boldsymbol{p}')^2+m^2}$. At this stage, we may already have some insights about the interaction. Our $\mathcal{L}_{int}$ is a product of three field operators and each of them involve two terms with annihilation and creation operators. This is why there are $2^3 = 8$ terms in Eq. (152). Among them, the last 6 terms show possible processes. For instance, the third term corresponds a process in which a particle a (pion) disappears and particles b and c (nucleon and anti-nuclen) emerges. So this term corresponds to the pair creation of $N\bar{N}$ by π. One can confirm the momentum is conserved in the process. The energy is not conserved yet and there are exponential factors instead. Later, we will see these exponentials turn into delta functions corresponding to the energy conservation for each processes in the evaluation the scattering matrix. The first two terms in Eq. (152) will not contribute to scattering matrices since they violate the energy conservation. The second and higher order terms in Eq. (90) will be involved in the later discussion with introducing some techniques to expand time ordered products.

● Meson Decay

Consider a process $\pi \to N\bar{N}$. This process is involved in the lowest order term in Eq. (93) through the third term in Eq. (152). We write Initial and final states as

$$
\begin{aligned}
|i\!> &= a^\dagger(\boldsymbol{k})\,|0\!> \;\overset{\rightarrow}{=}\; |\pi(\boldsymbol{k})\!>\,, \\
|f\!> &= b^\dagger(\boldsymbol{p}_N)c^\dagger(\boldsymbol{p}_{\bar{N}})\,|0\!> \;\overset{\rightarrow}{=}\; |N(\boldsymbol{p}_N)\bar{N}(\boldsymbol{p}_{\bar{N}})\!>
\end{aligned}
\tag{153}
$$

From Eq. (93), we read to the leading (first) order in g that

$$
\begin{aligned}
<\!f|\,S^{(1)}\,|i\!> &= -ig\int d^4x <\!N(\boldsymbol{p}_N)\bar{N}(\boldsymbol{p}_{\bar{N}})|\,:\!\psi^\dagger(x)\psi(x)\phi(x)\!:\,|\pi(\boldsymbol{k})\!> \\
&= -ig\int d^4x <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})\!:\!\psi^\dagger(x)\psi(x)\phi(x)\!:\!a^\dagger(\boldsymbol{k})\,|0\!>
\end{aligned}
\tag{154}
$$

Here we omitted indeces I on fields under understanding that we are in the interaction picture. In expanding fields as $\psi^\dagger \sim c + b^\dagger$, $\psi \sim b + c^\dagger$ and $\phi \sim a + a^\dagger$, we find only a term $\sim b^\dagger c^\dagger a$ contributes in Eq. (154). This corresponds to the third term in H_I in Eq. (152). We proceed from Eq. (154):

$$
\begin{aligned}
<\!f|\,S^{(1)}\,|i\!> &= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}'}{2k^{0\prime}}\int d^4x\; e^{i(p+p'-k')x} \\
&\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})[b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a(\boldsymbol{k}')]a^\dagger(\boldsymbol{k})\,|0\!> \\
&= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}\frac{d^3\boldsymbol{k}'}{2k^{0\prime}}(2\pi)^4\delta^4(p+p'-k') \\
&\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})[b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')a(\boldsymbol{k}')]a^\dagger(\boldsymbol{k})\,|0\!> \\
&= \frac{-ig}{\sqrt{(2\pi)^9}}\int \frac{d^3\boldsymbol{p}}{2p^0}\frac{d^3\boldsymbol{p}'}{2p^{0\prime}}(2\pi)^4\delta^4(p+p'-k) \\
&\qquad <\!0|\,b(\boldsymbol{p}_N)c(\boldsymbol{p}_{\bar{N}})b^\dagger(\boldsymbol{p})c^\dagger(\boldsymbol{p}')\,|0\!> \\
&= \frac{-ig}{\sqrt{(2\pi)^9}}(2\pi)^4\delta^4(p_N+p_{\bar{N}}-k)
\end{aligned}
\tag{155}
$$

Reaction rate is defined in the same way as that for scatterting as

$$
R_{fi} = \int d\Phi_f|\,<\!f|\,T\,|i\!>|^2\,,
\tag{156}
$$

where Φ_f and T are defined in Eqs. (**??**) and (**??**) respectively [9] . For decays of a particle with the mass M into n particles, differential decay width is defined as

$$
d\Gamma = \frac{(2\pi)^3}{2M}dR_{fi} = \frac{(2\pi)^3}{2M}\prod_i^n \frac{d^3\boldsymbol{p}_i}{2p_i^0}(2\pi)^4\delta^4(P_f-P_i)|\,<\!f|\,T\,|i\!>|^2
\tag{157}
$$

[9] A note on dimensions: $dim[d\Phi_f(<f|)^2] = 1/E^4$, $dim[T] = E^4$ and $dim[\,|i>]^2 = 1/E^{2k}$ for an initial state composed of k particles. In total, $dim[R] = E^{4-2k}$. $k = 2$ and $dim[R] = E^0$ for scatterings. $k = 1$ and $dim[R] = E^2$ for particle decays.

In particular, for two particle decays,

$$
\begin{aligned}
d\Gamma_2 &= \frac{(2\pi)^3}{2M}\frac{d^3\boldsymbol{p}_1}{2p_1^0}d^4p_2\delta(p_2^2-m_2^2)(2\pi)^4\delta^4(P_f-P_i)|<f|\,T\,|i>|^2 \\
&= \frac{(2\pi)^7}{2M}\frac{P_1^{*2}dP_1^*d\Omega_1^*}{2p_1^{*0}}\delta(M^2+m_1^2-m_2^2-2Mp_1^{*0})|T_{fi}|^2 \\
&= \frac{(2\pi)^9}{32\pi^2M^2}P_1^*d\Omega_1^*|T_{fi}|^2 \qquad (158)
\end{aligned}
$$

We note that $dim|T_{fi}|^2 = E^2$. The way of writing the coefficient in Eq. (158) is chosen so that it is clear that a factor $(2\pi)^9$ is absorbed in the state normalizations in notations of some authors including [2], [3] and [6].

Let's come back to our toy model. From Eq. (155), we have $T_{fi} = g/\sqrt{(2\pi)^9}$. Substituting this in Eq. (158), we obtain

$$
\Gamma_2^{(1)} = \frac{g^2}{16\pi m^2}\lambda^{1/2}(m^2, M^2, M^2) \qquad (159)
$$

where a superscript (1) indicates the order of perturbation expansion in Eq. (93).

6 Interacting Fields (2/2)

Provided with $\mathcal{H}_{int}$ in terms of quantized fields in the interaction picture, we write

$$
\begin{aligned}
<f|\,S-1\,|i> &= -i\int d^4x' \;<f|\,\mathcal{H}_{int}(x')\,|i> \\
&\quad +\frac{(-i)^2}{2}\int d^4x_1'd^4x_2' \;<f|\,T[\mathcal{H}_{int}(x_1')\mathcal{H}_{int}(x_2')]\,|i> + \cdots
\end{aligned}
\qquad (160)
$$

To proceed beyond the second term requires some further preparations that we describe in the following.

6.1 Propagators of scalar fields

Let's take a real scalar field in Eqs (106, 104). We have

$$
\begin{aligned}
<0|\,\phi(x)\phi(y)\,|0> &= \int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3 4k^0k^{0\prime}}<0|\,a(\boldsymbol{k})a^\dagger(\boldsymbol{k}')\,|0>e^{-i(kx-k'y)} \\
&= \int\frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0}e^{-ik(x-y)} \overset{\to}{\equiv} D(x-y) \qquad (161)
\end{aligned}
$$

$$
\begin{aligned}
[\phi(x),\phi(y)] &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k'}}{(2\pi)^3 4k^0k^{0\prime}}\Big([a(\boldsymbol{k}),a^\dagger(\boldsymbol{k'})]e^{-i(kx-k'y)} \\
&\qquad\qquad +[a^\dagger(\boldsymbol{k}),a(\boldsymbol{k'})]e^{i(kx-k'y)}\Big) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0}\left(e^{-ik(x-y)}-e^{ik(x-y)}\right) \\
&= D(x-y)-D(y-x) \qquad (162)
\end{aligned}
$$

When $x-y$ is spacelike, there exists a Lorentz frame where $x^0=y^0$. Then the *r.h.s.* of Eq. (419) vanishes. The whole expression is Lorentz invariant and it must vanish for all $(x-y)^2<0$. Nevertheless, $D(x-y)$ itself does not vanish even $x-y$ is spacelike.

The Feynman propagator is defined as

$$
\begin{aligned}
<0|\,T[\phi(x)\phi(y)]\,|0> &= \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x) \qquad (163) \\
&= i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon} \qquad (164) \\
&\stackrel{\rightarrow}{=} \Delta_F(x-y)=\Delta_F(y-x)
\end{aligned}
$$

Proof of Eq.(164) ⇔ Eq. (163)

$$
\frac{1}{p^2-m^2+i\epsilon}=\frac{1}{2E_{\boldsymbol{p}}}\left(\frac{1}{p^0-E_{\boldsymbol{p}}+i\epsilon}-\frac{1}{p^0+E_{\boldsymbol{p}}-i\epsilon}\right), \qquad (165)
$$

$$
\begin{aligned}
\Delta_F(x-y) &= i\int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&\quad\times\int\frac{dp^0}{2\pi}e^{-ip^0(x^0-y^0)}\left(\frac{1}{p^0-E_{\boldsymbol{p}}+i\epsilon}-\frac{1}{p^0+E_{\boldsymbol{p}}-i\epsilon}\right) \\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\frac{-1}{2\pi i}\Big(\theta(x^0-y^0)(-2\pi i e^{-iE_{\boldsymbol{p}}(x^0-y^0)}) \\
&\quad -\theta(y^0-x^0)(+2\pi i e^{iE_{\boldsymbol{p}}(x^0-y^0)}\Big) \\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(\theta(x^0-y^0)e^{-ip(x-y)}+\theta(y^0-x^0)e^{ip(x-y)}\right) \\
&= \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x) \\
&\qquad\qquad\qquad\qquad \blacksquare \qquad (166)
\end{aligned}
$$

From the first line to the second, we have performed integrations in a complex p^0 plane. Contour of the integration should go through the real axis from $-\infty$

to ∞ and turn lower (upper) half plane (anti) clockwise when $x^0 - y^0 > (<)\, 0$. A factor $\pm 2\pi i$ appears as a result of the residue calculus.

For the complex scalar field prescribed in Eqs. (136, 137), we find

$$
\begin{aligned}
&<0|\, \varphi(x)\varphi(y)\, |0> = <0|\, \varphi^\dagger(x)\varphi^\dagger(y)\, |0> = 0\,, \\
&<0|\, T[\varphi^\dagger(x)\varphi(y)]\, |0> = <0|\, T[\varphi(x)\varphi^\dagger(y)]\, |0> = \Delta_F(x-y)
\end{aligned}
\tag{167}
$$

6.2 Prescription for time ordered products

Since the second and higher order terms in Eq. (93) are written in terms of time ordered products of H_I's and each H_I is given in terms of a normal product of fields like one in the first line of Eq. (152), we need a way to deal with time ordered products of fields. A technique to do this is provided by Wick's theorem, which we have described in Appendix C. With the Feynman propagator defined in Eq. (164), Wick's theorem (427) for the real scalar field reads

$$
\begin{aligned}
T[\phi_1\phi_2\cdots] &= \,:\phi_1\phi_2\cdots: \\
&+ \sum_{i<j} :\cdots \overbracket{\phi_i \cdots \phi_j} \cdots: \\
&+ \sum_{i<j,k<l} :\cdots \phi_i \cdots \phi_k \cdots \phi_j \cdots \phi_l \cdots: \\
&+ \cdots \text{ (all possible contracts)} \\
&= \,:\phi_1\phi_2\cdots: \\
&+ \sum_{i<j} \Delta_F(x_i - x_j) :\cdots \cancel{\phi_i} \cdots \cancel{\phi_j} \cdots: \\
&+ \sum_{i<j,k<l} \Delta_F(x_i - x_j)\Delta_F(x_k - x_l) :\cdots \cancel{\phi_i} \cdots \cancel{\phi_k} \cdots \cancel{\phi_j} \cdots \cancel{\phi_l} \cdots: \\
&+ \cdots
\end{aligned}
\tag{168}
$$

where ϕ_i stands for $\phi(x_i)$. For the complex scalar field, we have a similar formula as the above but contracts are taken only among φ and $\varphi^\dagger$ since ones among the same kind disappears as it can be read from Eq. (167). Let us examine an example in which normal products of complex scalar fields are involved inside a time ordered product:

$$
\begin{aligned}
T[:\varphi_1^\dagger\varphi_2::\varphi_3^\dagger\varphi_4:] &= \,:\varphi_1^\dagger\varphi_2\varphi_3^\dagger\varphi_4: + \,:\overbracket{\varphi_1^\dagger\varphi_2::\varphi_3^\dagger\varphi_4}: + :\varphi_1^\dagger\overbracket{\varphi_2::\varphi_3^\dagger}\varphi_4: \\
&\quad + :\overbracket{\varphi_1^\dagger\overbracket{\varphi_2::\varphi_3^\dagger}\varphi_4}: \\
&= \,:\varphi_1^\dagger\varphi_2\varphi_3^\dagger\varphi_4: + \Delta_F(x_1 - x_4):\varphi_2\varphi_3^\dagger: + \Delta_F(x_2 - x_3):\varphi_1^\dagger\varphi_4: \\
&\quad + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)
\end{aligned}
\tag{169}
$$

6.2.1 Scattering of Scalar Nucleons

Let's go back to the scalar Yukawa theory of the section 5.4 and consider nucleon scattering process $\psi\psi \to \psi\psi$.

$$
\begin{aligned}
|i\!> &= b^\dagger(\boldsymbol{p}_a) b^\dagger(\boldsymbol{p}_b)\,|0\!> \stackrel{\rightarrow}{=} |N_a N_b\!> \\
|f\!> &= b^\dagger(\boldsymbol{p}_1) b^\dagger(\boldsymbol{p}_2)\,|0\!> \stackrel{\rightarrow}{=} |N_1 N_2\!>
\end{aligned}
\tag{170}
$$

The first contribution to $S-1$ in Eq. (160) arise from the term of the second order in $\mathcal{H}_{int}$. It reads

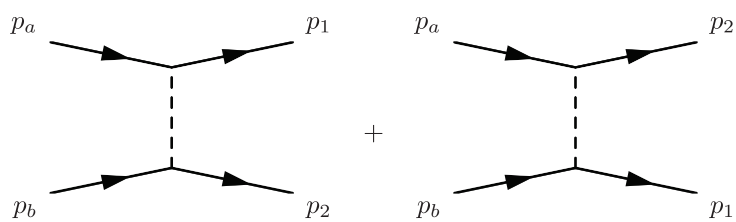
$$
\begin{aligned}
<\!f|\, S^{(2)}\,|i\!> &= \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 <\!f|\,T[:\!\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\!: \\
&\qquad\qquad \times :\!\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\!:]\,|i\!> \\
&= \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \Delta_F(x_1-x_2) \\
&\qquad <\!N_2N_1|\,:\!\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2)\!:\,|N_aN_b\!>\,,
\end{aligned}
\tag{171}
$$

The sandwich $<\!N_1N_2|\,\ldots\,|N_bN_a\!>$ reads

$$
\begin{aligned}
&<\!N_1N_2|\,:\!\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2)\!:\,|N_bN_a\!> \\
=& \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
&<\!0|\,b(\boldsymbol{p}_2)b(\boldsymbol{p}_1)\left\{b^\dagger(\boldsymbol{p}'_1)b^\dagger(\boldsymbol{p}'_2)b(\boldsymbol{p}'_a)b(\boldsymbol{p}'_b)\right\} b^\dagger(\boldsymbol{p}_a)b^\dagger(\boldsymbol{p}_b)\,|0\!> \\
=& \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
&<\!0|\,b(\boldsymbol{p}_2)\left\{2p_1^{0\prime}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)+b^\dagger(\boldsymbol{p}'_1)b(\boldsymbol{p}_1)\right\} b^\dagger(\boldsymbol{p}'_2) \\
&b(\boldsymbol{p}'_a)\left\{2p_b^{0\prime}\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_a)+b^\dagger(\boldsymbol{p}_a)b(\boldsymbol{p}'_b)\right\} b^\dagger(\boldsymbol{p}_b)\,|0\!> \\
=& \int \frac{d^3\boldsymbol{p}'_1 d^3\boldsymbol{p}'_2}{(2\pi)^3 4p_1^{0\prime}p_2^{0\prime}}\frac{d^3\boldsymbol{p}'_a d^3\boldsymbol{p}'_b}{(2\pi)^3 4p_a^{0\prime}p_b^{0\prime}} e^{i(p'_1x_1+p'_2x_2-p'_ax_1-p'_bx_2)} \\
&<\!0|\,\left\{4p_1^{0\prime}p_2^{0\prime}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_1)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_2)+4p_1^{0\prime}p_2^{0\prime}\delta^3(\boldsymbol{p}'_1-\boldsymbol{p}_2)\delta^3(\boldsymbol{p}'_2-\boldsymbol{p}_1)\right\} \\
&\left\{4p_a^{0\prime}p_b^{0\prime}\delta^3(\boldsymbol{p}'_a-\boldsymbol{p}_b)\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_a)+4p_a^{0\prime}p_b^{0\prime}\delta^3(\boldsymbol{p}'_a-\boldsymbol{p}_a)\delta^3(\boldsymbol{p}'_b-\boldsymbol{p}_b)\right\}\,|0\!> \\
=& \frac{1}{(2\pi)^6}\left\{e^{i(p_1x_1+p_2x_2)}+e^{i(p_2x_1+p_1x_2)}\right\}\left\{e^{-i(p_ax_1+p_bx_2)}+e^{-i(p_bx_1+p_ax_2)}\right\}
\end{aligned}
$$

Substituting this result in Eq. (171) and adopting the expression (164), we obtain

$$
\begin{aligned}
<f|\, S^{(2)}\, |i> \;&=\; \frac{(-ig)^2}{2(2\pi)^6} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \int d^4x_1 d^4x_2 \left\{ e^{i(p_1+k-p_a)x_1} e^{i(p_2-k-p_b)} \right. \\
&\quad + e^{i(p_2+k-p_a)x_1} e^{i(p_1-k-p_b)} + e^{i(p_1+k-p_b)x_1} e^{i(p_2-k-p_1)} \\
&\quad \left. + e^{i(p_2+k-p_b)x_1} e^{i(p_1-k-p_a)} \right\} \\
&= \frac{i(-ig)^2}{(2\pi)^6} \int \frac{(2\pi)^4 d^4k}{k^2 - m^2 + i\epsilon} \left\{ \delta^4(p_1 + k - p_a)\delta^4(p_2 - k - p_b) \right. \\
&\qquad\qquad \left. + \delta^4(p_2 + k - p_a)\delta^4(p_1 - k - p_b) \right\} \qquad (172) \\
&= \frac{i(-ig)^2}{(2\pi)^6} \left\{ \frac{1}{(p_1 - p_a)^2 - m^2} + \frac{1}{(p_2 - p_a)^2 - m^2} \right\} \\
&\quad \times (2\pi)^4 \delta^4(p_1 + p_2 - p_a - p_b) \\
&=
\end{aligned}
$$



(173)

The final expression is written in terms of two Feynman diagrams which correspond respectively to the two terms in Eq. (172). These diagrams are abbreviated a bit for a technical reason of the drawing. A fully drawn diagram may look like the following figure. Arrows on external lines indicate

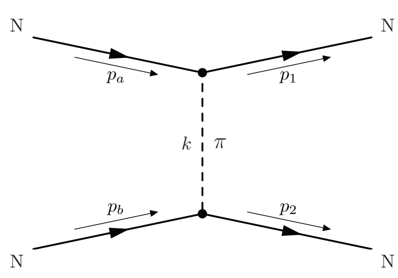


Figure 1: Fully drawn Feynman diagram for the first term in Eq. (173)

flows of conserved quantum numbers associated with particles like baryon number, flavor and so on. In the present case, it is the electric charge which is conserved in the Lagrangian density (145) as is stated above Eq. (151). In many situations, we draw a diagram like Fig. 1 without momentum arrows to represent the two diagrams in Eq. (173).

The correspondence between diagrams and equations are ensured by obeying the so called Feynman rule described as follows:

1. For each external lines, associate a factor $1/\sqrt{(2\pi)^3}$.

2. To each vertex, associate a factor

$$(-ig)(2\pi)^4\delta^4(\sum_{in} p_{in})\,, \tag{174}$$

   where p_{in} denotes a momentum flowing into the vertex and the sum is taken over all momenta connected to the vertex. Momenta flowing out from the vertex get a minus sign. For the upper vertex in Fig. (1), for instance, we read $\sum_{in} p_{in} = p_a - k - p_1$.

3. For each internal broken line, corresponding to a ϕ particle with momentum k, write a factor of

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon} \tag{175}$$

   When the internal line is solid, corresponding to a ψ particle, write the same factor with m replaced by the nucleon mass M.

The scattering amplitude $T_{fi} = <f|\,T\,|i>$ in Eq. (**??**) is now written as

$$\begin{aligned} T_{fi}^{(2)} &= \frac{(-ig)^2}{(2\pi)^6}\left\{\frac{1}{(p_1 - p_a)^2 - m^2} + \frac{1}{(p_2 - p_a)^2 - m^2}\right\} \\ &= \frac{(-ig)^2}{(2\pi)^6}\left(\frac{1}{t - m^2} + \frac{1}{u - m^2}\right) \end{aligned} \tag{176}$$

Finally, from Eq. (**??**), we write the scattering cross section in scalar Yukawa theory to the lowest (second) order as

$$\frac{d\sigma^{(2)}}{dt} = \frac{g^4}{16\pi\lambda(s, m_N^2, m_N^2)}\left(\frac{1}{t - m^2} + \frac{1}{u - m^2}\right)^2 \tag{177}$$

Exercise1: Examine the kinematical regions for t and u in this scattering.

An enhancement of the cross section at small $|t|$ follows from the mass pole of the exchanged pion. We also understand an enhancement at small $|u|$ is a consequence of the domination of t-channel exchanges and the indistinguishability of the two nucleons in the final state.

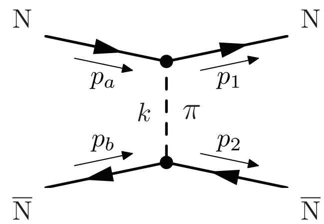
$N\overline{N} \to N\overline{N}$ amplitude

Initial and final states:

$$
\begin{aligned}
|i> &= b^\dagger(\boldsymbol{p}_a)c^\dagger(\boldsymbol{p}_b)\,|0> \ \overset{\rightarrow}{=}\ |N_a\overline{N}_b> \\
|f> &= b^\dagger(\boldsymbol{p}_1)c^\dagger(\boldsymbol{p}_2)\,|0> \ \overset{\rightarrow}{=}\ |N_1\overline{N}_2>
\end{aligned}
\tag{178}
$$

Since there are four (anti) nucleons in the initial and final states, we need at least four ψ (and $\psi^\dagger$) fields from interaction Hamiltonians and all ϕ fields should be contracted out. Thus, S-matrix elements to the lowest order is written in the same form as Eq. (171) with N_b and N_2 replaced by $\overline{N}_b$ and $\overline{N}_2$ respectively. Evaluating the sandwich factor as before, we get

$$<f|\,S^{(2)}\,|i> \quad =$$



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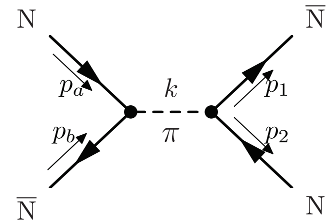

(179)

It reads

$$
\begin{aligned}
T_{fi}^{(2)} &= \frac{(-ig)^2}{(2\pi)^6}\left\{\frac{1}{(p_a-p_1)^2-m^2\cancel{+i\epsilon}}+\frac{1}{(p_a+p_b)^2-m^2+i\epsilon}\right\} \\
&= \frac{(-ig)^2}{(2\pi)^6}\left(\frac{1}{t-m^2}+\frac{1}{s-m^2+i\epsilon}\right)
\end{aligned}
\tag{180}
$$

Cross section is obtained by adopting Eq. (**??**) as before. We omit $i\epsilon$ in the first term in Eq. (180) since $t < 0$. In the second term, however, $s = m^2$ may occur when $m > 2M$. This is why we remained $i\epsilon$ in the second term. However, in this case, correction of the pion propagator due to higher order terms with nucleon loops will bring a finite imaginary part into the denominator of the propagator. Nevertheless, for the lowest order tree diagram, we keep the $i\epsilon$ term.

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Addendum: Derivation of Eq. (179)

It is yet instructive to show the derivation of Eq. (179). The way of evaluation here will be a little bit more skillful than one shown before. To evaluate Eq. (171) with the initial and final states replaced by the current case, we write

$$\psi(x) = b(x) + c^\dagger(x)\,, \quad \psi^\dagger(x) = b^\dagger(x) + c(x)\,, \tag{181}$$

referring the form in Eq. (149). In the following, we abbreviate arguments of fields and their parts by suffices. For instance, $\psi(x_i)$ is abbreviated by ψ_i and $b(x_i)$ by b_i. We will use the same abbreviation b_i either for $b(x_i)$ or for $b(\boldsymbol{p}_i)$ when there is little possibility of confusion. An abbreviation c_i is used in a similar manner.

