

# Mathe-Ergänzungskurs

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## Teil I

# Q1

## 1 Reelle Zahlenfolgen

### 1.1 Definitionen

**Definition 1.1** (Reelle Zahlenfolge).

$a: \mathbb{N} \rightarrow \mathbb{R}$  heißt reelle Zahlenfolge.

$$n \mapsto a(n) = a_n$$

**Definition 1.2** (Bildungsvorschrift). Als Bildungsvorschrift bezeichnet man

(a)  $a(n) = f(n)$  z.B.  $a(n) = n^2$  (explizit)

(b)  $a(n) = f(a_1, \dots, a_{n-1}, n)$  z.B.  $a(n+1) = a(n) + a(n-1)$  (rekursiv)

**Definition 1.3** (Monotonie). Eine beliebige Folge  $(a_n)$  ist...

1. ...monoton steigend genau dann, wenn

$$\forall n_1, n_2 \in \mathbb{N}: n_1 > n_2 \implies a_{n_1} \geq a_{n_2}.$$

2. ...monoton fallend genau dann, wenn

$$\forall n_1, n_2 \in \mathbb{N}: n_1 > n_2 \implies a_{n_1} \leq a_{n_2}.$$

3. ...streng monoton steigend genau dann, wenn

$$\forall n_1, n_2 \in \mathbb{N}: n_1 > n_2 \implies a_{n_1} > a_{n_2}.$$

4. ...streng monoton fallend genau dann, wenn

$$\forall n_1, n_2 \in \mathbb{N}: n_1 > n_2 \implies a_{n_1} < a_{n_2}.$$

**Definition 1.4** (Beschränktheit). Eine beliebige Folge  $(a_n)$  ist...

1. ...nach unten beschränkt genau dann, wenn

$$\exists a \in \mathbb{R}: \forall n \in \mathbb{N}: a_n \geq a.$$

2. ...nach oben beschränkt genau dann, wenn

$$\exists b \in \mathbb{R}: \forall n \in \mathbb{N}: a_n \leq b.$$

3. ...beschränkt genau dann, wenn sie nach oben und nach unten beschränkt ist.

**Definition 1.5** (Supremum). *Das Supremum einer beliebigen nach oben beschränkten Folge  $(a_n)$  ist die kleinste obere Schranke dieser Folge.*

**Definition 1.6** (Infimum). *Analog zum Supremum ist das Infimum einer beliebigen nach unten beschränkten Folge  $(a_n)$  die größte untere Schranke dieser Folge.*

**Definition 1.7** (Konvergenz). *Eine beliebige Folge  $(a_n)$  ist konvergent gegen  $g$  genau dann, wenn*

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n \geq N_\varepsilon: |a_n - g| < \varepsilon.$$

## 1.2 Satz (Der Grenzwert von $a_n = \frac{1}{n}$ ist 0)

**Satz 1.1.** *Sei  $(a_n)_{n=1}^\infty$  eine Folge mit der Bildungsvorschrift  $a_n = \frac{1}{n}$ . Dann gilt  $\lim_{n \rightarrow \infty} a_n = 0$*

*Beweis.* Die Behauptung ist per Definition der Konvergenz (Definition 1.7) äquivalent zu

$$\begin{aligned} & \forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n \geq N_\varepsilon: |a_n - 0| < \varepsilon \\ \iff & \forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n \geq N_\varepsilon: \left| \frac{1}{n} \right| < \varepsilon \end{aligned}$$

Diese Aussage gilt, weil es für jedes  $\varepsilon$  ein  $N_\varepsilon$  gibt, so dass für alle  $n > N_\varepsilon$  der Betrag von  $\frac{1}{n}$  kleiner als  $\varepsilon$  ist. Dieses  $N_\varepsilon$  lässt sich durch  $\left\lceil \frac{1}{\varepsilon} \right\rceil + 1$  berechnen. QED

## 1.3 Satz (rekursive Summenfolge = explizite)

**Satz 1.2.** *Seien  $a_1(n)$  und  $a_2(n)$  Folgen mit den Bildungsvorschriften*

$$\begin{aligned} a_1(n) &= a_1(n) + (n+1) & a_2(n) &= \sum_{k=0}^n k \\ a_1(0) &= 0. \end{aligned}$$

*Dann gilt  $\forall n: a_1(n) = a_2(n)$ .*

*Beweis.* Der Beweis wird durch vollständige Induktion geführt.

