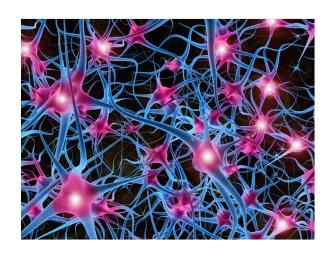
# Neural Networks: Multi-layer perceptron

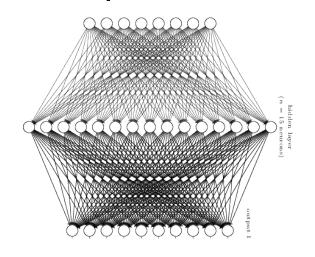
#### Objectives

- Neural net history
  - Biomimicry
  - Al Winter(s)
  - Enablers of current resurgence
- "Vanilla" neural networks: multilayer perceptrons
  - Parts of a neuron
  - Feed-forward
  - Backpropagation and gradient descent
- Provide examples of NN hyperparameters and training considerations
- Keras introduction (Python neural networks API)
- References

**Neural Net motivation** 

#### Our brain: the ultimate parallel computer





Number of parameters of biggest artificial neural net:

160,000,000,000 (<u>Digital Reasoning, 2015</u>)

The number of neurons in your brain:

The number of synapses in your brain:

1,000,000,000,000

The average number of synapses per neuron:

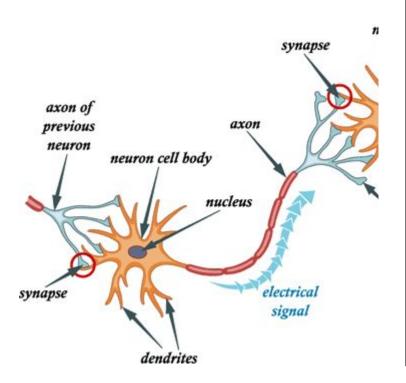
10,000

If we want a computer to be able to perform the same tasks as our brain, we should look to how the brain works for inspiration.

**Neural Net history** 

#### Biomimicry

## McCulloch-Pitts first neuron model in 1943



#### BULLETIN OF MATHEMATICAL BIOPHYSICS VOLUME 5, 1943

#### A LOGICAL CALCULUS OF THE IDEAS IMMANENT IN NERVOUS ACTIVITY

WARREN S. MCCULLOCH AND WALTER PITTS

From The University of Illinois, College of Medicine,
Department of Psychiatry at The Illinois Neuropsychiatric Institute,
and The University of Chicago

Because of the "all-or-none" character of nervous activity, neural events and the relations among them can be treated by means of propositional logic. It is found that the behavior of every net can be described in these terms, with the addition of more complicated logical means for nets containing circles; and that for any logical expression satisfying certain conditions, one can find a net behaving in the fashion it describes. It is shown that many particular choices among possible neurophysiological assumptions are equivalent, in the sense that for every net behaving under one assumption, there exists another net which behaves under the other and gives the same results, although perhaps not in the same time. Various applications of the calculus are discussed.

#### I. Introduction

Theoretical neurophysiology rests on certain cardinal assumptions. The nervous system is a net of neurons, each having a soma and an axon. Their adjunctions, or synapses, are always between the axon of one neuron and the soma of another. At any instant a neuron has some threshold, which excitation must exceed to initiate an impulse.

#### Improvements to MCP neuron

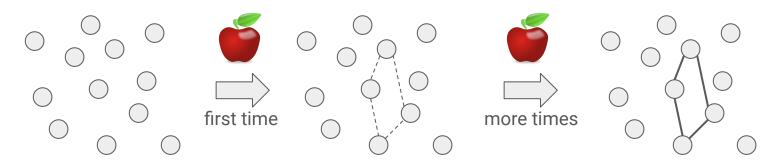
1949, Donald Hebb, neuropsychologist

"When an axon of cell A is near enough to excite cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased."

a.k.a.

"Cells that fire together wire together." - Hebb's rule

Repeated activity in response to a stimulus changes a neural network as it learns.

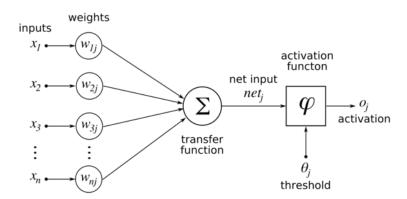


Neurons before learning about apple

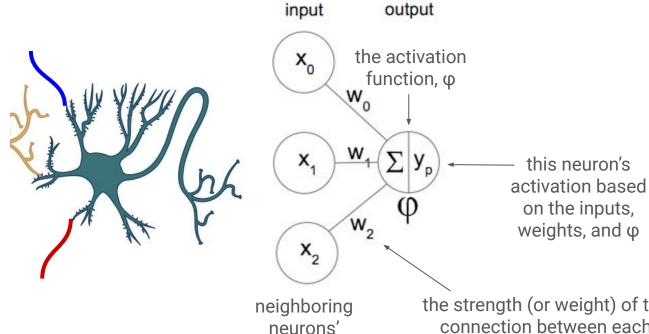
Apple has been learned

#### Improvements to MCP neuron

- 1957, Frank Rosenblatt invents the Perceptron
  - Initializes the weights (synaptic strength) to random values (e.g. -1.0 to 1.0)
  - Weights change during supervised learning according to the delta rule,  $\sim$  (yi yp). After a <u>certain</u> number of training passes through the whole training set (a.k.a. the number of <u>epochs</u>) stop changing the weights.
  - Implements a learning rate that affects how quickly weights can change.
  - Adds a bias to the activation function that shifts the location of the activation threshold and allows this location to be a learned quantity (by multiplying it by a weight).



