# Bayesian Inference

Chris (credit Frank, Ryan)

galvanize

# Objectives

1. Understand Frequentists vs. Bayesian
2. Be able to use Bayes' Rule
3. State Bayes theorem and the identify the:
    a. Prior
    b. Likelihood
    c. Posterior
    d. Normalizing Constant
4. Use Bayes theorem to update posterior repeatedly with new data points

# Warm up

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1. A coin is biased at 0.8 in favor of heads. What is the probability of flipping 6 or more heads in 8 of this coin?
2. Suppose we randomly select a driver from this group. What is the probability that the driver:

| **Age Group** | 18-25 | 26-39 | 40-55 | 55+ | |
|---|---|---|---|---|---|
| 0-1 Accidents | 100 | 150 | 250 | 75 | **575** |
| 2-3 Accidents | 150 | 25 | 125 | 25 | **325** |
| 3+ accidents | 50 | 25 | 25 | 0 | **100** |
| **Totals** | **300** | **200** | **400** | **100** | **1000** |

   a. Is in the 26-39 age group, given they have more than 3 accidents?
   b. Had 2-3 accidents, given they are in the 18-25 age group?
   c. Had 3+ accidents, given that they have had more than 1 accident and are between 26 and 55 years old?

# Intuition - Part 1

What is the probability that it rained in my city last night?

(No info is given about which city I'm currently in.)

$$P(\text{rain}) = 0.1$$

What is the probability that it rained in my city last night <u>given</u> that I'm in LA?

$$P(rain|LA) = 0.02$$

What is the probability that it rained in my city last night?

(No info is given about which city I'm currently in.)

$$P(\text{rain}) = 0.1$$

What is the probability that it rained in my city last night <u>given</u> that I live in LA <u>and</u> I see that the road is dry?

$$P(rain|LA, dry\ roads) = 0.001$$

# Intuition - Part 2

You arrive at a road that is closed for a race.
You see a runner wearing bib #77.

How many runners are in the race?

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You arrive at a road that is closed for a race.
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You see another runner wearing bib #4.
You see a pair of runners wearing bibs #68 & 3.

How many runners are in the race?

# Intuition - Part 2

You arrive at a road that is closed for a race.
You see a runner wearing bib #77.
You see another runner wearing bib #4.
You see a pair of runners wearing bibs #68 & 3.
You see a runner with bib #3043.

How many runners are in the race?

# Defining probability

<u>Frequentist Probability</u>

"Long Run" frequency of an outcome

<u>Subjective Probability</u>

A measure of degree of belief

Bayesians consider both types

# Frequentist vs. Bayesian

## Experiment 1:

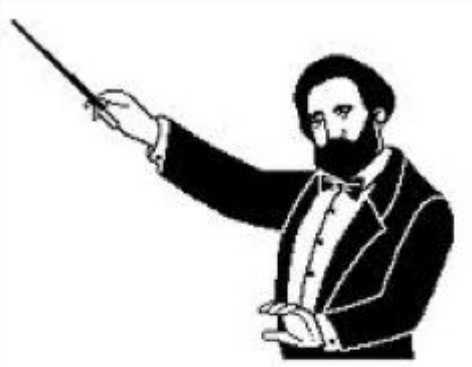
A fine classical musician says he's able to distinguish Haydn from Mozart. Small excerpts are selected at random and played for the musician. Musician makes 10 correct guesses in exactly 10 trials.

## Experiment 2:

Drunken man says he is psychic and can correctly guess what face of the coin will fall down, mid-air. Coins are tossed and the drunken man shouts out guesses while the coins are mid-air. Drunken man correctly guesses the outcomes of the 10 throws.

Adapted example from Jim Berger's book, The Likelihood Principle. Also adapted from Tammy Lee's slides.

# Frequentist vs. Bayesian



Frequentist: "They're both so skilled! I have **as much confidence** in musician's ability to distinguish Haydn and Mozart as I do the drunk's psychic ability to predict coin tosses"

Bayesian: "I'm not convinced by the drunken man…"

The Bayesian approach is to incorporate **prior knowledge** into the experimental results.