**Induktionsanfang:** Für  $n = 0$

$$a_1(0) = 0 \tag{1}$$

$$a_2(0) = \sum_{k=0}^0 k = 0 \tag{2}$$

$$(1) \wedge (2) \implies a_1(0) = a_2(0)$$

**Induktionsschritt:** Induktionshypothese:  $\exists n: a_1(n) = a_2(n)$   
 Zu zeigen ist, Ind. Hypot.  $\implies a_1(n+1) = a_2(n+1)$

$$\begin{aligned} a_1(n+1) &= a_1(n) + (n+1) \\ &= a_2(n) + (n+1) \quad \text{Ind. Hypot.} \\ &= \sum_{k=0}^n k + (n+1) \\ &= \sum_{k=0}^{n+1} k \\ &= a_2(n+1) \end{aligned}$$

QED

#### 1.4 Satz (Jede konvergente Folge ist beschränkt)

**Satz 1.3.** Sei  $(a_n)_{n=1}^{\infty}$  eine konvergente Folge mit dem Grenzwert  $a$ . Dann gilt

$$\exists m, M \in \mathbb{R}: \forall n \in \mathbb{N}: m < a_n < M.$$

*Beweis.* Da  $a_n$  gegen  $a$  konvergiert gilt

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n \geq N_\varepsilon: |a_n - a| < \varepsilon.$$

Da  $|a_n - a| < \varepsilon$  in der oberen Aussage äquivalent zu  $-x < a_n < x$  ist, gilt auch

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n \geq N_\varepsilon: -\varepsilon < a_n - a < \varepsilon.$$

Für jedes  $\varepsilon > 0$  existiert also ein  $N_\varepsilon$ , so dass  $a_n$  für alle  $n \geq N_\varepsilon$  beschränkt ist. Da es nur endlich viele Folgenglieder für  $n < N_\varepsilon$  gibt, lässt sich eine obere Grenze als

$$\max(\{a_n | n < N_\varepsilon\} \cup \{\varepsilon + a\})$$

und eine untere Grenze als

$$\min(\{a_n | n < N_\varepsilon\} \cup \{-\varepsilon + a\})$$

berechnen.

QED

## 1.5 Satz von Bolzano-Weierstraß (I und II)

**Satz 1.4** (Satz von Bolzano-Weierstraß I). *Jede beschränkte Folge hat eine konvergente Teilfolge.*

*Beweis.*  $(a_n)_{n=1}^\infty$  sei beschränkt durch  $m \leq a_n \leq M$  für alle  $n \in \mathbb{N}$ . Man teile das Intervall  $[m, M]$  in zwei Teile bei  $\frac{m+M}{2}$ .

1. Fall: Auf  $\frac{m+M}{2}$  liegen unendlich viele Folgeglieder.
2. Fall: In  $[m, \frac{m+M}{2}[$  liegen unendlich viele Folgeglieder.  
Dann beginne mit  $[m, \frac{m+M}{2}[$  von vorne.
3. Fall: In  $] \frac{m+M}{2}, M]$  liegen unendlich viele Folgeglieder.  
Dann beginne mit  $] \frac{m+M}{2}, M]$  von vorne.

Das Verfahren...

- (a) ... bricht mit Eintreten des ersten Falls ab und hat damit eine konvergente Teilfolge.
- (b) ... setzt sich unendlich fort und erzeugt eine Folge von Intervallen mit
  - $I_n \subset I_{n-1}, I_0 = [m, M]$ ,
  - Länge von  $I_n = \frac{M-m}{2^n} \xrightarrow{n \rightarrow \infty} 0$ ,
  - Jedes Intervall enthält unendlich viele Folgeglieder.