#### Perceptron



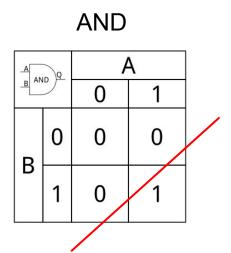
activations

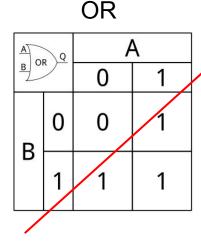
In	Inputs, X		Desired output,	
	X	21	у	
X <sub>0</sub>	<b>X</b> <sub>1</sub>	X <sub>2</sub>	y <sub>t</sub>	
1	0	0	0	
1	1	1	1	
1	0	1	0	
1	1	0	1	

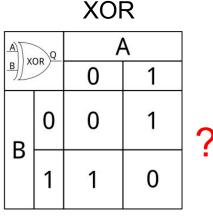
the strength (or weight) of the connection between each neighboring axon and this neuron.
After training, the weights are fixed.

#### Set-back: the XOR affair

- Perceptrons, an introduction to computational geometry (book by Minsky and Papert 1969)
  - From Wikipedia: "critics of the book state that the authors imply that, since a single artificial neuron is incapable of implementing some functions such as the <u>XOR</u> logical function, larger networks also have similar limitations, and therefore should be dropped."

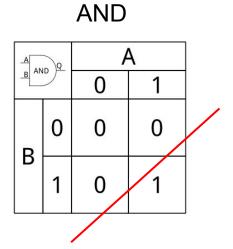


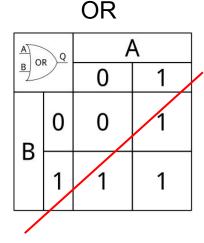


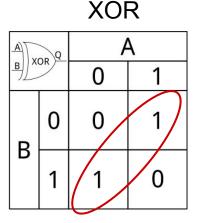


## XOR affair solution: go deeper (multi-layer)

- Perceptrons, an introduction to computational geometry (book by Minsky and Papert 1969)
  - From Wikipedia: "critics of the book state that the authors imply that, since a single artificial neuron is incapable of implementing some functions such as the <u>XOR</u> logical function, larger networks also have similar limitations, and therefore should be dropped."
  - Clarification: single layer perceptron networks are limited to being linear classifiers. Not true of deeper MLP (multi-layer perceptron) networks.



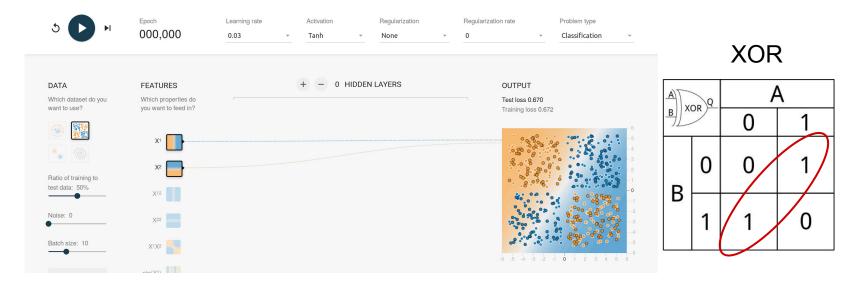




#### Prove it to yourself

"Single layer perceptron (no hidden layer) networks are limited to being linear classifiers. Not true of deeper MLP (multi-layer perceptron) networks."

https://playground.tensorflow.org/



How many hidden layers & neurons in the hidden layer are needed to learn XOR?

# Breakthrough for multi-layer networks

#### LEARNING INTERNAL REPRESENTATIONS BY ERROR PROPAGATION

David E. Rumelhart, Geoffrey E. Hinton, and Ronald J. Williams

September 1985

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ICS Report 8506

Parallel	NTARY NOTAT Distributed Pr Books/MII Pr	ocessing: Exploration	published in J. L. Moons in the Microstruc	cClelland, D. E. Rumelhart, & the cure of Cognition: Vol 1. Foundate	PDP Research Group, ions. Cambridge, MA:
17.	COSATI	CODES	18. SUBJECT TERMS	(Continue on reverse if necessary and	identify by block number)
FIELD	GROUP	SUB-GROUP		networks; perceptrons; ad achines; back propagation	
	This pa	per presents a ger		number)  perceptron learning procedure for crule, called the generalized d	_

error of the system's performance. The major theoretical contribution of the work is the procedure called error propagation, whereby the gradient can be determined by individual units of the network based only on locally available information. The major empirical contribution of the work is to show that the problem of local minima is not serious in this application of gradient descent.

