Runtime (and Memory) Analysis

Problem

- Code can take an unfeasibly long time to run.
- (Code can require more memory than available.)

Solution

- Analyze the runtime of your code to make it more efficient.
- (Analyze the memory requirement of your code to make it more efficient.)

Example 1

How long does this take to run?

```
for i in range(n):
    print(i)
```

Runtime is expressed as a function of n, the size of the code's input.

Example 1

We'll assume a unit of execution : step
to simplify things!

```
for i in range(n):
    print(i)
```

the loop is run once :
some time is spend on the
initialization and wrap up :
1 step

Python 2 : a list of
size n is created :
takes n steps

Python 3 : a generator is used (not a list)
getting the next element with
__next__() is called n times :
takes n steps

the print
function is
called n times :
takes n steps

total : 1 + 2n steps

Example 1

```
for i in range(n):  # 1 + n steps
    print(i)        # n steps
```

1 step $+ \, 2n$ steps $\approx 2n$ steps for large value of n

but does the 2 really matter
w/ our fuzzy definition of steps?

Example 2

How long does this take to run?

```
for i in range(n):
    for j in range(n):
        print(i, j)
```

$1 + n$ "steps"

$n \times (1+n)$

n^2

total runtime: $1 + n + n(1+n) + n^2$

$= 1 + 2n + 2n^2$

Example 2

```
for i in range(n):
    for j in range(n):
        print(i, j)
```

Example 2

```
for i in xrange(n):         # 1 + n steps
    for j in xrange(n):     # n * (1 + n) steps
        print(i, j)         # n^2 steps
```

1 step $+ 2n$ steps $+ 2n^2$ steps $\approx 2n^2$ steps for large value of n

Example 3

How long does this take to run?

```
def find_anagrams(words):
    anagrams = []
    for word1 in words:
        for word2 in words:
            if word1 != word2:
                for word in permutations(word1):
                    if word == list(word2):
                        anagrams.append(word1)
    return anagrams
```

Example 3

What's n? length of the list of words.

k as maximal/typical length of a word.

```
def find_anagrams(words):
    anagrams = []
    for word1 in words:
        for word2 in words:
            if word1 != word2:
                for word in permutations(word1):
                    if word == list(word2):
                        anagrams.append(word1)
    return anagrams
```

$\boxed{1}$

$\boxed{1 + n}$

$\boxed{n \times (1+n)}$

$\boxed{n^2 \times k}$

comparison of words of length k: k steps

$\boxed{n^2 \times k!}$

$k!$ permutations

$\boxed{n^2 \times k! \times k}$

$\boxed{n^2 \times k! \times k}$

$\boxed{1}$

Example 3

```
def find_anagrams(words):
  anagrams = []                             # 1
  for word1 in words:                       # 1 + n
    for word2 in words:                     # n * (1 + n)
      if word1 != word2:                    # n^2 * k
        for word in permutations(word1):    # n^2 * k!
          if word == list(word2):           # n^2 * k! * k
            anagrams.append(word1)          # n^2 * k! * k
  return anagrams                           # 1
```

$$\overset{3}{\cancel{2}} + 2n + n^2 + n^2 k + n^2 k! + 2n^2 (k+1)! \text{ steps}$$

Is this a useful expression?

$$2 + 2n + n^2 + n^2 k + n^2 k! + 2n^2 (k+1)! \text{ steps}$$

- (n, k) = (10, 5)
  - 2 + 20 + 1E2 + 5E2 + 12E4 + **12E5**
- (n, k) = (100, 10)
  - 2 + 200 + 1E4 + 1E5 + 4E10 + **8E11**

With large inputs, only the largest terms of the expression are relevant.

Simplifying Runtime Analysis: Big O Notation

- Often, a single term dominates the runtime expression.
- Runtime analysis can be simplified by eliminating lower order terms.
- Big O notation provides a principled way to do this.

