Mathe-Ergänzungskurs

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Teil I

Q1

1 Reelle Zahlenfolgen

1.1 Definitionen

Definition 1.1 (Reelle Zahlenfolge)**.**
$$\begin{aligned} a\colon \mathbb{N} &\to \mathbb{R} \\ n &\mapsto a(n) = a_n \end{aligned} \quad \textit{heißt reelle Zahlenfolge.}$$

Definition 1.2 (Bildungsvorschrift)**.** *Als Bildungsvorschrift bezeichnet man*

(a) $a(n) = f(n)$ *z.B.* $a(n) = n^2$ *(explizit)*

(b) $a(n) = f(a_1, \ldots, a_{n-1}, n)$ *z.B.* $a(n+1) = a(n) + a(n-1)$ *(rekursiv)*

Definition 1.3 (Monotonie)**.** *Eine beliebige Folge* (a_n) *ist...*

1. *...monton steigend genau dann, wenn*
$$\forall n_1, n_2 \in \mathbb{N}\colon n_1 > n_2 \implies a_{n_1} \geq a_{n_2}.$$

2. *...monoton fallend genau dann, wenn*
$$\forall n_1, n_2 \in \mathbb{N}\colon n_1 > n_2 \implies a_{n_1} \leq a_{n_2}.$$

3. *...streng monoton steigend genau dann, wenn*
$$\forall n_1, n_2 \in \mathbb{N}\colon n_1 > n_2 \implies a_{n_1} > a_{n_2}.$$

4. *...streng monoton fallend genau dann, wenn*
$$\forall n_1, n_2 \in \mathbb{N}\colon n_1 > n_2 \implies a_{n_1} < a_{n_2}.$$

Definition 1.4 (Beschränktheit)**.** *Eine beliebige Folge* (a_n) *ist...*

1. *...nach unten beschränkt genau dann, wenn*
$$\exists a \in \mathbb{R}\colon \forall n \in \mathbb{N}\colon a_n \geq a.$$

2. *...nach oben beschränkt genau dann, wenn*
$$\exists b \in \mathbb{R}\colon \forall n \in \mathbb{N}\colon a_n \leq b.$$

3. *...beschränkt genau dann, wenn sie nach oben und nach unten beschränkt ist.*

Definition 1.5 (Supremum)**.** *Das Supremum einer beliebigen nach oben beschränkten Folge* (a_n) *ist die kleinste obere Schranke dieser Folge.*

Definition 1.6 (Infimum)**.** *Analog zum Supremum ist das Infimum einer beliebigen nach unten beschränkten Folge* (a_n) *die größte untere Schranke dieser Folge.*

Definition 1.7 (Konvergenz)**.** *Eine beliebige Folge* (a_n) *ist konvergent gegen* g *genau dann, wenn*

$$\forall \varepsilon > 0 \colon \exists N_\varepsilon \in \mathbb{N} \colon \forall n \geq N_\varepsilon \colon |a_n - g| < \varepsilon.$$

1.2 Satz (Der Grenzwert von $a_n = \frac{1}{n}$ ist 0)

Satz 1.1. *Sei* $(a_n)_{n=1}^{\infty}$ *eine Folge mit der Bildungsvorschrift* $a_n = \frac{1}{n}$.
Dann gilt $\lim\limits_{n\to\infty} a_n = 0$

Beweis. Die Behauptung ist per Definition der Konvergenz (Definition 1.7) äquivalent zu

$$\begin{aligned} &\forall \varepsilon > 0 \colon \exists N_\varepsilon \in \mathbb{N} \colon \forall n \geq N_\varepsilon \colon |a_n - 0| < \varepsilon \\ \Longleftrightarrow\ &\forall \varepsilon > 0 \colon \exists N_\varepsilon \in \mathbb{N} \colon \forall n \geq N_\varepsilon \colon \left|\frac{1}{n}\right| < \varepsilon \end{aligned}$$

Diese Aussage gilt, weil es für jedes ε ein N_ε gibt, so dass für alle $n > N_\varepsilon$ der Betrag von $\frac{1}{n}$ kleiner als ε ist. Dieses N_ε lässt sich durch $\lceil \frac{1}{\varepsilon} \rceil + 1$ berechnen. QED

1.3 Satz (rekursive Summenfolge = explizite)

Satz 1.2. *Seien* $a_1(n)$ *und* $a_2(n)$ *Folgen mit den Bildungsforschriften*

$$\begin{aligned} &a_1(n) = a_1(n) + (n+1) \quad a_2(n) = \sum_{k=0}^{n} k \\ &a_1(0) = 0. \end{aligned}$$

Dann gilt $\forall n \colon a_1(n) = a_2(n)$.