What is the probability of colorblindedness?

$$P(Colorblind) = 0.0425$$
$$P(Colorblind|Female) = 0.005$$
$$P(Colorblind|Male) = 0.08$$
$$P(Male) = 0.5$$

What is the probability someone is male and colorblind?

Discuss.

# Review: Conditional Probabilities

$$P(M,C) \neq P(M) \times P(C)$$ NO!

$$P(M,C) = P(M) \times P(C|M)$$ Yes

But how do we solve:

$$P(Male|Colorblind)?$$

# Bayes' Rule

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$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

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# Bayes' Rule

posterior probability

likelihood

prior probability

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

marginal likelihood

# Bayes' Rule: Example

$$P(\text{psychic}|\text{correct}) = \frac{P(\text{correct}|\text{psychic})P(\text{psychic})}{P(\text{correct})}$$

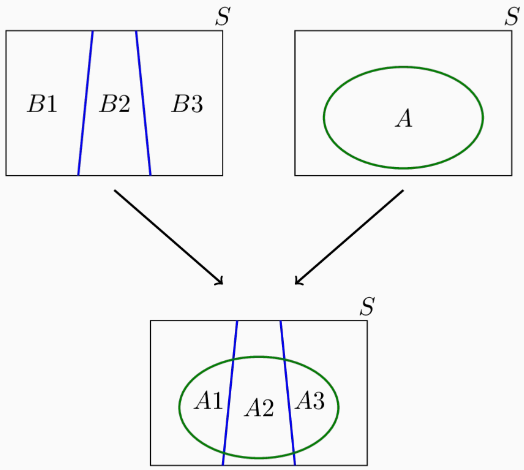
$$= \frac{1.0 * 0.0001}{0.5^{10}}$$

Very subjective!

$$= 10.2\%$$

# Exercise - Law of Total Probability

1. A box contains three coins: two regular coins and one fake two-headed coin (P(H)=1). Question 6 of this link.
   a. You pick a coin at random and toss it. What is the probability that it lands heads up?
   b. You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?




$$P(A) = P\left(\bigcup_i (A \cap B_i)\right) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)\,P(B_i)\,.$$

# Exercise, get yourself used to long, wordy questions

A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with measles.

Let **F** stand for an event of a child being sick with flu and **M** stand for an event of a child being sick with measles. Assume for simplicity that F ∪ M = Ω, i.e., that there no other maladies in that neighborhood.

A well-known symptom of measles is a rash (the event of having which we denote R). Assume that the probability of having a rash if one has measles is P(R | M) = 0.95.

However, occasionally children with flu also develop rash, and the probability of having a rash if one has flu is P(R | F) = 0.08. Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?