Considering the initial and final states in Eq. (178), we need to remain terms $\sim b^\dagger c^\dagger bc$ in the sandwich of the normal order product in Eq. (171). Since there are two $\psi^\dagger$'s and two ψ's, we have four ways to pick up necessary factors:

$$
\begin{aligned}
&<\overline{N}_2 N_1|:\psi^\dagger_{1'}\psi_{1'}\psi^\dagger_{2'}\psi_{2'}:|N_a\overline{N}_b> \\
&=<0|\, c_2 b_1 : (b^\dagger_{1'} + c_{1'})(b_{1'} + c^\dagger_{1'})(b^\dagger_{2'} + c_{2'})(b_{2'} + c^\dagger_{2'}) : b^\dagger_a c^\dagger_b\, |0> \\
&\text{(pick up relevant terms)} \\
&=<0|\, c_2 b_1 : \left(b^\dagger_{1'} b_{1'} c_{2'} c^\dagger_{2'} + c_{1'} c^\dagger_{1'} b^\dagger_{2'} b_{2'} + b^\dagger_{1'} c^\dagger_{1'} c_{2'} b_{2'} + c_{1'} b_{1'} b^\dagger_{2'} c^\dagger_{2'} \right) : b^\dagger_a c^\dagger_b\, |0> \\
&\text{(take normal ordering)} \\
&=<0|\, c_2 b_1 \left\{ b^\dagger_{1'} c^\dagger_{2'} b_{1'} c_{2'} + c^\dagger_{1'} b^\dagger_{2'} c_{1'} b_{2'} + b^\dagger_{1'} c^\dagger_{1'} c_{2'} b_{2'} + b^\dagger_{2'} c^\dagger_{2'} c_{1'} b_{1'} \right\} b^\dagger_a c^\dagger_b\, |0> \\
&\text{(b's and c's are commuting to each orher)} \\
&=<0|\, c_2 b_1 \left\{ b^\dagger_{1'} c^\dagger_{2'} c_{2'} b_{1'} + b^\dagger_{2'} c^\dagger_{1'} c_{1'} b_{2'} + b^\dagger_{1'} c^\dagger_{1'} c_{2'} b_{2'} + b^\dagger_{2'} c^\dagger_{2'} c_{1'} b_{1'} \right\} b^\dagger_a c^\dagger_b\, |0>
\end{aligned}
\tag{182}
$$

Using suffices $1'$ and $2'$, we are writing integration variables in Eq. (171) as $x_{1'}$ and $x_{2'}$ and they can be exchanged since Δ_F is an even function. Thus,

$$
\begin{aligned}
&\int d^4x_{1'} d^4x_{2'} \Delta_F(1'-2')\, <\overline{N}_2 N_1|:\psi^\dagger_{1'}\psi_{1'}\psi^\dagger_{2'}\psi_{2'}:|N_a\overline{N}_b> \\
&= 2\int d^4x_{1'} d^4x_{2'} \Delta_F(1'-2')\, <0|\, c_2 b_1 \left\{ \underset{p'_1}{b^\dagger_{1'}}\, \underset{p'_2}{c^\dagger_{2'}}\, \underset{p'_a}{c_{2'}}\, \underset{p'_b}{b_{1'}} + \underset{p'_1}{b^\dagger_{1'}}\, \underset{p'_2}{c^\dagger_{1'}}\, \underset{p'_a}{c_{2'}}\, \underset{p'_b}{b_{2'}} \right\} b^\dagger_a c^\dagger_b\, |0>
\end{aligned}
\tag{183}
$$

In the last line of the above equation, assignments for momentum variables to correspond to external lines are shown for convenience. For instance, p'_1 indicated below $b^\dagger_{1'}$ reads

$$
b^\dagger_{1'} = b^\dagger(x'_1) = \int \frac{d^3\boldsymbol{p}'_1}{\sqrt{(2\pi)^3 2p_1{}'^0}} b^\dagger(\boldsymbol{p}'_1) e^{ip'_1 x'_1} \tag{184}
$$

In our abbreviated notations, it then reads that, for instance,

$$
[b_1, b^\dagger_{1'}] = \frac{1}{\sqrt{(2\pi)^3}} e^{ip_1 x'_1}\,, \qquad [b_{1'}, b^\dagger_a] = \frac{1}{\sqrt{(2\pi)^3}} e^{-ip_a x'_1} \tag{185}
$$

Inside integrations over x'_1 and x'_2 with a even function Δ_F in Eq. (183), one can thus proceed Eq. (182) as

$$
\begin{aligned}
&<\overline{N}_2 N_1|:\psi^\dagger_{1'}\psi_{1'}\psi^\dagger_{2'}\psi_{2'}:|N_a\overline{N}_b>\\
&= 2<0|\,c_2\left([b_1,b^\dagger_{1'}]+b^\dagger_{1'}b_1\right)\left\{c^\dagger_{2'}c_{2'}\left([b_{1'},b^\dagger_a]+b^\dagger_a b_{1'}\right)+c^\dagger_{1'}c_{2'}\left([b_{2'},b^\dagger_a]+b^\dagger_a b_{2'}\right)\right\}c^\dagger_b\,|0>\\
&= 2<0|\,[b_1,b^\dagger_{1'}][c_2,c^\dagger_{2'}][b_{1'},b^\dagger_a][c_{2'},c^\dagger_b]+[b_1,b^\dagger_{1'}][c_2,c^\dagger_{1'}][b_{2'},b^\dagger_a][c_{2'},c^\dagger_b]\,|0>\\
&= \frac{2}{(2\pi)^6}\left\{e^{i(p_1x'_1+p_2x'_2-p_ax'_1-p_bx'_2)}+e^{i(p_1x'_1+p_2x'_1-p_ax'_2-p_bx'_2)}\right\}
\end{aligned}
\tag{186}
$$

We are now ready to proceed from Eq. (171). Adopting the expression (164) of Δ_F, we have

$$
\begin{aligned}
<f|\,S^{(2)}\,|i> &= \frac{(-ig)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\int d^4x'_1 d^4x'_2\;e^{-ik(x'_1-x'_2)}\\
&\qquad <\overline{N}_2N_1|:\psi^\dagger_{1'}\psi_{1'}\psi^\dagger_{2'}\psi_{2'}:|N_a\overline{N}_b>\\
&= \frac{(-ig)^2}{(2\pi)^6}\int d^4k\frac{i(2\pi)^4}{k^2-m^2+i\epsilon}\left\{\delta^4(p_1-p_a-k)\delta^4(p_2-p_b+k)\right.\\
&\qquad\left.+\,\delta^4(p_1+p_2-k)\delta^4(p_a+p_b-k)\right\}\\
&= i(2\pi)^4\delta^4(P_f-P_i)\frac{(-ig)^2}{(2\pi)^6}\int\frac{d^4k}{k^2-m^2+i\epsilon}\\
&\qquad\left\{\delta^4(p_1-p_a-k)+\delta^4(p_a+p_b-k)\right\}
\end{aligned}
\tag{187}
$$

Eq. (179) is thus recovered.

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<u>$N\overline{N}\to\pi\pi$ amplitude</u>

Initial and final states:

$$
\begin{aligned}
|i>&= b^\dagger(\boldsymbol{p}_N)c^\dagger(\boldsymbol{p}_{\bar N})\,|0> \;\stackrel{\to}{=}\; |N\overline{N}>\\
|f>&= a^\dagger(\boldsymbol{k}_1)a^\dagger(\boldsymbol{k}_2)\,|0> \;\stackrel{\to}{=}\; |\pi_1\pi_2>
\end{aligned}
\tag{188}
$$

Field decompositions:

$$
\psi = b + c^\dagger\,,\quad \phi = a + a^\dagger \tag{189}
$$

We are to be employing the same abbreviation rules as before. This time we should go back to the first equation in Eq. (171) for picking up one b from a ψ, one c from a $\psi^\dagger$ and two $a^\dagger$'s from two ϕ's in the T-product. The remaining pair of ψ and $\psi^\dagger$ are contracted to give a propagator. With a help of Wick's theorem in the form (417), we write

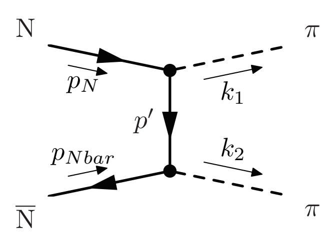
$$
\begin{aligned}
&<\pi_2\pi_1|\, T[:\psi^\dagger_{1'}\psi_{1'}\phi_{1'}::\psi^\dagger_{2'}\psi_{2'}\phi_{2'}:]\,|N\overline{N}> \\
&=<\pi_2\pi_1|\,:\psi^\dagger_{1'}\psi_{1'}\phi_{1'}\psi^\dagger_{2'}\psi_{2'}\phi_{2'}:+:\psi^\dagger_{1'}\psi_{1'}\phi_{1'}\psi^\dagger_{2'}\psi_{2'}\phi_{2'}:|N\overline{N}> \\
&=\{<\pi_2\pi_1|\,:\phi_{1'}\phi_{2'}:\}\left\{\psi_{1'}\psi^\dagger_{2'}:\psi^\dagger_{1'}\psi_{2'}:+\psi^\dagger_{1'}\psi_{2'}:\psi_{1'}\psi^\dagger_{2'}:\right\}|N\overline{N}>
\end{aligned}
\tag{190}
$$

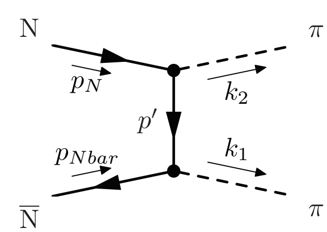
Propagator of the complex scalar fields is given in Eq. (167) and

$$
\begin{aligned}
&=\Delta_F(1'-2')<0|\left\{a_2a_1a^\dagger_{1'}a^\dagger_{2'}\right\}\{c_{1'}b_{2'}+b_{1'}c_{2'}\}\,b^\dagger_N c^\dagger_{\bar N}\,|0> \\
&=\Delta_F(1'-2')<0|\left\{[a_1,a^\dagger_{1'}][a_2,a^\dagger_{2'}]+[a_2,a^\dagger_{1'}][a_1,a^\dagger_{2'}]\right\}\left\{[b_{2'},b^\dagger_N][c_{1'},c^\dagger_{\bar N}]+[b_{1'},b^\dagger_N][c_{2'},c^\dagger_{\bar N}]\right\}|0> \\
&=\frac{1}{(2\pi)^6}\Delta_F(1'-2')\left\{e^{i(k_1x'_1+k_2x'_2)}+e^{i(k_2x'_1+k_1x'_2)}\right\}\left\{e^{-i(p_Nx'_2+p_{\bar N}x'_1)}+e^{-i(p_Nx'_1+p_{\bar N}x'_2)}\right\}
\end{aligned}
\tag{191}
$$

Inside integration over x'_1 and x'_2, Adopting Eq. (164) with m replaced by the nucleon mass M, we may write

$$
\begin{aligned}
<\pi_2\pi_1|\,S^{(2)}\,|N\overline{N}> \;&=\; \frac{(-ig)^2}{2(2\pi)^6}\int\frac{d^4p'}{(2\pi)^4}\frac{i}{p'^2-M^2+i\epsilon}\int d^4x'_1d^4x'_2\;e^{-ip'(x'_1-x'_2)} \\
&\quad\times 2\left\{e^{i(k_1x'_1+k_2x'_2-p_Nx'_1-p_{\bar N}x'_2)}+e^{i(k_1x'_1+k_2x'_2-p_Nx'_2-p_{\bar N}x'_1)}\right\} \\
&=\; \frac{(-ig)^2}{(2\pi)^6}\int d^4p'\frac{i(2\pi)^4}{p'^2-M^2+i\epsilon}\left\{\delta^4(k_1-p_N-p')\delta^4(k_2-p_{\bar N}+p')\right. \\
&\qquad\left.+\,\delta^4(k_1-p_{\bar N}-p')\delta^4(k_2-p_N+p')\right\}
\end{aligned}
$$

$=$ 

$+$ 

$$
=\; i(2\pi)^4\delta^4(P_f-P_i)\frac{(-ig)^2}{(2\pi)^6}\left\{\frac{1}{t-M^2+i\epsilon}+\frac{1}{u-M^2+i\epsilon}\right\} \tag{192}
$$

For a technical reason, we wrote p_{Nbar} in places of $p_{\bar N}$ in Feynman diagrams in Eq. (192). One may omit $i\epsilon$ in denominators of propagators in Eq. (192). The same comments in the last paragraph just after Eq. (177) is also relevant here by exchanging terms pion and nucleon there.

7 Dirac Fields

These fields describe massive, charged, spin 1/2 particles.
((Put the following line somewhere: Spinors belong to the fundamental representation of the Poincaré Group SL(2,C))).

7.1 Classical Free Field

Dirac equation

The Dirac equation is written as

$$i\partial_t\psi(x) = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\partial} + \beta m\right)\psi(x)\,. \tag{193}$$

Solution $\psi(x)$ of this equation satisfies the Klein-Gordon equation [10] if and only if coefficients $\boldsymbol{\alpha}$ and β satisfy

$$\{\alpha_i,\alpha_j\}_+ = 2\delta_{ij}\,,\quad \{\alpha_i,\beta\}_+ = 0\,,\quad \beta^2 = 1\,, \tag{194}$$

as one may confirm it by taking time derivative of the both hand sides of Eq. (193). It follows that they must be traceless:

$$\begin{aligned}\operatorname{tr}\alpha_i &= \operatorname{tr}\beta^2\alpha_i = \operatorname{tr}\beta\alpha_i\beta = -\operatorname{tr}\alpha_i = 0\,,\\ \operatorname{tr}\beta &= \operatorname{tr}\alpha_i^2\beta = \operatorname{tr}\alpha_i\beta\alpha_i = -\operatorname{tr}\beta = 0\,.\end{aligned}$$

The minimum dimension of α's and β as matrices to satisfy the conditions (194) is found to be 4. Introduce γ matrices by

$$\gamma^0 = \beta\,,\quad \gamma^i = \beta\alpha_i\ \ . \tag{195}$$

In terms of γ matrices, Eq. (194) reads

$$\{\gamma^\mu,\gamma^\nu\}_+ = 2g^{\mu\nu}\,. \tag{196}$$

This is called the Clifford algebra. This relationship means $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$ for $i = 1,2,3$. γ^0 is obviously traceless and γ^i's are also for

$$\operatorname{tr}\gamma^i = \operatorname{tr}\beta\alpha_i = \operatorname{tr}\alpha_i\beta = -\operatorname{tr}\beta\alpha_i = 0\,. \tag{197}$$

The Dirac equation is then written as

$$\left(i\partial\!\!\!/ - m\right)\psi(x) = 0 \tag{198}$$

[10] The Dirac equation is not the unique equation that is the first order in time derivative and whose solution satisfies the K-G eq. The Duffin-Kemmer equation also satisfies these requirements [6].

where $\not{a} \stackrel{\leftarrow}{=} \gamma_\mu a^\mu = \gamma \cdot a$ and $\not{\partial} = \gamma^0\partial_0 + \gamma^i\partial_i = \gamma^0\partial_0 + \boldsymbol{\gamma}\cdot\boldsymbol{\partial}$.

We may choose α_i's and β as hermitian matrices. In that case we have

$$\gamma^{0\dagger} = \gamma^0\,, \quad \gamma^{i\dagger} = -\gamma^i \quad \Longleftrightarrow \quad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{199}$$

Define the conjugate Dirac field by

$$\overline{\psi} = \psi^{*t}\gamma^0 \tag{200}$$

Taking complex conjugate and transpose of Eq. (198), we obtain

$$\begin{aligned}
0 &= \psi^{*t}(x)\left(-i\overleftarrow{\partial}\cdot\gamma^\dagger - m\right) \\
&= \psi^{*t}(x)(\gamma^0)^2\left(-i\overleftarrow{\partial}\cdot\gamma^\dagger - m\right) \\
&= \overline{\psi}(x)\left(-i\overleftarrow{\partial}\cdot\gamma^0\gamma^\dagger\gamma^0 - m\right)\gamma^0 \\
&= -\overline{\psi}(x)\left(i\overleftarrow{\partial}\cdot\gamma + m\right)\gamma^0
\end{aligned}$$

Thus, the conjugate Dirac field satisfies

$$\overline{\psi}(x)\left(i\overleftarrow{\not{\partial}} + m\right) = 0 \tag{201}$$

This equation is equivalent with Eq. (198) in the classical level provided we choose γ's in such a way that Eq. (199) holds.

Dirac matrices

We define

$$\gamma_5 \equiv \gamma^5 \stackrel{\leftarrow}{=} i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{202}$$

The γ_5 satisfies

$$\{\gamma_5, \gamma_\mu\}_+ = 0\,, \quad (\gamma_5)^2 = 1\,, \quad \gamma_5^\dagger = \gamma_5 \tag{203}$$

We also define

$$\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,, \tag{204}$$

which has a property $\sigma_{\mu\nu}^\dagger = \gamma^0\sigma_{\mu\nu}\gamma^0$. The reason why we are defining these quantities lays in the follwing fact: When d is the rank of an irreducible matrix representation of an algebra composed of hypercomplex numbers, the number of linearly independent elements of the algebra is $n = d^2$. Therefore, the minimal algebra composed of γ's has $4^2 = 16$ linearly independent elements. They are given as

$$\begin{cases} \Gamma^S = 1 \\ \Gamma^V_\mu = \gamma_\mu \\ \Gamma^T_{\mu\nu} = \sigma_{\mu\nu} \stackrel{\leftarrow}{=} \frac{i}{2}[\gamma_\mu, \gamma_\nu] \\ \Gamma^A_\mu = \gamma_5 \gamma_\mu \\ \Gamma^P = \gamma_5 \end{cases} \tag{205}$$

We are now presenting three unitary equivalent expressions of α_i's, β and γ's. First we write down the Pauli matrices here again. [11]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{206}$$

They satisfy relationships

$$\{\sigma_i, \sigma_j\}_+ = 2\delta_{ij}\,, \quad \sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij} \tag{207}$$

(1) Dirac representation

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3 \otimes 1\,, \qquad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} = i\sigma_2 \otimes \sigma_i \tag{208}$$

Elements of $\begin{bmatrix} \ddots \end{bmatrix}$ are 2×2 matrices. Expressions with the direct product makes calculations more transparent. For instance,

$$\begin{aligned} \alpha_i = \beta\gamma^i = \gamma^0\gamma^i = (\sigma_3 \otimes 1)(i\sigma_2 \otimes \sigma_i) = -i\sigma_2\sigma_3 \otimes \sigma_i = \sigma_1 \otimes \sigma_i \\ = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \end{aligned} \tag{209}$$

Other matrices are similarly obtained as

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \otimes 1\,, \quad \sigma_{0i} = -i\sigma_1 \otimes \sigma_i\,, \quad \sigma_{ij} = \epsilon_{ijk} \otimes \sigma_k \tag{210}$$

When we write the Dirac field as composed of two component large and small components,

$$\psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix} \tag{211}$$

[11] It was already given in Eq. (70).

the Dirac equation for these components reads

$$\begin{cases} i\partial_t\varphi(x) = m\varphi(x) + \frac{1}{i}\boldsymbol{\sigma}\cdot\boldsymbol{\partial}\chi(x) \\ i\partial_t\chi(x) = -m\chi(x) + \frac{1}{i}\boldsymbol{\sigma}\cdot\boldsymbol{\partial}\varphi(x) \end{cases} \tag{212}$$

(2) Majorna representation

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \sigma_1 \otimes \sigma_2\,, \quad \gamma^1 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3\,, \end{bmatrix} = 1 \otimes i\sigma_3 \\ \gamma^2 &= \begin{bmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = -i\sigma_2 \otimes \sigma_2\,, \quad \gamma^3 = \begin{bmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} = 1 \otimes -i\sigma_1\,, \end{aligned} \tag{213}$$

and $\gamma_5 = \sigma_3 \otimes \sigma_2$. In this representation, we have $(\gamma^\mu)^* = -\gamma^\mu$ and the Dirac equation becomes real. The Dirac fields are given as linear combinations of real sollutions. This representation is related with the Dirac one by

$$\gamma^\mu_{\text{Majorana}} = U\gamma^\mu_{\text{Dirac}}U^\dagger\,, \quad U = U^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{bmatrix} \tag{214}$$

(3) Chiral representation

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \sigma_1 \otimes -1\,, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0\,, \end{bmatrix} = i\sigma_2 \otimes \sigma_i \\ \gamma_5 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3 \otimes 1\,, \quad \sigma_{0i} = \begin{bmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{bmatrix} = -i\sigma_3 \otimes \sigma_i\,, \\ \sigma_{ij} &= \begin{bmatrix} \epsilon_{ijk}\sigma_k & 0 \\ 0 & \epsilon_{ijk}\sigma_k \end{bmatrix} = \epsilon_{ijk} \otimes \sigma_k\,, \end{aligned} \tag{215}$$

In this representation, spatial rotators and Lorentz boosters, namely, proper Lorentz transformations take forms as diagonal in the $\begin{bmatrix} \ddots \end{bmatrix}$ space so that φ and χ fields are transformed independently. This representation is related with the Dirac one by

$$\gamma^\mu_{\text{Chiral}} = U\gamma^\mu_{\text{Dirac}}U^\dagger\,, \quad U = \frac{1}{\sqrt{2}}(1 - \gamma_5\gamma^0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{216}$$

Lorentz transformation property

Under a proper homogeneous Lorentz transformation [12]

$$x^\mu \mapsto x^{\mu'} = L(\boldsymbol{\beta},\boldsymbol{\theta})^\mu_{\ \nu}\, x^\nu\,, \tag{217}$$

the Dirac field transforms as

$$\psi(x) \mapsto \psi'(x') = S(L)\psi(x)\,, \quad S(L) = \exp[-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}]\,, \tag{218}$$

where $\omega^{\nu\mu} = -\omega^{\mu\nu}$ and

$$\omega^{0i} = \xi_i\,, \quad \boldsymbol{\xi} = \xi\boldsymbol{\beta}/\beta\,, \quad \xi = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \tag{219}$$

$$\omega^{ij} = -\epsilon_{ijk}\theta_k \tag{220}$$

We may examine the properties under boosts and rotations separately by considering

$$S_{boost}(L(\boldsymbol{\beta},\mathbf{0})) = \exp[-\frac{i}{2}\sigma_{0i}\xi_i] \tag{221}$$

$$= \cosh\frac{\xi}{2} - \frac{\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}{\beta}\sinh\frac{\xi}{2} \tag{222}$$

and

$$S_{rotation}(L(\mathbf{0},\boldsymbol{\theta})) = \exp[\frac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}] \tag{223}$$

$$= \cos\frac{\theta}{2} + i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{\theta}\sin\frac{\theta}{2} \tag{224}$$

where three components of

$$\boldsymbol{\sigma} \stackrel{\leftarrow}{=} \frac{1}{2}\epsilon^{\uparrow ij}\sigma_{ij} = \frac{i}{2}\epsilon^{\uparrow ij}\gamma^i\gamma^j = \gamma_5\gamma^0\boldsymbol{\gamma} \tag{225}$$

are the spinor (fundamental) reperesentation of generators of the spatial rotation. They certainly satisfy $\{\sigma^i,\sigma^j\}_+ = 2\delta^{ij}$. Also, $S^\dagger_{boost}(L) = S_{boost}(L)$,

---

[12]

$$L_\mu^{\ \rho} L^\mu_{\ \sigma} = g^\rho_\sigma\,,$$

Denoting a matrix with its elements given by $L^\mu_{\ \nu}$ as L, we have

$$L_\mu^{\ \rho} = (L^{-1})^\rho_{\ \mu}$$

$S^\dagger_{rotation}(L) = S^{-1}_{rotation}(L)$ and $\gamma^0 S^\dagger(L)\gamma^0 = S^{-1}(L)$. The covariance of the Dirac equation (198) is guaranteed by a relationship

$$S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu \gamma^\nu \tag{226}$$

$S(L)$ has a property

$$\gamma^0 S^\dagger(L)\gamma^0 = S^{-1}(L)\,, \tag{227}$$

so that

$$\overline{\psi}(x) \mapsto \overline{\psi}'(x') = \psi^{*t}(x)S^\dagger(L)\gamma^0 = \overline{\psi}(x)S^{-1}(L) \tag{228}$$

and $\overline{\psi}\psi$ is a Lorentz scalar.