Beweis. Der Beweis wird durch vollständige Induktion geführt.
Induktionsanfang: Für $n = 0$

$$a_1(0) = 0 \tag{1}$$

$$a_2(0) = \sum_{k=0}^{0} k = 0 \tag{2}$$

$$(1) \wedge (2) \implies a_1(0) = a_2(0)$$

Induktionsschritt: Induktionshypothese: $\exists n\colon a_1(n) = a_2(n)$
Zu zeigen ist, Ind. Hypot. $\implies a_1(n+1) = a_2(n+1)$

$$\begin{aligned}
a_1(n+1) &= a_1(n) + (n+1) \\
&= a_2(n) + (n+1) \quad \text{Ind. Hypot.} \\
&= \sum_{k=0}^{n} k + (n+1) \\
&= \sum_{k=0}^{n+1} k \\
&= a_2(n+1)
\end{aligned}$$

QED

1.4 Satz (Jede konvergente Folge ist beschränkt)

Satz 1.3. *Sei* $(a_n)_{n=1}^{\infty}$ *eine konvergente Folge mit dem Grenzwert* a*. Dann gilt*

$$\exists m, M \in \mathbb{R}\colon \forall n \in \mathbb{N}\colon m < a_n < M.$$

Beweis. Da a_n gegen a konvergiert gilt

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall n \geq N_\varepsilon\colon |a_n - a| < \varepsilon.$$

Da $|a_n - a| < \varepsilon$ in der oberen Aussage äquivalent zu $-x < a_n < x$ ist, gilt auch

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall n \geq N_\varepsilon\colon -\varepsilon < a_n - a < \varepsilon.$$

Für jedes $\varepsilon > 0$ existiert also ein N_ε, so dass a_n für alle $n \geq N_\varepsilon$ beschränkt ist. Da es nur endlich viele Folgenglieder für $n < N_\varepsilon$ gibt, lässt sich eine obere Grenze als

$$max(\{a_n | n < N_\varepsilon\} \cup \{\varepsilon + a\})$$

und eine untere Grenze als

$$min(\{a_n | n < N_\varepsilon\} \cup \{-\varepsilon + a\}\})$$

berechnen. QED

1.5 Satz von Bolzano-Weierstraß (I und II)

Satz 1.4 (Satz von Bolzano-Weierstraß I)**.** *Jede beschränkte Folge hat eine konvergente Teilfolge.*

Beweis. $(a_n)_{n=1}^{\infty}$ sei beschränkt durch $m \leq a_n \leq M$ für alle $n \in \mathbb{N}$. Man teile das Intervall $[n, M]$ in zwei Teile bei $\frac{m+M}{2}$.

1. Fall: Auf $\frac{m+M}{2}$ liegen unendlich viele Folgeglieder.

2. Fall: In $[m, \frac{m+M}{2}[$ liegen unendlich viele Folgeglieder.
   Dann beginne mit $[m, \frac{m+M}{2}[$ von vorne.

3. Fall: In $]\frac{m+M}{2}, M]$ liegen undenlich viele Folgeglieder.
   Dann beginne mit $]\frac{m+M}{2}, M]$ von vorne.

Das Verfahren...

(a) ... bricht mit Eintreten des ersten Falls ab und hat damit eine konvergente Teilfolge.

(b) ... setzt sich unendlich fort und erzeugt eine Folge von Intervallen mit

- $I_n \subset I_{n-1}, I_0 = [m; M]$,
- Länge von $I_n = \frac{M-m}{2^n} \overset{n\to\infty}{\to} 0$,
- Jedes Intervall enthält unendlich viele Folgeglieder.

Zu dieser Intervallschachtelung gehört genau eine reelle Zahl. Nimmt man aus jedem Intervall das Folgeglied mit dem kleinsten Index, welches noch nicht vorher ausgewählt wurde, erhält man eine Teilfolge, die gegen diese Zahl konvergiert. QED

Satz 1.5 (Satz von Bolzano-Weierstraß II)**.** *Jede beschränkte und monotone Folge ist konvergent.*

Beweis. O.B.d.A (Ohne Beschränkung der Allgemeinheit) sei $(a_n)_{n=1}^{\infty}$ monoton wachsend. Sei *sup* das Supremum von (a_n).