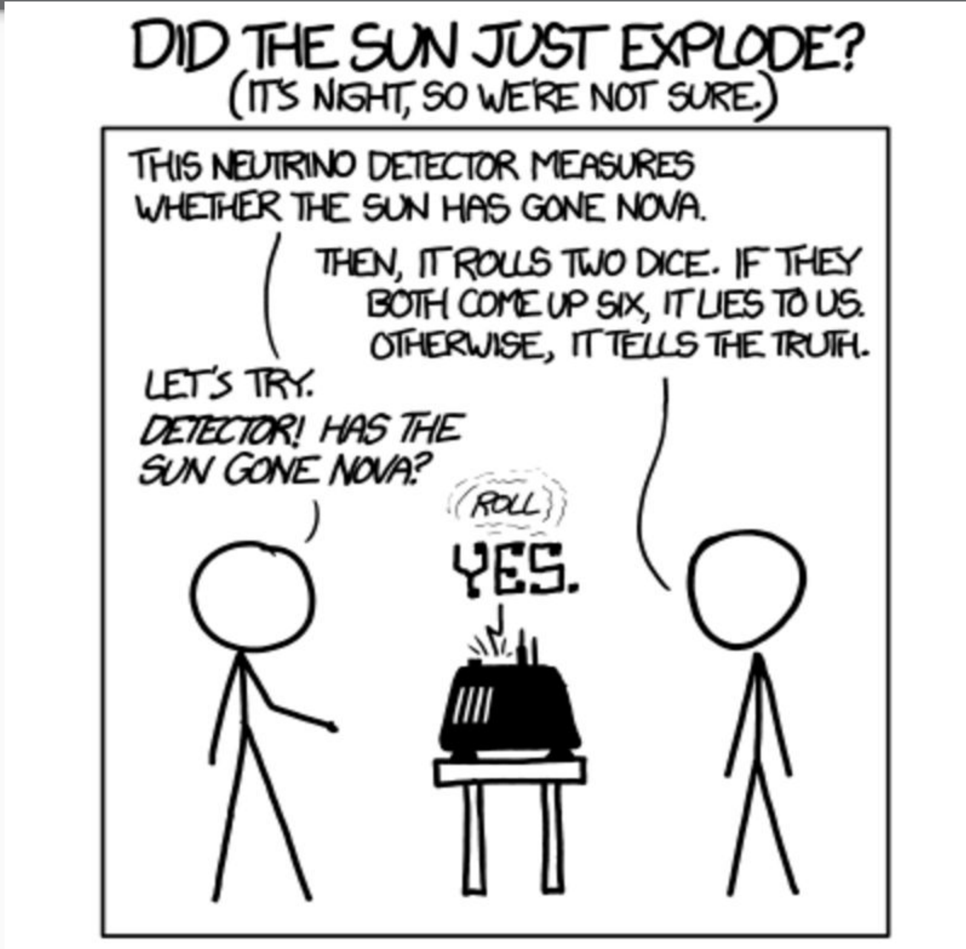
**Solution.**

We use Bayes's formula.

$$P(M \mid R) = \frac{P(R \mid M)P(M)}{(P(R \mid M)P(M) + P(R \mid F)P(F))} = \frac{0.95 \times 0.10}{(0.95 \times 0.10 + 0.08 \times 0.90)} \simeq 0.57.$$

Which is nowhere close to 95% of P(R—M).

# xkcd: Frequentists vs. Bayesians (#1132)



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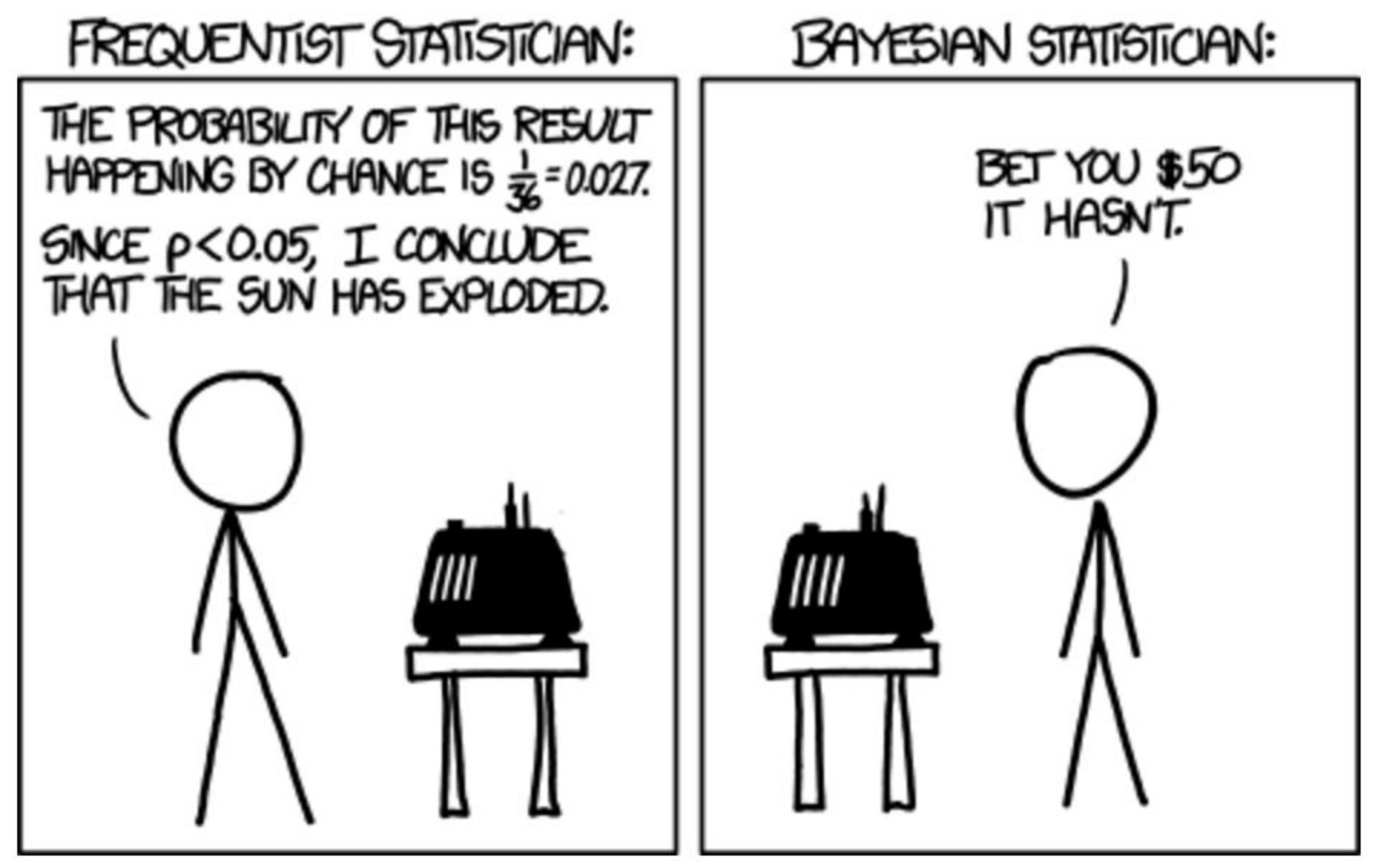