For later use, we write down a particular form of S_{boost}. For a Lorentz transformation

$$L(-\boldsymbol{\beta}) = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}\cdot \\ \gamma\boldsymbol{\beta} & 1+\hat{\boldsymbol{\beta}}(\gamma-1)\hat{\boldsymbol{\beta}}\cdot \end{pmatrix}\,, \tag{229}$$

which transforms $p^{(0)} = (m, \mathbf{0}) \mapsto p = Lp^{(0)} = (E, \boldsymbol{p})$, we have

$$\begin{aligned} S_{boost}(L) &= \tfrac{1}{\sqrt{2m(E+m)}}[\not{p}\gamma^0 + m]\,, \\ S^{-1}_{boost}(L) &= \tfrac{1}{\sqrt{2m(E+m)}}[\gamma^0\not{p} + m]\,, \end{aligned} \tag{230}$$

Since L^{-1} corresponds to the change of the sign of the spacial momentum, we have $S^{-1}_{boost}(L) = S_{boost}(L^{-1})$.

The Dirac spinors

We write solutions of the Dirac equation (198) as

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}2p^0}\left[\psi^{(+)}(\boldsymbol{p})e^{-ipx} + \psi^{(-)}(\boldsymbol{p})e^{ipx}\right]\,, \tag{231}$$

where $p^0=\sqrt{\boldsymbol{p}^2+m^2}$. The Dirac equation demands [13]

$$(\not p - m)\psi^{(+)}(\boldsymbol{p}) = 0\,, \quad (\not p + m)\psi^{(-)}(\boldsymbol{p}) = 0 \tag{232}$$

For a Lorentz transformation $L: p^{(0)} = (m, \boldsymbol{0}) \mapsto p$, we write $\psi^{(\pm)}(\boldsymbol{p}) = S(L)\psi^{(\pm)}(\boldsymbol{0})$. Eq. (441) reads

$$S^{-1}(L)\not p S(L) = p_\mu L^\mu{}_\nu \gamma^\nu = L_\mu{}^\rho L^\mu{}_\nu p^{(0)}_\rho \gamma^\nu = g^\rho_\nu p^{(0)}_\rho \gamma^\nu = m\gamma^0$$

and we have

$$(\gamma^0 - 1)\psi^{(+)}(\boldsymbol{0}) = 0\,, \quad (\gamma^0 + 1)\psi^{(-)}(\boldsymbol{0}) = 0$$

Each of these equations have two linearly independent solutions for $\gamma_0^2 = 1$, $\operatorname{tr}\gamma_0 = 0$ and $\operatorname{rank}(\gamma_0) = 4$. Denoting normalized solutions as $u^{(\alpha)}(\boldsymbol{p})$ and $v^{(\alpha)}(\boldsymbol{p})$, $\alpha = 1, 2$ respectively for $\psi^{(+)}(\boldsymbol{p})$ and $\psi^{(-)}(\boldsymbol{p})$, we write

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{\alpha=1,2} \left[b_\alpha(\boldsymbol{p}) u^{(\alpha)}(\boldsymbol{p}) e^{-ipx} + d^*_\alpha(\boldsymbol{p}) v^{(\alpha)}(\boldsymbol{p}) e^{ipx} \right] \tag{233}$$

Eq. (232) reads

$$(\not p - m)u^{(\alpha)}(\boldsymbol{p}) = 0\,, \quad (\not p + m)v^{(\alpha)}(\boldsymbol{p}) = 0 \tag{234}$$

For conjugate (adjoint) spinors defined in Eq. (200) we have

$$\overline{u}^{(\alpha)}(\boldsymbol{p})(\not p - m) = 0\,, \quad \overline{v}^{(\alpha)}(\boldsymbol{p})(\not p + m) = 0 \tag{235}$$

[13] As a solution of the K-G eq., we can write

$$\begin{aligned}
\psi(x) &= \int \frac{d^4p}{\sqrt{(2\pi)^3}} \delta(p^2 - m^2) \left[\theta(p^0) + \theta(-p^0)\right] \psi(p) e^{-ipx} \\
&= \int \frac{d^4p}{\sqrt{(2\pi)^3}} \delta(p^2 - m^2) \Big[\underbrace{\psi(p)\theta(p^0)}_{\psi^+} e^{-ipx} + \underbrace{\psi(-p)\theta(p^0)}_{\psi^-} e^{ipx} \Big] \\
&= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \left[\psi^+(\boldsymbol{p}) e^{-ipx} + \psi^-(\boldsymbol{p}) e^{ipx} \right]
\end{aligned}$$

The Dirac eq. (198) requires

$$(\not p - m)\psi(p)\big|_{p^2=m^2} = 0\,,$$

from which Eq. (232) follows.

We choose their normalization [14]as

$$
\begin{aligned}
\overline{u}^{(\alpha)}(\boldsymbol{p})u^{(\beta)}(\boldsymbol{p}) = 2m\delta^{\alpha\beta}\,, \quad \overline{v}^{(\alpha)}(\boldsymbol{p})v^{(\beta)}(\boldsymbol{p}) = -2m\delta^{\alpha\beta}\,, \\
\overline{u}^{(\alpha)}(\boldsymbol{p})v^{(\beta)}(\boldsymbol{p}) = 0\,, \quad \overline{v}^{(\alpha)}(\boldsymbol{p})u^{(\beta)}(\boldsymbol{p}) = 0
\end{aligned}
\tag{236}
$$

Energy state projection

$$
\hat{\Omega}_+(\boldsymbol{p}) = \frac{\not{p}+m}{2m}\,, \quad \hat{\Omega}_-(\boldsymbol{p}) = \frac{-\not{p}+m}{2m}
\tag{237}
$$

Spin state projection

The spin states are defined in the rest frame of particle. Consider the generator of spatial rotation given in Eq. (440). For $s^{(0)} = (0, \boldsymbol{s})$ with a spatial 3 vector $\boldsymbol{s}$, with helps of Eq. (441) and a relation

$$
S^{-1}(L)\gamma_5 S(L) = \det(L)\gamma_5\,,
\tag{238}
$$

we have

$$
\begin{aligned}
S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L) &= S^{-1}(L)\gamma_5\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{s}S(L) \\
&= \det(L)\gamma_5 L^0{}_\mu\gamma^\mu L^i{}_\nu\gamma^\nu s^i \\
&= \det(L)\gamma_5 (p^{(0)}_\rho/m)L^\rho{}_\mu\gamma^\mu(-s^{(0)}_\sigma)L^\sigma{}_\nu\gamma^\nu \\
&= \det(L)\gamma_5 (L^{-1})_\mu{}^\rho(p^{(0)}_\rho/m)\gamma^\mu(L^{-1})_\nu{}^\sigma\gamma^\nu(-s^{(0)}_\sigma)\,,
\end{aligned}
$$

where $p^{(0)} = (m, \mathbf{0})$. Then, for $L = L(\boldsymbol{\beta}) : p = (E, \boldsymbol{p}) \mapsto p^{(0)}$, we have

$$
\begin{aligned}
S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L) &= -(1/m)\det(L)\gamma_5 p_\mu\gamma^\mu\gamma^\nu s_\nu\,, \\
&= \det(L)\frac{\gamma_5\not{s}\not{p}}{m}\,,
\end{aligned}
\tag{239}
$$

where we have used a fact $\not{p}\not{s} = -\not{s}\not{p}$ for $p\cdot s = 0$. Thus, we have for a proper Lorentz transformation that

$$
\frac{\gamma_5\not{s}\not{p}}{m}u(\boldsymbol{p}) = S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L)S^{-1}(L)u(\mathbf{0}) = S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}u(\mathbf{0})
\tag{240}
$$

and similar relationship for v. We set two linearly independent spinors in the rest frame as eigenstates of $\boldsymbol{\sigma}\cdot\boldsymbol{s}$ for $\boldsymbol{s}^2 = 1$ and write

$$
\begin{aligned}
&\boldsymbol{\sigma}\cdot\boldsymbol{s}\; u(\mathbf{0},\boldsymbol{s}) = u(\mathbf{0},\boldsymbol{s}) \\
&\boldsymbol{\sigma}\cdot\boldsymbol{s}\; u(\mathbf{0},-\boldsymbol{s}) = -u(\mathbf{0},-\boldsymbol{s}) \\
&-\boldsymbol{\sigma}\cdot\boldsymbol{s}\; v(\mathbf{0},\boldsymbol{s}) = v(\mathbf{0},\boldsymbol{s}) \\
&-\boldsymbol{\sigma}\cdot\boldsymbol{s}\; v(\mathbf{0},-\boldsymbol{s}) = -v(\mathbf{0},-\boldsymbol{s})
\end{aligned}
\tag{241}
$$

---

[14]Izykson: $\{b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')\}_+ = \frac{2E}{2m}\delta^3(\boldsymbol{p}-\boldsymbol{p}')$, $\bar{u}^\alpha u^\beta = \delta^{\alpha\beta}$,
$\psi(x) \sim \int d^3\boldsymbol{p}\frac{2m}{2E}[bue^{-ipx}\ldots]$

Hioki, Tong: $\{b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')\}_+ = (2\pi)^3 2E\delta^3(\boldsymbol{p}-\boldsymbol{p}')$, $\bar{u}^\alpha u^\beta = 2m\delta^{\alpha\beta}$,
$\psi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E}[bue^{-ipx}\ldots]$

dim $\psi = E^{3/2}$

Since $S^{-1}(L)u(\mathbf{0}, \boldsymbol{s}) = u(\boldsymbol{p}, \boldsymbol{s})$ and so on, we have a set of similar relations by replacing $\mathbf{0}$ by $\boldsymbol{p}$ and $\boldsymbol{\sigma}\cdot\boldsymbol{s}$ by $\gamma_5 \not{s}\not{p}/m$. Considering relationships $(\not{p}/m)u(\boldsymbol{p}) = u(\boldsymbol{p})$ and $(-\not{p}/m)v(\boldsymbol{p}) = v(\boldsymbol{p})$, an operator matirx

$$\hat{\Sigma}(s) = \frac{1 + \gamma_5 \not{s}}{2} \tag{242}$$

projects out spinstates along $\boldsymbol{s}$ as

$$\begin{aligned}
&\hat{\Sigma}(s)u(\boldsymbol{p}, \boldsymbol{s}) = u(\boldsymbol{p}, \boldsymbol{s}) \\
&\hat{\Sigma}(s)u(\boldsymbol{p}, -\boldsymbol{s}) = 0 \\
&\hat{\Sigma}(s)v(\boldsymbol{p}, \boldsymbol{s}) = v(\boldsymbol{p}, \boldsymbol{s}) \\
&\hat{\Sigma}(s)v(\boldsymbol{p}, -\boldsymbol{s}) = 0
\end{aligned} \tag{243}$$

In the Dirac representation,

$$\boldsymbol{\sigma} \quad = \quad 1 \otimes \boldsymbol{\sigma}_{Pauli} = \begin{bmatrix} \boldsymbol{\sigma}_{Pauli} & 0 \\ 0 & \boldsymbol{\sigma}_{Pauli} \end{bmatrix} \tag{244}$$

For $p = Lp^{(0)}$, $p^{(0)} = (m, \mathbf{0})$,

$$\hat{\Sigma}(s)u(\boldsymbol{p}) = \hat{\Sigma}(s)S_{boost}(L)u(\mathbf{0}) = \frac{1 + \gamma_5 \not{s}}{2} \frac{\not{p}\gamma^0 + m}{\sqrt{2m(E+m)}} \tag{245}$$

$$\mathcal{L} = i\bar{\psi}(x)\partial\!\!\!/\psi(x) - m\bar{\psi}(x)\psi(x) \tag{246}$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$.
(Comment on independent ψ and $\overline{\psi}$ treatment.)

$$\left(i\partial\!\!\!/ + m\right)\psi(x) = 0\,, \quad \bar{\psi}(x)\left(i \overleftarrow{\partial\!\!\!/} + m\right) = 0 \tag{247}$$

7.2 Quantized Free Field

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}2p^0} \sum_{s=\pm 1} \left[c(\boldsymbol{p},s)u(\boldsymbol{p},s)e^{-ipx} + d^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s)e^{ipx}\right] \tag{248}$$

$$\{c(\boldsymbol{p},s), c^\dagger(\boldsymbol{p}',s')\} = \{d(\boldsymbol{p},s), d^\dagger(\boldsymbol{p}',s')\} = 2p^0\delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}') \tag{249}$$

$$\begin{array}{l} (p\!\!\!/ - m)u(\boldsymbol{p},s) = 0\,, \quad \bar{u}(\boldsymbol{p},s)(p\!\!\!/ - m) = 0\,, \\ (p\!\!\!/ + m)v(\boldsymbol{p},s) = 0\,, \quad \bar{v}(\boldsymbol{p},s)(p\!\!\!/ + m) = 0\,, \end{array} \tag{250}$$

Propagetor

$$\begin{aligned} S_F(q) &= i\int d^4x e^{iqx} <0|\, T[\psi(x)\bar{\psi}(0)]\,|0> \\ &= \frac{-1}{q\!\!\!/ - m + i\epsilon} = -\frac{q\!\!\!/ + m}{q^2 - m^2 + i\epsilon} \end{aligned} \tag{251}$$

7.3 Dirac Yukawa Theory

Let us consider a system

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi - g\phi\overline{\psi}\psi\,, \tag{252}$$

$$\mathcal{L}_\psi = \overline{\psi}(i\partial\!\!\!/ - m)\psi\,, \quad \mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 \tag{253}$$

with a Dirac nucleon field ψ with the mass m and a scalar pion field ϕ with the mass μ. The interaction term $\mathcal{L}_{int} = -g\phi\overline{\psi}\psi$ shows they interact each other through the Yukawa coupling.

<u>$NN \to NN$</u>
Initial and final states:

$$\begin{array}{l} |i>= c^\dagger_{r_a}(\boldsymbol{p}_a)c^\dagger_{r_b}(\boldsymbol{p}_b)\,|0> \overrightarrow{\equiv} |N_a, N_b> \\ |f>= c^\dagger_{r_1}(\boldsymbol{p}_1)c^\dagger_{r_2}(\boldsymbol{p}_2)\,|0> \overrightarrow{\equiv} |N_1, N_2> \end{array} \tag{254}$$

S-matrix element of the lowest order

$$
\begin{aligned}
<f|\, S^{(2)}\,|i> &= \frac{(-ig)^2}{2}\int d^4x_1' d^4x_2' <N_2N_1|\, T[:\phi_{1'}\overline{\psi}_{1'}\psi_{1'}::\phi_{2'}\overline{\psi}_{2'}\psi_{2'}:]\,|N_aN_b> \\
&= \frac{(-ig)^2}{2}\int d^4x_1' d^4x_2' \Delta_F(x_1'-x_2') <N_2N_1|\, T[:\overline{\psi}_{1'}\psi_{1'}::\overline{\psi}_{2'}\psi_{2'}:]\,|N_aN_b>
\end{aligned}
\tag{255}
$$

Arguments of fields are indicated by suffices. Wick's theorem reads

$$
<N_2N_1|\, T[:\overline{\psi}_{1'}\psi_{1'}::\overline{\psi}_{2'}\psi_{2'}:]\,|N_aN_b>=<N_2N_1|\,:\overline{\psi}_{1'}\psi_{1'}\overline{\psi}_{2'}\psi_{2'}:|N_aN_b> \tag{256}
$$

Decompose Dirac fields as

$$
\psi = \psi^{(+)}(c) + \psi^{(-)}(d^\dagger)\,, \quad \overline{\psi} = \overline{\psi^{(+)}}(c^\dagger) + \overline{\psi^{(-)}}(d) \tag{257}
$$

where arguments should be understood just as symbols to remember definitions of decomposed fields given below:

$$
\left\{
\begin{aligned}
&\psi^{(+)}_{l\alpha}(c) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{r=1,2} c_r(\boldsymbol{p}) u^{(r)}_\alpha(\boldsymbol{p}) e^{-ipx_l} \;\stackrel{\rightarrow}{=} c_{l(r,p)} u^{(r,p)}_\alpha \\
&\overline{\psi^{(+)}}_{l\alpha}(c^\dagger) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{r=1,2} c^\dagger_r(\boldsymbol{p}) \bar{u}^{(r)}_\alpha(\boldsymbol{p}) e^{ipx_l} \;\stackrel{\rightarrow}{=} c^\dagger_{l(r,p)} \bar{u}^{(r,p)}_\alpha \\
&\psi^{(-)}_{l\alpha}(d^\dagger) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{r=1,2} d^\dagger_r(\boldsymbol{p}) v^{(r)}_\alpha(\boldsymbol{p}) e^{ipx_l} \;\stackrel{\rightarrow}{=} d^\dagger_{l(r,p)} v^{(r,p)}_\alpha \\
&\overline{\psi^{(-)}}_{l\alpha}(d) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{r=1,2} d_r(\boldsymbol{p}) \bar{v}^{(r)}_\alpha(\boldsymbol{p}) e^{-ipx_l} \;\stackrel{\rightarrow}{=} d_{l(r,p)} \bar{v}^{(r,p)}_\alpha
\end{aligned}
\right.
\tag{258}
$$

Indexes of spinors are explicitly shown. Last equations in each line define symbols to abbreviate expressions in the left hand side. Substituting the decomposition (257) in Eq. (256), it becomes

$$
\begin{aligned}
&<N_2N_1|\,:\overline{\psi}_{1'\alpha}\psi_{1'\alpha}\overline{\psi}_{2'\beta}\psi_{2'\beta}:|N_aN_b> \\
&=<N_2N_1|\,:\overline{\psi^{(+)}}_{1'\alpha}\psi^{(+)}_{1'\alpha}\overline{\psi^{(+)}}_{2'\beta}\psi^{(+)}_{2'\beta}:|N_aN_b> \\
&=<N_2N_1|\,\overline{\psi^{(+)}}_{1'\alpha}\overline{\psi^{(+)}}_{2'\beta}\psi^{(+)}_{2'\beta}\psi^{(+)}_{1'\alpha}\,|N_aN_b> \\
&=<0|\,c_2c_1\left\{c^\dagger_{1'(1')}c^\dagger_{2'(2')}c_{2'(b')}c_{1'(a')}\right\}c^\dagger_a c^\dagger_b\,|0>\;\bar{u}^{(1')}_\alpha \bar{u}^{(2')}_\beta u^{(b')}_\beta u^{(a')}_\alpha\,,
\end{aligned}
\tag{259}
$$

In the last expression, we have used abbreviations

$$
\begin{aligned}
&c_1 = c_{r_1}(\boldsymbol{p}_1) \\
&c_{1'(a')} = c_{1'(r_{a'}p_{a'})}\,, \quad u^{(a')}_\alpha = u^{(r_{a'},p_{a'})}_\alpha
\end{aligned}
\tag{260}
$$

$$(259)\ <0|\cdots|0> \;=\; <0|\,c_2\left\{[c_1,c^\dagger_{1'(1')}]_+ - c^\dagger_{1'(1')}c_1\right\}c^\dagger_{2'(2')}$$
$$c_{2'(b')}\left\{[c_{1'(a')},c^\dagger_a]_+ - c^\dagger_a c_{1'(a')}\right\}c^\dagger_b\,|0>$$
$$= <0|\left\{[c_1,c^\dagger_{1'(1')}]_+[c_2,c^\dagger_{2'(2')}]_+ - [c_2,c^\dagger_{1'(1')}]_+[c_1,c^\dagger_{2'(2')}]_+\right\}$$
$$\left\{[c_{1'(a')},c^\dagger_a]_+[c_{2'(b')},c^\dagger_b]_+ - [c_{2'(b')},c^\dagger_a]_+[c_{1'(a')},c^\dagger_b]_+\right\}|0> \tag{261}$$

where $[\cdots]_+$ stands for anti-commutator. We have, for instance

$$[c_1,c^\dagger_{1'(a')}]_+ = \frac{1}{\sqrt{(2\pi)^3}}\sum_{(a')}\delta_{(a')1}e^{ip_{a'}x_{1'}}\times$$
$$[c_{1'(a')},c^\dagger_b]_+ = \frac{1}{\sqrt{(2\pi)^3}}\sum_{(a')}\delta_{(a')b}e^{-ip_{a'}x_{1'}}\times \tag{262}$$

where

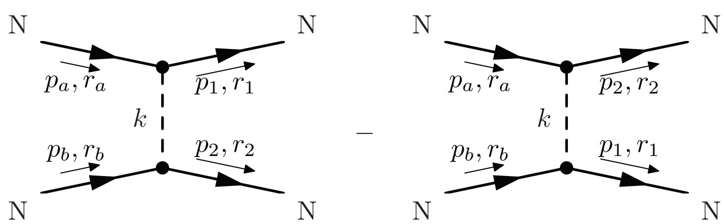
$$\sum_{(a')}\delta_{(a')a}\times \stackrel{\leftarrow}{=} \int d^3\boldsymbol{p}_{a'}\sum_{r_{a'}}\delta^3(\boldsymbol{p}_{a'}-\boldsymbol{p}_a)\delta_{r_{a'}r_a}\times \tag{263}$$

and the multiplication at the end means factors behind are involved in the summation.

$$(259) \;=\; \frac{1}{(2\pi)^6}\sum_{(1')(2')(a')(b')}\left\{(\delta_{1(1')}\delta_{2(2')}-\delta_{2(1')}\delta_{1(2')})(\delta_{(a')a}\delta_{(b')b}-\delta_{(b')a}\delta_{(a')b})\right\}\times$$
$$e^{i(p_{1'}x_{1'}+p_{2'}x_{2'}-p_{a'}x_{1'}-p_{b'}x_{2'})}\bar{u}^{(1')}_\alpha u^{(a')}_\alpha \bar{u}^{(2')}_\beta u^{(b')}_\beta \tag{264}$$

Remembering that Δ_F is an even function,

$$
\begin{aligned}
S_{fi}^{(2)} &= \frac{(-ig)^2}{2}\int d^4x_1'd^4x_2'\Delta_F(x_1'-x_2')\frac{1}{(2\pi)^6}\sum_{(1')(2')(a')(b')} \\
&\quad \left\{\delta_{(1')1}\delta_{(2')2}\delta_{(a')a}\delta_{(b')b}+\delta_{2(1')}\delta_{1(2')}\delta_{(a')b}\delta_{(b')a}-\delta_{(1')1}\delta_{(2')2}\delta_{(a')b}\delta_{(b')a}-\delta_{2(1')}\delta_{1(2')}\delta_{(a')a}\delta_{(b')b}\right\} \\
&\quad e^{i(p_{1'}x_{1'}+p_{2'}x_{2'}-p_{a'}x_{1'}-p_{b'}x_{2'})}\bar{u}_\alpha^{(1')}u_\alpha^{(a')}\bar{u}_\beta^{(2')}u_\beta^{(b')} \\
&= \frac{(-ig)^2}{(2\pi)^6}\int d^4x_1'd^4x_2'\Delta_F(x_1'-x_2')\sum_{(1')(2')(a')(b')}\left\{\delta_{(1')1}\delta_{(2')2}\delta_{(a')a}\delta_{(b')b}-\delta_{2(1')}\delta_{1(2')}\delta_{(a')a}\delta_{(b')b}\right\} \\
&\quad e^{i(p_{1'}x_{1'}+p_{2'}x_{2'}-p_{a'}x_{1'}-p_{b'}x_{2'})}\bar{u}_\alpha^{(1')}u_\alpha^{(a')}\bar{u}_\beta^{(2')}u_\beta^{(b')} \\
&= \frac{(-ig)^2}{(2\pi)^6}\int d^4x_1'd^4x_2'\Delta_F(x_1'-x_2')\Big\{e^{i(p_1x_{1'}+p_2x_{2'}-p_ax_{1'}-p_bx_{2'})}\bar{u}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}u_\beta^{(b)} \\
&\qquad -e^{i(p_2x_{1'}+p_1x_{2'}-p_ax_{1'}-p_bx_{2'})}\bar{u}_\alpha^{(2)}u_\alpha^{(a)}\bar{u}_\beta^{(1)}u_\beta^{(b)}\Big\} \\
&= \frac{(-ig)^2}{(2\pi)^6}\int d^4k\frac{i(2\pi)^4}{k^2-\mu^2+i\epsilon}\Big\{\delta^4(p_1-p_a-k)\delta^4(p_2-p_b+k)\bar{u}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}u_\beta^{(b)} \\
&\qquad -\delta^4(p_2-p_a-k)\delta^4(p_1-p_b+k)\bar{u}_\alpha^{(2)}u_\alpha^{(a)}\bar{u}_\beta^{(1)}u_\beta^{(b)}\Big\} \\
&=
\end{aligned}
$$



$$
= i(2\pi)^4\delta^4(P_f-P_i)\frac{(-ig)^2}{(2\pi)^6}\left\{\frac{\bar{u}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}u_\beta^{(b)}}{(p_1-p_a)^2-\mu^2+i\epsilon}-\frac{\bar{u}_\alpha^{(2)}u_\alpha^{(a)}\bar{u}_\beta^{(1)}u_\beta^{(b)}}{(p_2-p_a)^2-\mu^2+i\epsilon}\right\} \tag{265}
$$

$$
T_{fi}^{(2)} = \frac{(-ig)^2}{(2\pi)^6}\left\{\frac{\bar{u}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}u_\beta^{(b)}}{(p_1-p_a)^2-\mu^2+i\epsilon}-\frac{\bar{u}_\alpha^{(2)}u_\alpha^{(a)}\bar{u}_\beta^{(1)}u_\beta^{(b)}}{(p_2-p_a)^2-\mu^2+i\epsilon}\right\} \tag{266}
$$

Though the meaning of our short-hand notations are stated before, we write here again that $u^{(1)}$ in the above equation, for instance, stands for $u^{r_1}(\boldsymbol{p}_1)$.