Weil *sup* das Supremum von (a_n) ist, gilt

$$\forall n \in \mathbb{N} \colon a_n \leq sup \wedge \forall \varepsilon > 0 \colon \exists N_\varepsilon \in \mathbb{N} \colon sup - \varepsilon < a_{N_\varepsilon}.$$

Wir zeigen nun, dass (a_n) gegen *sup* konvergiert, mit anderen Worten:

$$\forall \varepsilon > 0 \colon \exists N_\varepsilon \colon \forall n > N_\varepsilon \colon |a_n - sup| < \varepsilon. \tag{1}$$

Es gilt

$$\begin{aligned} & |a_n - sup| < \varepsilon \\ \iff & sup - a_n < \varepsilon \qquad \text{weil } sup > a_n \\ \iff & sup - \varepsilon < a_n \end{aligned}$$

Aussage (1) ist also wahr genau dann, wenn

$$\forall \varepsilon > 0\colon \exists N_\varepsilon\colon \forall n > N_\varepsilon\colon sup - \varepsilon < a_n.$$

Laut Definition des Supremums (1.5) gilt $\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon sup - \varepsilon < a_{N_\varepsilon}$. Weil (a_n) monoton wachsend ist, gilt auch $\forall n > N_\varepsilon\colon a_{N_\varepsilon} \leq a_n$. Daraus folgt, dass (1) wahr ist und (a_n) gegen *sup* konvergiert. QED

1.6 Cauchy-Folgen

Definition 1.8. *Eine Folge* $(a_n)_{n=1}^\infty$ *heißt Cauchy-Folge (altmodisch auch Fundamentalfolge), wenn gilt:*

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m, n \in \mathbb{N}\colon m, n \geq N_\varepsilon \implies |a_m - a_n| < \varepsilon$$

Lemma 1.6 (Dreiecksungleichung)**.**

$$\forall x, y \in \mathbb{R}\colon |x + y| \leq |x| + |y|$$

Beweis. Seien $x, y \in \mathbb{R}$ und beliebig, aber fest. Da ein beliebiges $a \in \mathbb{R}$ entweder positiv ($a = |a|$) oder negativ ($a = -|a|$) ist (und $-a$ auch) gilt:

$$a \leq |a| \wedge -a \leq |a| \tag{1}$$

Für $x + y$ müssen zwei Fälle überprüft werden:

1. Fall: $x + y \geq 0$

$$\begin{aligned} |x + y| &= x + y \\ (1) \implies x + y &\leq |x| + |y| \end{aligned}$$

2. Fall: $x_y < 0$

$$\begin{aligned} |x + y| &= -x - y \\ (1) \implies -x - y &\leq |x| + |y| \end{aligned}$$

QED

Satz 1.7. *In den reellen Zahlen (in jeder topologisch abgeschlossenen Menge mit Abstandsbegriff) sind Konvergenz und Cauchy-Eigenschaft äquivalent.*

Beweis. ($\implies$) Sei $(a_n)_{n=1}^\infty$ konvergent gegen a. Zu zeigen ist, dass (a_n) eine Cauchy-Folge ist:

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m, n \in \mathbb{N}\colon m, n \geq N_\varepsilon \implies |a_m - a_n| < \varepsilon$$

Wir wissen, dass (a_n) gegen a konvergiert. Es gilt also

$$\forall \varepsilon > 0\colon \exists N_\varepsilon\colon \forall n > N_\varepsilon\colon |a_n - a| < \varepsilon.$$

Weil die Aussage für alle ε (also auch für $\frac{\varepsilon}{2}$) gilt, finden wir auch ein N_ε und ein M_ε, so dass gilt

$$\forall \varepsilon > 0\colon \forall n > N_\varepsilon, m > M_\varepsilon\colon |a_n - a| + |a_m - a| < \varepsilon.$$

Laut der Dreiecksungleichung gilt also auch

$$\forall \varepsilon > 0\colon \forall n > N_\varepsilon, m > M_\varepsilon\colon |a_m - a + a - a_n| \leq |a_n - a| + |a_m - a| < \varepsilon.$$

Das impliziert

$$\forall \varepsilon > 0\colon \forall n > N_\varepsilon, m > M_\varepsilon\colon |a_m - a_n| < \varepsilon.$$

Das ist äquivalent zur Definition der Cauchy-Folge, weil man ein K_ε bestimmen kann, welches größer oder gleich N_ε und M_ε ist. Man wähle also $K_\varepsilon := max\,\{N_\varepsilon, M_\varepsilon\}$. Dann gilt

$$\forall \varepsilon > 0\colon \forall n, m > K_\varepsilon\colon |a_m - a_n| < \varepsilon.$$

Damit ist (a_n) eine Cauchy-Folge.