## Breakout

A couple has two children.
*At least* one of the children is a girl.

What is the probability that both children are girls?

# Use Bayesian Inference

A simple question:
Is this coin fair?

1. Can you use hypothesis testing?
2. What if you can't get all the data at once?
3. What if you are an alien and you don't know how coin works?

# Bayesian Updates

$A$: the distribution associated with the probability of flipping heads (success)

$B$: the results of our flips (number of heads, number of trials)

$A$: the unknown truth that governs our observations

$B$: the observations that we gradually gather

likelihood (the data gathered, #H #T)

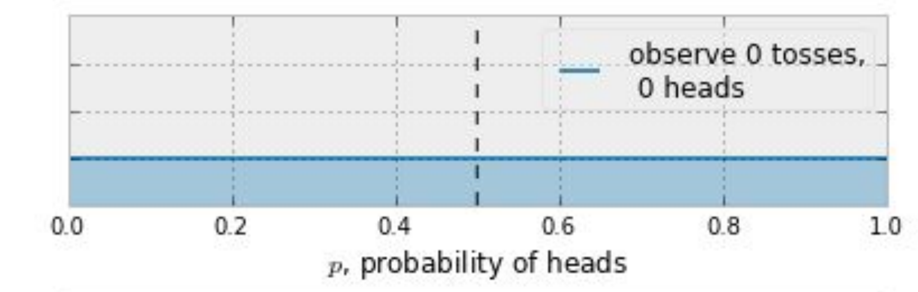
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

prior probability (our belief about the probability of heads, initially, but after collecting data it's the old posterior)

posterior probability (the probability of heads given the flips)

the marginal probability of the flips (ensures that total probability of posterior sums to 1)

# Bayesian Updates




No data

Uniform prior -> Uniform posterior

*A*: the distribution associated with the probability of flipping heads (successes)

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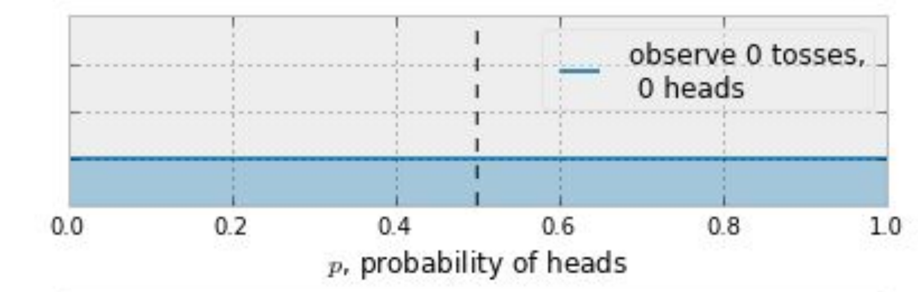


