PRIMITIVE RECURSIVE DEPENDENT TYPE THEORY
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TAKEAWAY

MLTT with natural numbers, but without Π-types, is primitive recursive.

PRIMITIVE RECURSION

Definition

The *basic primitive recursive functions* are constant functions, the successor function and projections of type $\mathbb{N}^n \to \mathbb{N}$. A *primitive recursive function* is obtained by finite applications of composition of the basic p.r. functions and the *primitive recursion operator*

$$\text{primrec} : \mathbb{N} \to (\mathbb{N} \times \mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}$$
$$\text{primrec}(g, h, 0) = g$$
$$\text{primrec}(g, h, k + 1) = h(k, \text{primrec}(g, h, k)).$$

MOTIVATION FOR CONSERVATIVE EXTENSION

- PRA as base theory for reverse mathematics [Simpson, 2009] and formal metatheory [Kleene, 1952]

- Theorems encoded in base system

- More expressive base system -> less encoding

- Syntax closer to proof assistants -> enables formal verification

BEYOND PRIMITIVE RECURSION

The *Ackermann function $A : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$* given by

$$A(0) = (n \mapsto n + 1)$$

$$A(m + 1) = \begin{cases} 0 & \mapsto A(m, 1) \\ n + 1 & \mapsto A(m, A(m + 1, n)) \end{cases}$$

grows faster than any p.r. function. It requires elimination into a function type.

PRIMITIVE RECURSIVE DEPENDENT TYPE THEORY

Takeaway

MLTT with natural numbers, but without Π-types, is primitive recursive.

Definition

Let T be a restriction of MLTT with a universe U_0 closed under Σ- and intensional identity types (but not Π-types), containing finite types, and a closed type N with standard elimination principle

$$\frac{n : N \vdash X(n) : U_0 \qquad \vdash g : X(0) \qquad n : N, x : X(n) \vdash h(n, x) : X(n+1)}{n : N \vdash \mathrm{ind}_{g,h}(n) : X(n)}$$

for U_0-small type families. Larger universes U_α may contain Π-types, and

$$\Pi_{n:N} X(n) : U_1.$$

Theorem

The definable terms

$$n : N \vdash f(n) : N$$

in T are exactly the primitive recursive functions.

POTENTIAL FURTHER EXTENSIONS

▶ Syntactically different standard natural numbers type with large elimination principle

▶ Finitary inductive types and type families, finitary induction-recursion, e.g. lists

▶ Primitive recursive universe of types – judgemental variant of internal p.r. Gödel encoding of the codes in U_0

▶ Comonadic modality $\square$ for simultaneous recursion on $\square N \times N$ (c.f. [Hofmann, 1997])

▶ Primitive Recursive Homotopy/Cubical Type Theory – not clear how to adapt our adequacy proof

RELATED WORK

- ▶ Calculus of Primitive Recursive Constructions [Herbelin and Patey, 2014] – PRTT has function types in higher universes, closer to Agda syntax

- ▶ MLTT with recursion operators [Paulson, 1986]

- ▶ Partial recursive functions via inductive domain predicates [Bove, 2003]

- ▶ Coinductive types of partial elements [Bove and Capretta, 2007]

PROOF OF CONSERVATIVITY BY LOGICAL RELATIONS
IDEA: SYNTHETIC TAIT COMPUTABILITY

Given a lex functor

$$\rho : T \to \mathrm{Set}$$

we can extend along the Yoneda embedding

and use the internal language of the Artin gluing

$$\mathrm{Set} \downarrow \hat{\rho}$$

to prove statements about objects $\rho(X)$ [Sterling, 2021].

PROOF OF CONSERVATIVITY BY LOGICAL RELATIONS
INGREDIENTS

- Standard model
$$[\![-]\!]_{\mathrm{Set}} : \mathrm{T} \to \mathrm{Set}, \qquad [\![\mathrm{N}]\!]_{\mathrm{Set}} = \mathbb{N}.$$

- Model
$$[\![-]\!]_{\mathcal{R}} : \mathrm{T} \to \mathcal{R}$$

  in a topos where
$$\mathcal{R}([\![\mathrm{N}]\!]_{\mathcal{R}}, [\![\mathrm{N}]\!]_{\mathcal{R}})$$

  are exactly the primitive recursive functions $\mathbb{N} \to \mathbb{N}$.

- Model
$$[\![-]\!]_{\mathrm{Set}\downarrow\hat{\rho}} : \mathrm{T} \to \mathrm{Set} \downarrow \hat{\rho}$$

  with
$$\rho(X) = \Gamma([\![X]\!]_{\mathcal{R}}) \times [\![X]\!]_{\mathrm{Set}}.$$

- Canonicity:
$$\mathbb{N} \cong \Gamma(\mathrm{N})$$

Any term

$$n : \mathbf{N} \vdash f(n) : \mathbf{N}$$

of T is interpreted in $\mathrm{Set} \downarrow \hat{\rho}$ as

$$
\begin{array}{ccc}
\mathbb{N} \cong \Gamma(\mathbf{N}) & \xrightarrow{\quad \Gamma(f) \quad} & \Gamma(\mathbf{N}) \cong \mathbb{N} \\
\Delta_{\mathbf{N}} \downarrow & \llbracket f \rrbracket_{\mathrm{Set} \downarrow \hat{\rho}} & \downarrow \Delta_{\mathbf{N}} \\
\mathbb{N} \times \mathbb{N} \cong \rho(\mathbf{N}) & \xrightarrow[\Gamma(\widehat{\llbracket f \rrbracket_{\mathcal{R}}}) \times \widehat{\llbracket f \rrbracket_{\mathrm{Set}}}]{} & \rho(\mathbf{N}) \cong \mathbb{N} \times \mathbb{N}.
\end{array}
$$

Since $\widehat{\llbracket f \rrbracket_{\mathcal{R}}}$ is primitive recursive, so is $\llbracket f \rrbracket_{\mathrm{Set}}$.

□

QUESTIONS?

Thank you!

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