$(\Longleftarrow)$ Sei (b_n) eine Cauchy-Folge. Dann gilt

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m, n \geq N_\varepsilon\colon |b_m - b_n| < \varepsilon.$$

Weil diese Aussage für <u>alle</u> $m, n \geq N_\varepsilon$ gilt, gilt sie auch für
$n = N_\varepsilon, m \geq N_\varepsilon$. Das bedeutet, dass alle b_m nicht weiter von b_{N_ε} entfernt sind als ε. Oder auch

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m \geq N_\varepsilon\colon |b_m - b_{N_\varepsilon}| < \varepsilon.$$

Diese Aussage sei nicht zu verwechseln mit der Definition der Konvergenz. Der entscheidende Unterschied ist, dass b_{N_ε} kein fester Wert ist. Was allerdings aus dieser Aussage folgt ist, dass (b_m) für $m \geq N_\varepsilon$ beschränkt ist. Die Folge ist auch für alle $m < N_\varepsilon$ beschränkt, weil es nur endlich viele Folgeglieder mit diesem Kriterium gibt. Es lässt sich also eine obere Schranke als

$$max\,(\{b_n | n < N_\varepsilon\} \cup \{b_{N_\varepsilon} + \varepsilon\})$$

und eine untere Schranke als

$$min\,(\{b_n | n < N_\varepsilon\} \cup \{b_{N_\varepsilon} - \varepsilon\})$$

berechnen.

Hinweis
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QED

1.7 Einschachtelungssatz/Sandwichlemma

Satz 1.8 (Einschachtelungssatz/Sandwichlemma)**.** *Seien $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ und $(c_n)_{n=1}^{\infty}$ beliebige Folgen mit $\forall n \in \mathbb{N}\colon a_n \leq b_n \leq c_n$ und $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = g$. Dann konvergiert auch (b_n) gegen g.*

Beweis. Zu zeigen ist

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall n > N_\varepsilon\colon |b_n - g| < \varepsilon.$$

Gegeben ist

$$\forall \varepsilon > 0\colon \exists N_\varepsilon, M_\varepsilon \in \mathbb{N}\colon \forall n > N_\varepsilon, m > M_\varepsilon\colon |a_n - g| < \varepsilon \wedge |c_m - g| < \varepsilon.$$

Es existiert also für alle $\varepsilon > 0$ ein N_ε und ein M_ε, so dass für $K_\varepsilon = max\,\{N_\varepsilon, M_\varepsilon\}$ gilt

$$\forall k > K_\varepsilon\colon |a_k - g| < \varepsilon \wedge |c_k - g| < \varepsilon.$$

Das und $\forall n \in \mathbb{N}\colon a_n \leq b_n \leq c_n$ implizieren

$$\begin{aligned} &\forall k > K_\varepsilon\colon \; -\varepsilon < a_k - g \leq b_k - g \leq c_k - g < \varepsilon \\ \Longrightarrow\; &\forall k > K_\varepsilon\colon \; -\varepsilon < b_k - g < \varepsilon \\ \Longleftrightarrow\; &\forall k > K_\varepsilon\colon |b_k - g| < \varepsilon \end{aligned}$$

(b_n) ist also ebenfalls konvergent gegen g. QED

1.8 Teilfolgekriterium

Satz 1.9 (Teilfolgekriterium)**.** *Eine Folge $(a_n)_{n=1}^{\infty}$ konvergiert genau dann gegen g, wenn jede Teilfolge von (a_n) ebenfalls gegen g konvergiert.*

Beweis. ($\Longleftarrow$) Wenn jede Teilfolge von (a_n) gegen g konvergiert, konvergiert auch (a_n) gegen g, weil (a_n) eine Teilfolge von (a_n) ist.