link to paper

# Back propagation

#### LEARNING INTERNAL REPRESENTATIONS BY ERROR PROPAGATION

David E. Rumelhart, Geoffrey E. Hinton, and Ronald J. Williams

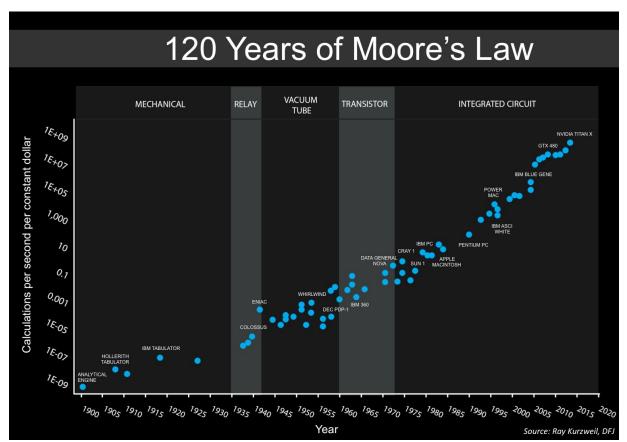
September 1985

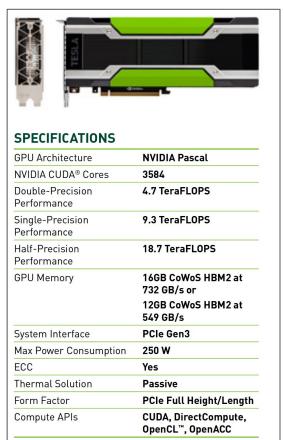
ICS Report 8506



This paper presents a generalization of the perceptron learning procedure for learning the correct sets of connections for arbitrary networks. The rule, called the generalized delta rule, is a simple scheme for implementing a gradient descent method for finding weights that minimize the sum squared error of the system's performance. The major theoretical contribution of the work is the procedure called error propagation, whereby the gradient can be determined by individual units of the network based only on locally available information. The major empirical contribution of the work is to show

#### Breakthrough: compute power





Steve Jurvetson, Moore's Law over 120 Years, CC BY 2.0

### Signs of the current revolution

2009: Microsoft researcher Li Deng invited neural nets pioneer <u>Geoffrey Hinton</u> to visit. Impressed with his research, Deng's group experimented with neural nets for speech recognition. "We were achieving more than 30% improvements in accuracy with the very first prototypes."

2011 & 2012: Microsoft and Google introduced deep-learning technology into their commercial speech-recognition products.

2012: At a workshop in Florence, Italy, the founder of the annual <u>ImageNet</u> computer-vision contest announced that two of Hinton's students had identified objects with almost twice the accuracy of the nearest competitor. "It was a spectacular result," recounts Hinton, "and convinced lots and lots of people who had been very skeptical before."

But beware: an A.I. Winter may yet be coming. Hype precedes it.

## (Artificial) Neural networks

#### Paraphrased from Wikipedia:

- Artificial neural networks (ANNs) are a family of models inspired by biological neural networks
- Systems of interconnected 'neurons' which exchange messages...
- The connections have numeric weights that can be tuned based on experience, making neural nets ... capable of learning.

"Like other machine learning methods – systems that learn from data – neural networks have been used to solve a wide variety of tasks, like computer vision and speech recognition, that are hard to solve using ordinary rule-based programming."

#### ANNs we will talk about

ANN	Description Past capstones	Picture of architecture
Multilayer perceptron (MLP)	Standard algorithm for supervised learning. Used for pattern, speech, image recognition.  Smarter than Nate Silver Predicting Solar Power Plant Insolation	Input First Second Output layer layer layer
Autoencoder	Used for unsupervised learning of efficient codings, usually for dimensionality reduction, though recently to make generative models.  Tasty-Tweets Solving LSAT Puzzles with seq2seq Wedfuly Recommender	Output Decode Hidden Encode

#### ANNs we will talk about

ANN	Description Past capstones	Picture of architecture
Convolutional neural network	Node connections inspired by visual cortex. Uses kernels to aggregate/transform information in network. Used for image, video recognition, nlp  EyeNet, Curb Appeal, Wildflower Classifier ChefNet Brain Tumor Segmentation Detect Depression in Speech	Ource pixel (0) (0) (0) (0) (0) (0) (0) (0) (0) (0)
Recurrent neural network	Connections between nodes can be cyclical, giving the network memory. Used for sequences: handwriting, speech recognition, time series.  RadStaffer GOT Book Generator ABC Music	Input layer Output layer

## ANN topics we will talk about

Topic	Description Past capstones	Picture
Reinforcement Learning	Use NNs feature extraction and prediction abilities to learn from an environment gitMunny - Crypto Trading with Deep Q Networks Sphero Navigation	Environment  Reward  Interpreter  State  Agent
Transfer Learning	Leverage vetted NN architectures and pre-trained weights to get good performance on limited data <u>Dog Breed Classifier</u>	

