# Primitive Recursive Dependent Type Theory

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# TAKEAWAY

MLTT with natural numbers, but without $\Pi$-types, is primitive recursive.

# Primitive Recursion

**Definition**

The *basic primitive recursive functions* are constant functions, the successor function and projections of type $\mathbb{N}^n \to \mathbb{N}$. A *primitive recursive function* is obtained by finite applications of composition of the basic p.r. functions and the *primitive recursion operator*

$$\begin{aligned}\mathrm{primrec} : \mathbb{N} \to (\mathbb{N} \times \mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}\\ \mathrm{primrec}(g, h, 0) &= g\\ \mathrm{primrec}(g, h, k+1) &= h(k, \mathrm{primrec}(g, h, k)).\end{aligned}$$

## Motivation for Conservative Extension

- PRA as base theory for reverse mathematics [Simpson, 2009] and formal metatheory [Kleene, 1952]
- Theorems encoded in base system
- More expressive base system -> less encoding
- Syntax closer to proof assistants -> enables formal verification

# Beyond Primitive Recursion

## A Nonexample

The *Ackermann function* $A : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ given by

$$A(0) = (n \mapsto n + 1)$$
$$A(m+1) = \begin{cases} 0 & \mapsto A(m, 1) \\ n+1 & \mapsto A(m, A(m+1, n)) \end{cases}$$

grows faster than any p.r. function. It requires elimination into a function type.

# Primitive Recursive Dependent Type Theory

## Takeaway

MLTT with natural numbers, but without Π-types, is primitive recursive.

## Definition

Let T be a restriction of MLTT with a universe $U_0$ closed under Σ- and intensional identity types (but not Π-types), containing finite types, and a closed type N with standard elimination principle

$$\frac{n : \mathrm{N} \vdash X(n) : U_0 \qquad \vdash g : X(0) \qquad n : \mathrm{N}, x : X(n) \vdash h(n,x) : X(n+1)}{n : \mathrm{N} \vdash \mathrm{ind}_{g,h}(n) : X(n)}$$

for $U_0$-small type families. Larger universes $U_\alpha$ may contain Π-types, and

$$\Pi_{n:\mathrm{N}} X(n) : U_1.$$

## Theorem

The definable terms

$$n : \mathrm{N} \vdash f(n) : \mathrm{N}$$

in T are exactly the primitive recursive functions.

# Potential Further Extensions

- Syntactically different standard natural numbers type with large elimination principle
- Finitary inductive types and type families, finitary induction-recursion, e.g. lists
- Primitive recursive universe of types – judgemental variant of internal p.r. Gödel encoding of the codes in $U_0$
- Comonadic modality $\Box$ for simultaneous recursion on $\Box N \times N$ (c.f. [Hofmann, 1997])
- Primitive Recursive Homotopy/Cubical Type Theory – not clear how to adapt our adequacy proof

# Related Work

- Calculus of Primitive Recursive Constructions [Herbelin and Patey, 2014] – PRTT has function types in higher universes, closer to Agda syntax
- MLTT with recursion operators [Paulson, 1986]
- Partial recursive functions via inductive domain predicates [Bove, 2003]
- Coinductive types of partial elements [Bove and Capretta, 2007]

# Proof of Conservativity by Logical Relations
## Idea: Synthetic Tait Computability

Given a lex functor

$$\rho : \mathrm{T} \to \mathrm{Set}$$

we can extend along the Yoneda embedding

$$\begin{tikzcd} \hat{\mathrm{T}} \arrow[r, "\hat{\rho}"] & \mathrm{Set} \\ \mathrm{T} \arrow[u, hook] \arrow[ur, "\rho"'] & \end{tikzcd}$$

and use the internal language of the Artin gluing

$$\mathrm{Set} \downarrow \hat{\rho}$$

to prove statements about objects $\rho(X)$ [Sterling, 2021].

# Proof of Conservativity by Logical Relations

## Ingredients

- Standard model
$$[\![-]\!]_{\mathrm{Set}} : \mathrm{T} \to \mathrm{Set}, \qquad [\![\mathrm{N}]\!]_{\mathrm{Set}} = \mathbb{N}.$$
- Model
$$[\![-]\!]_{\mathcal{R}} : \mathrm{T} \to \mathcal{R}$$
in a topos where
$$\mathcal{R}([\![\mathrm{N}]\!]_{\mathcal{R}}, [\![\mathrm{N}]\!]_{\mathcal{R}})$$
are exactly the primitive recursive functions $\mathbb{N} \to \mathbb{N}$.
- Model
$$[\![-]\!]_{\mathrm{Set}\downarrow\hat{\rho}} : \mathrm{T} \to \mathrm{Set} \downarrow \hat{\rho}$$
with
$$\rho(X) = \Gamma([\![X]\!]_{\mathcal{R}}) \times [\![X]\!]_{\mathrm{Set}}.$$
- Canonicity:
$$\mathbb{N} \cong \Gamma(\mathrm{N})$$

# Proof of Conservativity by Logical Relations

## Externalisation

Any term

$$n : \mathrm{N} \vdash f(n) : \mathrm{N}$$

of T is interpreted in $\mathrm{Set} \downarrow \hat{\rho}$ as

$$\begin{tikzcd}
\mathbb{N} \cong \Gamma(\mathrm{N}) \arrow[rr, "\Gamma(f)"] \arrow[d, "\Delta_{\mathrm{N}}"'] \arrow[drr, phantom, "{[\![f]\!]_{\mathrm{Set}\downarrow\hat{\rho}}}"] & & \Gamma(\mathrm{N}) \cong \mathbb{N} \arrow[d, "\Delta_{\mathrm{N}}"] \\
\mathbb{N} \times \mathbb{N} \cong \rho(\mathrm{N}) \arrow[rr, "\Gamma(\widehat{[\![f]\!]_{\mathcal{R}}}) \times \widehat{[\![f]\!]_{\mathrm{Set}}}"'] & & \rho(\mathrm{N}) \cong \mathbb{N} \times \mathbb{N}.
\end{tikzcd}$$

Since $\widehat{[\![f]\!]_{\mathcal{R}}}$ is primitive recursive, so is $[\![f]\!]_{\mathrm{Set}}$.

$\square$

# Questions?

Thank you!

Slides & Draft: `jsvb.xyz`

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