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Uniform prior -> Uniform posterior
**Note that they are both continuous.**

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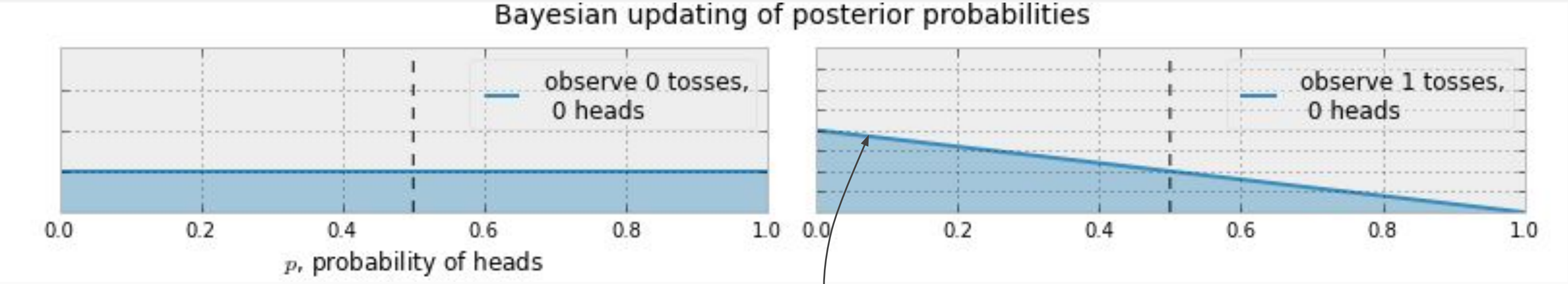



Likelihood: 0 successes, 1 trial (Binomial)
Posterior = Likelihood * Prior / Marginal

**Posterior becomes the *new prior* for the next posterior calculation**

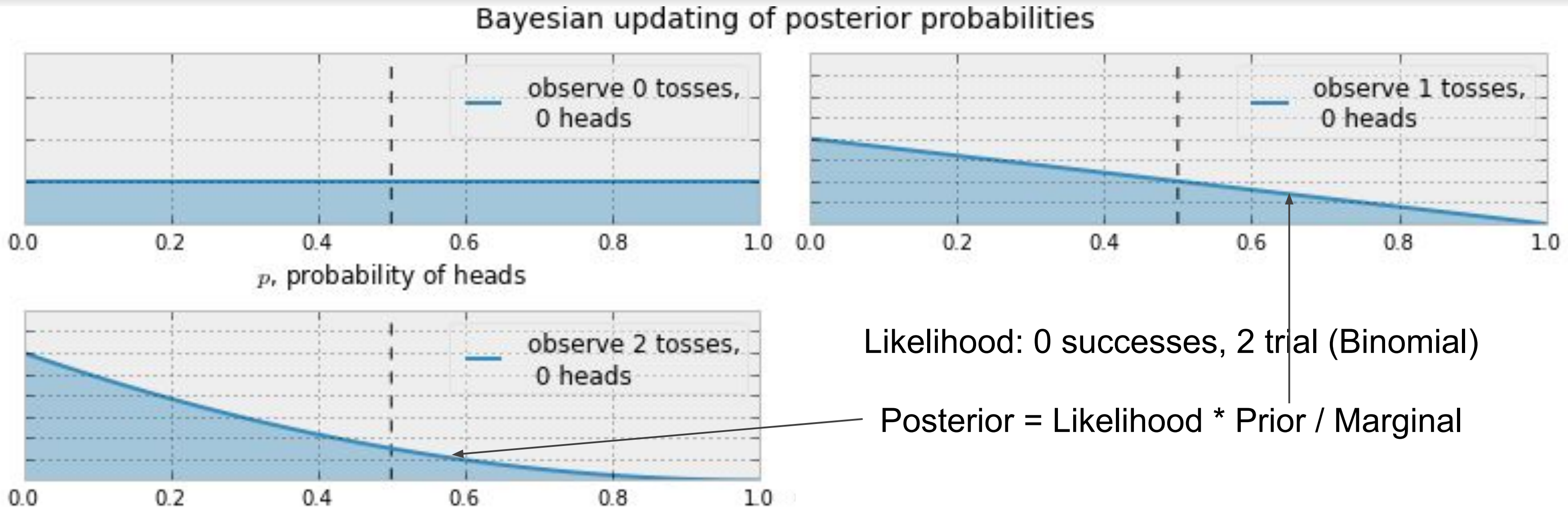
*A*: the distribution associated with the probability of flipping heads (successes)