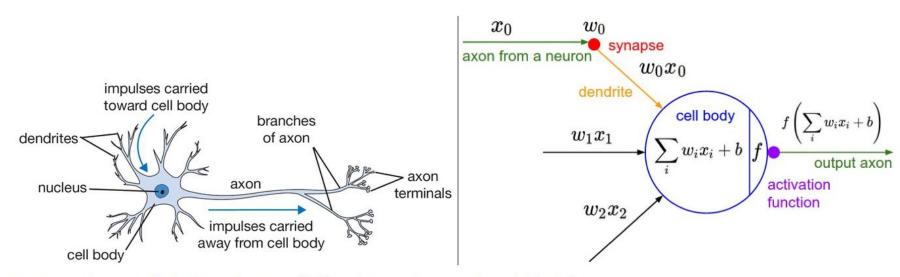
## multilayer perceptron

"Vanilla" neural network:

#### The parts

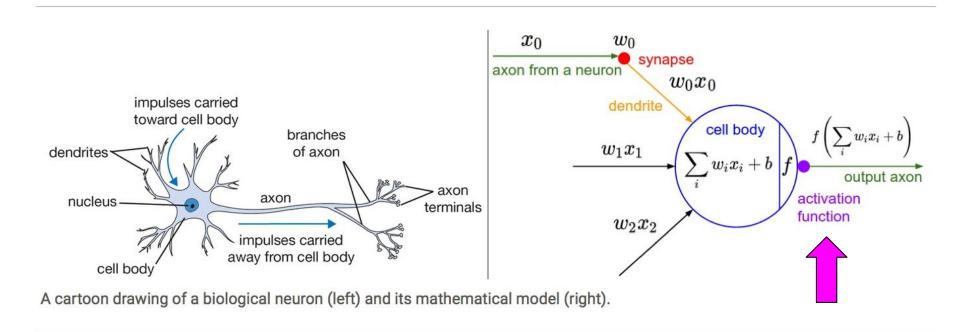
- Computation unit: perceptron (neuron)
  - Input, weights, summation, activation function, prediction
- Use a single neuron to explain:
  - Feed-forward
  - Backpropagation (get gradients)
  - Gradient descent and its flavors (stochastic, batch, minibatch)
  - Extrapolate to multilayer networks
- Gradient descent solution optimizers
- Multi-layer network
- MLP code-it-from-scratch pseudocode

### A perceptron (neuron, neurode, node, unit)

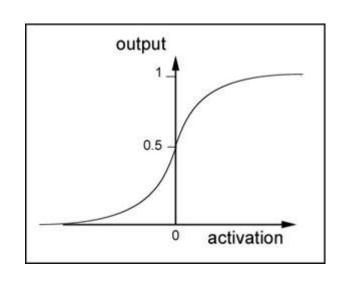


A cartoon drawing of a biological neuron (left) and its mathematical model (right).

### A perceptron (neuron, neurode, node, unit)



#### Sigmoid activation function, and its derivative



$$y = \frac{1}{1 + e^{-x}}$$

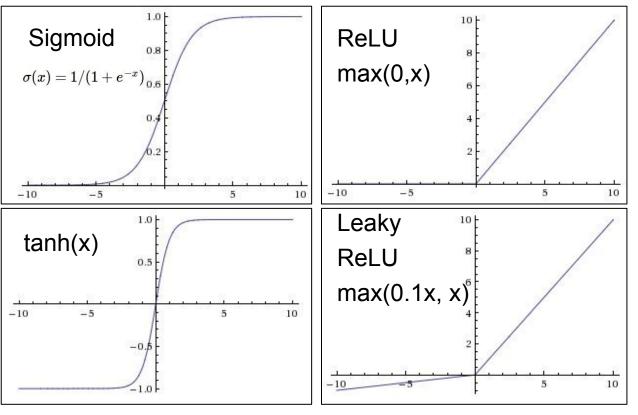
quotient rule:

$$derivate\left(\frac{f}{g}\right) = \left(\frac{f'g - g'f}{g^2}\right)$$

$$\frac{dy}{dx} = \frac{0(1 + e^{-x}) - (-1 * e^{-x})}{(1 + e^{-x})^2}$$

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = (1-y)y$$

#### Activation functions - a subset



Your choice of an activation function for a given layer of your network should be informed by literature and your experience.

Questions to consider when selecting:

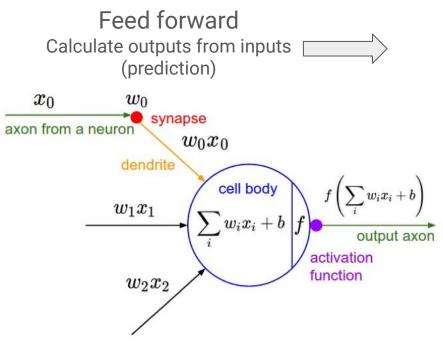
- 1) Benefit of zero mean?
- Does derivative exist everywhere and does it give useful information?
- 3) Easy to compute?
- 4) Weight and bias initializations differ for these - take care!