Big O Definition

Let f and g be two functions: $\mathbb{N} \to \mathbb{R}^+$. We say that

$$f(n) \in O(g(n))$$

(read f is Big-"O" of g)

if there exists a constant $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$f(n) \leq cg(n)$$

Big O Intuitive Summary

$$f = O(g)$$

"f is bounded by g" (roughly speaking)

- f is bounded by some constant multiple of g
- (only true for sufficiently large input values)

Practical runtime analysis only cares about large input values and approximate bounds.

Big O Conventions

Be Precise

- Technically, if a function is $O(n)$, then it is also $O(n^2)$.
  - But a more precise upper bound is more useful.
- Therefore, give the lowest upper bound possible.
  - I.e., don't say $O(n^2)$ when you could instead say $O(n)$.

Ignore Constants

- If a function is $O(n)$, then it is also $O(2n)$.
- Saying $O(2n)$ is highly unconventional.

Big O Example

What's the big O analysis of this expression? (from the earlier anagrams example)

$$2 + 2n + n^2 + n^2k + n^2k! + 2n^2(k+1)! \text{ steps}$$

$$\underbrace{n^2(k+1)!}_{k \text{ is bounded so is } (k+1)! \text{ then}} \left(\frac{2}{n^2(k+1)!} + \frac{2}{n(k+1)!} + \frac{1}{(k+1)!} + \frac{1}{n^2(k+1)} + 2 \right)$$

$n \to +\infty$

$$0 \qquad 0 \qquad 0 \qquad 0 \qquad 2$$

$$\boxed{O(n^2)}$$

Big O Example

Example 4

How long does this take to run?

```
def find_anagrams(words):
    anagrams = []
    d = defaultdict(list)
    for word in words:
        d[tuple(sorted(word))].append(word)
    for _, words in d.items():
        if len(words) > 1:
            anagrams.extend(words)
    return anagrams
```

Example 4

```
def find_anagrams(words):
    anagrams = []                    O(1)
    d = defaultdict(list)            O(1)
    for word in words:               O(n)
        d[tuple(sorted(word))].append(word)   O(n k log k)
    for _, words in d.items():
        if len(words) > 1:
            anagrams.extend(words)
    return anagrams                  O(1)
```

k log k

k

1

1

not trivial to know how often these lines are executed

but this one runs at most n times

O(n)

so ~ O(n k log k)

or

O(n) if we consider c bounded

Example 4

```python
def find_anagrams(words):
 anagrams = []                          # O(1)
 d = defaultdict(list)                  # O(1)
 for word in words:                     # O(n)
  d[tuple(sorted(word))].append(word) # O(n * k * log(k))
  # line is executed n times; then in series,
  # - sorted() is O(k * log(k))
  # - d[] is O(1)
  # - .append() is O(1)
 for _, words in d.items():             # v
  if len(words) > 1:                    # v
   anagrams.extend(words)               # O(n)
 return anagrams                        # O(~~n~~ 1)
```

Common runtimes to remember

- $O(1)$ - constant
- $O(n)$ - linear
- $O(nlog(n))$ - "n log n" – e.g. sorting
- $O(n^2)$ - quadratic
- $O(n^3)$ - cubic – e.g. matrix multiplication
- $O(2^n)$ - exponential (base is not necessarily 2)
- $O(n!)$ - factorial

oh boy!

Runtime in practical terms

n	Linear runtime	Quadratic runtime
100	1 s	1s
1000	10 s	100 s
10,000	100 s	10,000 s = 167 min
100,000	1000 s = 17 min	1,000,000 s = 11 days

Quadratic (n^2) algorithms are already really slow.

Linear (n) or $nlog(n)$ are ideal; anything bigger is usually too slow for large data.

Back to example 1

How much memory does this take to run?

one variable

```
for i in range(n):
    print(i)
```

Python 2: list of size n $O(n)$
Python 3: generator $O(1)$

Memory is expressed as a function of n, the size of the code's input.