Zu dieser Intervallschachtelung gehört genau eine reelle Zahl. Nimmt man aus jedem Intervall das Folgeglied mit dem kleinsten Index, welches noch nicht vorher ausgewählt wurde, erhält man eine Teilfolge, die gegen diese Zahl konvergiert. QED

**Satz 1.5** (Satz von Bolzano-Weierstraß II). *Jede beschränkte und monotone Folge ist konvergent.*

*Beweis.* O.B.d.A (Ohne Beschränkung der Allgemeinheit) sei  $(a_n)_{n=1}^\infty$  monoton wachsend. Sei  $\sup$  das Supremum von  $(a_n)$ .

Weil  $\sup$  das Supremum von  $(a_n)$  ist, gilt

$$\forall n \in \mathbb{N}: a_n \leq \sup \wedge \forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \sup - \varepsilon < a_{N_\varepsilon}.$$

Wir zeigen nun, dass  $(a_n)$  gegen  $\sup$  konvergiert, mit anderen Worten:

$$\forall \varepsilon > 0: \exists N_\varepsilon: \forall n > N_\varepsilon: |a_n - \sup| < \varepsilon. \quad (1)$$

Es gilt

$$\begin{aligned} & |a_n - \sup| < \varepsilon \\ \iff & \sup - a_n < \varepsilon \quad \text{weil } \sup > a_n \\ \iff & \sup - \varepsilon < a_n \end{aligned}$$

Aussage (1) ist also wahr genau dann, wenn

$$\forall \varepsilon > 0: \exists N_\varepsilon: \forall n > N_\varepsilon: \sup - \varepsilon < a_n.$$

Laut Definition des Supremums (1.5) gilt  $\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \sup - \varepsilon < a_{N_\varepsilon}$ . Weil  $(a_n)$  monoton wachsend ist, gilt auch  $\forall n > N_\varepsilon: a_{N_\varepsilon} \leq a_n$ . Daraus folgt, dass (1) wahr ist und  $(a_n)$  gegen  $\sup$  konvergiert. QED

## 1.6 Cauchy-Folgen

**Definition 1.8.** Eine Folge  $(a_n)_{n=1}^\infty$  heißt Cauchy-Folge (altmodisch auch Fundamentalfolge), wenn gilt:

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m, n \in \mathbb{N}: m, n \geq N_\varepsilon \implies |a_m - a_n| < \varepsilon$$

**Lemma 1.6** (Dreiecksungleichung).

$$\forall x, y \in \mathbb{R}: |x + y| \leq |x| + |y|$$

*Beweis.* Seien  $x, y \in \mathbb{R}$  und beliebig, aber fest. Da ein beliebiges  $a \in \mathbb{R}$  entweder positiv ( $a = |a|$ ) oder negativ ( $a = -|a|$ ) ist (und  $-a$  auch) gilt:

$$a \leq |a| \wedge -a \leq |a| \tag{1}$$

Für  $x + y$  müssen zwei Fälle überprüft werden:

1. Fall:  $x + y \geq 0$

$$\begin{aligned} |x + y| &= x + y \\ (1) \implies x + y &\leq |x| + |y| \end{aligned}$$

2. Fall:  $x + y < 0$

$$\begin{aligned} |x + y| &= -x - y \\ (1) \implies -x - y &\leq |x| + |y| \end{aligned}$$

QED

**Satz 1.7.** In den reellen Zahlen (in jeder topologisch abgeschlossenen Menge mit Abstandsbegriff) sind Konvergenz und Cauchy-Eigenschaft äquivalent.

*Beweis.* ( $\implies$ ) Sei  $(a_n)_{n=1}^\infty$  konvergent gegen  $a$ . Zu zeigen ist, dass  $(a_n)$  eine Cauchy-Folge ist:

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m, n \in \mathbb{N}: m, n \geq N_\varepsilon \implies |a_m - a_n| < \varepsilon$$

Wir wissen, dass  $(a_n)$  gegen  $a$  konvergiert. Es gilt also

$$\forall \varepsilon > 0: \exists N_\varepsilon: \forall n > N_\varepsilon: |a_n - a| < \varepsilon.$$