*B*: the results of our flips (number of heads, number of trials)

MathJax

$$
\begin{aligned}
P(p|one\ tail) &\approx \frac{P(one\ tail|p)P(p)}{P(one\ tail)} \\
&\approx 1 - p
\end{aligned}
$$

# Bayesian Updates

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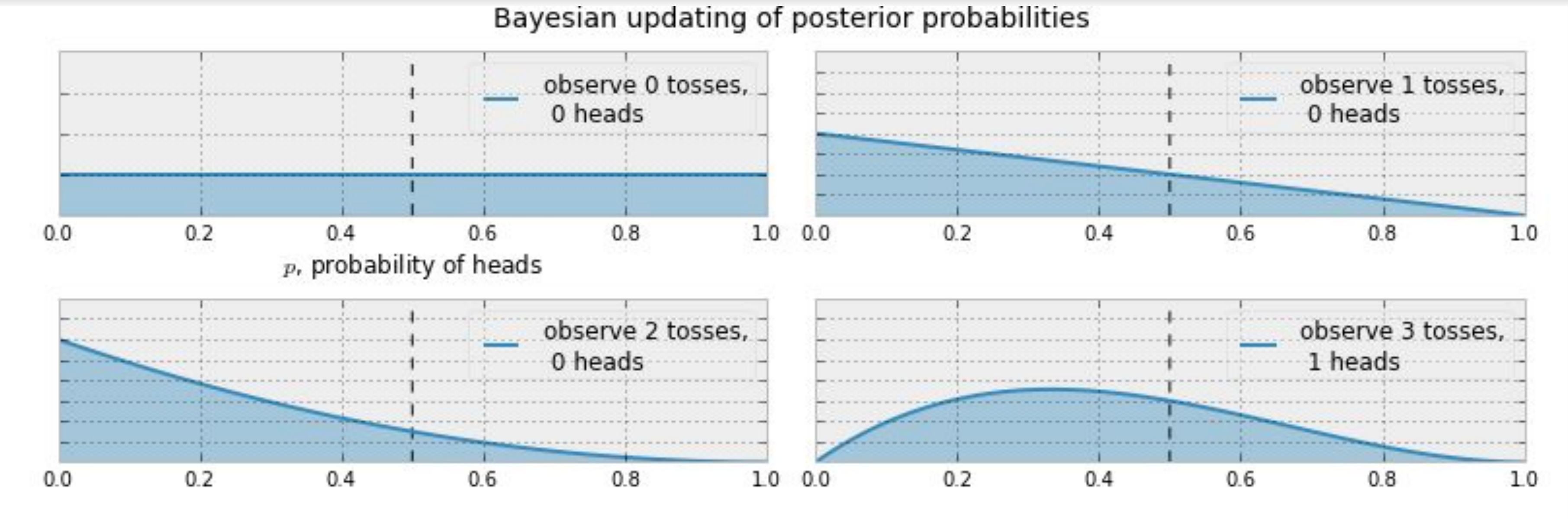