Python 2: $O(n)$
Python 3: $O(1)$

Back to example 2

How much memory does this take to run?

```
for i in range(n):
    for j in range(n):
        print(i, j)
```

2 variables

Python 2: list of size n

$O(n)$

not $O(n^2)$

as only 2 lists (one for i, one for j) exist at a given time

Python 3: generators

$O(1)$

Back to example 3

Example 4

How much memory does this take to run?

What's the size for anagrams (dictionary of lists)?

$\sum |list| = n$

$\to \boxed{O(n)}$

```
def find_anagrams(words):
    anagrams = []
    for word1 in words:
        for word2 in words:
            if word1 != word2:
                for word in permutations(word1):
                    if word == list(word2):
                        anagrams.append(word1)
    return anagrams
```

What's the size of anagrams? list of at most all words

$\boxed{O(n) \text{ or } O(nk)}$

(or $O(nk)$)

Back to example 4

How long does this take to run?

```python
def find_anagrams(words):
    anagrams = []
    d = defaultdict(list)
    for word in words:
        d[tuple(sorted(word))].append(word)
    for _, words in d.items():
        if len(words) > 1:
            anagrams.extend(words)
    return anagrams
```

Important data structures

- Lists
- Dictionaries
- Graphs
  - Trees

Runtime of typical Python list functions

- Appending $O(1)$
- Adding to the beginning or middle $O(n)$
- Popping from the end $O(1)$
- Popping from the beginning or middle $O(n)$
- Looking up by index $O(1)$
- Searching an unsorted list $O(n)$
- Searching a sorted list $O(\log n)$ "bisection"

Runtime of typical Python list functions (Time Complexity)

- Appending:
  - $O(1)$
- Adding to the beginning or middle:
  - $O(n)$ (have to shift elements over!)
- Popping from the end:
  - $O(1)$
- Popping from the beginning or middle:
  - $O(n)$ (again, have to shift elements over!)
- Looking up by index:
  - $O(1)$
- Searching an unsorted list:
  - O(n) (have to look at every item)
- Searching a sorted list:
  - $O(log n)$ (binary search)

Runtime of typical Python dictionary functions

- Inserting an item $O(1)$
- Removing an item $O(1)$
- Looking up by key $O(1)$
- Looking up by value $O(n)$

Runtime of typical Python dictionary functions (Time Complexity)

- Inserting an item:
  - $O(1)$
- Removing an item:
  - $O(1)$
- Looking up by key:
  - $O(1)$
- Looking up by value:
  - $O(n)$ (have to look at every item)

More

Algorithms Notes for Professionals book

- Chapter 2: Algorithm Complexity
- Chapter 22: Big-O Notation

Important algorithms: Sorting

Algorithm	Best	Average	Worst	Memory
Bubble sort	n	n^2	n^2	1
Selection sort	n^2	n^2	n^2	1
Insertion sort	$n2$	n^2	n^2	1
Merge sort	$nlog(n)$	$nlog(n)$	$nlog(n)$	n (worst)
In-place merge sort	-	-	$n(log(n))^2$	1
Quicksort	$nlog(n)$	$nlog(n)$	n^2	$logn$ (average) n (worst)
Heapsort	$nlog(n)$	$nlog(n)$	$nlog(n)$	1

Important algorithms: Sorting

Algorithms Notes for Professionals book

- Chapters 23: Sorting
  - Chapters 24: Bubble Sort
  - Chapters 25: Merge Sort
  - Chapters 26: Insertion Sort
  - Chapters 27: Bucket Sort
  - Chapters 28: Quick Sort
  - Chapters 29: Counting Sort
  - Chapters 30: Heap Sort
  - Chapters 31: Cycle Sort
  - Chapters 32: Odd-Even Sort
  - Chapters 33: Selection Sort

Other important algorithms

Algorithms Notes for Professionals book

- Chapter 3: Graph
  - Chapter 4: Graph Traversals
  - Chapter 5: Dijkstra's Algorithm
- Chapter 34: Trees
  - Chapter 35: Binary Search Trees
  - Chapter 37: Binary Tree traversals
  - Chapter 41: Breadth-First Search
  - Chapter 41: Depth First Search