Weil die Aussage für alle  $\varepsilon$  (also auch für  $\frac{\varepsilon}{2}$ ) gilt, finden wir auch ein  $N_\varepsilon$  und ein  $M_\varepsilon$ , so dass gilt

$$\forall \varepsilon > 0: \forall n > N_\varepsilon, m > M_\varepsilon: |a_n - a| + |a_m - a| < \varepsilon.$$

Laut der Dreiecksungleichung gilt also auch

$$\forall \varepsilon > 0: \forall n > N_\varepsilon, m > M_\varepsilon: |a_m - a + a - a_n| \leq |a_n - a| + |a_m - a| < \varepsilon.$$

Das impliziert

$$\forall \varepsilon > 0: \forall n > N_\varepsilon, m > M_\varepsilon: |a_m - a_n| < \varepsilon.$$

Das ist äquivalent zur Definition der Cauchy-Folge, weil man ein  $K_\varepsilon$  bestimmen kann, welches größer oder gleich  $N_\varepsilon$  und  $M_\varepsilon$  ist. Man wähle also  $K_\varepsilon := \max\{N_\varepsilon, M_\varepsilon\}$ . Dann gilt

$$\forall \varepsilon > 0: \forall n, m > K_\varepsilon: |a_m - a_n| < \varepsilon.$$

Damit ist  $(a_n)$  eine Cauchy-Folge.

( $\Leftarrow$ ) Sei  $(b_n)$  eine Cauchy-Folge. Dann gilt

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m, n \geq N_\varepsilon: |b_m - b_n| < \varepsilon.$$

Weil diese Aussage für alle  $m, n \geq N_\varepsilon$  gilt, gilt sie auch für  $n = N_\varepsilon, m \geq N_\varepsilon$ . Das bedeutet, dass alle  $b_m$  nicht weiter von  $b_{N_\varepsilon}$  entfernt sind als  $\varepsilon$ . Oder auch

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m \geq N_\varepsilon: |b_m - b_{N_\varepsilon}| < \varepsilon.$$

Diese Aussage sei nicht zu verwechseln mit der Definition der Konvergenz. Der entscheidende Unterschied ist, dass  $b_{N_\varepsilon}$  kein fester Wert ist. Was allerdings aus dieser Aussage folgt ist, dass  $(b_m)$  für  $m \geq N_\varepsilon$  beschränkt ist. Die Folge ist auch für alle  $m < N_\varepsilon$  beschränkt, weil es nur endlich viele Folgenglieder mit diesem Kriterium gibt. Es lässt sich also eine obere Schranke als  $\max\{b_n | n < N_\varepsilon\}$  und eine untere Schranke als  $\min\{b_n | n < N_\varepsilon\}$  berechnen.

#### Hinweis

An dieser Stelle fehlt (im Moment) etwas. Falls Sie dazu beitragen möchten dieses Dokument zu vervollständigen, können Sie das auf [GitHub](#) tun. Diese Stelle befindet sich auf Zeile 385 der Datei `Ma-EK.tex`

QED



## 1.7 Einschachtelungssatz/Sandwichlemma

**Satz 1.8** (Einschachtelungssatz/Sandwichlemma). *Seien  $(a_n)_{n=1}^\infty$ ,  $(b_n)_{n=1}^\infty$  und  $(c_n)_{n=1}^\infty$  beliebige Folgen mit  $\forall n \in \mathbb{N}: a_n \leq b_n \leq c_n$  und  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = g$ . Dann konvergiert auch  $(b_n)$  gegen  $g$ .*

*Beweis.* Zu zeigen ist

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall n > N_\varepsilon: |b_n - g| < \varepsilon.$$

Gegeben ist

$$\forall \varepsilon > 0: \exists N_\varepsilon, M_\varepsilon \in \mathbb{N}: \forall n > N_\varepsilon, m > M_\varepsilon: |a_n - g| < \varepsilon \wedge |c_m - g| < \varepsilon.$$