#### Karpathy:

ReLU > Leaky ReLU > tanh (but be careful of ReLU learning rates)

and: **beware of sigmoid!** (exploding, vanishing gradient issues, slower to compute)

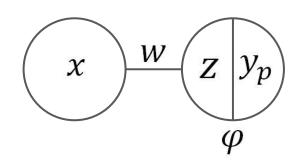
#### Computations



Back propagation & gradient descent

**During training**, use back-propagation to calculate the gradient of the cost function with respect to each weight, then use gradient descent to change the weights so that predicted output is closer to train output.

### Computations - Feed forward (in 1 perceptron)

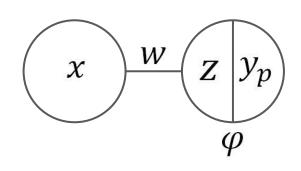


#### Given

Input,  $\chi$ Weight, WProduct, z=wxActivation function,  $\varphi$ 

What is the prediction,  $y_p$  ?

## Computations - Feed forward (in 1 perceptron)



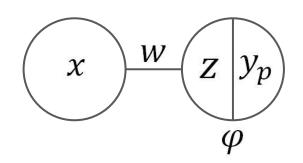
#### Given

Input,  $\chi$ Weight, WProduct, z = wxActivation function,  $\varphi$ 

What is the prediction,  $y_p$  ?

$$y_p = \varphi(z)$$
where
$$z = wx$$

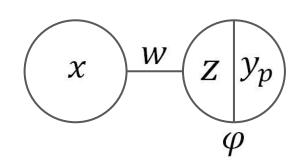
#### Computations: Backpropagation (calculate the gradient)



Given Cost function, 
$$E = \frac{1}{2}(y_t - y_p)^2$$

What is the gradient of the cost function with respect to the weight?  $\frac{\partial E}{\partial w}$ 

#### Computations: Backpropagation (calculate the gradient)



Given Cost function,  $E = \frac{1}{2}(y_t - y_p)^2$ 

What is the gradient of the cost function with respect to the weight?

$$\frac{\partial E}{\partial w}$$

Hint: Use the power rule and the chain rule

Backpropagation (1)

∂w

 $\partial E \partial y_p \partial \varphi \partial z$ 

 $\partial y_p \partial \varphi \partial z \partial w$ 

power rule & chain rule power rule

chain rule

(Leibniz notation)

Cost function,

sigmoid function (see earlier slide)

derivative of

 $E = \frac{1}{2} \big( y_t - y_p \big)^2$ What is the gradient of the cost function with respect to the

Given

weight?

power rule  $-(y_t - y_p) \cdot 1 \cdot \varphi(z) (1 - \varphi(z)) \cdot x$  Backpropagation (2)  $\frac{\partial E}{\partial w} = \frac{\partial E}{\partial y_p} \frac{\partial y_p}{\partial \varphi} \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial w}$ 

$$\begin{array}{c|c}
\hline
x & w \\
\hline
\varphi
\end{array}$$
Civen

$$\frac{\partial E}{\partial w} = -(y_t - y_p) \cdot 1 \cdot \varphi(z) (1 - \varphi(z)) \cdot x$$

$$\frac{\partial E}{\partial w} = -x(y_t - y_p)\varphi(z)(1 - \varphi(z))$$
Cost function.

$$E = \frac{1}{2} \left( y_t - y_p \right)^2$$

What is the gradient of the cost 
$$\frac{\partial E}{\partial w}$$
 function with respect to the weight?

#### Computations: Gradient descent to update the weight

 $\frac{\text{Given}}{\text{Gradient,}} \frac{\partial E}{\partial w}$ 

Learning rate, lpha

How should the weight be updated?  $\Delta w$ 

### Computations: Gradient descent to update the weight

$$(x)$$
  $w$   $(z)$   $y_p$   $\varphi$ 

z = wx

 $\Delta w = -\alpha \frac{\partial E}{\partial w}$ 

Given Gradient, 
$$\frac{\partial E}{\partial w}$$

Learning rate,  $\alpha$ 

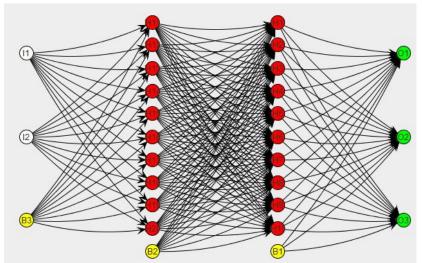
 $\frac{\partial E}{\partial w} = -x(y_t - \varphi(wx))\varphi(wx)(1 - \varphi(wx))$ 

How should the weight be updated?  $\Delta w$ 

 $\Delta w = \alpha x (y_t - \varphi(wx)) \varphi(wx) (1 - \varphi(wx))$ 

#### Computations

Feed forward
Calculate outputs from inputs (prediction)





**During training**, use back-propagation to calculate the gradient of the cost function with respect to each weight, then use gradient descent to change the weights so that predicted output is closer to train output.

