1 Introdcution

2 Theory

2.1 Jaynes-Cummings Hamiltonian

Rabi cycles are a phenomenon that occurs when an atom interacts with light. Both, Serge Haroche and David Wineland, investigated and made use of the shifts in energylevels and relative phases of states that appear when a laser shines in on an atom. In order give a useful and complete description of Raby cycles, one has to take into account that the energy levels of the atom and of the photon field are quantized. The first "fully quantized" approach for the case of coherent light in a cavity and a two level system was given by Edwin Jaynes and Fred Cummings in 1963 [1]. To describe the system (following [2]) we introduce the total Hamiltonian, consisting of the Hamiltonian for the quantized photon field for a single mode, the Hamiltonian of the two level system consisting of a ground state $|g\rangle$ and an excited state $|e\rangle$ and the interaction Hamiltonian

$$\hat{H} = \hat{H}_{\text{field}} + \hat{H}_{\text{atom}} + \hat{H}_{\text{int}}, \tag{1}$$

where, neglecting the zero point energies of field and atom,

$$\hat{H}_{\text{field}} = \hbar\omega_{\text{f}}\, \hat{a}^\dagger \hat{a} \tag{2}$$

$$\hat{H}_{\text{atom}} = \frac{\hbar\omega_a}{2}\, \hat{\sigma}_3 \tag{3}$$

$$\hat{H}_{\text{int}} = \hbar\omega_{\text{int}} \left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \left(\hat{a} + \hat{a}^\dagger\right). \tag{4}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators that appear in the quantization of the electromagnetic field[1], the Pauli matrix $\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$ acts as the inversion operator of the atom, and the associated Pauli matrices $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ project any state $|g\rangle$ ($|e\rangle$) to $|e\rangle$ ($|g\rangle$) respectively and can thus be seen as atomic transition operators. It is now useful to look at the time evolution of the terms in the total Hamiltonian (1). To do so we switch to the interaction picture, using the Hamiltonian without interaction $\hat{H}_0 = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}}$ to describe the time evolution of the operators. For the annihilation operator we have

$$\hat{a}(t) = e^{i\hat{H}_0 t/\hbar}\, \hat{a}(0)\, e^{-i\hat{H}_0 t/\hbar}$$

[1] To solve the free field equation of the elmg. field one usually uses the Fourier ansatz

$$A^\mu(x) = \int \mathrm{d}\tilde{k} \sum_{\lambda=0}^{3} \left[a_\lambda(\vec{k})\epsilon_\lambda^\mu(k)\mathrm{e}^{-ikx} + a_\lambda^\dagger(\vec{k})\epsilon_\lambda^\mu(k)^*\mathrm{e}^{+ikx}\right],$$

where λ goes over all possible polarizations. When demanding that A^μ and its conjugate field π^ν obey the canonical quantization relation, the resulting commutator relations for a_λ and $a_\lambda^\dagger$ allow the interpretation as creation and annihilation operators.

using the Baker-Campbell-Hausdorff formula, we obtain

$$= \hat{a}(0) + \frac{it}{\hbar}\left[\hat{H}_0, \hat{a}\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 \left[\hat{H}_0, \left[\hat{H}_0, \hat{a}\right]\right] + \ldots$$

The first commutator is $\left[\hat{H}_0, \hat{a}\right] = -\hbar\omega_f \hat{a}$ as $\hat{a}$ commutes with $\hat{H}_{\text{atom}}$ and $\left[\hat{a}^\dagger, \hat{a}\right] = -1$. The nested commutators will thus only add higher orders of ω_f. The time evolution then becomes

$$\begin{aligned} &= \hat{a}(0)\left(1 - it\omega_f + (it\omega_f)^2 + \ldots\right) \\ &= \hat{a}(0)e^{-i\omega_f t}.^{2} \end{aligned} \tag{5}$$