Es existiert also für alle  $\varepsilon > 0$  ein  $N_\varepsilon$  und ein  $M_\varepsilon$ , so dass für  $K_\varepsilon = \max\{N_\varepsilon, M_\varepsilon\}$  gilt

$$\forall k > K_\varepsilon: |a_k - g| < \varepsilon \wedge |c_k - g| < \varepsilon.$$

Das und  $\forall n \in \mathbb{N}: a_n \leq b_n \leq c_n$  implizieren

$$\begin{aligned} & \forall k > K_\varepsilon: -\varepsilon < a_k - g \leq b_k - g \leq c_k - g < \varepsilon \\ \implies & \forall k > K_\varepsilon: -\varepsilon < b_k - g < \varepsilon \\ \iff & \forall k > K_\varepsilon: |b_k - g| < \varepsilon \end{aligned}$$

$(b_n)$  ist also ebenfalls konvergent gegen  $g$ .

QED

## 1.8 Teilfolgekriterium

**Satz 1.9** (Teilfolgekriterium). *Eine Folge  $(a_n)_{n=1}^\infty$  konvergiert genau dann gegen  $g$ , wenn jede Teilfolge von  $(a_n)$  ebenfalls gegen  $g$  konvergiert.*

*Beweis.* (  $\Leftarrow$  ) Wenn jede Teilfolge von  $(a_n)$  gegen  $g$  konvergiert, konvergiert auch  $(a_n)$  gegen  $g$ , weil  $(a_n)$  eine Teilfolge von  $(a_n)$  ist.

(  $\Rightarrow$  ) Indirekter Beweis.

**Annahme:** Es gibt eine Teilfolge, die nicht gegen  $g$  konvergiert. Dann existiert eine streng monoton steigende Folge von natürlichen Zahlen  $(n_k)_{k=1}^\infty$ , so dass die Folge  $b_k = a_{n_k}$  eine Teilfolge von  $(a_n)$  ist, für die gilt

$$\begin{aligned} & \neg \forall \varepsilon > 0: \exists K_\varepsilon \in \mathbb{N}: \forall k > K_\varepsilon: |b_k - g| < \varepsilon \\ \iff & \neg \forall \varepsilon > 0: \exists K_\varepsilon \in \mathbb{N}: \forall k > K_\varepsilon: |a_{n_k} - g| < \varepsilon \end{aligned} \tag{1}$$

Weil  $(a_n)$  gegen  $g$  konvergiert, gilt

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m > N_\varepsilon: |a_m - g| < \varepsilon.$$

Weil  $n_m \geq m > N_\varepsilon$  (strenge Monotonie von  $(n_k)$ ), gilt

$$\forall \varepsilon > 0: \exists N_\varepsilon \in \mathbb{N}: \forall m > N_\varepsilon: |a_{n_m} - g| < \varepsilon,$$

was im Widerspruch zu (1) steht.

QED

## 1.9 Grenzwertsätze für Folgen

**Satz 1.10** (Grenzwertsätze). *Seien  $(a_n)$  und  $(b_n)$  konvergente Folgen mit*

$$\lim_{n \rightarrow \infty} a_n = a \wedge \lim_{n \rightarrow \infty} b_n = b.$$

*Dann gelten folgende Aussagen.*

1.  $\lim_{n \rightarrow \infty} a_n + b_n = a + b$
2.  $\lim_{n \rightarrow \infty} a_n - b_n = a - b$
3.  $\lim_{n \rightarrow \infty} a_n \cdot b_n = a \cdot b$
4.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0 \wedge \exists N_\varepsilon: \forall n \geq N_\varepsilon: b_n \neq 0$

*Beweis.*

1. Weil  $(a_n)$  und  $(b_n)$  gegen  $a$  bzw.  $b$  konvergieren ( $|a_n - a|$  und  $|b_n - b|$  werden beliebig klein), gilt

$$\begin{aligned} & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |a_n - a| + |b_n - b| < \varepsilon \\ \implies & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |a_n - a + b_n - b| < \varepsilon \quad \text{Lemma 1.6} \\ \iff & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |a_n + b_n - (a + b)| < \varepsilon \end{aligned}$$

Das bedeutet, dass der Grenzwert von  $a_n + b_n = a + b$  ist. Damit ist 1. bewiesen.