Feynman rules for Dirac fermions

1. For each incoming (outgoing) fermion with momentum $\boldsymbol{p}$ and polarization r, associate $u^{(r)}(\boldsymbol{p})/\sqrt{(2\pi)^3}$ ($\bar{u}^{(r)}(\boldsymbol{p})/\sqrt{(2\pi)^3}$). For each incoming (outgoing) anti-fermion, associate $\bar{v}^{(r)}(\boldsymbol{p})/\sqrt{(2\pi)^3}$ ($v^{(r)}(\boldsymbol{p})/\sqrt{(2\pi)^3}$).

2. To each vartex, associate a factor $(-ig)(2\pi)^4\delta^4(\sum_{in} p_{in})$.

3. For each internal fermion line, write a factor of

$$\int \frac{d^4p'}{(2\pi)^4}\frac{i(\not{p}'+m)}{p'^2-m^2+i\epsilon} \tag{267}$$

<u>$N\bar{N} \to N\bar{N}$</u>

$$\begin{aligned} S_{fi}^{(2)} &= \frac{(-ig)^2}{(2\pi)^6}\int d^4k \frac{i(2\pi)^4}{k^2-\mu^2+i\epsilon}\Big\{\delta^4(p_1-p_a-k)\delta^4(p_2-p_b+k)\bar{u}_\alpha^{(1)}u_\alpha^{(a)}v_\beta^{(2)}\bar{v}_\beta^{(b)} \\ &\qquad +\delta^4(p_1+p_2-k)\delta^4(p_a+p_b+k)\bar{v}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}v_\beta^{(b)}\Big\} \end{aligned} \tag{268}$$

$$T_{fi}^{(2)} = \frac{(-ig)^2}{(2\pi)^6}\left\{\frac{\bar{u}_\alpha^{(1)}u_\alpha^{(a)}v_\beta^{(2)}\bar{v}_\beta^{(b)}}{(p_1-p_a)^2-\mu^2+i\epsilon}+\frac{\bar{v}_\alpha^{(1)}u_\alpha^{(a)}\bar{u}_\beta^{(2)}v_\beta^{(b)}}{(p_a+p_b)^2-\mu^2+i\epsilon}\right\} \tag{269}$$

<u>$N\bar{N} \to \pi\pi$</u>

$$T_{fi}^{(2)} = \frac{(-ig)^2}{(2\pi)^6}\left\{\frac{\bar{v}_\alpha^{(b)}[(\not{p}_a-\not{p}_1)+m]u_\alpha^{(a)}}{(p_1-p_a)^2-m^2+i\epsilon}+\frac{\bar{v}_\alpha^{(b)}[(\not{p}_a-\not{p}_2)+m]u_\alpha^{(a)}}{(p_2-p_a)^2-m^2+i\epsilon}\right\} \tag{270}$$

8 Photon Fields

8.1 Classical Theory

Maxwell's equations, a set of classical field equations of the electromagnetism, in a vacuum, are written [15]in a Lorentz covariant form as

$$\partial_\nu F^{\nu\mu}(x) = \frac{1}{c} j^\mu(x)\,, \tag{271}$$

$$\frac{1}{2}\epsilon^{\mu\rho\sigma\tau}\partial_\rho F_{\sigma\tau}(x) = 0\,, \tag{272}$$

where $j^\mu(x) = (c\rho, \boldsymbol{j})$ is the four charge current density and $\epsilon^{\mu\rho\sigma\tau}$ is the Levi-Civita tensor in the 4 dimension. The field strength tensor $F^{\mu\nu}$ is defined in terms of the four electromagnetic potential $A^\mu(x) = (\phi, \boldsymbol{A})$ as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{273}$$

which is apparently an antisymmetric tensor. The field strength tensor is related with the electric and magnetic fields as

$$\begin{cases} F^{\uparrow 0} = \boldsymbol{E}\,, \\ F^{ij} = \epsilon^{ijk} B_k \ \ (\Leftrightarrow\ \boldsymbol{B} = -\frac{1}{2}\epsilon^{\uparrow ij} F^{ij})\,, \end{cases} \tag{274}$$

or explicitly in a matrix form as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{275}$$

In the first equation in Eq. (274), an upper arrow as a superscript represents cartesian components of a spatial vector in the right hand side. [16] With these correspondences, we may easily reproduce the familiar form of Maxwell's equations in a vacuum:

$$\begin{cases} \boldsymbol{\partial}\cdot\boldsymbol{E} = \rho & \text{Coulomb} \\ \boldsymbol{\partial}\times\boldsymbol{B} - \partial_0\boldsymbol{E} = \frac{1}{c}\boldsymbol{j} & \text{Ampère-Maxwell} \\ \boldsymbol{\partial}\times\boldsymbol{E} + \partial_0\boldsymbol{B} = \boldsymbol{0} & \text{Faraday} \\ \boldsymbol{\partial}\cdot\boldsymbol{B} = 0 & \text{Coulomb (magnetic)} \end{cases}$$

[15] We use the Heaviside-Lorentz Gauss unit system of the electrodynamics and write the light speed c explicitly for a while.

[16] This notation is convenient in dealing with calculus among spatial vectors.

It follows from the definition (273) that the set of homogeneous Maxwell's equations (272) is satisfied automatically and becomes an identity. This identity is called the Bianchi identity. Now, problems of electromagnetism reduce themselves to solving inhomogeneous equations (271). Substituting Eq. (273) into Eq. (271), we have

$$(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu = j^\mu \tag{276}$$

Here and hereafter, we set $c = 1$ as usual. A famous problem (and the sake) of the electromagnetism is that the operator acting on A_ν in the left hand side is not invertible. [17]This is due to a symmetry of Eq. (276) under the gauge transformation of the second kind:

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu \Lambda\,, \tag{277}$$

where Λ is an arbitrary second order differentiable function. One can confirm that the physically observable fields $F^{\mu\nu}$ in Eq. (273) are certainly invariant under this transformation. This is why we are insisted (and allowed) to pick a representative of the field A^μ among those connected by gauge transformations and, as a result, obtain two degrees of freedom corresponding to photon polarizations from the four of the Lorentz vector A^μ.

The Lagrangian density which reproduces Maxwell's equations (276) may be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu\,. \tag{278}$$

The first term is manifestly gauge invariant. The second term is not manifest but the gauge invariance is guarantied in the level of action through the current conservation law $\partial \cdot j = 0$, which is a consistency condition of the Maxwell equation (276). To fix a gauge, we impose condition(s) on the field A_μ and modify the first term of the Lagrangian density (278) accordingly. This modification is not accounted for as a change by a total derivative, which does not change the field equation. We need to modify the field equation itself to make it solvable. In the classical theory, this can be achieved as described below. In the quantum theory, however, an special care is required to set condition(s) on the field.

Before discussing particular choices of a gauge, we make notes on the Lagrangian density (278). The first term has a few different expressions as

[17]It is equivalent to say Green's function satisfying

$$(\partial^2 g^\mu_\nu - \partial_\nu \partial^\mu) G^\nu_\lambda(x - x') = g^\mu_\lambda \delta(x - x')$$

does not exist. If it does, one get $\partial_\lambda \delta(x - x') = \partial_\nu (\partial^2 g^\mu_\nu - \partial_\nu \partial^\mu) G^\nu_\lambda = 0$, which is mathematically contradicting.

follows:

$$\begin{aligned}
\mathcal{L}_{F^2} &\equiv -\frac{1}{4}F^2 \\
&= -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} = -\frac{1}{2}\left[(\partial_\mu A_\nu)^2 - \partial_\mu A_\nu \partial^\nu A^\mu\right] , \qquad (279)\\
&= \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) , \qquad (280)\\
&= \frac{1}{2}\left[A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu - \partial_\mu(A_\nu F^{\mu\nu})\right] . \qquad (281)
\end{aligned}$$

The equivalence among these expressions can be checked in a straightforward manner. [18]In these expressions, $F^{\mu\nu}$ abbreviates its expression in terms of the field A. Eq. (280) shows through Eq. (274) that $\mathcal{L}_{F^2}$ has no kinetic term proportional with $\dot{A}_0$ so that canonical conjugate momontum Π_0 associated with A_0 is identically zero. This may cause a problem when one apply canonical quantization method. The canonical conjugate momentum $\Pi^\uparrow$ associated with $A_\uparrow = -\boldsymbol{A}$ is immediately obtained from Eq. (280) as

$$\boldsymbol{\Pi} = -\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{A}}} = \boldsymbol{E} . \qquad (283)$$

Finally, Eq. (281) shows explicitly that Maxwell's equations are deduced from $\mathcal{L}_{F^2}$ by requiring action to be unchanged by varying A_ν. [19]

Hamiltonian density is written in a variety of expressions as

$$\begin{aligned}
\mathcal{H}_{F^2} &= \Pi^i \dot{A}_i - \mathcal{L}_{F^2} \\
&= (F^{i0})^2 - \partial^i A^0 F^{i0} - \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) , \\
&= \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) - A^0 \boldsymbol{\partial}\cdot\boldsymbol{E} + \partial_i\left\{A^0 E^i\right\} , \qquad (284)\\
&= -\frac{1}{2}\left[(\partial^i A^0)^2 - (\partial^0 A^i)^2\right] + \frac{1}{2}\partial_i A_j F^{ij} . \qquad (285)
\end{aligned}$$

Lorentz gauge

The gauge condition is written as

$$\partial_\mu A^\mu = 0 . \qquad (286)$$

[18]It may help for doing so to write

$$\boldsymbol{B}^2 = \frac{1}{2}(F^{ij})^2 = \partial_i A_j F^{ij} . \qquad (282)$$

[19]In such a treatment, we adopt the variational principle considering $\mathcal{L}$ is a functional of the field A solely and its derivatives are not independent variables. We must take variation of both of A in places before and after differential operators. For the form of Eq. (281), we make use of partial integrations twice.

Since

$$\partial_\mu F^{\mu\nu} = \partial^2 A^\nu - \partial_\nu(\partial_\mu A^\mu)\,, \tag{287}$$

the field equation (276) reduces to the massless Klein-Gordon equation $\partial^2 A^\nu = j^\nu$ under this condition. Making use of an expression Eq. (279), Lagrangian density under the Lorentz condition reads

$$\begin{aligned}\mathcal{L}_{F^2} &= -\frac{1}{2}\left[(\partial_\mu A_\nu)^2 + \cancel{A_\nu \partial^\nu \partial_\mu A^\mu} - \partial_\mu(A_\nu \partial^\nu A^\mu)\right] \\ &= \mathcal{L}_{(\partial A)^2} + \text{total derivative}\,,\end{aligned} \tag{288}$$

where we have defined a new Lagrangian density

$$\mathcal{L}_{(\partial A)^2} = -\frac{1}{2}(\partial_\mu A_\nu)^2\,, \tag{289}$$

that leads to the massless Klein-Gordon equation. Appart from the explicit way to modify $\mathcal{L}$ as in the above, the Lorentz gauge condition can be imposed in terms of Lagrangian multiplier η as follows:

$$\begin{aligned}\mathcal{L}_{Lor} &= \mathcal{L}_{F^2} - \frac{\eta}{2}(\partial \cdot A)^2 \\ &= \frac{1}{2}\left[A_\nu \partial_\mu F^{\mu\nu} + \eta A_\nu \partial^\nu \partial^\mu A_\mu\right] + \text{total derivative}\,, \\ &= \frac{1}{2}\left[A_\nu \partial^2 A^\nu - (1-\eta) A_\nu \partial^\mu \partial^\nu A_\mu\right] + \text{total derivative}\,,\end{aligned} \tag{290}$$

that induces

$$\begin{aligned}0 &= \partial_\mu F^{\mu\nu} + \eta \partial^\nu \partial^\mu A_\mu - j^\nu \\ &= \partial^2 A^\nu - (1-\eta)\partial^\nu \partial^\mu A_\mu - j^\nu\end{aligned} \tag{291}$$

for the full Lagrangian density $\mathcal{L} = \mathcal{L}_{Lor} - j^\mu A_\mu$. Euler-Lagrange equation for η results in the Lorentz condition and field equations of the system is composed of that and the massless Klein-Gordon equation.

With the Lorentz condition, the gauge is not completely fixed. We still have freedom of making gauge transformation for an arbitrary function Λ satisfying $\partial^2\Lambda = 0$.

Coulomb gauge

$$\boldsymbol{\partial} \cdot \boldsymbol{A} = 0 \tag{292}$$

Inhomogeneous Maxwell's equations read

$$\partial_\mu F^{\mu 0} = -\boldsymbol{\partial}^2 A^0 - \partial_0 \cancel{\boldsymbol{\partial} \cdot \boldsymbol{A}} = \rho \tag{293}$$

$$\partial_\mu F^{\mu\uparrow} = \partial^2 \boldsymbol{A} + \boldsymbol{\partial}\partial_0 A^0 + \boldsymbol{\partial}\cancel{(\boldsymbol{\partial} \cdot \boldsymbol{A})} = \boldsymbol{j} \tag{294}$$

The solution of the first (nondynamical) equation for A^0 is given as

$$A^0(x) = \frac{1}{4\pi}\int d^3\boldsymbol{x}'\,\frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{295}$$

The solution is unique under the boundary condition that A^0 and ρ vanish at spatial infinity. Cancelling out terms with $\partial_0 A_0$ in the Lagrangian density $\mathcal{L}_{F^2}$, we read from Eq. (279) that

$$\begin{aligned}
\mathcal{L}_{F^2} &= -\frac{1}{2}\left[(\partial_\mu A_0)^2 - (\partial_\mu A_i)^2 - \partial_0 A_\nu \partial^\nu A_0 - \partial_i A_\nu \partial^\nu A^i\right] \\
&= -\frac{1}{2}\left[-(\partial_i A_0)^2 - (\partial_\mu A_i)^2 - \partial_0 A^i \partial_i A_0 - \partial_i A_\nu \partial^\nu A^i\right] \\
&= -\frac{1}{2}\left[\left\{A_0 \boldsymbol{\partial}^2 A_0 - \partial_i(A_0\partial_i A_0)\right\} - \left\{A_i \partial^2 A^i + \partial_\mu(A_i\partial^\mu A_i)\right\}\right. \\
&\qquad \left. +2\left\{A^i\partial_i\partial_0 A_0 - \partial_0(A^i\partial_i A_0)\right\} + \left\{A^i\partial_i\partial_j A^j - \partial_i(A^j\partial_j A^i)\right\}\right] .
\end{aligned}$$

where we have used abbreviations $(a_i)^2 = a_i a_i = \boldsymbol{a}^2$ while $(a_\mu)^2 = a_\mu a^\mu = a^2$. The last expression in the above equation has a form easy to take variations in A. In the Hamiltonian density of a form equivalent to Eq. (285),

$$\mathcal{H}_{F^2} = \frac{1}{2}(\partial^0 A^i)^2 + \frac{1}{2}\partial_i A_j F^{ij} + \frac{1}{2}A^0 \boldsymbol{\partial}^2 A^0 - \frac{1}{2}\partial_i\left(A^0\partial_i A^0\right) ,$$

we may substitute the solution (295) of non-dynamical variable A^0 in the third term to write

$$\mathcal{H}_{F^2,\mathrm{Coul}} = \frac{1}{2}(\partial^0 A^i)^2 + \frac{1}{2}\partial_i A_j F^{ij} - \frac{1}{2}A^0\rho - \frac{1}{2}\partial_i\left(A^0\partial_i A^0\right) .$$

Adding the source term $j\cdot A = \rho A^0 - \boldsymbol{j}\cdot\boldsymbol{A}$, we obtain the full Hamiltonian density in the Coulomb gauge as

$$\mathcal{H} = \frac{1}{2}(\partial^0 A^i)^2 + \frac{1}{2}\partial_i A_j F^{ij} + \mathcal{H}_{\mathrm{Coul}} - \boldsymbol{j}\cdot\boldsymbol{A} - \frac{1}{2}\partial_i\left(A^0\partial_i A^0\right) , \tag{296}$$

where

$$\mathcal{H}_{\mathrm{Coul}} = \frac{1}{2}A^0\rho = \frac{1}{8\pi}\int d^3\boldsymbol{x}'\,\frac{\rho(\boldsymbol{x},t)\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} . \tag{297}$$

The last term in Eq. (296) is a total derivative and will not contribute to the Hamiltonian.

The gauge condition (292) is equivalent with writing

$$\boldsymbol{A} = \left(1 - \frac{\boldsymbol{\partial}\boldsymbol{\partial}\cdot}{\boldsymbol{\partial}^2}\right)\boldsymbol{A} . \tag{298}$$

From the current conservation law and Eq. (294), we have

$$\boldsymbol{\partial} \cdot \boldsymbol{j} = -\partial_0 \rho = \boldsymbol{\partial}^2 \partial_0 A^0 \tag{299}$$

and

$$\boldsymbol{j}_T \overset{\leftarrow}{=} \left(1 - \frac{\boldsymbol{\partial}\boldsymbol{\partial}\cdot}{\boldsymbol{\partial}^2}\right) \boldsymbol{j} = \boldsymbol{j} - \boldsymbol{\partial}\partial_0 A^0 \tag{300}$$

so that Eq. (294) can be written [20]as

$$\partial^2 \boldsymbol{A} = \boldsymbol{j}_T \,. \tag{301}$$

When there is no source current, we have $A^0 = 0$ from Eq. (295) and as the dynamical field equation $\partial^2 \boldsymbol{A} = \boldsymbol{0}$. In this case the Coulomb gauge condition (292) can be realized as a further restriction $\boldsymbol{\partial}^2 \Lambda = \boldsymbol{\partial} \cdot \boldsymbol{A}$ for a given A in the Lorentz condition case. Now A^0 and longitudinal component of $\boldsymbol{A}$ are dropped from dynamical degree of freedom and we have 2 degree of freedom remaining. They will be identified as the two polarization states of the photon.

8.2 Quantization of Free Field

Coulomb gauge

The dynamical field equation for free field is

$$\partial^2 \boldsymbol{A} = \boldsymbol{0} \,, \tag{302}$$

for which we may write the solution as

$$\boldsymbol{A}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} \left[\boldsymbol{a}(\boldsymbol{k}) e^{-ikx} + \boldsymbol{a}^\dagger(\boldsymbol{k}) e^{ikx}\right] , \tag{303}$$

where $\omega = k^0 = |\boldsymbol{k}|$ and $\boldsymbol{a}^\dagger$ is complex (hermite) conjugate of $\boldsymbol{a}$ in the classical (quantum) level. [21]The gauge condition (292) reads

$$\boldsymbol{k} \cdot \boldsymbol{a}(\boldsymbol{k}) = 0 \,. \tag{305}$$

[20]An expression of $\boldsymbol{j}_T$ and $\boldsymbol{j}_L = \boldsymbol{j} - \boldsymbol{j}_T$ in terms of integration is given in [12]. We note that $A^0 = 0$ when $\rho = 0$ for our boundary condition at spatial infinity and thus $\boldsymbol{j}_T = \boldsymbol{0}$ for the case of no source $j^\mu = 0$.

[21]Expressions below follow:

$$\begin{aligned} \dot{\boldsymbol{A}} &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} (-i\omega) \left[\boldsymbol{a} e^{-ikx} - \boldsymbol{a}^\dagger e^{ikx}\right] \\ F^{ij} &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} \left[(-ik^i a^j + ik^j a^i) e^{-ikx} + (ik^i a^{j\dagger} - ik^j a^{i\dagger}) e^{ikx}\right] \\ \boldsymbol{B} &= \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} i\boldsymbol{k} \times \left[\boldsymbol{a} e^{-ikx} - \boldsymbol{a}^\dagger e^{ikx}\right] \end{aligned} \tag{304}$$

Considering there are two independent dynamical degrees of freedom, we introduce polarization 3 vectors $\boldsymbol{e}^{(\lambda)}$, $\lambda = 1, 2$ which satisfy the following three conditions:

$$\begin{aligned}
&\boldsymbol{k} \cdot \boldsymbol{e}^{(\lambda)} = 0\,, \\
&\boldsymbol{e}^{(\lambda)*} \cdot \boldsymbol{e}^{(\lambda')} = \delta^{\lambda\lambda'}\,, \\
&\textstyle\sum_\lambda \boldsymbol{e}^{(\lambda)} \boldsymbol{e}^{(\lambda)*} \cdot = 1 - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}} \cdot \;.
\end{aligned} \tag{306}$$

where we wrote $\boldsymbol{k}/|\boldsymbol{k}| = \hat{\boldsymbol{k}}$. Decomposing $\boldsymbol{a}$ with these basis, we write

$$\boldsymbol{A}(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} \sum_{\lambda=1,2} \left[a_\lambda(\boldsymbol{k}) \boldsymbol{e}^{(\lambda)} e^{-ikx} + a_\lambda^\dagger(\boldsymbol{k}) \boldsymbol{e}^{(\lambda)*} e^{ikx} \right]. \tag{307}$$

In a coordinate system where the wave vector $\boldsymbol{k}$ lays in the direction of the 3rd axix, one may choose $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ as unit vectors in directions of the 1st and 2nd axes, respectively. One may also choose two polarization vectors $\boldsymbol{e}^{(\pm)} = (1, \pm i, 0)/\sqrt{2}$ for right- and left- hanaded circular polarizations. [22]

On applying the canonical quantization method, one needs to take constraints of the gauge condition on canonical variables $\boldsymbol{A}$ and $\boldsymbol{\Pi} = \boldsymbol{E}$ into account. Since $\boldsymbol{\partial} \cdot \boldsymbol{A} = \boldsymbol{\partial} \cdot \boldsymbol{E} = 0$ shold hold, these variables are restricted in a space projected by an operator in Eq. (298). Therefore we write the equal-time commutation relation among them as

$$\begin{aligned}
[A^i(\boldsymbol{x}, t), \Pi^j(\boldsymbol{y}, t)] &= \left(\delta^{ij} - \frac{\partial^i \partial^j}{\boldsymbol{\partial}^2} \right) i\delta^3(\boldsymbol{x} - \boldsymbol{y}) \\
&= i \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2} \right) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}
\end{aligned} \tag{308}$$

One may check the consistency with the constraints by taking divergences of the both hand sides. Other commutation relations are not altered since the right hand sides are proportional to 0:

$$[A^i(\boldsymbol{x}, t), A^j(\boldsymbol{y}, t)] = [\Pi^i(\boldsymbol{x}, t), \Pi^j(\boldsymbol{y}, t)] = 0 \tag{309}$$

Substituting Eq. (307) in an expression $\boldsymbol{\Pi} = -\partial_0 \boldsymbol{A}$ and Eqs. (308, 309), we find

$$\begin{aligned}
&[a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] = [a_\lambda^\dagger(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')] = 0\,, \\
&[a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k}')] = 2\omega\delta^3(\boldsymbol{k} - \boldsymbol{k}')\delta_{\lambda\lambda'}\,,
\end{aligned} \tag{310}$$

for $\lambda = 1, 2$. Now we can interpret $a_\lambda^\dagger(\boldsymbol{k})$ and $a_\lambda(\boldsymbol{k})$ as creation and annihilation operators for dynamical photons of definite polarization represented by λ.

[22] [13] takes $\boldsymbol{e}^{(\pm)} = \mp i(1, \pm i, 0)/\sqrt{2}$.