($\Longrightarrow$) Indirekter Beweis.
Annahme: Es gibt eine Teilfolge, die nicht gegen g konvergiert. Dann existiert

eine streng monoton steigende Folge von natürlichen Zahlen $(n_k)_{k=1}^{\infty}$, so dass die Folge $b_k = a_{n_k}$ eine Teilfolge von (a_n) ist, für die gilt

$$\begin{aligned} &\neg\forall\varepsilon > 0\colon \exists K_\varepsilon \in \mathbb{N}\colon \forall k > K_\varepsilon\colon |b_k - g| < \varepsilon \\ \iff &\neg\forall\varepsilon > 0\colon \exists K_\varepsilon \in \mathbb{N}\colon \forall k > K_\varepsilon\colon |a_{n_k} - g| < \varepsilon \end{aligned} \tag{1}$$

Weil (a_n) gegen g konvergiert, gilt

$$\forall\varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m > N_\varepsilon\colon |a_m - g| < \varepsilon.$$

Weil $n_m \geq m > N_\varepsilon$ (strenge Monotonie von (n_k)), gilt

$$\forall\varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m > N_\varepsilon\colon |a_{n_m} - g| < \varepsilon,$$

was im Widerspruch zu (1) steht. QED

1.9 Grenzwertsätze für Folgen

Satz 1.10 (Grenzwertsätze)**.** *Seien (a_n) und (b_n) konvergente Folgen mit*

$$\lim_{n\to\infty} a_n = a \wedge \lim_{n\to\infty} b_n = b.$$

Dann gelten folgende Aussagen.

1. $\lim\limits_{n\to\infty} a_n + b_n = a + b$
2. $\lim\limits_{n\to\infty} a_n - b_n = a - b$
3. $\lim\limits_{n\to\infty} a_n \cdot b_n = a \cdot b$
4. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0 \wedge \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon b_n \neq 0$

Beweis.

1. Weil (a_n) und (b_n) gegen a bzw. b konvergieren ($|a_n - a|$ und $|b_n - b|$ werden beliebig klein), gilt

   $$\begin{aligned} &\forall\varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |a_n - a| + |b_n - b| < \varepsilon \\ \Longrightarrow &\forall\varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |a_n - a + b_n - b| < \varepsilon \qquad \text{Lemma 1.6} \\ \iff &\forall\varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |a_n + b_n - (a + b)| < \varepsilon \end{aligned}$$

   Das bedeutet, dass der Grenzwert von $a_n + b_n = a + b$ ist. Damit ist 1. bewiesen.

2. Weil (a_n) und (b_n) gegen a bzw. b konvergieren, gilt analog zu 1.:

$$\begin{aligned}
&\forall \varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |a_n - a| + |b_n - b| < \varepsilon \\
\Longrightarrow\ &\forall \varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |(a_n - a) - (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon \quad \text{Lemma 1.6} \\
\Longleftrightarrow\ &\forall \varepsilon > 0\colon \exists N_\varepsilon\colon \forall n \geq N_\varepsilon\colon |(a_n - b_n) - (a - b)| < \varepsilon
\end{aligned}$$

Damit ist der Grenzwert von $a_n - b_n = a - b$.

3. Weil die Folge (a_n) konvergent ist, ist sie laut Satz 1.3 auch beschränkt. Es existiert also ein $s \in \mathbb{R}^+$, für die also gilt $\forall n \in \mathbb{N}\colon |a_n| \leq s$.

   Wir müssen zeigen, dass für alle $\varepsilon > 0$ ein N_ε existiert, so dass

$$\forall n \geq N_\varepsilon\colon |a_n b_n - ab| < \varepsilon.$$

   Weil a_n gegen a konvergiert, existiert ein N_1, so dass

$$\forall n \geq N_1\colon |a_n - a| < \frac{\varepsilon}{2(|b| + 1)} \tag{1}$$

   und analog dazu ein N_2, so dass

$$\forall n \geq N_2\colon |b_n - b| < \frac{\varepsilon}{2s}. \tag{2}$$

   Wenn wir N_ε jetzt als $max\{N_1, N_2\}$ definieren, können wir die zu zeigende Ungleichung weiter Umformen.