2. Weil  $(a_n)$  und  $(b_n)$  gegen  $a$  bzw.  $b$  konvergieren, gilt analog zu 1.:

$$\begin{aligned} & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |a_n - a| + |b_n - b| < \varepsilon \\ \implies & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |(a_n - a) - (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon \quad \text{Lemma 1.6} \\ \iff & \forall \varepsilon > 0: \exists N_\varepsilon: \forall n \geq N_\varepsilon: |(a_n - b_n) - (a - b)| < \varepsilon \end{aligned}$$

Damit ist der Grenzwert von  $a_n - b_n = a - b$ .

3. Weil die Folge  $(a_n)$  konvergent ist, ist sie laut Satz 1.3 auch beschränkt. Es existiert also ein  $s \in \mathbb{R}^+$ , für die also gilt  $\forall n \in \mathbb{N}: |a_n| \leq s$ .

Wir müssen zeigen, dass für alle  $\varepsilon > 0$  ein  $N_\varepsilon$  existiert, so dass

$$\forall n \geq N_\varepsilon: |a_n b_n - ab| < \varepsilon.$$

Weil  $a_n$  gegen  $a$  konvergiert, existiert ein  $N_1$ , so dass

$$\forall n \geq N_1: |a_n - a| < \frac{\varepsilon}{2(|b| + 1)} \quad (1)$$

und analog dazu ein  $N_2$ , so dass

$$\forall n \geq N_2: |b_n - b| < \frac{\varepsilon}{2s}. \quad (2)$$

Wenn wir  $N_\varepsilon$  jetzt als  $\max\{N_1, N_2\}$  definieren, können wir die zu zeigende Ungleichung weiter Umformen.

$$\begin{aligned} & \forall n \geq N_\varepsilon: |a_n b_n - ab| < \varepsilon \\ \iff & \forall n \geq N_\varepsilon: |a_n b_n - a_n b + a_n b - ab| < \varepsilon \\ \iff & \forall n \geq N_\varepsilon: |a_n(b_n - b)| + |b(a_n - a)| < \varepsilon && \text{Dreiecksungleichung} \\ \iff & \forall n \geq N_\varepsilon: |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| < \varepsilon \\ \iff & s \cdot \frac{\varepsilon}{2s} + |b| \cdot \frac{\varepsilon}{2(|b| + 1)} < \varepsilon && \text{Ungleichungen 1, 2} \\ \iff & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon && \text{Weil } \frac{|b|}{|b| + 1} < 1 \\ \iff & \varepsilon \leq \varepsilon \end{aligned}$$

Weil  $\varepsilon \leq \varepsilon$  offensichtlich gilt, ist  $\lim_{n \rightarrow \infty} a_n b_n = ab$

4. Um 4. zu beweisen, muss nur bewiesen werden, dass  $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{b}$ , da  $\frac{a}{b} = (\frac{1}{b}) \cdot a$  und der Rest aus 3. folgt.

$$\lim_{x \rightarrow b} \frac{1}{x} = \frac{1}{b},$$

weil  $\frac{1}{x}$  stetig in  $b$  ist ( $b \neq 0$  ist gegeben). Aus der Definition für Stetigkeit über Folgen folgt, dass für alle Folgen  $(x_n)$ , die gegen  $b$  konvergieren — also auch für  $(b_n)$  — gilt:

$$\lim_{n \rightarrow \infty} \frac{1}{x_n} = \frac{1}{b}$$

QED

## 1.10 Übungsaufgaben

1. Finden Sie den Grenzwert der jeweiligen Folge.

(a)  $a_n = \frac{\frac{1}{n} + \frac{1}{n^2} + 1}{\frac{1}{n^3} + \frac{1}{n} + 3}$

(b)  $b_n = \frac{n^2 + n - 1}{n^3 + 1}$

$$(c) \quad c_n = \frac{\left(\frac{1}{3}\right)^n + \left(\frac{1}{4}\right)^n + 1^n}{\left(\frac{1}{5}\right)^n + 2}$$

$$(d) \quad d_n = \frac{3^n + 5^n + 7^n}{2^n + 3^n + 7^n}$$

2. Begründen Sie folgende Sätze mithilfe einer Skizze oder durch einen Beweis.

- (a) Eine Folge kann nur einen Grenzwert haben. (*Hinweis: Indirekter Beweis*)
- (b) Sei  $(a_n)$  eine beliebige Folge für die  $\forall n \in \mathbb{N}: a_n > 0$  gilt. Falls die Folge konvergiert kann  $\lim_{n \rightarrow \infty} a_n < 0$  nicht stimmen.