The Hamiltonian corresponding to Eq. (296) before taking normal product reads [23]

$$H = \int \frac{d^3\boldsymbol{k}}{2\omega}\frac{\omega}{2}(\boldsymbol{a}^\dagger\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{a}^\dagger) + H_{\text{Coul}} - \int d^3\boldsymbol{x}\boldsymbol{j}\cdot\boldsymbol{A} \tag{311}$$

To obtain the Feynman propagator in this gauge, we first derive the

[23]Denoting $d^3\tilde{\boldsymbol{k}} = d^3\boldsymbol{k}/\{\sqrt{(2\pi)^3}2\omega\}$, $\boldsymbol{a}' = \boldsymbol{a}(\boldsymbol{k}')$ and $\omega' = k'^0 = |\boldsymbol{k}'|$, we write

$$(\partial_0 A_i)^2 = \int d^3\tilde{\boldsymbol{k}}d^3\tilde{\boldsymbol{k}}'(-i\omega)(-i\omega')\left[\boldsymbol{a}e^{-ikx}-\boldsymbol{a}^\dagger e^{ikx}\right]\cdot\left[\boldsymbol{a}'e^{-ik'x}-\boldsymbol{a}'^\dagger e^{ik'x}\right]$$

$$\begin{aligned}\int d^3\boldsymbol{x}\boldsymbol{B}^2 &= \int d^3\tilde{\boldsymbol{k}}d^3\tilde{\boldsymbol{k}}'\int d^3\boldsymbol{x}\, i\boldsymbol{k}\times\left[\boldsymbol{a}e^{-ikx}-\boldsymbol{a}^\dagger e^{ikx}\right]\cdot i\boldsymbol{k}'\times\left[\boldsymbol{a}'e^{-ik'x}-\boldsymbol{a}'^\dagger e^{ik'x}\right]\\
&= -\int d^3\tilde{\boldsymbol{k}}d^3\tilde{\boldsymbol{k}}'\int d^3\boldsymbol{x}\,\Big[\{(\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}\cdot\boldsymbol{a}')-(\boldsymbol{k}'\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{a}')\}e^{-i(k+k')x}\\
&\qquad +\{(\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}^\dagger\cdot\boldsymbol{a}'^\dagger)-(\boldsymbol{k}'\cdot\boldsymbol{a}^\dagger)(\boldsymbol{k}\cdot\boldsymbol{a}'^\dagger)\}e^{i(k+k')x}\\
&\qquad -\{(\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}\cdot\boldsymbol{a}'^\dagger)-(\boldsymbol{k}'\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{a}'^\dagger)\}e^{-i(k-k')x}\\
&\qquad -\{(\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}^\dagger\cdot\boldsymbol{a}')-(\boldsymbol{k}'\cdot\boldsymbol{a}^\dagger)(\boldsymbol{k}\cdot\boldsymbol{a}')\}e^{i(k-k')x}\Big]\end{aligned}$$

Adding the above two reads

$$\begin{aligned}&\int d^3\boldsymbol{x}\left\{(\partial_0 A_i)^2+\boldsymbol{B}^2\right\}\\
&= -\int d^3\tilde{\boldsymbol{k}}d^3\tilde{\boldsymbol{k}}'\int d^3\boldsymbol{x}\,\Big[\{(\omega\omega'+\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}\cdot\boldsymbol{a}')-(\boldsymbol{k}'\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{a}')\}e^{-i(k+k')x}+\cdots\\
&\qquad -\{(\omega\omega'+\boldsymbol{k}\cdot\boldsymbol{k}')(\boldsymbol{a}\cdot\boldsymbol{a}'^\dagger)-(\boldsymbol{k}'\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{a}'^\dagger)\}e^{-i(k-k')x}+\cdots\Big]\\
&= -\int\frac{d^3\boldsymbol{k}}{4\omega^2}\left[\{(\omega^2-\boldsymbol{k}^2)(\boldsymbol{a}(\boldsymbol{k})\cdot\boldsymbol{a}(-\boldsymbol{k}))-(\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{k}))(-\boldsymbol{k}\cdot\boldsymbol{a}(-\boldsymbol{k}))\}e^{-2i\omega t}+\cdots\right.\\
&\qquad \left.-\{(\omega^2+\boldsymbol{k}^2)(\boldsymbol{a}\cdot\boldsymbol{a}^\dagger)-(\boldsymbol{k}\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{a}^\dagger)\}+\cdots\right]\\
&= \int\frac{d^3\boldsymbol{k}}{2\omega}\omega\left(\boldsymbol{a}^\dagger\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{a}^\dagger\right)\end{aligned}$$

This leads to Eq. (311).

following expression:

$$
\begin{aligned}
D^{ij}(x-y) &\overset{\leftarrow}{=} <0|\, A^i(x)A^j(y)\, |0> \\
&= \int d^3\tilde{\boldsymbol{k}} d^3\tilde{\boldsymbol{k}}' <0|\, \left[a^i(\boldsymbol{k}), a^{j\dagger}(\boldsymbol{k}')\right] \,|0> e^{-ikx+ik'y} \\
&= \int d^3\tilde{\boldsymbol{k}} d^3\tilde{\boldsymbol{k}}' <0|\, \left[a_\lambda(\boldsymbol{k}), a_{\lambda'}{}^\dagger(\boldsymbol{k}')\right] \,|0> e^{(\lambda)i} e^{(\lambda')j*} e^{-ikx+ik'y} \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 4\omega^2} 2\omega \sum_\lambda e^{(\lambda)i} e^{(\lambda)j*} e^{-ik(x-y)} \qquad (312) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega} (\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}) e^{-ik(x-y)} . \qquad (313)
\end{aligned}
$$

The Feynman propagator is then obtained as [24]

$$
\begin{aligned}
D_T^{ij}(x-y) &\overset{\leftarrow}{=} <0|\, T\left[A^i(x)A^j(y)\right] \,|0> \\
&= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} \left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right) e^{-ik\cdot(x-y)} ,
\end{aligned}
\qquad (314)
$$

where we have distinguished this propagator with a subscript T to indicate this one is for transverse photons.

Lorentz gauge

We consider a prescription where we start with the Lagrangian density in Eq. (290) with $\eta = 1/\alpha = 1$ (Feynman gauge). Thus, our Lagrangian density is written as

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{F^2} - \frac{1}{2}(\partial_\mu A^\mu)^2 \\
&= \mathcal{L}_{(\partial A)^2} + \frac{1}{2}\partial_\nu (A_\mu \partial^\mu A^\nu - A_\nu \partial_\mu A^\mu) . \qquad (315)
\end{aligned}
$$

This Lagrangian density is equivalent with $\mathcal{L}_{F^2}$ provided the Lorentz gauge condition is imposed. That means we modify the Lagrangian density explicitly so that the EOM is solvable. We quantize the field first without

[24]The derivation goes similar way as that for scalar fields. For checking the result, it is straightforward to obtain

$$D_T^{ij}(x-y) = \theta(x^0 > y^0) D^{ij}(x-y) + \theta(y^0 > x^0) D^{ij}(y-x)$$

by performing the contour integration of the variable k_0 in Eq. (314), constraints and later impose the gauge condition. Since the field equation is the massless Klein-Goldon equation

$$\partial^2 A^\mu = j^\mu \,, \tag{316}$$

we write the solution for the free field as

$$A_\mu(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2\omega}} \sum_{\lambda=0}^{3} \left[a_\lambda(\boldsymbol{k}) e_\mu^{(\lambda)} e^{-ikx} + a_\lambda^\dagger(\boldsymbol{k}) e_\mu^{(\lambda)*} e^{ikx} \right] . \tag{317}$$

As the field A_μ is not constrained, the polarization four vectors $e_\mu^{(\lambda)}(\boldsymbol{k})$ can be chosen as 4 linearly independent basis:

$$\begin{cases} g^{\mu\nu} e_\mu^{(\lambda)*} e_\nu^{(\lambda')} = g^{\lambda\lambda'} & \text{(orthonormal)} \\ g_{\lambda\lambda'} e_\mu^{(\lambda)} e_\nu^{(\lambda')*} = g_{\mu\nu} & \text{(complete)} \end{cases} \tag{318}$$

In a standard choice of the polarization vectors, $e^{(3)}$ is taken along the spacial momentum vector $\boldsymbol{k}$ and other two are in the plane orthogonal to k:

$$e^{(0)} = \begin{pmatrix} 1 \\ \boldsymbol{0} \end{pmatrix} \quad e^{(1)} = \begin{pmatrix} 0 \\ \boldsymbol{e}_1 \end{pmatrix} \quad e^{(2)} = \begin{pmatrix} 0 \\ \boldsymbol{e}_2 \end{pmatrix} \quad e^{(3)} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{k}} \end{pmatrix} \tag{319}$$

where $k = (k^0, \boldsymbol{k}) = \omega(1, \hat{\boldsymbol{k}})$ and $\boldsymbol{e}_2 = \hat{\boldsymbol{k}} \times \boldsymbol{e}_1$ with $\boldsymbol{e}_1$ being an arbitrary chosen spacial unit vector perpendicular to $\hat{\boldsymbol{k}}$. In these expressions, we have to notice that spatial components of $e_\mu^{(\lambda)}$ have opposite signatures from ones indicated in the above which are ones for contravariant vectors $e^{(\lambda)\mu}$ in our notation. Another standard choice of the polarization vector basis is to take the transverse ones as

$$e^{(\pm)} = \frac{1}{\sqrt{2}} \left(e^{(1)} \pm i e^{(2)} \right) \tag{320}$$

to represent the two helicities of the photon.

For our Lagrangian density (315), we have components of the momentum canonically conjugate to A_μ as

$$\begin{aligned} \pi^0 &= \frac{\partial L}{\partial \dot{A}_0} = -\partial_\mu A^\mu \,, \\ \pi^i &= \frac{\partial L}{\partial \dot{A}_i} = \partial^i A^0 - \partial^0 A^i = E^i \,. \end{aligned} \tag{321}$$

Since the free field A is not constrained, we write canonical quantization conditions as usual [25]as

$$\begin{aligned}
&[A^\mu(\boldsymbol{x},t), A^\nu(\boldsymbol{y},t)] = [\pi^\mu(\boldsymbol{x},t), \pi^\nu(\boldsymbol{y},t)] = 0\,, \\
&[A^\mu(\boldsymbol{x},t), \pi^\nu(\boldsymbol{y},t)] = i g^{\mu\nu}\delta^3(\boldsymbol{x}-\boldsymbol{y})\,.
\end{aligned} \tag{322}$$

In terms of creation and annihilation operators in Eq. (317), these relationships are equivalent with

$$\begin{aligned}
&[a_\lambda(\boldsymbol{k}), a_{\lambda'}(\boldsymbol{k'})] = \left[a^\dagger_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k'})\right] = 0\,, \\
&\left[a_\lambda(\boldsymbol{k}), a^\dagger_{\lambda'}(\boldsymbol{k'})\right] = -g_{\lambda\lambda'}2\omega\delta^3(\boldsymbol{k}-\boldsymbol{k'})\,.
\end{aligned} \tag{323}$$

The signature of the *r.h.s.* in the second equation for $\lambda = \lambda' = 0$ is quite twisted for it makes the norm of a timelike polarized photon state negative. Here is the place where we need to impose constraints on the field by virtue of the gauge freedom to surpress such an oscillation. However the Lorentz gauge condition in Eq. (286) as an operator equation is not consistent with the second equation in Eq. (322) as a derivative of the delta function in the *r.h.s.* is not vanishing.

Gupta and Bleuler solved this problem by imposing the Lorentz condition (286) not as an operator equation but as a restriction on the physical Hilbert space;

$$<\Psi'_{Phys}|\,\partial_\mu A^\mu\,|\Psi_{Phys}>= 0\,. \tag{324}$$

We may decompose any state $|\Psi>$ of the Hilbert space H into a direct product of states $|\psi_T>$ containing transverse photons and states $|\phi>$ containing the timelike and longitudinal photons. Since

$$\partial\cdot A \sim (a_3 - a_0)e^{-ikx} + h.c.\,,$$

for our choice (319) of basis, the condition (324) reduces to

$$(a_3 - a_0)\,|\phi>= 0 \tag{325}$$

Regarding the signature in Eq. (323), the number operator in the Fock space of $|\phi>$ may be defined as

$$N' = \int \frac{d^3\boldsymbol{k}}{2\omega}\left[a^\dagger_3(\boldsymbol{k})a_3(\boldsymbol{k}) - a^\dagger_0(\boldsymbol{k})a_0(\boldsymbol{k})\right] \tag{326}$$

[25]Since A_μ and π^μ are canonically conjugate to each other, we write

$$[A_\mu(\boldsymbol{x},t), \pi^\nu(\boldsymbol{y},t)] = i\delta^\nu_\mu\delta^3(\boldsymbol{x}-\boldsymbol{y})\,.$$

Uprising the subscript of A_μ, one obtain Eq. (322).

and the condition (325) requires

$$
\begin{aligned}
<\phi'|\, N' \,|\phi> &= <\phi'| \left\{ \int \frac{d^3\boldsymbol{k}}{2\omega} \left[a_3^\dagger(\boldsymbol{k}) - a_0^\dagger(\boldsymbol{k}) \right] a_3(\boldsymbol{k}) \,|\phi> \right\} \\
&= \int \frac{d^3\boldsymbol{k}}{2\omega} \left\{ <\phi'| \left[a_3^\dagger(\boldsymbol{k}) - a_0^\dagger(\boldsymbol{k}) \right] \right\} a_3(\boldsymbol{k}) \,|\phi> \\
&= 0
\end{aligned}
\tag{327}
$$

Thus, $<\phi'|\,\phi>= 0$ unless both of $|\phi>$ and $|\phi'>$ are eigenstates of N' belonging to the eigenvalue 0, namely, the vacuum state $|0>_L$ of timelike and longitudinal photons. This also mean $||\,|\phi>|| = 0$ when the state $|\phi>$ involves more than one timelike or longitudinal photon. Such states will never contribute to expectation values of gauge invariant physical observables [26]and we may just disregard them by imposing $|\phi>\equiv |0>_L$.

The propagator in the Feynman gauge is evaluated from Eqs. (317) and (323) with a help of Eq. (323) as

$$
<0|\, T[A_\mu(x)A_\nu(y)\,|0>= \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2 + \imath\epsilon} e^{-ik\cdot(x-y)} \tag{328}
$$

In the course of our discussion, we could deal $\eta = 1/\alpha$ in Eq. (291) more generally as an arbitrary fixed number. Such a way of fixing a gauge is called the covariant α gauge. In this gauge, the photon propagator is written as [27]

$$
D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + \imath\epsilon} \left(g^{\mu\nu} + (\alpha - 1) \frac{k^\mu k^\nu}{k^2 + i\epsilon} \right) e^{-ik\cdot(x-y)} \tag{330}
$$

Physical results should not depend on the choice of the value of α and one may choose it so that computations get simple ,or, leave it arbitrary so that one can check the correctness of computations.

---

[26]Such a quantity $\mathcal{O}$ will behave as

$$
\mathcal{O} \sim \sum_{i=1}^{3} a_i^\dagger(\boldsymbol{k}) a_i(\boldsymbol{k}) - a_0^\dagger(\boldsymbol{k}) a_0(\boldsymbol{k})
$$

and the longitudinal and timelike photons cancel among themselves for

$$
<\phi|\, a_3^\dagger a_3 \,|\phi>=<\phi|\, a_0^\dagger a_0 \,|\phi>
$$

is ensured by the condition (325).

[27]It is straightforward to observe

$$
\left[\partial^2 g^{\mu\nu} - (1 - 1/\alpha)\, \partial^\mu \partial^\nu \right] D_{F\nu\rho}(x) = i g_\rho^\nu \delta^4(x) \tag{329}
$$

showing $D_F^{\mu\nu}$ is certainly the Green's function of Eq. (291) .

8.3 Interacting Photon Field

Let us consider QED, namely the photon field interacting with the massive Dirac field. We write the Lagrangian density as

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{F^2} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}}\,, \tag{331}$$

where

$$\mathcal{L}_{F^2} = -\frac{1}{4}F^2\,, \quad \mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\partial\!\!\!/ - m)\psi\,. \tag{332}$$

To determine $\mathcal{L}_{\text{int}}$, we require that $\mathcal{L}_{\text{QED}}$ is invariant under the following gauge transformations:

$$\psi(x) \mapsto \psi'(x) = e^{-ie\lambda(x)}\psi(x)\,, \quad \psi^\dagger(x) \mapsto \psi^{\dagger\prime}(x) = \psi^\dagger(x)e^{ie\lambda(x)}\,, \tag{333}$$

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu\lambda(x)\,, \tag{334}$$

Due to the locality of the transformation, $\mathcal{L}_{\text{Dirac}}$ is not invariant since,

$$\overline{\psi}\partial_\mu\psi \mapsto \overline{\psi}'\partial_\mu\psi' = \overline{\psi}\left(\partial_\mu - ie(\partial_\mu\lambda)\right)\psi \neq \overline{\psi}\left(\partial_\mu\right))\,\psi \tag{335}$$

We may make this differential bilinear form invariant by replacing the derivative by a covariant derivative, which is defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu\,. \tag{336}$$

We surely find

$$\overline{\psi}D_\mu\psi \mapsto \overline{\psi}'D'_\mu\psi' = \overline{\psi}D_\mu\psi\,,$$

for

$$D'_\mu = D_\mu + ie\partial_\mu\lambda(x)\,. \tag{337}$$

This substitution of a partial derivative by the covariant derivative is called minimum substitution. Applying the substitution in $\mathcal{L}_{\text{Dirac}}$, we obtain

$$\overline{\psi}(iD\!\!\!\!/ - m)\psi = \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi\,. \tag{338}$$

Thus, in Eq. (331),

$$\mathcal{L}_{\text{int}} = -e\overline{\psi}A\!\!\!/\psi \tag{339}$$

gives a gauge invariant $\mathcal{L}_{\text{QED}}$ which we were seeking for. This form of $\mathcal{L}_{\text{int}}$ is called the minimal coupling. Now we have an explicit form of the full Lagrangian density. However, we already know it is not solvable because of the lack of the photon propagator. Therefore we introduce a gauge fixing term which violates the gauge invariance but it will be recovered for all physically meaningful quantities. We write our Lagrangian density as

$$\begin{aligned}\mathcal{L}_{\text{QED, GF}} &= \mathcal{L}_{F^2} + \mathcal{L}_{GF} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}}\,, \\ &= -\frac{1}{4}F^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi\,. \end{aligned} \tag{340}$$

Identifying $\mathcal{L}_{int}$ as in Eq. (339), the photon and Dirac fields in the interaction picture obey free field equations for which Feynman propagators are given by Eq. (251) and Eq. (330), respectively.

8.4 Scattering Amplitudes

In the following computations, we will use the Feynman gauge ($\alpha = 1$).

$e^-e^- \to e^-e^-$ (Møller scattering)
Initial and final states:

$$\begin{aligned}|i\!> &= c^\dagger_{r_a}(\boldsymbol{p}_a)c^\dagger_{r_b}(\boldsymbol{p}_b)\,|0\!> \stackrel{\rightarrow}{=} |e^-_a, e^-_b> \\ |f\!> &= c^\dagger_{r_1}(\boldsymbol{p}_1)c^\dagger_{r_2}(\boldsymbol{p}_2)\,|0\!> \stackrel{\rightarrow}{=} |e^-_1, e^-_2> \end{aligned} \tag{341}$$

We use notations used in 7.3 to write the second order S-matrix as

$$\begin{aligned}<\!f|\,S^{(2)}\,|i\!> &= \frac{(-i)^2}{2}\int d^4x_{1'}d^4x_{2'}\;\;<\!f|\,T[\mathcal{H}_{int}(x'_1)\mathcal{H}_{int}(x'_2)]\,|i\!> \\ &= \frac{(-ie)^2}{2}\int d^4x'_1 d^4x'_2 \\ &\quad < e^-_2, e^-_1|\,T\left[:\overline{\psi}_{1'}A\!\!\!/_{1'}\psi_{1'}::\overline{\psi}_{2'}A\!\!\!/_{2'}\psi_{2'}:\right]\,|e^-_a e^-_b> \\ &= \frac{(-ie)^2}{2}\int d^4x_{1'}d^4x_{2'}\;<\!0|\,T\left[A^\mu_{1'}A^\nu_{2'}\right]\,|0\!> \\ &\quad < e^-_2, e^-_1|\,T\left[:\overline{\psi}_{1'}\gamma_\mu\psi_{1'}::\overline{\psi}_{2'}\gamma_\nu\psi_{2'}:\right]\,|e^-_a e^-_b> \\ &= \frac{(-ie)^2}{2}\int d^4x_{1'}d^4x_{2'}D^{\mu\nu}_F(x_{1'}-x_{2'})(\gamma_\mu)_{\alpha\beta}(\gamma_\nu)_{\gamma\delta} \\ &\quad < e^-_2, e^-_1|\,T\left[:\overline{\psi}_{1'\alpha}\psi_{1'\beta}::\overline{\psi}_{2'\gamma}\psi_{2'\delta}:\right]\,|e^-_a e^-_b> \end{aligned} \tag{342}$$

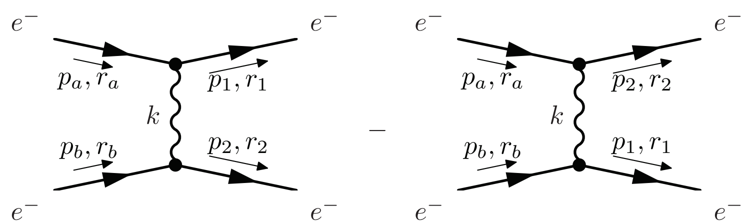
$$\begin{aligned}&< e^-_2, e^-_1|\,T\left[:\overline{\psi}_{1'\alpha}\psi_{1'\beta}::\overline{\psi}_{2'\gamma}\psi_{2'\delta}:\right]\,|e^-_a e^-_b> \\ &=< e^-_2, e^-_1|\,:\overline{\psi}^{(+)}_{1'\alpha}\,\psi^{(+)}_{1'\beta}\,\overline{\psi}^{(+)}_{2'\gamma}\,\psi^{(+)}_{2'\delta}:|e^-_a e^-_b> \\ &=< e^-_2, e^-_1|\,\overline{\psi}^{(+)}_{1'\alpha}\,\overline{\psi}^{(+)}_{2'\gamma}\,\psi^{(+)}_{2'\delta}\,\psi^{(+)}_{1'\beta}\,|e^-_a e^-_b> \\ &=< 0|\,c_2c_1\left\{c^\dagger_{1'(1')}c^\dagger_{2'(2')}c_{2'(b')}c_{1'(a')}\right\}c^\dagger_a c^\dagger_b\,|0\!>\,\bar{u}^{(1')}_\alpha\bar{u}^{(2')}_\gamma u^{(b')}_\delta u^{(a')}_\beta \end{aligned} \tag{343}$$

The last expression can be compared with Eq. (259). Proceeding further we obtain an expression quite similar with Eq. (264):

$$(343) \;=\; \frac{1}{(2\pi)^6} \sum_{(1')(2')(a')(b')} \left\{ (\delta_{(1')1}\delta_{(2')2} - \delta_{2(1')}\delta_{1(2')})(\delta_{(a')a}\delta_{(b')b} - \delta_{(a')b}\delta_{(b')a}) \right\} \times$$
$$e^{i(p_{1'}x_{1'}+p_{2'}x_{2'}-p_{a'}x_{1'}-p_{b'}x_{2'})}\bar{u}_\alpha^{(1')} u_\beta^{(a')} \bar{u}_\gamma^{(2')} u_\delta^{(b')} \tag{344}$$

In 7.3, we have made use of a fact that boson propagator $\Delta_F(x)$ is an even function. $D_F^{\mu\nu}(x)$ in Eq. (342) is also even and symmetric about suffices μ and ν as it is observed in Eq. (330). Setting $\alpha = 1$, we have

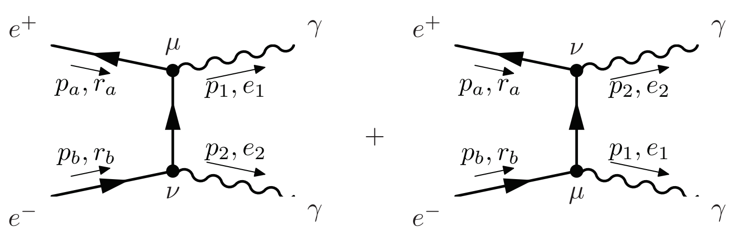
$$S_{fi}^{(2)} \;=\; \frac{(-ie)^2}{(2\pi)^6} \int d^4k \frac{-ig^{\mu\nu}(2\pi)^4}{k^2 + \imath\epsilon}$$
$$\left\{ \delta^4(p_1 - p_a - k)\delta^4(p_2 - p_b + k) \left[\bar{u}^{(1)}\gamma_\mu u^{(a)}\right] \left[\bar{u}^{(2)}\gamma_\nu u^{(b)}\right] \right.$$
$$\left. -\delta^4(p_2 - p_a - k)\delta^4(p_1 - p_b + k) \left[\bar{u}^{(2)}\gamma_\mu u^{(a)}\right] \left[\bar{u}^{(1)}\gamma_\nu u^{(b)}\right] \right\}$$
$$=$$



(345)

$$T_{fi}^{(2)} = -\frac{(-ig)^2}{(2\pi)^6} \left\{ \frac{\left[\bar{u}^{(1)}\gamma_\mu u^{(a)}\right]\left[\bar{u}^{(2)}\gamma^\mu u^{(b)}\right]}{(p_1 - p_a)^2 + i\epsilon} - \frac{\left[\bar{u}^{(2)}\gamma_\mu u^{(a)}\right]\left[\bar{u}^{(1)}\gamma^\mu u^{(b)}\right]}{(p_2 - p_a)^2 + i\epsilon} \right\} \tag{346}$$

<u>$e^+e^- \to \gamma\gamma$ (Annihilation)</u>

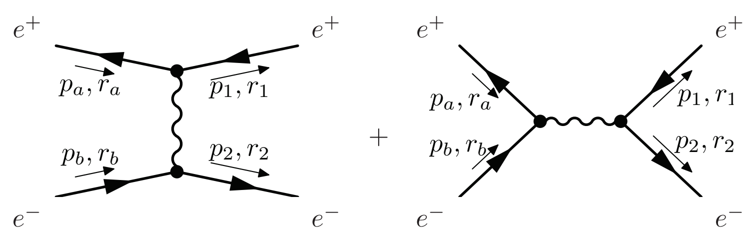


$$= \frac{(-ie)^2}{(2\pi)^6} \left[\frac{\bar{v}^{(a)} \not{e}_2 (\not{p}_a - \not{p}_1 + m) \not{e}_1 u^{(b)}}{(p_a - p_1)^2 - m^2 + i\epsilon} + \frac{\bar{v}^{(a)} \not{e}_1 (\not{p}_a - \not{p}_2 + m) \not{e}_2 u^{(b)}}{(p_a - p_2)^2 - m^2 + i\epsilon} \right] \tag{347}$$

Feynman rules for Photons coupled with Dirac charge e

1. At each vertex, associate $-ie\gamma_\mu$.
2. For each internal photon line, associate
$$-\frac{ig^{\mu\nu}}{k^2+i\epsilon}$$
3. Specify a polarization vector e for each external photons.