#### Computations

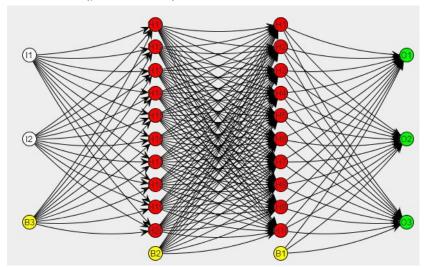
**Goal:** Minimize the error or loss function - RSS (regression), misclassification rate (classification) - by changing the weights in the model.

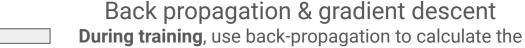
**Back propagation**, the recursive application of the chain rule back along the computational network, allows the calculation of the local gradients, enabling....

**Gradient descent** to be used to find the changes in the weights required to minimize the loss function.

The idea is the same for a single perceptron, or for a multi-layer network.

Feed forward
Calculate outputs from inputs (prediction)





gradient of the cost function with respect to each weight, then use gradient descent to change the weights so that predicted output is closer to train output.

#### MLP pseudocode from scratch

In groups of 4, write some pseudocode to do this!

#### MLP pseudocode from scratch

```
# showing training using stochastic gradient descent
# training (attempt to converge on weights)
for a desired number of epochs:
    for each row of X, y in inputs, targets:
        Feed-forward to find:
             node activations
            prediction
        Calculate loss
        Backpropagate to find the gradient of the loss w.r.t. the weights
        Use gradient descent to update the weights
    print the training error
# now that the weights are trained
for all test data:
    Feed-foward to find the predictions
print the test error
```

#### Batch, Mini-Batch, and SGD in pseudocode

```
loop maxEpochs times
 for-each data item
   compute a gradient for each weight and bias
   accumulate gradient
 end-for
 use accumulated gradients to update each weight and bias
end-loop
                                                loop maxEpochs times
                                                  loop until all data items used
                                                    for-each batch of items
                                                      compute a gradient for each weight and bias
                                                      accumulate gradient
                                                    end-batch
                                                    use accumulated gradients to update each weight and bias
                                                  end-loop all item
                                                end-loop
loop maxEpochs times
```

for-each data item
compute gradients for each weight and bias
use gradients to update each weight and bias
end-for
end-loop

#### Batch, Mini-Batch, and SGD in pseudocode

```
loop maxEpochs times
                                                 Batch
 for-each data item
   compute a gradient for each weight and bias
   accumulate gradient
 end-for
 use accumulated gradients to update each weight and bias
end-loop
                                                loop maxEpochs times
                                                                                                  Mini-Batch
                                                  loop until all data items used
                                                    for-each batch of items
                                                      compute a gradient for each weight and bias
                                                      accumulate gradient
                                                    end-batch
                                                    use accumulated gradients to update each weight and bias
                                                  end-loop all item
                                                end-loop
```

```
loop maxEpochs times

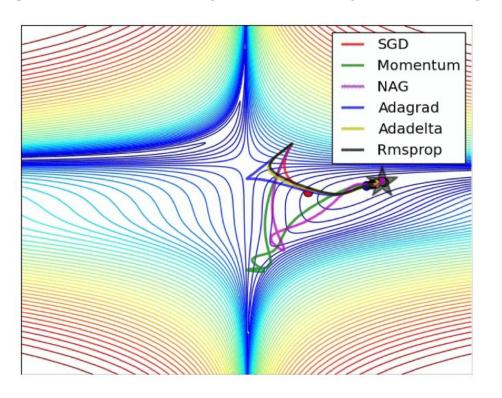
for-each data item

compute gradients for each weight and bias

use gradients to update each weight and bias
end-for
end-loop
```

#### Ok - gradient descent, but how?

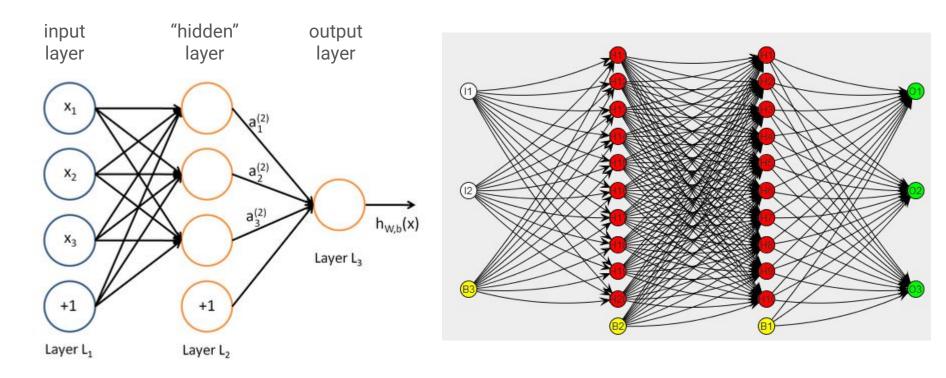
Different gradient descent optimizers: <a href="http://cs231n.github.io/neural-networks-3/">http://cs231n.github.io/neural-networks-3/</a>