$$\begin{aligned}
&\forall n \geq N_\varepsilon\colon |a_n b_n - ab| < \varepsilon \\
\Longleftrightarrow\ &\forall n \geq N_\varepsilon\colon |a_n b_n - a_n b + a_n b - ab| < \varepsilon \\
\Longleftarrow\ &\forall n \geq N_\varepsilon\colon |a_n(b_n - b)| + |b(a_n - a)| < \varepsilon && \text{Dreiecksungleichung} \\
\Longleftarrow\ &\forall n \geq N_\varepsilon\colon |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| < \varepsilon \\
\Longleftarrow\ &s \cdot \frac{\varepsilon}{2s} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} < \varepsilon && \text{Ungleichungen 1, 2} \\
\Longleftarrow\ &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon && \text{Weil } \frac{|b|}{|b|+1} < 1 \\
\Longleftrightarrow\ &\varepsilon \leq \varepsilon
\end{aligned}$$

   Weil $\varepsilon \leq \varepsilon$ offensichtlich gilt, ist $\lim\limits_{n\to\infty} a_n b_n = ab$

4. Um 4. zu beweisen, muss nur bewiesen werden, dass $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$, da $\frac{a}{b} = (\frac{1}{b}) \cdot a$ und der Rest aus 3. folgt.

   $\lim_{x\to b} \frac{1}{x} = \frac{1}{b}$, gilt weil $\frac{1}{x}$ stetig in b ist ($b \neq 0$ ist gegeben). Aus der Definition für Stetigkeit über Folgen folgt, dass für alle Folgen (x_n), die gegen b konvergieren — also auch für (b_n) — gilt:

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{b}$$

QED

1.10 Übungsaufgaben

1. Finden Sie den Grenzwert der jeweiligen Folge.
   (a) $a_n = \frac{\frac{1}{n}+\frac{1}{n^2}+1}{\frac{1}{n^3}+\frac{7}{n}+3}$
   (b) $b_n = \frac{n^2+n-1}{n^3+1}$
   (c) $c_n = \frac{\left(\frac{1}{3}\right)^n+\left(\frac{1}{4}\right)^n+1^n}{\left(\frac{1}{5}\right)^n+2}$
   (d) $d_n = \frac{3^n+5^n+7^n}{2^n+3^n+7^n}$
2. Begründen Sie folgende Sätze mithilfe einer Skizze oder durch einen Beweis.
   (a) Eine Folge kann nur einen Grenzwert haben. *(Hinweis: Indirekter Beweis)*
   (b) Sei (a_n) eine beliebige Folge für die $\forall n \in \mathbb{N}$: $a_n > 0$ gilt. Falls die Folge konvergiert kann $\lim\limits_{n\to\infty} a_n < 0$ nicht stimmen.

1.10.1 Lösungen

1.
   (a) Wir wissen, dass $\frac{1}{n}$ mit $n \in \mathbb{N}$ gegen 0 konvergiert (Satz 1.1).

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}+1}{\frac{1}{n^3}+\frac{7}{n}+3} && \text{Satz 1.1}\\ &= \lim_{n\to\infty} \frac{1}{3} && \text{Grenzwertsätze}\\ &= \frac{1}{3}\end{aligned}$$

Hinweis
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1.11 Bestimmte Divergenz

Definition 1.9 (Bestimmte Dviergenz). *Die Folge $(a_n)_{n=1}^{\infty}$ heißt genau dann* <u>*bestimmt*</u> *divergent gegen ∞, wenn gilt*

$$\forall k\colon \exists N_k\colon \forall n \geq N_k\colon a_n > k.$$

(a_n) heißt bestimmt divergent gegen $-\infty$ genau dann, wenn gilt

$$\forall k\colon \exists N_k\colon \forall n \geq N_k\colon a_n < k.$$

Satz 1.11 (Rechenregeln für bestimmte Divergenz (RRbD))**.** *Seien (a_n) und (b_n) zwei beliebe Folgen, die bestimmt gegen ∞ konvergieren und (c_n) eine beliebe Folge, die gegen ein beliebiges $c \in \mathbb{R}$ konvergiert. Dann gelten folgende Aussagen.*