### 1.10.1 Lösungen

1.

- (a) Wir wissen, dass  $\frac{1}{n}$  mit  $n \in \mathbb{N}$  gegen 0 konvergiert (Satz 1.1).

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{1}{n^2} + 1}{\frac{1}{n^3} + \frac{7}{n} + 3} && \text{Satz 1.1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} && \text{Grenzwertsätze} \\ &= \frac{1}{3} \end{aligned}$$

#### Hinweis

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### 1.11 Bestimmte Divergenz

**Definition 1.9** (Bestimmte Divergenz). Die Folge  $(a_n)_{n=1}^{\infty}$  heißt genau dann bestimmt divergent gegen  $\infty$ , wenn gilt

$$\forall k: \exists N_k: \forall n \geq N_k: a_n > k.$$

$(a_n)$  heißt bestimmt divergent gegen  $-\infty$  genau dann, wenn gilt

$$\forall k: \exists N_k: \forall n \geq N_k: a_n < k.$$

**Satz 1.11** (Rechenregeln für bestimmte Divergenz (RRbD)). Seien  $(a_n)$  und  $(b_n)$  zwei beliebige Folgen, die bestimmt gegen  $\infty$  konvergieren und  $(c_n)$  eine beliebige Folge, die gegen ein beliebiges  $c \in \mathbb{R}$  konvergiert. Dann gelten folgende Aussagen.

1.  $\lim_{n \rightarrow \infty} a_n + b_n = \infty$

$$2. \lim_{n \rightarrow \infty} -a_n - b_n = -\infty$$

$$3. \lim_{n \rightarrow \infty} a_n \cdot b_n = \infty$$

$$4. \lim_{n \rightarrow \infty} a_n \cdot (-b_n) = -\infty$$

$$5. \lim_{n \rightarrow \infty} a_n + c_n = \infty$$

$$6. \lim_{n \rightarrow \infty} a_n \cdot c_n = \begin{cases} \infty, & c > 0 \\ -\infty, & c < 0 \end{cases}$$

$$7. \lim_{n \rightarrow \infty} \frac{a_n}{c_n} = \begin{cases} \infty, & c > 0 \\ -\infty, & c < 0 \\ \infty, & c = 0 \wedge \forall n \in \mathbb{N}: a_n > 0 \\ -\infty, & c = 0 \wedge \forall n \in \mathbb{N}: a_n < 0 \end{cases}$$

$$8. \lim_{n \rightarrow \infty} \frac{c_n}{a_n} = 0$$

## 2 Grenzwerte und Funktionen

### 2.1 Stetigkeit

**Definition 2.1** (Stetigkeit über Folgen). *Eine Funktion  $f: A \rightarrow \mathbb{R}$  heißt stetig in  $x_0 \in A$  genau dann, wenn für alle Folgen  $(x_n)_{n=1}^\infty$  mit*

$$\begin{aligned} &\forall n: x_n \in A \\ &\wedge \lim_{n \rightarrow \infty} x_n = x_0 \end{aligned}$$

*gilt  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .*

**Definition 2.2** ( $\varepsilon$ - $\delta$ -Definition für Stetigkeit). *Eine Funktion  $f: A \rightarrow \mathbb{R}$  heißt stetig in  $x_0 \in A$  genau dann, wenn*

$$\forall \varepsilon > 0: \exists \delta > 0: \forall x \in A \setminus \{x_0\}: |x_0 - x| < \delta \implies |f(x_0) - f(x)| < \varepsilon.$$

## Teil II

# Anhang

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