$e^+e^- \to e^+e^-$ (Bhabha scattering)



$$= \quad -\frac{(-ie)^2}{(2\pi)^6}\left(-\frac{\left[\bar{v}^{(1)}\gamma^\mu v^{(a)}\right]\left[\bar{u}^{(2)}\gamma_\mu u^{(b)}\right]}{(p_a-p_1)^2+i\epsilon}+\frac{\left[\bar{v}^{(a)}\gamma^\mu u^{(b)}\right]\left[\bar{v}^{(1)}\gamma_\mu u^{(2)}\right]}{(p_a+p_b)^2+i\epsilon}\right) \tag{348}$$

$e^-\gamma \to e^-\gamma$ (Compton scattering)
* add text here

9 Massive Vector Fields

neutral or charged Spin 1 particle $(\Box + m^2)\,\varphi_\mu(x) = 0$

9.0.1 Lagrangian

For the complex field, terms which can appear in Lagrangian density are $\varphi^*_\mu \varphi^\mu$, $\partial_\mu \varphi^*_\nu \cdot \partial^\mu \varphi^\nu$, $\partial_\mu \varphi^*_\nu \cdot \partial^\nu \varphi^\mu$ and $\partial^\mu \varphi^*_\mu \cdot \partial_\nu \varphi^\mu$. We want Klein-Gordon equation as the EOM.

Proca (1936) proposed a Lagrangian density which leads to a positive definite Hamiltonian density:

$$\begin{aligned} \mathcal{L}_{\text{Pr}} &= -\frac{1}{2} F^*_{\mu\nu} F^{\mu\nu} + m^2 \varphi^*_\mu \varphi^\mu \\ & \quad (= -\partial_\mu \varphi^*_\nu F^{\mu\nu} = -F^*_{\mu\nu} \partial^\mu \varphi^\nu)\,, \end{aligned} \tag{349}$$

where

$$F_{\mu\nu} \equiv \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu \tag{350}$$

$\dim[\varphi] = \dim[m] = E$. For the case of real field, remove conjugation symbol * and divide each term by 2.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*_\nu)} = -F^{\mu\nu}\,, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\nu)} = -F^{*\mu\nu} \tag{351}$$

Euler-Lagrange equation reads

$$\partial_\mu F^{\mu\nu} = -m^2 \varphi^\nu\,, \qquad \partial^\mu F^*_{\mu\nu} = -m^2 \varphi^*_\nu \tag{352}$$

Taking divergence in both hand sides, we get the Lorentz condition

$$\partial_\nu \varphi^\nu = 0\,, \qquad \partial^\nu \varphi^*_\nu = 0 \tag{353}$$

when $m^2 \neq 0$. This condition suppresses irrelevant degrees of freedom in φ_μ and φ^*_μ by one for each and these fields come to describe spin 1 fields correctly. Owing to this condition, field equations in Eq. (352) become a pair of Klein-Goldon equations. Conjugate momenta are given as

$$\varphi^*_\mu \longleftrightarrow \pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^*_\mu} = F^{\mu 0}\,, \qquad \varphi_\mu \longleftrightarrow \pi^{*\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\mu} = F^{*\mu 0} \tag{354}$$

One observes that $\pi^0 = \pi^{*0} = 0$ as it was happen in the electromagnetism. In this case, however, one can make use of Eqs. (352) and (354) to solve φ^0 and φ^{*0} in terms of other fields as [28]

$$\varphi^0 = -\frac{\partial_\mu \pi^\mu}{m^2}\,, \qquad \varphi^{*0} = -\frac{\partial_\mu \pi^{*\mu}}{m^2} \tag{355}$$

[28] Time derivatives of fields are expressed in terms of π and π^* from Eq. (354) as

$$\partial_0 \varphi^i = -\pi^i - \partial_i \varphi^0\,, \qquad \partial_0 \varphi^{*i} = -\pi^{*i} - \partial_i \varphi^{*0}$$

For the Lagrangian density (349) can be written as

$$\mathcal{L}_{\rm Pr} \;=\; -\partial_\mu\varphi^*_\nu\cdot\partial^\mu\varphi^\nu + m^2\varphi^*_\mu\varphi^\mu - \varphi^*_\nu\partial^\nu\partial_\mu\varphi^\mu + \partial_\mu\{\varphi^*_\nu\partial^\nu\varphi^\mu\}\,, \quad (356)$$

we can also employ

$$\mathcal{L}_{\rm KG} \;=\; -\partial_\mu\varphi^*_\nu\cdot\partial^\mu\varphi^\nu + m^2|\varphi|^2 \quad (357)$$

as our Lagrangian density together with the Lorentz condition (353) as a subsidiary condition. In this case, we have Klein-Gordon equations as EOM's and non-vanishing π^0 as we have already seen in the electromagnetism.

In the following, however, we employ Stueckelberg's Lagrangian given by

$$\mathcal{L}_{\rm St} \;=\; \mathcal{L}_{\rm Pr} - \frac{1}{\alpha}(\partial\cdot\varphi^*)(\partial\cdot\varphi) \quad (358)$$

which includes a "gauge fixing term" although the gauge invariance is already violated by the mass term in $\mathcal{L}_{\rm Pr}$. The involvement of this term is said [9] necessary for the theory to be renormalizable. It reads

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*_\nu)} = -F^{\mu\nu} - \frac{1}{\alpha}g^{\mu\nu}(\partial\cdot\varphi) \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\nu)} = -F^{*\mu\nu} - \frac{1}{\alpha}g^{\mu\nu}(\partial\cdot\varphi^*)\,. \quad (359)$$

Field equations become [29]

$$\begin{aligned}(\Box + m^2)\varphi^\mu + \left(\frac{1}{\alpha} - 1\right)\partial^\mu(\partial\cdot\varphi) = 0 \\ (\Box + m^2)\varphi^{*\mu} + \left(\frac{1}{\alpha} - 1\right)\partial^\mu(\partial\cdot\varphi^*) = 0\end{aligned} \quad (362)$$

The Hamiltonian density is then written as

$$\begin{aligned}\mathcal{H} &= \pi^\mu\dot\varphi^*_\mu + \pi^*_\mu\dot\varphi^\mu - \mathcal{L} \\ &= |\boldsymbol{\pi}|^2 + \frac{|\boldsymbol{\partial}\cdot\boldsymbol{\pi}|^2}{m^2} + |\boldsymbol{\partial}\times\boldsymbol{\varphi}|^2 + m^2|\boldsymbol{\varphi}|^2 + \text{total derivative}\end{aligned}$$

which is obviously positive definite.

[29]Taking the divergence of the field equation (362), we have

$$\frac{1}{\alpha}\left[\Box + \alpha m^2\right](\partial\cdot\varphi) = 0\,, \qquad \text{(c.c.)}\,. \quad (360)$$

Namely, the field $(\partial\cdot\varphi)$ is a Klein-Gordon field with mass squared αm^2. We assume $\alpha > 0$ in the following.

It would be also noted that the field

$$\varphi^T_\mu = \left(g_{\mu\nu} + \frac{1}{\alpha m^2}\partial_\mu\partial_\nu\right)\varphi^\nu \quad (361)$$

extracts the component of spin 1 from a mixture φ^ν composed of spin 1 and spin 0 components. This tansverse component satisfies $\partial\cdot\varphi^T = 0$ by virtue of Eq. (360).

and conjugate momenta are now written as

$$\begin{aligned}
\varphi^*_\mu \longleftrightarrow \pi^\mu &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^*_\mu} = F^{\mu 0} - \delta^{\mu 0}\frac{1}{\alpha}(\partial\cdot\varphi) \\
\varphi_\mu \longleftrightarrow \pi^{*\mu} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\mu} = F^{*\mu 0} - \delta^{\mu 0}\frac{1}{\alpha}(\partial\cdot\varphi^*)
\end{aligned} \tag{363}$$

Though fields φ^μ and $\varphi^{*\mu}$ are independent of each other, expression for one is obvious from another and we omit to write expressions for the later in the following. The solution to the Stueckelberg's field equation (362) is writen as

$$\begin{aligned}
\varphi_\mu(x) &= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}} \sum_{\lambda=0}^{3} \frac{1}{2p^0_{(\lambda)}} \Big[a_\lambda(\boldsymbol{p})\,\tilde{\epsilon}^{(\lambda)}_\mu(\boldsymbol{p}) e^{-ip_{(\lambda)}x} \\
&\qquad\qquad + b^\dagger_\lambda(\boldsymbol{p})\,\tilde{\epsilon}^{(\lambda)*}_\mu(\boldsymbol{p}) e^{ip_{(\lambda)}x} \Big] \\
&= \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}} \left(\sum_{\lambda=1}^{3} \frac{1}{2p^0} \Big[a_\lambda\,\tilde{\epsilon}^{(\lambda)}_\mu e^{-ipx} + b^\dagger_\lambda\,\tilde{\epsilon}^{(\lambda)*}_\mu e^{ipx} \Big] \right. \\
&\qquad\qquad \left. + \frac{1}{2p^0_\alpha} \Big[a_0\,\tilde{\epsilon}^{(0)}_\mu e^{-ip_\alpha x} + b^\dagger_0\,\tilde{\epsilon}^{(0)*}_\mu e^{ip_\alpha x} \Big] \right)
\end{aligned} \tag{364}$$

where

$$p_{(\lambda)} = \begin{cases} p \equiv (p^0, \boldsymbol{p})\,, \quad p^0 \equiv \sqrt{\boldsymbol{p}^2 + m^2}\,, \quad p^2 = m^2 \quad \text{for } \lambda = 1,2,3 \\ p_\alpha \equiv (p^0_\alpha, \boldsymbol{p})\,, \quad p^0_\alpha \equiv \sqrt{\boldsymbol{p}^2 + \alpha m^2}\,, \quad p^2_\alpha = \alpha m^2 \quad \text{for } \lambda = 0 \end{cases} \tag{365}$$

and polarization vectors $\tilde{\epsilon}^{(\lambda)}$ are written as

$$\begin{cases} \tilde{\epsilon}^{(0)}_\mu = p_{\alpha\mu}/m \\ \tilde{\epsilon}^{(i)}_\mu = \Lambda^\mu{}_\nu(\boldsymbol{\beta}) e^{(i)\mu}\,, \quad i = 1,2,3 \end{cases} \tag{366}$$

For notations of the second equation, we refer to Appendix E.

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Addendum: Solution of the Stueckelberg's field equation

Let us confirm that the expression (364) satisfies the field equation (362). First, we prepare components:

$$\begin{aligned}
\partial\cdot\varphi &= -i\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\sum_{\lambda=0}^{3}\frac{1}{2p^0_{(\lambda)}}\left[a_\lambda\,(\tilde{\epsilon}^{(\lambda)}\cdot p_{(\lambda)})e^{-ip_{(\lambda)}x}-b^\dagger_\lambda\,(\tilde{\epsilon}^{(\lambda)*}\cdot p_{(\lambda)})e^{ip_{(\lambda)}x}\right]\\
&= -i\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\left(\sum_{\lambda=1}^{3}\frac{1}{2p^0}\left[a_\lambda\,(\tilde{\epsilon}^{(\lambda)}\cdot p)e^{-ipx}-b^\dagger_\lambda\,(\tilde{\epsilon}^{(\lambda)*}\cdot p)e^{ipx}\right]\right.\\
&\qquad\left.+\frac{1}{2p^0_\alpha}\left[a_0\,(\tilde{\epsilon}^{(0)}\cdot p_\alpha)e^{-ip_\alpha x}-b^\dagger_0\,(\tilde{\epsilon}^{(0)*}\cdot p_\alpha)e^{ip_\alpha x}\right]\right)
\end{aligned}\tag{367}$$

$$\partial^\mu(\partial\cdot\varphi) = -\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\sum_{\lambda=0}^{3}\frac{p^\mu_{(\lambda)}}{2p^0_{(\lambda)}}\left[a_\lambda\,(\tilde{\epsilon}^{(\lambda)}\cdot p_{(\lambda)})e^{-ip_{(\lambda)}x}+b^\dagger_\lambda\,(\tilde{\epsilon}^{(\lambda)*}\cdot p_{(\lambda)})e^{ip_{(\lambda)}x}\right]\tag{368}$$

$$\begin{aligned}
\Box(\partial\cdot\varphi) &= i\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\sum_{\lambda=0}^{3}\frac{p^2_{(\lambda)}}{2p^0_{(\lambda)}}\left[a_\lambda\,(\tilde{\epsilon}^{(\lambda)}\cdot p_{(\lambda)})e^{-ip_{(\lambda)}x}-b^\dagger_\lambda\,(\tilde{\epsilon}^{(\lambda)*}\cdot p_{(\lambda)})e^{ip_{(\lambda)}x}\right]\\
&= i\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\left(\sum_{\lambda=1}^{3}\frac{m^2}{2p^0}\left[a_\lambda\,(\tilde{\epsilon}^{(\lambda)}\cdot p)e^{-ipx}-b^\dagger_\lambda\,(\tilde{\epsilon}^{(\lambda)*}\cdot p)e^{ipx}\right]\right.\\
&\qquad\left.+\frac{\alpha m^2}{2p^0_\alpha}\left[a_0\,(\tilde{\epsilon}^{(0)}\cdot p_\alpha)e^{-ip_\alpha x}-b^\dagger_0\,(\tilde{\epsilon}^{(0)*}\cdot p_\alpha)e^{ip_\alpha x}\right]\right)
\end{aligned}\tag{369}$$

Second, we evaluate the *l.h.s.* of Eq. (362):

$$\begin{aligned}
&(\Box+m^2)\varphi_\mu+\left(\frac{1}{\alpha}-1\right)\partial_\mu(\partial\cdot\varphi)\\
&=-\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\sum_{\lambda=0}^{3}\frac{1}{2p^0_{(\lambda)}}\left[a_\lambda\left\{(p^2_{(\lambda)}-m^2)\tilde{\epsilon}^{(\lambda)}_\mu+(\frac{1}{\alpha}-1)p_{(\lambda)\mu}(\tilde{\epsilon}^{(\lambda)}\cdot p_{(\lambda)})\right\}e^{-ip_{(\lambda)}x}\right.\\
&\qquad\left.+b^\dagger_\lambda\left\{(p^2_{(\lambda)}-m^2)\tilde{\epsilon}^{(\lambda)*}_\mu+(\frac{1}{\alpha}-1)p_{(\lambda)\mu}(\tilde{\epsilon}^{(\lambda)*}\cdot p_{(\lambda)})\right\}e^{ip_{(\lambda)}x}\right]\\
&=-\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}}\left(\sum_{l=1}^{3}\frac{1}{2p^0}\left[a_l(\frac{1}{\alpha}-1)p_\mu(\tilde{\epsilon}^{(l)}\cdot p)e^{-ipx}+b^\dagger_l(\frac{1}{\alpha}-1)p_\mu(\tilde{\epsilon}^{(l)*}\cdot p)e^{ipx}\right]\right.\\
&\qquad+\frac{1}{2p^0_\alpha}\left[a_0(\alpha-1)\left\{m^2\tilde{\epsilon}^{(0)}_\mu-\frac{1}{\alpha}p_{\alpha\mu}(\tilde{\epsilon}^{(0)}\cdot p_\alpha)\right\}e^{-ip_\alpha x}\right.\\
&\qquad\left.\left.+b^\dagger_0(\alpha-1)\left\{m^2\tilde{\epsilon}^{(0)*}_\mu-\frac{1}{\alpha}p_{\alpha\mu}(\tilde{\epsilon}^{(0)*}\cdot p_\alpha)\right\}e^{ip_\alpha x}\right]\right)
\end{aligned}\tag{370}$$

In the first line of the last expression, we have from Eq. (366) that $(\tilde{\epsilon}^{(l)}\cdot p)=(\tilde{\epsilon}^{(l)*}\cdot p)=0$ and $(\tilde{\epsilon}^{(0)}\cdot p_\alpha)=(\tilde{\epsilon}^{(0)*}\cdot p_\alpha)=\alpha m$ in the second and third lines

so that the last expression vanishes as the whole. Thus, we have shown that the expression (364) together with Eqs. (365, 366) surely satisfies Stueckelberg's field equation (362).
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We quantize the field (364) by requiring canonical commutation relations which reads

$$\begin{aligned} &[a_\lambda(\boldsymbol{p}), a^\dagger_{\lambda'}(\boldsymbol{p}')] = [b_\lambda(\boldsymbol{p}), b^\dagger_{\lambda'}(\boldsymbol{p}')] = -g_{\lambda\lambda'}\, 2p^0_{(\lambda)}\, \delta^3(\boldsymbol{p}-\boldsymbol{p}) \\ &\text{others} = 0 \end{aligned} \tag{371}$$

We are now at a position to compute the propagator.

$$\begin{aligned} &<\!0|\, \varphi_\mu(x)\varphi^\dagger_\nu(y)\, |0\!>\big|_{x^0>y^0} \\ &= \int \frac{d^3\boldsymbol{p} d^3\boldsymbol{p}'}{(2\pi)^3} \sum_{\lambda,\lambda'=0}^{3} \frac{1}{4p^0_{(\lambda)}p^0_{(\lambda')}} <\!0|\, a_\lambda(\boldsymbol{p})\tilde{\epsilon}^{(\lambda)}_\mu(\boldsymbol{p})e^{-ip_{(\lambda)}x} \\ &\qquad\qquad a^\dagger_{\lambda'}(\boldsymbol{p}')\tilde{\epsilon}^{(\lambda')}_\nu(\boldsymbol{p}')e^{ip'_{(\lambda')}y}\, |0\!> \\ &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_{\lambda,\lambda'=0}^{3} \frac{-g_{\lambda\lambda'}}{2p^0_{(\lambda)}} \tilde{\epsilon}^{(\lambda)}_\mu(\boldsymbol{p})\tilde{\epsilon}^{(\lambda')}_\nu(\boldsymbol{p})e^{-ip_{(\lambda)}(x-y)} \\ &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \left(\frac{-1}{2p^0}\left[g_{\mu\nu} - g_{00}\frac{p_\mu p_\nu}{m^2}\right] e^{-ip(x-y)} - g_{00}\frac{p_{\alpha\mu}p_{\alpha\nu}/m^2}{2p^0_\alpha} e^{-p_\alpha(x-y)} \right) \end{aligned} \tag{372}$$

and

$$\begin{aligned} D_F^{\mu\nu}(x-y) &\stackrel{\leftarrow}{=} <\!0|\, T\left[\varphi_\mu(x)\varphi^\dagger_\nu(y)\right]\, |0\!> \\ &= -i\int \frac{d^4p}{(2\pi)^4}\left(\frac{g_{\mu\nu} - p_\mu p_\nu/m^2}{p^2-m^2+i\epsilon} + \frac{p_\mu p_\nu/m^2}{p^2-\alpha m^2+i\epsilon} \right) e^{-ip(x-y)} \end{aligned} \tag{373}$$

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10 Derivative Couplings

So far, we are evaluating S-matrix elements making use of Dyson's formula (160) where interaction Hamiltonian density in the interaction picture is denoted just as $\mathcal{H}_{int}$. This is however a shorthand for $(\mathcal{H}_{int})_I$ if we faithfully follow notations used in Eqs. (93) and (84). This shorthand had been allowed since $(\mathcal{H}_{int})_I$ could be obtained just by replacing Heisenberg operator fields involved in $\mathcal{H}_{int} = -\mathcal{L}_{int}$ by ones in the interaction picture. However, this blessed situation relies on a fact that we have been considering only $\mathcal{L}_{int}$ that does not involve derivatives of fields. Let us consider a theory with field $\varphi(x)$ described by $\mathcal{L} = \mathcal{L}_0(\varphi, \dot{\varphi}) + \mathcal{L}_{int}$, where we omit spatial derivatives of φ included in $\mathcal{L}$. When $\mathcal{L}_{int}$ does not involve $\dot{\varphi}$, the conjugate momentum is given by $\pi = \partial\mathcal{L}_0/\partial\dot{\varphi}$ and the Hamiltonian density reads $\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} = \mathcal{H}_0 - \mathcal{L}_{int}$, where $\mathcal{H}_0 = \dot{\varphi}\partial\mathcal{L}_0/\partial\dot{\varphi} - \mathcal{L}_0$. Thus we may write $\mathcal{H}_{int} = -\mathcal{L}_{int}(\varphi)$ and understand that $(\mathcal{H}_{int})_I = -\mathcal{L}_{int}(\varphi_I)$. This is the situation we have been engaged so far. For derivative couplings at play, we have $\mathcal{L}_{int} = \mathcal{L}_{int}(\varphi, \dot{\varphi})$ and it turns out [30]that $(\mathcal{H}_{int})_I = -\mathcal{L}_{int}(\varphi_I, \dot{\varphi}_I) + (1/2)\left(\partial\mathcal{L}_{int}/\partial\dot{\varphi}_I\right)^2$ for systems with $\partial\mathcal{L}_0/\partial\dot{\varphi} = \dot{\varphi}$ and $\mathcal{L}_{int}$ depending only linearly on $\dot{\varphi}$. The appearance of the second term in $(\mathcal{H}_{int})_I$ is rather messy.

There is another problem caused by the presence of derivative couplings. In evaluating S-matrix, we will meet vacuum expectation value of the T-product of derivatives of fields. It turns out [31]that

$$
\begin{aligned}
<0|\, T\left[\partial_\mu\varphi(x)\partial_\nu\varphi(y)\right] |0> &= \partial_{x^\mu}\partial_{y^\nu} <0|\, T\left[\varphi(x)\varphi(y)\right] |0> \\
&\quad -i\delta_{\mu 0}\delta_{\nu 0}\delta^4(x-y)
\end{aligned}
\tag{374}
$$

The second term in the *r.h.s* is, again, rather messy. It turns out, however, that the above mentioned two problems cancel each other resulting in a simple prescription. Namely, we may keep to write $\mathcal{H}_{int} = -\mathcal{L}_{int}(\varphi_I, \partial_\mu\varphi_I)$ in Eq. (160) with a compensation that we must replace T products by T* products defined by

$$
T^*\left[\partial_\mu\varphi(x)\partial_\nu\varphi(y)\right] \equiv \partial_{x^\mu}\partial_{y^\nu} T\left[\varphi(x)\varphi(y)\right] \tag{375}
$$

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Addendum1. Effect of derivative couplings on the form of $(\mathcal{H}_{int})_I$

Field operators $\varphi_I(x)$ in the interaction picture follow free field equations corresponding to a Lagrangian density $\mathcal{L}_0(\varphi_I(x), \dot{\varphi}_I(x))$. The momentum field conjugate to $\varphi_I(x)$ is given as $\pi_I(x) = \partial\mathcal{L}_0/\partial\dot{\varphi}_I(x)$ and the free Hamiltonian density in

[30]See addendum 1 below which follows ref. [8].

[31]See addendum 2 below.

this picture $(\mathcal{H}_0)_I = \pi_I \dot{\varphi}_I - \mathcal{L}_0(\varphi_I(x), \dot{\varphi}_I(x))$ govern the time evolution of $\varphi_I(x)$. As we have discussed in 6.2, operators in the interaction picture $\mathcal{O}_I$ are related with ones in the Heisenberg picture, $\mathcal{O}_H$, by an unitary time evolution operator U as $\mathcal{O}_I = U\mathcal{O}_H U^{-1}$.