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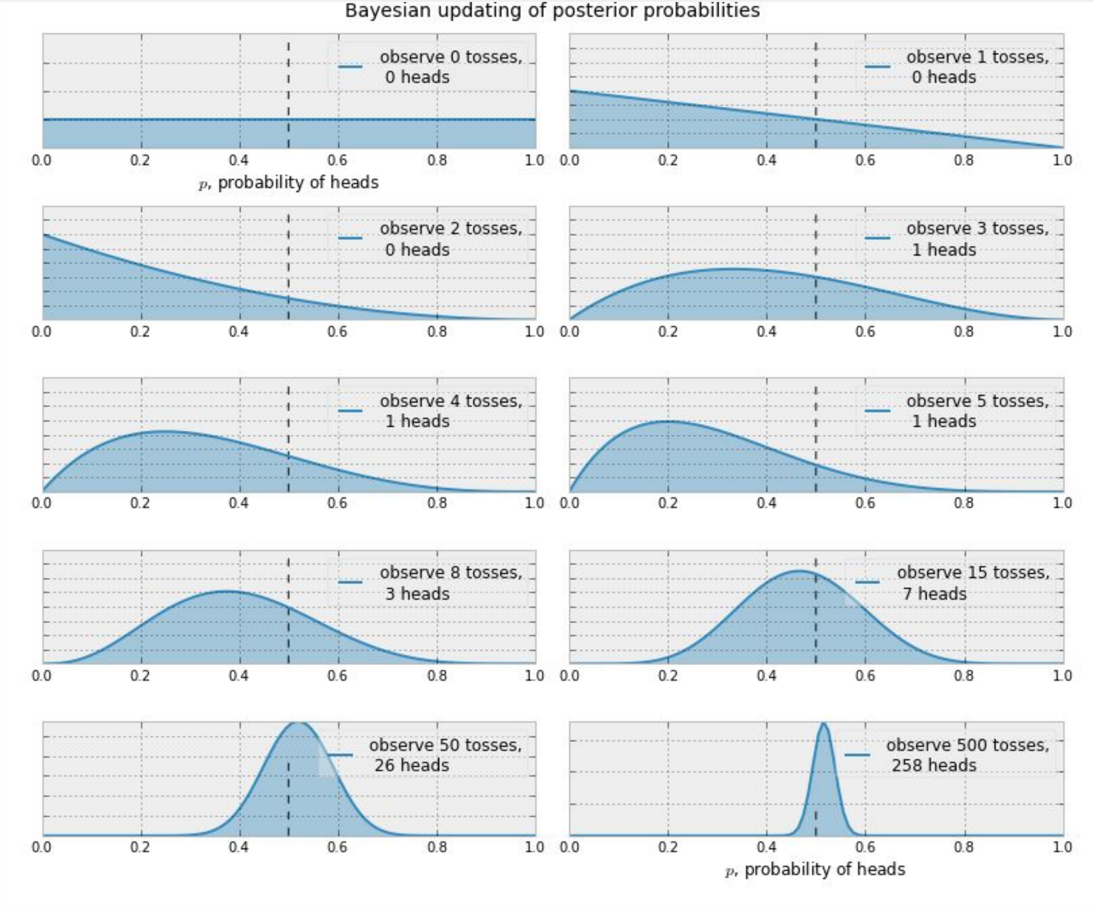

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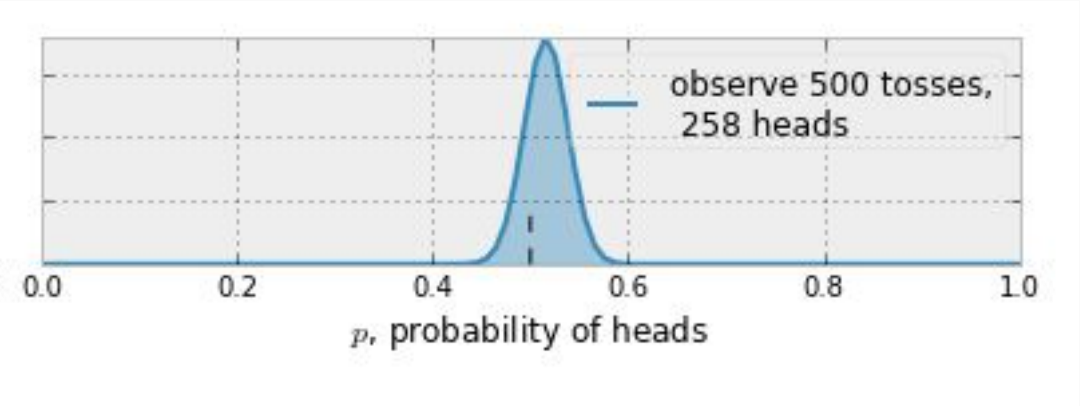

# Bayesian Updating Visual

[Seeing Theory - Bayesian]

# Bayesian Updating in Code

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`bayesian_updating_simple_example.ipynb`