See also:

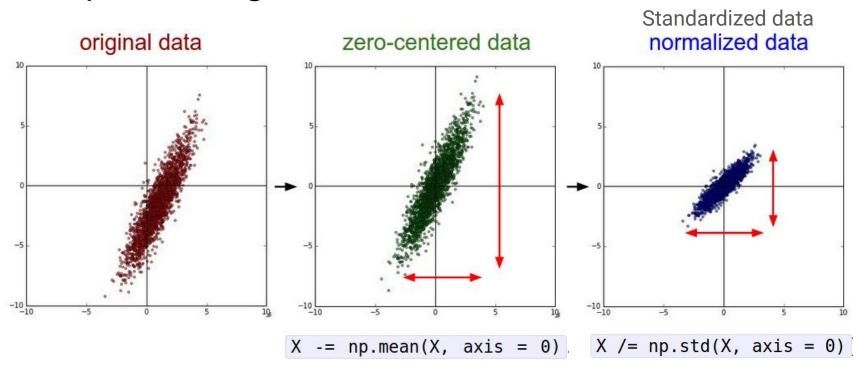
https://keras.io/optimizers/

## Multilayer perceptron architecture



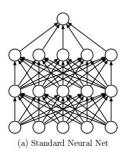
# Training

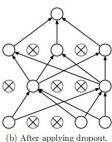
#### Pre-processing



Karpathy advice for images: subtract the mean image only, can do it per channel

- Most are particular to your application: read the literature and start with something that has been shown to work well.
- Structure: the number of hidden layers, the number of nodes in each layer
- Activation functions
- Weight and bias initialization (for weights Karpathy recommends Xavier init.)
- Training method: Loss function, learning rate, batch size, number of epochs
- Regularization: Likely needed! (NN very complex models that will overfit)
- weight decay (L1 & L2, see <a href="here">here</a>), early stopping, dropout
- Random seed





Yes, it's a lot of hyperparameters.

Yes, it's a lot of hyperparameters.

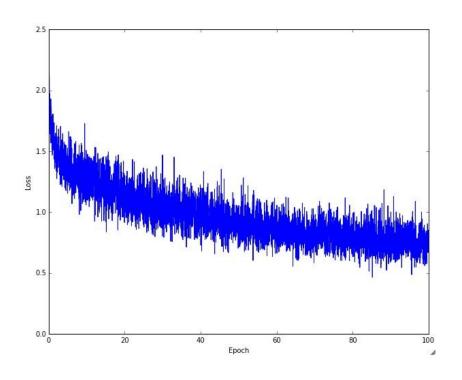
How do you figure out the right ones?

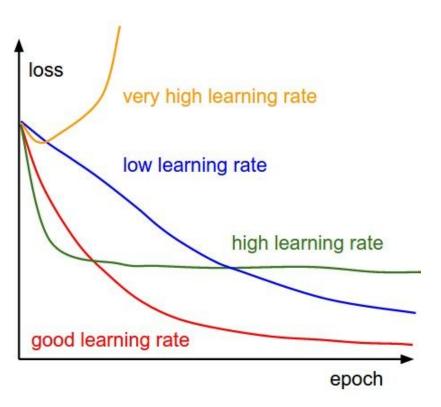
Yes, it's a lot of hyperparameters.

How do you figure out the right ones?

- 1) Search the literature for similar use case and architecture to define a baseline model.
- Once you have that, start varying the hyperparameters and check performance using cross-validation.

#### Monitor training process





#### Potential problems

- Instability, difficulty visualizing the training process, interpretability
- The vanishing (or exploding) gradient problem during backpropagation:

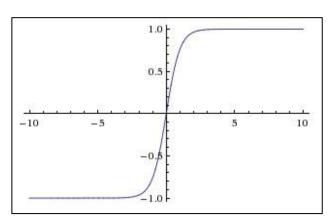
$$\frac{\partial E}{\partial w_{ij}} = \frac{\partial E}{\partial o_j} \frac{\partial o_j}{\partial net_j} \frac{\partial net_j}{\partial w_{ij}} \qquad input$$

$$\Delta w_{ij} = -\alpha \left( \frac{\partial E}{\partial w_{ij}} \right) \qquad o_3 \qquad w_{32} \qquad net_2 \qquad o_2 \qquad w_{21} \qquad net_1 \qquad o_1 \qquad w_{10} \qquad net_0 \qquad o_0 = y_p$$

$$\Delta w_{32} = -\alpha \left( \frac{\partial E}{\partial w_{32}} \right)$$

$$\frac{\partial E}{\partial w_{32}} \propto \varphi'(net_0) \cdot \varphi'(net_1) \cdot \varphi'(net_2)$$

The weights farthest away from the output can have very large, or very small, updates.



