PRIMITIVE RECURSIVE DEPENDENT TYPE THEORY HOTT/UF 2024

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TAKEAWAY

MLTT with natural numbers, but without Π -types, is primitive recursive.

PRIMITIVE RECURSION

Definition

The *basic primitive recursive functions* are constant functions, the successor function and projections of type $\mathbb{N}^n \to \mathbb{N}$. A *primitive recursive function* is obtained by finite applications of composition of the basic p.r. functions and the *primitive recursion operator*

$$\begin{split} \text{primrec} : \mathbb{N} &\to (\mathbb{N} \times \mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N} \\ & \text{primrec}(g, h, 0) = g \\ & \text{primrec}(g, h, k + 1) = h(k, \text{primrec}(g, h, k)). \end{split}$$

MOTIVATION FOR CONSERVATIVE EXTENSION

- ▶ PRA as base theory for reverse mathematics [Simpson, 2009] and formal metatheory [Kleene, 1952]
- ► Theorems encoded in base system
- ► More expressive base system -> less encoding
- ▶ Syntax closer to proof assistants -> enables formal verification

BEYOND PRIMITIVE RECURSION

A NONEXAMPLE

The *Ackermann function* $A : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ given by

$$A(0) = (n \mapsto n+1)$$

$$A(m+1) = \begin{cases} 0 & \mapsto A(m,1) \\ n+1 & \mapsto A(m,A(m+1,n)) \end{cases}$$

grows faster than any p.r. function. It requires elimination into a function type.

PRIMITIVE RECURSIVE DEPENDENT TYPE THEORY

Takeaway

MLTT with natural numbers, but without Π -types, is primitive recursive.

Definition

Let T be a restriction of MLTT with a universe U_0 closed under Σ - and intensional identity types (but not Π -types), containing finite types, and a closed type N with standard elimination principle

$$\frac{n: \mathbb{N} \vdash X(n): \mathbb{U}_0 \qquad \vdash g: X(0) \qquad n: \mathbb{N}, x: X(n) \vdash h(n, x): X(n+1)}{n: \mathbb{N} \vdash \operatorname{ind}_{g,h}(n): X(n)}$$

for U_0 -small type families. Larger universes U_α may contain Π -types, and

$$\Pi_{n:N}X(n): U_1.$$

Theorem

The definable terms

$$n: N \vdash f(n): N$$

in T are exactly the primitive recursive functions.

POTENTIAL FURTHER EXTENSIONS

- ▶ Syntactically different standard natural numbers type with large elimination principle
- Finitary inductive types and type families, finitary induction-recursion, e.g. lists
- ▶ Primitive recursive universe of types judgemental variant of internal p.r. Gödel encoding of the codes in U₀
- ightharpoonup Comonadic modality \square for simultaneous recursion on $\square N \times N$ (c.f. [Hofmann, 1997])
- Primitive Recursive Homotopy/Cubical Type Theory not clear how to adapt our adequacy proof

RELATED WORK

- ► Calculus of Primitive Recursive Constructions [Herbelin and Patey, 2014] PRTT has function types in higher universes, closer to Agda syntax
- ▶ MLTT with recursion operators [Paulson, 1986]
- ▶ Partial recursive functions via inductive domain predicates [Bove, 2003]
- ► Coinductive types of partial elements [Bove and Capretta, 2007]

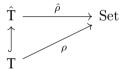
PROOF OF CONSERVATIVITY BY LOGICAL RELATIONS

IDEA: SYNTHETIC TAIT COMPUTABILITY

Given a lex functor

$$\rho: T \to Set$$

we can extend along the Yoneda embedding



and use the internal language of the Artin gluing

Set
$$\downarrow \hat{\rho}$$

to prove statements about objects $\rho(X)$ [Sterling, 2021].

PROOF OF CONSERVATIVITY BY LOGICAL RELATIONS

INGREDIENTS

► Standard model

$$[\![-]\!]_{\operatorname{Set}}:T\to\operatorname{Set},\qquad [\![N]\!]_{\operatorname{Set}}=\mathbb{N}.$$

► Model

$$\llbracket -
rbracket_{\mathcal{R}} : T \to \mathcal{R}$$

in a topos where

$$\mathcal{R}([\![N]\!]_{\mathcal{R}},[\![N]\!]_{\mathcal{R}})$$

are exactly the primitive recursive functions $\mathbb{N} \to \mathbb{N}$.

► Model

$$[-]_{\operatorname{Set}\downarrow\hat{\rho}}: T \to \operatorname{Set}\downarrow\hat{\rho}$$

with

$$\rho(X) = \Gamma([\![X]\!]_{\mathcal{R}}) \times [\![X]\!]_{\mathrm{Set}}.$$

► Canonicity:

$$\mathbb{N} \cong \Gamma(\mathbb{N})$$

PROOF OF CONSERVATIVITY BY LOGICAL RELATIONS

EXTERNALISATION

Any term

$$n: \mathbf{N} \vdash f(n): \mathbf{N}$$

of T is interpreted in Set $\downarrow \hat{\rho}$ as

$$\mathbb{N} \cong \Gamma(\mathbb{N}) \xrightarrow{\Gamma(f)} \Gamma(\mathbb{N}) \cong \mathbb{N}$$

$$\Delta_{\mathbb{N}} \downarrow \qquad \qquad \downarrow \Delta_{\mathbb{N}}$$

$$\mathbb{N} \times \mathbb{N} \cong \rho(\mathbb{N}) \xrightarrow{\Gamma(\widehat{\mathbb{M}_{\mathbb{R}}}) \times \widehat{\mathbb{M}_{\mathrm{Set}}}} \rho(\mathbb{N}) \cong \mathbb{N} \times \mathbb{N}.$$

Since $\widehat{[f]}_{\mathcal{R}}$ is primitive recursive, so is $[\![f]\!]_{\text{Set}}$.

QUESTIONS?

Thank you!

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