1. $\lim\limits_{n\to\infty} a_n + b_n = \infty$
2. $\lim\limits_{n\to\infty} -a_n - b_n = -\infty$
3. $\lim\limits_{n\to\infty} a_n \cdot b_n = \infty$
4. $\lim\limits_{n\to\infty} a_n \cdot (-b_n) = -\infty$
5. $\lim\limits_{n\to\infty} a_n + c_n = \infty$
6. $\lim\limits_{n\to\infty} a_n \cdot c_n = \begin{cases} \infty, & c > 0 \\ -\infty, & c < 0 \end{cases}$
7. $\lim\limits_{n\to\infty} \frac{a_n}{c_n} = \begin{cases} \infty, & c > 0 \\ -\infty, & c < 0 \\ \infty, & c = 0 \land \forall n \in \mathbb{N} \colon a_n > 0 \\ -\infty, & c = 0 \land \forall n \in \mathbb{N} \colon a_n < 0 \end{cases}$
8. $\lim\limits_{n\to\infty} \frac{c_n}{a_n} = 0$

2 Grenzwerte und Funktionen

2.1 Stetigkeit

Definition 2.1 (Stetigkeit über Folgen)**.** *Eine Funktion $f \colon A \to \mathbb{R}$ heißt stetig in $x_0 \in A$ genau dann, wenn für alle Folgen $(x_n)_{n=1}^{\infty}$ mit*

$$\begin{gathered} \forall n \colon x_n \in A \\ \land \lim_{n\to\infty} x_n = x_0 \end{gathered}$$

gilt $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

Definition 2.2 ($\varepsilon-\delta$-Definition für Stetigkeit)**.** *Eine Funktion $f \colon A \to \mathbb{R}$ heißt stetig in $x_0 \in A$ genau dann, wenn*

$$\forall \varepsilon > 0 \colon \exists \delta > 0 \colon \forall x \in A \setminus \{x_0\} : |x_0 - x| < \delta \implies |f(x_0) - f(x)| < \varepsilon.$$

3 Metrische Räume

Definition 3.1 (Metrischer Raum)**.** *Seien $M \neq \emptyset$ und $d \colon M \times M \to [0; \infty[$ eine Metrik. M und d heißen zusammen metrischer Raum.*

3.1 Verallgemeinerungen der vorigen Abschnitte

Definition 3.2 (Allgemeine Grenzwerte und Konvergenz). *Sei $(a_n)_{n=1}^{\infty}$ eine Folge in M. Dann heißt (a_n) konvergent gegen den Grenzwert $a \in M$*

Definition 3.3 (Cauchy-Folge). *Eine Folge $(a_n)_{n=1}^{\infty}, \forall n\colon a_n \in M$ heißt Cauchy-Folge, wenn gilt:*

$$\forall \varepsilon > 0\colon \exists N_\varepsilon \in \mathbb{N}\colon \forall m, n > N_\varepsilon\colon d(a_m, a_n) < \varepsilon.$$

Definition 3.4 (Banach-Raum). *Ein metrischer Raum (M, d), in dem jede Cauchy-Folge konvergiert, heißt Banach-Raum oder vollständiger, metrischer Raum.*

Definition 3.5 (Stetigkeit). *Seien (X, d_x) und (Y, d_y) metrische Räume. Dann heißt $f\colon X \to Y$ stetig in $x_0 \in X$, wenn gilt:*

$$\forall \varepsilon > 0\colon \exists \delta > 0\colon \forall x \in X\colon d_x(x, x_0) < \delta \implies d_y(f(x), f(x_0)) < \varepsilon.$$

3.2 Der Fixpunktsatz von Banach

Satz 3.1 (Fixpunktsatz von Banach). *Seien (X, d) ein vollständiger, metrischer Raum und $M \subseteq X$ eine nichtleere, abgeschlossene Menge. Sei außerdem $\varphi\colon M \to M$ eine Lipschitz-stetige Funktion mit $L < 1$ (eine Kontraktion). Dann kann man zu jedem $x_0 \in M$ eine Folge definieren mit dem ersten Folgeglied x_0 und $x_{n+1} = \varphi(x_n)$ und es gibt ein $\xi \in M$ mit $\varphi(\xi) = \xi$, den Fixpunkt, sodass $\lim\limits_{n\to\infty} x_n = \xi$.*

Beweis. 1. $(x_n)_{n=1}^{\infty}$ ist eine Cauchy-Folge.
Seien x_n, x_{n+1} zwei aufeinanderfolgende Folgeglieder. Dann gilt

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq L \cdot d(x_n, x_{n-1}) \leq \cdots \leq L^n \cdot d(x_1, x_n).$$

Also $d(x_{n+1}, x_n) \overset{n\to\infty}{\longrightarrow} 0$. Wegen der Dreiecksungleichung (Lemma 1.6) gilt:

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k})$$

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Teil II

Anhang

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