Writing the full Lagrangian density as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}(\varphi(x), \dot{\varphi}(x))$, the conjugate momentum is given by $\pi = \partial\mathcal{L}_0/\partial\dot{\varphi} + \partial\mathcal{L}_{int}/\partial\dot{\varphi}$. The second term contributes when derivative couplings are involved in $\mathcal{L}_{int}$. For simplicity, we consider the case such like as the scalar field where $\partial\mathcal{L}_0/\partial\dot{\varphi} = \dot{\varphi}$ so that $\pi = \dot{\varphi} + \partial\mathcal{L}_{int}/\partial\dot{\varphi}$ and $\pi_I = \dot{\varphi}_I$. The full Hamiltonian density is defined by $\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}(\varphi(x), \dot{\varphi}(x))$ and it is written in terms of Heisenberg fields. Converting $\mathcal{H}$ into the interaction picture as $\mathcal{H}_I = U\mathcal{H}U^{-1}$, one meets an object $U^{-1}\dot{\varphi}U \overset{\rightarrow}{=} (\dot{\varphi})_I$ which is different from $\dot{\varphi}_I = \partial_0(\varphi_I)$. We write

$$
\begin{aligned}
(\dot{\varphi})_I &= \pi_I - U\frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}}(\varphi, \dot{\varphi})U^{-1} \\
&= \dot{\varphi}_I - \frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I}(\varphi_I, (\dot{\varphi})_I) \\
&= \dot{\varphi}_I - \frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I}(\varphi_I, \dot{\varphi}_I) + \Delta_2
\end{aligned} \tag{376}
$$

where Δ_2 represents terms of the second and higher orders of $\mathcal{L}_{int}$ and it appears only when $\mathcal{L}_{int}$ involves quadratic and higher orders of $\dot{\varphi}$. The second term in $\mathcal{H}$ is converted as

$$
\begin{aligned}
&\mathcal{L}(\varphi_I, \dot{\varphi}_I - \frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I} + \Delta_2) \\
&= \mathcal{L}(\varphi_I, \dot{\varphi}_I) + \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_{I\alpha}}\left\{-\frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I} + \Delta_2\right\}_\alpha \\
&\qquad + \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial\dot{\varphi}_{I\alpha}\partial\dot{\varphi}_{I\beta}}\left\{-\frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I} + \Delta_2\right\}_\alpha \left\{-\frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_I} + \Delta_2\right\}_\beta
\end{aligned} \tag{377}
$$

where an extra suffix α or β is added on the field to correspond to cases of fields with multiple components. With our assumption on the form of $\mathcal{L}_0$, we have

$$
\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_{I\alpha}} = \pi_I + \frac{\partial\mathcal{L}_{int}}{\partial\dot{\varphi}_{I\alpha}}, \qquad \frac{\partial^2\mathcal{L}}{\partial\dot{\varphi}_{I\alpha}\partial\dot{\varphi}_{I\beta}} = \delta^{\alpha\beta} + \frac{\partial^2\mathcal{L}_{int}}{\partial\dot{\varphi}_{I\alpha}\partial\dot{\varphi}_{I\beta}} \tag{378}
$$

Then we write,

$$\begin{aligned}
\mathcal{H}_I &= \pi_I(\dot{\varphi})_I - \mathcal{L}(\varphi_I, (\dot{\varphi})_I) \\
&= \pi_I \left\{ \dot{\varphi}_I - \frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} + \Delta_2 \right\} - \mathcal{L}(\varphi_I, \dot{\varphi}_I) - \left\{ \pi_I + \frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} \right\}_\alpha \left\{ -\frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} + \Delta_2 \right\}_\alpha \\
&\quad -\frac{1}{2} \left\{ -\frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} + \Delta_2 \right\}^2 - \frac{\partial^2 \mathcal{L}_{int}}{\partial \dot{\varphi}_I^2} \times \mathcal{O}(\mathcal{L}_{int}^2) \\
&= \pi_I \dot{\varphi}_I - \mathcal{L}(\varphi_I, \dot{\varphi}_I) - \frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_{I\alpha}} \left\{ -\frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} + \Delta_2 \right\}_\alpha \\
&\quad -\frac{1}{2} \left\{ -\frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} + \Delta_2 \right\}^2 - \frac{\partial^2 \mathcal{L}_{int}}{\partial \dot{\varphi}_I^2} \times \mathcal{O}(\mathcal{L}_{int}^2) \\
&= (\mathcal{H}_0)_I - \mathcal{L}_{int}(\varphi_I, \dot{\varphi}_I) + \frac{1}{2} \left(\frac{\partial \mathcal{L}_{int}}{\partial \dot{\varphi}_I} \right)^2 - \left\{ \frac{(\Delta_2)^2}{2} + \frac{\partial^2 \mathcal{L}_{int}}{\partial \dot{\varphi}_I^2} \times \mathcal{O}(\mathcal{L}_{int}^2) \right\}
\end{aligned} \tag{379}$$

Thus, we confirm a statement on the form of $(\mathcal{H}_{int})_I$ as it is mentioned in the text.

Addendum2. Derivation of Eq. (374)

From the definition of T product in Eq. (88), we have

$$\partial_{x^\mu} T[\varphi(x)\varphi(y)] = T[\partial_\mu \varphi(x) \cdot \varphi(y)] + \delta(x^0 - y^0)[\varphi(x), \varphi(y)] \tag{380}$$

The second term in the *r.h.s.* vanishes in the scheme of canonical quantization. Thus, single derivative does not make any trouble. However,

$$\begin{aligned}
\partial_{y^\nu} T[\partial_\mu \varphi(x) \cdot \varphi(y)] &= T[\partial_\mu \varphi(x) \partial_\nu \varphi(y)] + \delta_{\nu 0} \delta(x^0 - y^0)[\varphi(y), \partial_\mu \varphi(x)] \\
&= T[\partial_\mu \varphi(x) \partial_\nu \varphi(y)] + i \delta_{\mu 0} \delta_{\nu 0} \delta^4(x - y)
\end{aligned} \tag{381}$$

and we obtain Eq. (374) by substituting the relationship (380) in the *l.h.s* and taking vacuum expectation values of the both hand sides.

-----------.-----------.-----------.-----------.-----------

Let us consider the Dirac Yukawa theory discussed in 7.3 with a derivative coupling $\mathcal{L}_{int} = -g\overline{\psi}\mathcal{O}^\mu \psi \partial_\mu \phi$, where $\mathcal{O}^\mu$ denotes a Dirac matrix such as γ^μ for vector current and $\gamma^5\gamma^\mu$ for pseudo vector current. For $NN \to NN$ scattering amplitude, we set the same initial and final states as in Eq. (254).

The second order S-matrix reads

$$
\begin{aligned}
S^{(2)}_{fi} &= \frac{(-ig)^2}{2}\int d^4x_{1'}d^4x_{2'} < N_2N_1|\, T^*[:\overline{\psi}_{1'}\mathcal{O}^\mu\psi_{1'}\partial_{x^\mu_{1'}}\phi_{1'}::\overline{\psi}_{2'}\mathcal{O}^\nu\psi_{2'}\partial_{x^\nu_{2'}}\phi_{2'}:]\,|N_aN_b>\\
&= \frac{(-ig)^2}{2}\int d^4x_{1'}d^4x_{2'} <0|\, T^*[\partial_{x^\mu_{1'}}\phi_{1'}\partial_{x^\nu_{2'}}\phi_{2'}]\,|0>\\
&\qquad\qquad\qquad < N_2N_1|\, T[:\overline{\psi}_{1'}\mathcal{O}^\mu\psi_{1'}::\overline{\psi}_{2'}\mathcal{O}^\nu\psi_{2'}:]\,|N_aN_b>
\end{aligned}
\tag{382}
$$

The last factor in the second equation is nothing but one in Eq. (256) with addional Dirac matrices $\mathcal{O}^\mu$ and $\mathcal{O}^\nu$ between bilinear forms of Dirac spinors and the computation of this factor goes along quite similar way as that through Eq. (259) to (264). For the factor of the vacuum expectation value, we write

$$
\begin{aligned}
<0|\, T^*[\partial_{1'\mu}\phi_{1'}\partial_{2'\nu}\phi_{2'}]\,|0> &= <0|\, \partial_{1'\mu}\partial_{2'\nu}T[\phi_{1'}\phi_{2'}]\,|0>\\
&= \partial_{1'\mu}\partial_{2'\nu}\Delta_F(1'-2')\\
&= i\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^2-\mu^2+i\epsilon}e^{-ik(x_{1'}-x_{2'})}
\end{aligned}
\tag{383}
$$

where we have used Eq. (164). Combining them, we would write

$$
\begin{aligned}
T^{(2)}_{fi} &= \frac{(-ig)^2}{(2\pi)^6}\int d^4k\frac{k_\mu k_\nu}{k^2-\mu^2+i\epsilon}\left\{\delta^4(p_1-p_a-k)\left[\bar{u}^{(1)}\mathcal{O}^\mu u^{(a)}\right]\left[\bar{u}^{(2)}\mathcal{O}^\nu u^{(b)}\right]\right.\\
&\qquad\left. -\,\delta^4(p_2-p_a-k)\left[\bar{u}^{(2)}\mathcal{O}^\mu u^{(a)}\right]\left[\bar{u}^{(1)}\mathcal{O}^\nu u^{(b)}\right]\right\}
\end{aligned}
\tag{384}
$$

Take your time to compare this formula with Eqs. (265, 266). and also with Eqs. (345, 346).

11 Rarita-Schwinger Fields

Lorentz group
spin 3/2 Field Δ_μ
field equations and propagators

With an obvious shorthand notation, we write

$$\begin{aligned}
[\mu\nu\alpha] &= \mu\nu\alpha - \nu\mu\alpha + \alpha\mu\nu - \alpha\nu\mu + \nu\alpha\mu - \mu\alpha\nu \\
&= \mu\nu\alpha - (\{\nu,\mu\} - \mu\nu)\alpha + \alpha\mu\nu - \alpha(\{\nu,\mu\} - \mu\nu) \\
&\quad + \nu(\underline{\{\alpha,\mu\}} - \mu\alpha) - (\underline{\{\mu,\alpha\}} - \alpha\mu)\nu \\
&= 2\mu\nu\alpha + 2\alpha\mu\nu - 4\{\mu,\nu\}\alpha - (2\{\mu,\nu\} - \mu\nu)\alpha + \alpha\mu\nu \\
&= 3\mu\nu\alpha + 3\alpha\mu\nu - 6\{\mu,\nu\}\alpha
\end{aligned} \tag{385}$$

where

$$\{\mu,\nu\} = 2g^{\mu\nu} \tag{386}$$

and we read

$$-\frac{1}{3!}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]} = g^{\mu\nu}\gamma^{\alpha} - \frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha} + \gamma^{\alpha}\gamma^{\mu}\gamma^{\nu}\right) \tag{387}$$

12 Effective Lagrangian Theory

12.1 Oh-Nakayama

Why K should couple via a derivative?

13 Local Gauge Symmetries

U(1), color SU(3), ...

14 Renormalization

15 Ward Identity

Appendices

A A proof of Noether's theorem

First we define the Lie derivative. Given $\mathcal{L}[\varphi_\alpha(x), \partial^\mu \varphi_\alpha(x)]$, write an infinitesimal space-time transformation with s infinitesimal parameters $\delta\omega_j$ as

$$\begin{array}{l} x^\mu \mapsto x^{\mu'} = x^\mu + \delta x^\mu \,, \quad \delta x^\mu = \sum_{j=1}^s X_j^\mu \delta\omega_j \\ \varphi_\alpha(x) \mapsto \varphi'_\alpha(x') = \varphi_\alpha(x) + \delta\varphi_\alpha(x) \,, \quad \delta\varphi_\alpha(x) = \sum_{j=1}^s \Phi_{\alpha j}(x)\delta\omega_j \end{array} \tag{388}$$

$\delta\varphi_\alpha(x)$ contains change of the functional form and one due to the change in coordinates. Define the Lie derivative as the change of functional form:

$$\begin{aligned} \bar{\delta}\varphi_\alpha(x) &\overset{\leftarrow}{=} \varphi'_\alpha(x) - \varphi_\alpha(x) \\ &= \varphi'_\alpha(x) - \{\varphi'_\alpha(x') - \delta\varphi_\alpha(x)\} \\ &= \delta\varphi_\alpha(x) - \{\varphi'_\alpha(x') - \varphi'_\alpha(x)\} \\ &= \delta\varphi_\alpha(x) - \partial_\mu\varphi'_\alpha(x) \cdot \delta x^\mu + \mathcal{O}(\delta^2) \\ &= \delta\varphi_\alpha(x) - \partial_\mu\varphi_\alpha(x) \cdot \delta x^\mu + \mathcal{O}(\delta^2) \qquad (389) \\ &= \sum_{j=1}^s \left(\Phi_{\alpha j}(x) - \partial_\mu\varphi_\alpha(x) X_j^\mu\right) \delta\omega_j + \mathcal{O}(\delta^2) \qquad (390) \end{aligned}$$

$\bar{\delta}$ commutes with ∂^μ. On the other hand, $\delta\varphi_{\alpha:\mu} = \delta\partial_\mu\varphi_\alpha$ differs from $\partial_\mu\delta\varphi_\alpha$. The former one appears in

$$\varphi'_{\alpha:\mu}(x') = \varphi_{\alpha:\mu}(x) + \delta\varphi_{\alpha:\mu}(x)$$

and the equality between

$$\begin{aligned} \bar{\delta}\varphi_{\alpha:\mu}(x) &= \varphi'_{\alpha:\mu}(x) - \varphi_{\alpha:\mu}(x) \\ &= \delta\varphi_{\alpha:\mu}(x) - \partial_\nu\varphi_{\alpha:\mu}(x) \cdot \delta x^\nu + \mathcal{O}(\delta^2) \end{aligned}$$

and

$$\partial_\mu\bar{\delta}\varphi_\alpha = \partial_\mu\delta\varphi_\alpha(x) - \partial_\mu\varphi_{\alpha:\nu}(x) \cdot \delta x^\nu - \varphi_{\alpha:\nu}(x) \cdot \partial_\mu\delta x^\nu + \mathcal{O}(\delta^2)$$

leads to

$$\delta\partial_\mu\varphi_\alpha(x) = \partial_\mu\delta\varphi_\alpha(x) - \varphi_{\alpha:\nu}(x) \cdot \partial_\mu\delta x^\nu \tag{391}$$

Variation of the action

$$\mathcal{A} = \int d^4x \mathcal{L}(x) \tag{392}$$

reads

$$
\begin{aligned}
\delta\mathcal{A} &= \delta\int_\Omega \mathcal{L}[\varphi_\alpha(x),\partial^\mu\varphi_\alpha(x)]d^4x \\
&= \int_{\Omega'}\mathcal{L}[\varphi'_\alpha(x'),\varphi'_{\alpha:\mu}(x')]d^4x' - \int_\Omega \mathcal{L}[\varphi_\alpha(x),\varphi_{\alpha:\mu}(x)]d^4x
\end{aligned}
$$

First we consider Jacobian. For the transformation in Eq. (388), we have

$$\partial_\nu x^{\mu'} = \delta^\mu_\nu + \partial_\nu\delta x^\mu$$

and

$$d^4x' = \frac{\partial(x'_0,x'_1,x'_2,x'_3)}{\partial(x_0,x_1,x_2,x_3)}d^4x = (1+\partial_\mu\delta x^\mu)d^4x + \mathcal{O}(\delta^2) \tag{393}$$

to write

$$\delta d^4x = d^4x' - d^4x = \partial_\mu\delta x^\mu\cdot d^4x \tag{394}$$

Then we write, up to the first order in δ,

$$
\begin{aligned}
\delta\mathcal{A} &= \int_{\Omega'}\mathcal{L}'(x')(1+\delta)d^4x - \int_\Omega\mathcal{L}(x)d^4x \\
&= \int_\Omega\left[\mathcal{L}'(x')-\mathcal{L}(x)\right]d^4x + \int_\Omega\mathcal{L}'(x')\delta d^4x \\
&= \int_\Omega\delta\mathcal{L}(x)d^4x + \int_\Omega\mathcal{L}'(x')\partial_\mu\delta x^\mu\cdot d^4x \\
&= \int_\Omega d^4x\left(\delta\mathcal{L}(x)+\mathcal{L}(x)\partial_\mu\delta x^\mu\right)
\end{aligned} \tag{395}
$$

From Eq. (389),

$$
\begin{aligned}
\delta\mathcal{L}(x) &\overset{\leftarrow}{=} \mathcal{L}'(x')-\mathcal{L}(x) \\
&= \bar{\delta}\mathcal{L}(x) + \partial_\mu\mathcal{L}(x)\cdot\delta x^\mu
\end{aligned} \tag{396}
$$

Now,

$$
\begin{aligned}
\bar{\delta}\mathcal{L}(x) &= \mathcal{L}'(x)-\mathcal{L}(x) \\
&= \frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\bar{\delta}\varphi_\alpha + \frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\bar{\delta}\varphi_{\alpha:\mu} \\
&= \frac{\partial\mathcal{L}}{\partial\varphi_\alpha}\bar{\delta}\varphi_\alpha + \frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\partial_\mu\bar{\delta}\varphi_\alpha \\
&= \left(\frac{\partial\mathcal{L}}{\partial\varphi_\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\right)\bar{\delta}\varphi_\alpha + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\bar{\delta}\varphi_\alpha\right) \\
&= [\mathcal{L}]_{\varphi_\alpha}\bar{\delta}\varphi_\alpha + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha:\mu}}\bar{\delta}\varphi_\alpha\right),
\end{aligned}
$$

where we have defined

$$[\mathcal{L}]_{\varphi_\alpha} \stackrel{\leftarrow}{\equiv} \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \,,$$

which disappears when the Euler-Lagrange equation is satisfied. Going back to Eq. (396),

$$\delta \mathcal{L} \;=\; [\mathcal{L}]_{\varphi_\alpha} \bar{\delta}\varphi_\alpha + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \bar{\delta}\varphi_\alpha \right) + \partial_\mu \mathcal{L}(x) \cdot \delta x^\mu$$

Putting into Eq. (395),

$$\delta \mathcal{A} \;=\; \int_\Omega d^4x \left\{ [\mathcal{L}]_{\varphi_\alpha} \bar{\delta}\varphi_\alpha + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \bar{\delta}\varphi_\alpha + \mathcal{L}(x)\delta x^\mu \right) \right\} \tag{397}$$

Since Ω is arbitrary, we have for fields satisfying the Euler-Lagrange equation that

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \bar{\delta}\varphi_\alpha + \mathcal{L}(x)\delta x^\mu \right) \equiv \partial_\mu \Theta^\mu(x) \tag{398}$$

Noether current

$$\begin{aligned} \Theta^\mu(x) \;&\stackrel{\leftarrow}{\equiv}\; \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \bar{\delta}\varphi_\alpha + \mathcal{L}(x)\delta x^\mu \\ &=\; \sum_{j=1}^{s} \left[\frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \left(\Phi_{\alpha j}(x) - \partial_\nu \varphi_\alpha(x) X_j^\nu \right) + \mathcal{L}(x) X_j^\mu \right] \delta\omega_j \\ &\equiv\; \sum_{j=1}^{s} \Theta_j^\mu(x) \delta\omega_j \end{aligned}$$

$$\Theta_j^\mu(x) \;\stackrel{\leftarrow}{\equiv}\; \frac{\partial \mathcal{L}}{\partial \varphi_{\alpha:\mu}} \left(\Phi_{\alpha j}(x) - \partial_\nu \varphi_\alpha(x) X_j^\nu \right) + \mathcal{L}(x) X_j^\mu \tag{399}$$

Eq. (397) reads,

$$\frac{\delta \mathcal{A}}{\delta \omega_j} \;=\; \int_\Omega d^4x \partial_\mu \Theta_j^\mu(x)$$

Invariance of the $\mathcal{A}$ction leads to an equation of conservation Noether currents:

$$\partial_\mu \Theta_j^\mu(x) = 0 \tag{400}$$

Noether charges

$$C_j = \int \Theta_j^0(x) d^4x \,, \quad \text{for } j = 1, \ldots, s \tag{401}$$

B Symmetries

B.1 Isotropic spin

C Wick's Theorem

We write our field as

$$\varphi(x) = a(x) + b^\dagger(x)\,, \tag{402}$$

where $a(x)$ includes an annihilation operator and $b^\dagger(x)$ includes a creation operator. Particularly, we write

$$a(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} a(\boldsymbol{k}) e^{-ikx}\,, \quad b^\dagger(x) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}} b^\dagger(\boldsymbol{k}) e^{ikx} \tag{403}$$

Usual notation of hermite conjugates is applied to these objects and, for instance, meanings to write $\varphi^\dagger(x)$ and $b(x)$ are obvious. The creation and annihilation operators satisfy,

$$\begin{array}{c} [a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')] = [b(\boldsymbol{k}), b^\dagger(\boldsymbol{k}')] = 2k^0 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \\ \text{all other } [\cdots] \text{ among } a, a^\dagger, b, b^\dagger \text{ are } 0. \end{array} \tag{404}$$

$[\cdots]$ denotes the (anti-) commutator when we are considering a boson (fermion) field. Fields of different kinds are assumed to be commuting to each other. We introduce a signature factor s_f which reads $+1$ for bosons and -1 for fermions. In the following, we abbreviate different arguments by suffices. For instance, $\varphi(x_1)$ is denoted by φ_1. a_1 can mean both $a(x_1)$ and $a(\boldsymbol{k}_1)$. Also, we will write Φ to represent either of φ and $\varphi^\dagger$.

C.1 Normal ordered product

Denoted by $:\cdots:$. All creation operators stack to the left and all annihilation operators stack to the right. When the field is fermionic, an exchange of neighboring operators produce a s_f. For instance,

$$:a_1 a_1^\dagger b_2^\dagger b_2: = s_f^2 a_1^\dagger b_2^\dagger a_1 b_2, \tag{405}$$

where suffices stand for different arguments of x or $\boldsymbol{k}$. Another example reads,

$$\begin{aligned} :\varphi_1\varphi_2\varphi_3\varphi_4: \;=\;& a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger + s_f^3 a_2^\dagger a_3^\dagger a_4^\dagger a_1 + \dots \\ & + s_f^4 a_3^\dagger a_4^\dagger a_1 a_2 + s_f^3 a_2^\dagger a_4^\dagger a_1 a_3 + \dots \\ & (2^4 \text{ terms}) \end{aligned} \tag{406}$$

It is important to note that $<0|:\cdots:|0>=0$. A relationship

$$:a_1 a_2^\dagger: = s_f a_2^\dagger a_1 = a_1 a_2^\dagger - [a_1, a_2^\dagger] \tag{407}$$

remind us a caution for wrong manipulations such as follows. Assuming the linearity of $:\cdots:$ and a relationship $:c\text{-}number: = c\text{-}number$, we might write

$$\textbf{Wrong!} \quad ::a_1 a_2^\dagger:: = :a_1 a_2^\dagger - [a_1, a_2^\dagger]: = :a_1 a_2^\dagger: + c\text{-}number \tag{408}$$

which contradicts with the idempotency $::abc\ldots:: = :abc\ldots:$. We made it wrong when we replace a symbol $[a_1, a_2^\dagger]$ by a *c-number*. The $:\cdots:$ is an operation on symbols and one can not replace them by using mathematical equations. In fact, we have

$$:[a_1, a_2^\dagger]: = :a_1 a_2^\dagger - s_f a_2^\dagger a_1: = 0 \tag{409}$$

and thus recover the idempotency by keeping expressions in symbols. The same caution is applied to relationships

$$::\Phi_1\Phi_2\cdots::\Phi_a\Phi_b\cdots:: = :\Phi_1\Phi_2\cdots\Phi_a\Phi_b\cdots:, \quad \text{etc.} \tag{410}$$

where each Φ stands for a φ or $\varphi^\dagger$. The following is a frequently used relationship:

$$:\Phi_1\Phi_2: = \Phi_1\Phi_2 - <0|\,\Phi_1\Phi_2\,|0> \ . \tag{411}$$

C.2 Time ordered product

Denoted by $T[\cdots]$. This is also an idempotent manipulation on a series of symbols and a caution as one in the normal ordered product is necessary. In this product, a product of fields are ordered in such a way that a field on the left has the time variable larger than the right one.

$$T[\Phi_a\Phi_b\cdots] = s_f(1,2,\cdots;a,b,\cdots)\Phi_1\Phi_2\cdots \tag{412}$$

where Φ_l denotes one of $\varphi(x_l)$ or $\varphi^\dagger(x_l)$, $t_1 > t_2 > \cdots$ and $s_f(1,2,\cdots;a,b,\cdots)$ $= +1$ (-1) when Φ is fermionic and a permutation $(a,b,\cdots) \to (1,2,\cdots)$ is even (odd). When Φ is bosonic, one can omit s_f. In more general form, we write the definition of T-product as

$$T[\Phi_1\Phi_2\cdots] = \sum_P \theta(t_{P_1} > t_{P_2} > \cdots) s_f(P_1,P_2,\cdots;1,2,\cdots)\Phi_{P_1}\Phi_{P_2}\cdots \tag{413}$$

where the sum is taken over all permutations of $(1, 2, \dots) \to (P_1, P_2, \dots)$. Particularly,

$$
\begin{aligned}
T[\varphi_1\varphi_2] &= \theta(t_1 > t_2)\varphi_1\varphi_2 + \theta(t_2 > t_1)s_f\varphi_2\varphi_1 \\
&= \theta(t_1 > t_2)\left\{(a_1 + b_1^\dagger)(a_2 + b_2^\dagger)\right\} + \theta(t_2 > t_1)s_f\left\{(a_2 + b_2^\dagger)(a_1 + b_1^\dagger)\right\} \\
&= \theta(t_1 > t_2)\left\{a_1a_2 + \underline{[a_1, b_2^\dagger] + s_f b_2^\dagger a_1} + b_1^\dagger(a_2 + b_2^\dagger)\right\} + \theta(t_2 > t_1)s_f\{1 \leftrightarrow 2\} \\
&= \theta(t_1 > t_2)\left\{:\!\varphi_1\varphi_2\!: + [a_1, b_2^\dagger]\right\} + \theta(t_2 > t_1)s_f\{1 \leftrightarrow 2\} \\
&= \,:\!\varphi_1\varphi_2\!: + \theta(t_1 > t_2)[a_1, b_2^\dagger] + \theta(t_2 > t_1)s_f[a_2, b_1^\dagger] \\
&= \,:\!\varphi_1\varphi_2\!: + <0|\, T[\varphi_1\varphi_2]\, |0>
\end{aligned}
$$

In deriving the last line, we have employed the expression in the first line. Though this result itself is useless since $<0|\,T[\varphi_1\varphi_2]\,|0>= 0$ for complex scalar fields, remember that the T-product is a manipulation on symbols. This means the above equation can be generalized to arbitrary field to write

$$
T[\Phi_1\Phi_2] = \,:\!\Phi_1\Phi_2\!: + <0|\, T[\Phi_1\Phi_2]\, |0> \tag{414}
$$

and we can substitute either φ or $\varphi^\dagger$ to each of Φ to write, for instance,

$$
T[\varphi_1^\dagger\varphi_2] = \,:\!\varphi_1^\dagger\varphi_2\!: + <0|\, T[\varphi_1^\dagger\varphi_2]\, |0>
$$

which relates a T-product to the propagator of complex scalar fields.