Implementing a neural network

in Python using Keras

#### **Keras**

Models

About Keras models

Sequential





About Keras lavers

Core Layers

Convolutional Layers

Recurrent Layers

**Embedding Layers** 

Advanced Activations Layers

Normalization Layers

Noise layers

Layer wrappers

Writing your own Keras layers

Preprocessing

Sequence Preprocessin

**Text Preprocessing** 

Image Preprocessing

Objectives

Optimizers

Activations

#### Keras: Deep Learning library for Theano and TensorFlow

#### You have just found Keras.

Keras is a minimalist, highly modular neural networks library, written in Python and capable of running on top of either TensorFlow or Theano. It was developed with a focus on enabling fast experimentation. Being able to go from idea to result with the least possible delay is key to doing good research.

Use Keras if you need a deep learning library that:

- allows for easy and fast prototyping (through total modularity, minimalism, and extensibility).
- supports both convolutional networks and recurrent networks, as well as combinations of the two.
- supports arbitrary connectivity schemes (including multi-input and multi-output training).
- · runs seamlessly on CPU and GPU.

#### Examples

Read the documentation at Ke

Keras is compatible with: Pyth

Here are a few examples to get you started!

In the examples folder, you will also find example models for real datasets:

- CIFAR10 small images classification: Convolutional Neural Network (CNN) with realtime data augmentation
- IMDB movie review sentiment classification: LSTM over sequences of words
- Reuters newswires topic classification: Multilayer Perceptron (MLP)
- . MNIST handwritten digits classification: MLP & CNN
- · Character-level text generation with LSTM

...and more.

#### Overview of example code...

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```
Compare
3 layer numpy.py
3 layer keras.py
                        X = np.array([[0,0,1],
                                      [0,1,1],
                                      [1,0,1],
                                      [1,1,1]
                        y = np.array([[0]],
                                      [1],
                                      [1],
                                      [0])
```

#### References

- http://cs231n.stanford.edu/ Andrej Karpathy slides, lectures on youtube
- <a href="http://playground.tensorflow.org">http://playground.tensorflow.org</a> Visualize neural network training online
- A.I. vs. Machine Learning vs. Deep Learning
- Goodfellow et. al. <u>Deep Learning Book (online)</u>

#### Objectives

- Neural net history
  - Biomimicry
  - Al Winter(s)
  - Enablers of current resurgence
- "Vanilla" neural networks: multilayer perceptrons
  - Parts of a neuron
  - Feed-forward
  - Backpropagation and gradient descent
- Provide examples of NN hyperparameters and training considerations
- Keras introduction (Python neural networks API)
- References

# Appendix

#### Activation function derivatives

	Propagation	Back-propagation
Sigmoid	$y_S = \frac{1}{1 + e^{-x_S}}$	$\left[\frac{\partial E}{\partial x}\right]_{S} = \left[\frac{\partial E}{\partial y}\right]_{S} \frac{1}{(1 + e^{x_{S}})(1 + e^{-x_{S}})}$
Tanh	$y_s = \tanh(x_s)$	$\left[\frac{\partial E}{\partial x}\right]_S = \left[\frac{\partial E}{\partial y}\right]_S \frac{1}{\cosh^2 x_S}$
ReLu	$y_s = \max(0, x_s)$	$\left[\frac{\partial E}{\partial x}\right]_{S} = \left[\frac{\partial E}{\partial y}\right]_{S} \mathbb{I}\{x_{S} > 0\}$
Ramp	$y_s = \min(-1, \max(1, x_s))$	$\left[\frac{\partial E}{\partial x}\right]_{S} = \left[\frac{\partial E}{\partial y}\right]_{S} \mathbb{I}\{-1 < x_{S} < 1\}$

http://www.marekrei.com/blog/26-things-i-learned-in-the-deep-learning-summer-school/

#### Loss function derivatives

		Propagation	Back-propagation
Square		$y = \frac{1}{2}(x-d)^2$	$\frac{\partial E}{\partial x} = (x - d)^T \frac{\partial E}{\partial y}$
Log	$c = \pm 1$	$y = \log(1 + e^{-cx})$	$\frac{\partial E}{\partial x} = \frac{-c}{1 + e^{cx}} \frac{\partial E}{\partial y}$
Hinge	$c = \pm 1$	$y = \max(0, m - cx)$	$\frac{\partial E}{\partial x} = -c  \mathbb{I}\{cx < m\} \frac{\partial E}{\partial y}$
LogSoftMax	$c=1\dots k$	$y = \log(\sum_k e^{x_k}) - x_c$	$\left[\frac{\partial E}{\partial x}\right]_{S} = \left(e^{x_{S}}/\sum_{k} e^{x_{k}} - \delta_{SC}\right) \frac{\partial E}{\partial y}$
MaxMargin	$c=1\dots k$	$y = \left[\max_{k \neq c} \{x_k + m\} - x_c\right]_+$	$\left[\frac{\partial E}{\partial x}\right]_{S} = \left(\delta_{sk^*} - \delta_{sc}\right) \mathbb{I}\{E > 0\} \frac{\partial E}{\partial y}$