Correspondingly we obtain for the other operators

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega_f t} \tag{6}$$
$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_a t} \tag{7}$$

and therefore the full interaction Hamiltonian becomes

$$\begin{aligned} \hat{H}_{\text{int}}(t) = \hbar\omega_{\text{int}} \big(&\hat{\sigma}_+\hat{a}\, e^{i(\omega_a-\omega_f)t} + \hat{\sigma}_+\hat{a}^\dagger\, e^{i(\omega_a+\omega_f)t} \\ &+\hat{\sigma}_-\hat{a}\, e^{-i(\omega_a+\omega_f)t} + \hat{\sigma}_-\hat{a}^\dagger\, e^{-i(\omega_a-\omega_f)t}\big). \end{aligned} \tag{8}$$

Assuming that the photon frequency ω_f is close to the transition frequency of the atom ω_a, i.e. $|\omega_a - \omega_f| \ll \omega_a + \omega_f$, the rotating wave approximation can be applied. This means that all terms in $\hat{H}_{\text{int}}$ that oscillate with $\omega_a + \omega_f$ are neglected. Doing this and transforming back to the Schrödinger picture leaves us with the total Hamiltonian

$$\hat{H}_{\text{tot}} = \hbar\omega_f \hat{a}^\dagger\hat{a} + \frac{\hbar\omega_a}{2}\hat{\sigma}_3 + \hbar\omega_{\text{int}}\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \tag{9}$$

The last two terms can be seen as processes in which the atom aborbs or emmits one photon from the field while respectively changing its internal energy state.

2.2 Rabi Oscillations

Having found a Hamiltonian that describes the interaction of an atom with light, it is now interesting to investigate how this interaction influences the dynamics of the system. The original states of atom and fiel will no longer be eigenstates of the system

[2]By looking at the preceeding footnote and considering that kx is a scalar product of four-vectors but the integration over $\mathrm{d}\tilde{k}$ is only over the three spatial components, we see that the time evolution behaviour is already built-in in this ansatz.

and thus undergo a continuous oscillation, called Rabi oscillation. Bot, Serge Haroche and David Wineland have made experimental use of this effect in order to prepare and manipulate states. To describe the dynamics it is first useful to split the Hamiltonian in (9) into two parts, namely

$$\hat{H}_I = \hbar\omega_f \underbrace{\left(\hat{a}^\dagger \hat{a} + |e\rangle\langle e|\right)}_{\text{excitation number } \hat{N}_e} + \hbar\left(\frac{\omega_a}{2} - \omega_f\right)\underbrace{\left(|e\rangle\langle e| + |g\rangle\langle g|\right)}_{\text{e}^- \text{ number projector } \hat{P}_e} \tag{10}$$

$$\hat{H}_{II} = -\hbar\underbrace{(\omega_a - \omega_f)\,|g\rangle\langle g|}_{\equiv\Delta} + \hbar\omega_{\text{int}}\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \tag{11}$$

The first part $\hat{H}_I$ commutes with $\hat{H}_{\text{tot}}$ thus it is conserved over time and any interesting dynamics of the system are described by the second part. Let us now consider a state

$$|\psi(t)\rangle = C_1(t)\,|e\rangle\,|n\rangle + C_2(t)\,|g\rangle\,|n+1\rangle \tag{12}$$

with initial conditions $C_1(0) = 1$ and $C_2(0) = 0$. The time evolution is described by the time dependend Schrödinger equation $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}_{II}|\psi(t)\rangle$. In the resonant case ($\Delta = 0$) this can be exactly solved and yields

$$C_1(t) = \cos\left(\omega_{\text{int}}\sqrt{n+1}\,t\right) \tag{13}$$

$$C_2(t) = -i\sin\left(\omega_{\text{int}}\sqrt{n+1}\,t\right) \tag{14}$$

2.3 Dressed States

References

1. Jaynes, E. T. & Cummings, F. W. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proceedings of the IEEE* **51,** 89–109 (1963).

2. Gerry, C. & Knight, P. *Introductory quantum optics* (Cambridge university press, 2005).