C.3 Wick's theorem

We write Eq. (414) as

$$
T[\Phi_1\Phi_2] = \,:\!\Phi_1\Phi_2\!: + \overbracket{\Phi_1\Phi_2} \tag{415}
$$

and call the last term a contract of fields. Wick's theorem gives a way to express a T-product of more than two fields in terms of contracts and normal ordered products:

$$
\begin{aligned}
T[\Phi_1\Phi_2\cdots] &= \,:\!\Phi_1\Phi_2\cdots\!: \\
&+ \sum_{i<j} s_f(i,j,1,2,\cdots;1,2,\cdots){:}\cdots\Phi_i\cdots\Phi_j\cdots{:} \\
&+ \sum_{i<j,k<l} s_f(i,j,k,l,1,2,\cdots;1,2,\cdots){:}\cdots\Phi_i\cdots\Phi_k\cdots\Phi_j\cdots\Phi_l\cdots{:} \\
&+ \cdots \text{ (all possible contracts)} \\
&= \,:\!\Phi_1\Phi_2\cdots\!: \\
&+ \sum_{i<j} s_f(i,j,1,2,\cdots;1,2,\cdots) <0|\,T[\Phi_i\Phi_j]\,|0> :\cdots\cancel{\Phi_i}\cdots\cancel{\Phi_j}\cdots: \\
&+ \sum_{i<j,k<l} s_f(i,j,k,l,1,2,\cdots;1,2,\cdots) \\
&\qquad <0|\,T[\Phi_i\Phi_j]\,|0><0|\,T[\Phi_k\Phi_l]\,|0> :\cdots\cancel{\Phi_i}\cdots\cancel{\Phi_k}\cdots\cancel{\Phi_j}\cdots\cancel{\Phi_l}\cdots: \\
&+ \cdots
\end{aligned}
\tag{416}
$$

Each of Φ can be either φ or $\varphi^\dagger$. Since contracts appear to compensate the difference between T-product and normal ordered product, no contracts should be taken among fields inside a normal orderd product. For instance,

$$
\begin{aligned}
T[{:}\Phi_1\Phi_2{::}\Phi_3\Phi_4{:}] &= {:}\Phi_1\Phi_2\Phi_3\Phi_4{:} + s_f{:}\Phi_1\Phi_2\Phi_3\Phi_4{:} + s_f^2{:}\Phi_1\Phi_2\Phi_3\Phi_4{:} \\
&+ {:}\Phi_1\Phi_2\Phi_3\Phi_4{:} + s_f{:}\Phi_1\Phi_2\Phi_3\Phi_4{:} \\
&+ s_f{:}\Phi_1\Phi_2\Phi_3\Phi_4{:} + {:}\Phi_1\Phi_2\Phi_3\Phi_4{:}
\end{aligned}
\tag{417}
$$

C.3.1 Propagator

Let's take a real scalar field.

$$\begin{aligned}
<0|\,\varphi(x)\varphi(y)\,|0> &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3 4k^0k^{0\prime}} <0|\,a(\boldsymbol{k})a^\dagger(\boldsymbol{k}')\,|0>\, e^{-i(kx-k'y)} \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} e^{-ik(x-y)} \overset{\rightarrow}{=} D(x-y) \qquad (418)
\end{aligned}$$

$$\begin{aligned}
[\varphi(x),\varphi(y)] &= \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^3 4k^0k^{0\prime}} \Big([a(\boldsymbol{k}), a^\dagger(\boldsymbol{k}')]e^{-i(kx-k'y)} \\
&\qquad\qquad + [a\dagger(\boldsymbol{k}), a(\boldsymbol{k}')]e^{i(kx-k'y)}\Big) \\
&= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3 2k^0} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right) \\
&= D(x-y) - D(y-x) \qquad (419)
\end{aligned}$$

When $x-y$ is spacelike, there exists a Lorentz frame where $x^0 - y^0$. Then the *r.h.s.* of Eq. (419) vanishes. The whole expression is Lorentz invariant and it must vanish for all $(x-y)^2 < 0$. Nevertheless, $D(x-y)$ itself does not vanish even $x-y$ is spacelike.

The Feynman propagator is defined as

$$\begin{aligned}
<0|\,T[\varphi(x)\varphi(y)]\,|0> &= \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x) \qquad (420) \\
&= i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon} \qquad (421) \\
&\overset{\rightarrow}{=} \Delta_F(x-y)
\end{aligned}$$

Proof of Eq.(421) ⇔ Eq. (420)

$$\frac{1}{p^2-m^2+i\epsilon} = \frac{1}{2E_{\boldsymbol{p}}}\left(\frac{1}{p^0-E_{\boldsymbol{p}}+i\epsilon} - \frac{1}{p^0+E_{\boldsymbol{p}}-i\epsilon}\right), \qquad (422)$$

$$
\begin{aligned}
\Delta_F(x-y) &= i\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&\quad \times \int \frac{dp^0}{2\pi} e^{-ip^0(x^0-y^0)} \left(\frac{1}{p^0 - E_{\boldsymbol{p}} + i\epsilon} - \frac{1}{p^0 + E_{\boldsymbol{p}} - i\epsilon} \right) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \frac{-1}{2\pi i} \Big(\theta(x^0-y^0)(-2\pi i e^{-iE_{\boldsymbol{p}}(x^0-y^0)}) \\
&\quad -\theta(y^0-x^0)(+2\pi i e^{iE_{\boldsymbol{p}}(x^0-y^0)}) \Big) \\
&= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(\theta(x^0-y^0) e^{-ip(x-y)} + \theta(y^0-x^0) e^{ip(x-y)} \right) \\
&= \theta(x^0-y^0) D(x-y) + \theta(y^0-x^0) D(y-x)
\end{aligned}
$$

$$\blacksquare \tag{423}$$

C.3.2 T product

We decompose a real scalar field as

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x) \tag{424}$$

where $\varphi^{(+)}$ ($\varphi^{(-)}$) is the term which contain annihilation (creation) operator in Eq. (149). If $x^0 > y^0$,

$$
\begin{aligned}
T[\varphi(x)\varphi(y)] &= (\varphi^{(+)}(x) + \varphi^{(-)}(x))(\varphi^{(+)}(y) + \varphi^{(-)}(y)) \\
&= \varphi^{(+)}(x)\varphi^{(+)}(y) + ([\varphi^{(+)}(x), \varphi^{(-)}(y)] + \varphi^{(-)}(y)\varphi^{(+)}(x)) \\
&\quad + \varphi^{(-)}(x)\varphi^{(+)}(y) + \varphi^{(-)}(x)\varphi^{(-)}(y) \\
&= :\varphi(x)\varphi(y): + D(x-y)
\end{aligned}
$$

and if $y^0 > x^0$,

$$T[\varphi(x)\varphi(y)] = :\varphi(x)\varphi(y): + D(y-x)$$

Then, for arbitrary x^0 and y^0,

$$T[\varphi(x)\varphi(y)] = :\varphi(x)\varphi(y): + \Delta_F(x-y) \tag{425}$$

A similar evaluation shows for a complex scalar field that

$$T[\varphi(x)\varphi^\dagger(y)] = :\varphi(x)\varphi^\dagger(y): + \Delta_F(x-y) \tag{426}$$

•Wick's theorem

$$
\begin{aligned}
& T[\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\dots] \\
= \;& {:}\varphi_1\varphi_2\varphi_3\varphi_4\dots{:} \\
+ \;& \sum_{k<l} \Delta_F(x_k - x_l){:}\varphi_1 \dots \cancel{\varphi_k} \dots \cancel{\varphi_l} \dots{:} \\
+ \;& \sum_{k<l}\sum_{m<n} \Delta_F(x_k - x_l)\Delta_F(x_m - x_n){:}\dots \cancel{\varphi_k} \dots \cancel{\varphi_m} \dots \cancel{\varphi_l} \dots \cancel{\varphi_n} \dots{:} \\
+ \;& \cdots
\end{aligned}
\tag{427}
$$

D Dirac Matrices and Spinors

Majorana and Chiral representations

In the following, two more representations for Dirac matrices are given.

Majorna representation

$$
\begin{aligned}
\gamma^0 &= \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = \sigma_1 \otimes \sigma_2 \,, \quad \gamma^1 = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \,, \end{bmatrix} = 1 \otimes i\sigma_3 \\
\gamma^2 &= \begin{bmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{bmatrix} = -i\sigma_2 \otimes \sigma_2 \,, \quad \gamma^3 = \begin{bmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} = 1 \otimes -i\sigma_1 \,,
\end{aligned}
\tag{428}
$$

and $\gamma_5 = \sigma_3 \otimes \sigma_2$. In this representation, we have $(\gamma^\mu)^* = -\gamma^\mu$ and the Dirac equation becomes real. The Dirac fields are given as linear combinations of real sollutions. This representation is related with the Dirac one by

$$
\gamma^\mu_{\text{Majorana}} = U\gamma^\mu_{\text{Dirac}}U^\dagger \,, \quad U = U^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{bmatrix}
\tag{429}
$$

Chiral representation

$$
\begin{aligned}
\gamma^0 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \sigma_1 \otimes -1 \,, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \,, \end{bmatrix} = i\sigma_2 \otimes \sigma_i \\
\gamma_5 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3 \otimes 1 \,, \quad \sigma_{0i} = \begin{bmatrix} -i\sigma_i & 0 \\ 0 & i\sigma_i \end{bmatrix} = -i\sigma_3 \otimes \sigma_i \,, \\
\sigma_{ij} &= \begin{bmatrix} \epsilon_{ijk}\sigma_k & 0 \\ 0 & \epsilon_{ijk}\sigma_k \end{bmatrix} = \epsilon_{ijk} \otimes \sigma_k \,,
\end{aligned}
\tag{430}
$$

In this representation, spatial rotators and Lorentz boosters, namely, proper Lorentz transformations take forms as diagonal in the $\begin{bmatrix} \ddots \end{bmatrix}$ space so that

φ and χ fields are transformed independently. This representation is related with the Dirac one by

$$\gamma^\mu_{\rm Chiral} = U\gamma^\mu_{\rm Dirac}U^\dagger\,, \quad U = \frac{1}{\sqrt{2}}(1-\gamma_5\gamma^0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{431}$$

Lorentz transformation property

Under a proper homogeneous Lorentz transformation [32]

$$x^\mu \mapsto x^{\mu'} = L(\boldsymbol{\beta},\boldsymbol{\theta})^\mu_{\ \nu}\, x^\nu\,, \tag{432}$$

the Dirac field transforms as

$$\psi(x) \mapsto \psi'(x') = S(L)\psi(x)\,, \quad S(L) = \exp[-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}]\,, \tag{433}$$

where $\omega^{\nu\mu} = -\omega^{\mu\nu}$ and

$$\omega^{0i} = \xi_i\,, \quad \boldsymbol{\xi} = \xi\boldsymbol{\beta}/\beta\,, \quad \xi = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \tag{434}$$

$$\omega^{ij} = -\epsilon_{ijk}\theta_k \tag{435}$$

We may examine the properties under boosts and rotations separately by considering

$$S_{boost}(L(\boldsymbol{\beta},\mathbf{0})) = \exp[-\frac{i}{2}\sigma_{0i}\xi_i] \tag{436}$$

$$= \cosh\frac{\xi}{2} - \frac{\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}{\beta}\sinh\frac{\xi}{2} \tag{437}$$

and

$$S_{rotation}(L(\mathbf{0},\boldsymbol{\theta})) = \exp[\frac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\theta}] \tag{438}$$

$$= \cos\frac{\theta}{2} + i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{\theta}\sin\frac{\theta}{2} \tag{439}$$

---

[32]

$$L_\mu^{\ \rho}L^\mu_{\ \sigma} = g^\rho_\sigma\,,$$

Denoting a matrix with its elements given by $L^\mu_{\ \nu}$ as L, we have

$$L_\mu^{\ \rho} = (L^{-1})^\rho_{\ \mu}$$

where three components of

$$\boldsymbol{\sigma} \overset{\leftarrow}{=} \frac{1}{2}\epsilon^{\uparrow ij}\sigma_{ij} = \frac{i}{2}\epsilon^{\uparrow ij}\gamma^i\gamma^j = \gamma_5\gamma^0\boldsymbol{\gamma} \tag{440}$$

are the spinor (fundamental) reperesentation of generators of the spatial rotation. They certainly satisfy $\{\sigma^i, \sigma^j\}_+ = 2\delta^{ij}$. Also, $S^\dagger_{boost}(L) = S_{boost}(L)$, $S^\dagger_{rotation}(L) = S^{-1}_{rotation}(L)$ and $\gamma^0 S^\dagger(L)\gamma^0 = S^{-1}(L)$. The covariance of the Dirac equation (198) is guaranteed by a relationship

$$S^{-1}(L)\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu \tag{441}$$

$S(L)$ has a property

$$\gamma^0 S^\dagger(L)\gamma^0 = S^{-1}(L)\,, \tag{442}$$

so that

$$\overline{\psi}(x) \mapsto \overline{\psi}'(x') = \psi^{*t}(x)S^\dagger(L)\gamma^0 = \overline{\psi}(x)S^{-1}(L) \tag{443}$$

and $\overline{\psi}\psi$ is a Lorentz scalar.

For later use, we write down a particular form of S_{boost}. For a Lorentz transformation

$$L(-\boldsymbol{\beta}) = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}\cdot \\ \gamma\boldsymbol{\beta} & 1 + \hat{\boldsymbol{\beta}}(\gamma-1)\hat{\boldsymbol{\beta}}\cdot \end{pmatrix}\,, \tag{444}$$

which transforms $p^{(0)} = (m, \mathbf{0}) \mapsto p = Lp^{(0)} = (E, \boldsymbol{p})$, we have

$$\begin{aligned} S_{boost}(L) &= \tfrac{1}{\sqrt{2m(E+m)}}[\not{p}\gamma^0 + m]\,, \\ S^{-1}_{boost}(L) &= \tfrac{1}{\sqrt{2m(E+m)}}[\gamma^0\not{p} + m]\,, \end{aligned} \tag{445}$$

Since L^{-1} corresponds to the change of the sign of the spacial momentum, we have $S^{-1}_{boost}(L) = S_{boost}(L^{-1})$.

Projections

Energy state projection

$$\hat{\Omega}_+(\boldsymbol{p}) = \frac{\not{p}+m}{2m}\,, \quad \hat{\Omega}_-(\boldsymbol{p}) = \frac{-\not{p}+m}{2m} \tag{446}$$

For a Lorentz transformation $L : p^{(0)} = (m, \mathbf{0}) \mapsto p$, Eq. (441) reads

$$S^{-1}(L)\not{p}S(L) = p_\mu L^\mu{}_\nu\gamma^\nu = L_\mu{}^\rho L^\mu{}_\nu p^{(0)}_\rho\gamma^\nu = g^\rho_\nu p^{(0)}_\rho\gamma^\nu = m\gamma^0$$

Spin state projection

The spin states are defined in the rest frame of particle. Consider the generator of spatial rotation given in Eq. (440). For $s^{(0)} = (0, \boldsymbol{s})$ with a spatial 3 vector $\boldsymbol{s}$, with helps of Eq. (441) and a relation

$$S^{-1}(L)\gamma_5 S(L) = \det(L)\gamma_5\,, \tag{447}$$

we have

$$\begin{aligned}
S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L) &= S^{-1}(L)\gamma_5\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{s}S(L) \\
&= \det(L)\gamma_5 L^0{}_\mu\gamma^\mu L^i{}_\nu\gamma^\nu s^i \\
&= \det(L)\gamma_5(p^{(0)}_\rho/m)L^\rho{}_\mu\gamma^\mu(-s^{(0)}_\sigma)L^\sigma{}_\nu\gamma^\nu \\
&= \det(L)\gamma_5(L^{-1})_\mu{}^\rho(p^{(0)}_\rho/m)\gamma^\mu(L^{-1})_\nu{}^\sigma\gamma^\nu(-s^{(0)}_\sigma)\,,
\end{aligned}$$

where $p^{(0)} = (m, \boldsymbol{0})$. Then, for $L = L(\boldsymbol{\beta}) : p = (E, \boldsymbol{p}) \mapsto p^{(0)}$, we have

$$\begin{aligned}
S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L) &= -(1/m)\det(L)\gamma_5 p_\mu\gamma^\mu\gamma^\nu s_\nu\,, \\
&= \det(L)\frac{\gamma_5 \not{s}\not{p}}{m}\,,
\end{aligned} \tag{448}$$

where we have used a fact $\not{p}\not{s} = -\not{s}\not{p}$ for $p\cdot s = 0$. Thus, we have for a proper Lorentz transformation that

$$\frac{\gamma_5\not{s}\not{p}}{m}u(\boldsymbol{p}) = S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}S(L)S^{-1}(L)u(\boldsymbol{0}) = S^{-1}(L)\boldsymbol{\sigma}\cdot\boldsymbol{s}u(\boldsymbol{0}) \tag{449}$$

and similar relationship for v. We set two linearly independent spinors in the rest frame as eigenstates of $\boldsymbol{\sigma}\cdot\boldsymbol{s}$ for $\boldsymbol{s}^2 = 1$ and write

$$\begin{aligned}
&\boldsymbol{\sigma}\cdot\boldsymbol{s}\; u(\boldsymbol{0},\boldsymbol{s}) = u(\boldsymbol{0},\boldsymbol{s}) \\
&\boldsymbol{\sigma}\cdot\boldsymbol{s}\; u(\boldsymbol{0},-\boldsymbol{s}) = -u(\boldsymbol{0},-\boldsymbol{s}) \\
&-\boldsymbol{\sigma}\cdot\boldsymbol{s}\; v(\boldsymbol{0},\boldsymbol{s}) = v(\boldsymbol{0},\boldsymbol{s}) \\
&-\boldsymbol{\sigma}\cdot\boldsymbol{s}\; v(\boldsymbol{0},-\boldsymbol{s}) = -v(\boldsymbol{0},-\boldsymbol{s})
\end{aligned} \tag{450}$$

Since $S^{-1}(L)u(\boldsymbol{0},\boldsymbol{s}) = u(\boldsymbol{p},\boldsymbol{s})$ and so on, we have a set of similar relations by replacing $\boldsymbol{0}$ by $\boldsymbol{p}$ and $\boldsymbol{\sigma}\cdot\boldsymbol{s}$ by $\gamma_5\not{s}\not{p}/m$. Considering relationships $(\not{p}/m)u(\boldsymbol{p}) = u(\boldsymbol{p})$ and $(-\not{p}/m)v(\boldsymbol{p}) = v(\boldsymbol{p})$, an operator matirx

$$\hat{\Sigma}(s) = \frac{1+\gamma_5\not{s}}{2} \tag{451}$$

projects out spinstates along $\boldsymbol{s}$ as

$$\begin{aligned}
&\hat{\Sigma}(s)u(\boldsymbol{p},\boldsymbol{s}) = u(\boldsymbol{p},\boldsymbol{s}) \\
&\hat{\Sigma}(s)u(\boldsymbol{p},-\boldsymbol{s}) = 0 \\
&\hat{\Sigma}(s)v(\boldsymbol{p},\boldsymbol{s}) = v(\boldsymbol{p},\boldsymbol{s}) \\
&\hat{\Sigma}(s)v(\boldsymbol{p},-\boldsymbol{s}) = 0
\end{aligned} \tag{452}$$

In the Dirac representation,

$$\boldsymbol{\sigma} \;=\; 1 \otimes \boldsymbol{\sigma}_{Pauli} = \begin{bmatrix} \boldsymbol{\sigma}_{Pauli} & 0 \\ 0 & \boldsymbol{\sigma}_{Pauli} \end{bmatrix} \tag{453}$$

For $p = Lp^{(0)}$, $p^{(0)} = (m, \mathbf{0})$,

$$\hat{\Sigma}(s)u(\boldsymbol{p}) = \hat{\Sigma}(s)S_{boost}(L)u(\mathbf{0}) = \frac{1+\gamma_5 \not{s}}{2}\frac{\not{p}\gamma^0 + m}{\sqrt{2m(E+m)}} \tag{454}$$

<u>Propagator</u>

$$iS_{\xi\eta} \;\equiv\; \{\psi_\xi(x), \overline{\psi}_\eta(y)\} \tag{455}$$

$$iS(x-y) \;=\; \cdots = (i\not{\partial}_x + m)\left[D(x-y) - D(y-x)\right] \tag{456}$$

$$S_F(x-y) \;\overset{\leftarrow}{\equiv}\; <0|\, T[\psi(x)\overline{\psi}(y)\,|0> \tag{457}$$

$$= \; i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{\not{p} - m + i\epsilon} \tag{458}$$

E Polarization vectors

Polarization vectors $\epsilon_\mu^{(\lambda)}(\boldsymbol{p})$ for a free Klein-Goldon vector field are chosen as 4 linearly independent basis:

$$\begin{cases} g^{\mu\nu}\epsilon_\mu^{(\lambda)^*}\epsilon_\nu^{(\lambda')} = g^{\lambda\lambda'} & \text{(orthonormal)} \\ g_{\lambda\lambda'}\epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda')^*} = g_{\mu\nu} & \text{(complete)} \end{cases} \tag{459}$$

An obvious choice is $\epsilon^{(\lambda)\mu} = \epsilon^{(\lambda)*\mu} = e^{(\lambda)\mu}$ where

$$e^{(0)\mu} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}^\mu, \quad e^{(i)\mu} = \begin{pmatrix} 0 \\ \boldsymbol{e}^{(i)} \end{pmatrix}^\mu \; i = 1, 2, 3 \tag{460}$$

and $\boldsymbol{e}^{(i)}$ are 3 orthnormal spacial vectors. The choice of $\boldsymbol{e}^{(i)}$ may depend on the direction of the momentum $\boldsymbol{p}$. In particular, we may choose $\boldsymbol{e}^{(3)} = \hat{\boldsymbol{p}} \equiv \boldsymbol{p}/|\boldsymbol{p}|$ and $\boldsymbol{e}_2 = \hat{\boldsymbol{p}} \times \boldsymbol{e}_1$ with $\boldsymbol{e}_1$ being an arbitrary chosen spacial unit vector perpendicular to $\hat{\boldsymbol{p}}$. Regarding with the choice of transverse polarization basis, it is also common to employ

$$e^{(\pm)} = \frac{1}{\sqrt{2}}\left(e^{(1)} \pm ie^{(2)}\right) \tag{461}$$

to represent the two circular polarizations. The problem of indefinite norm of the massless Klein-Gordon field is avoided by suppressing excitations in the time direction in terms of subsidiary conditions, which usually take a form of gauge conditions. In a consistent scheme, the longitudinal excitation is also dropped and only transverse modes, which satisfy $p \cdot \epsilon^{(T)} = 0$, are survived as physical excitations.

In cases of massive fields, excitations of time components are excluded from dynamical degrees of freedom by virtue of the Lorentz condition, which is the unique Lorentz invariant restriction on the field. It is therefore essential to have polarization vectors which satisfy $\epsilon^{(\uparrow)} \cdot p = 0$. Since there exists a rest frame for each momentum $p = (p^0, \boldsymbol{p}) = m(\gamma, \gamma\boldsymbol{\beta})$, we may write

$$\epsilon^{(\lambda)\mu}(\boldsymbol{p}) = \Lambda^{\mu}{}_{\nu}(\boldsymbol{\beta})\, e^{(\lambda)\mu}, \quad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}\cdot \\ \gamma\boldsymbol{\beta} & 1 + \hat{\boldsymbol{\beta}}(\gamma - 1)\hat{\boldsymbol{\beta}}\cdot \end{pmatrix}, \tag{462}$$

for which we surely have above mentinoed relationship. Also, it is straightforward to confirm relationships in Eq. (459) and

$$\begin{aligned} g_{ij}\epsilon^{(i)\mu}\epsilon^{(j)*\nu} &= g^{\mu\nu} - g^{00}\epsilon^{(0)\mu}\epsilon^{(0)*\nu} \\ &= g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{m^2} \end{aligned} \tag{463}$$

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