Econometrics in Economics / Introduction to Econometrics

Refresher on stats and probability

based on Wooldridge (2019), App A-C; Stock and Watson (2020), Ch. 2–3

Patrick Schmidt (based on slides by **Simon Heß** and **Daniel Gutknecht**)

Winter 23

Road map

last part:

- what is econometrics?
- example: student test performance and class size
- causality and experimental vs observational data

this part:

- refresher on statistics and probability

The landscape of statistics? Where are we?

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inferential statistics / statistical inference:

- use data analyses to learn about (to “infer”) properties of the underlying probability distribution
  - key aspect: Uncertainty quantification.

mathematical statistics:

- the tools (probability theory) to carry out statistical analyses

This course's focus: inferential statistics using samples of data

- start from an **economic model**: **unobserved** parameters of interest $\mapsto$ **observables** (data)
  - solving the model for the parameters gives us a way to estimate it

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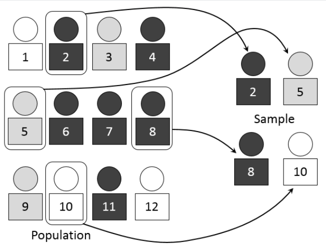

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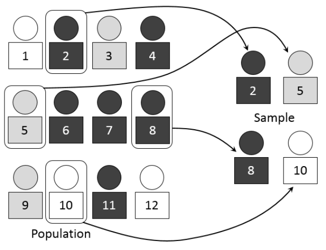

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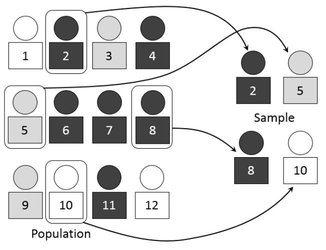

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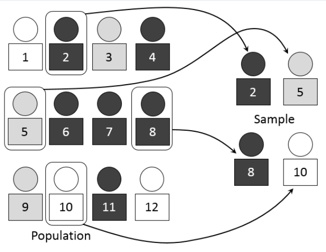

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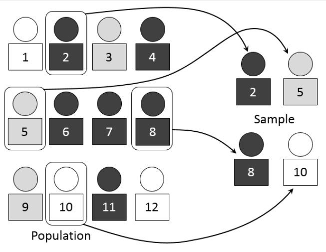

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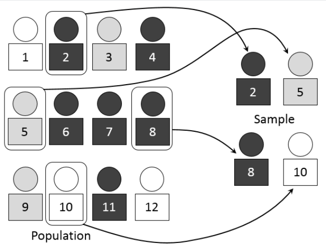

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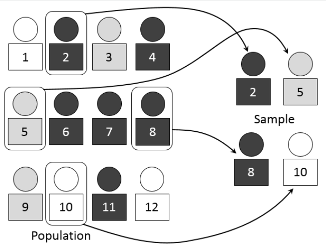

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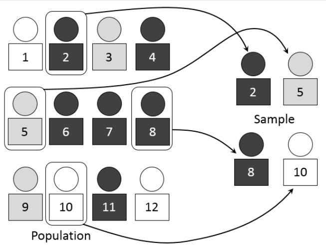
   - quantifying uncertainty is possible and important



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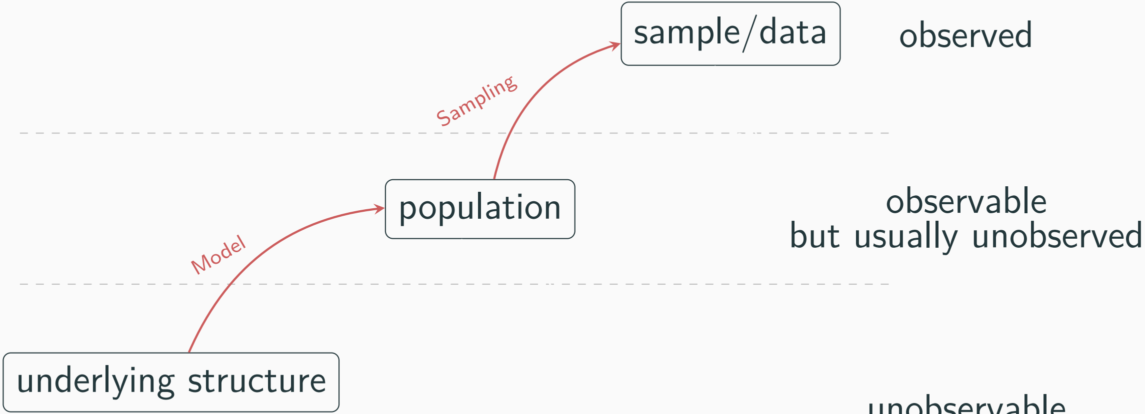

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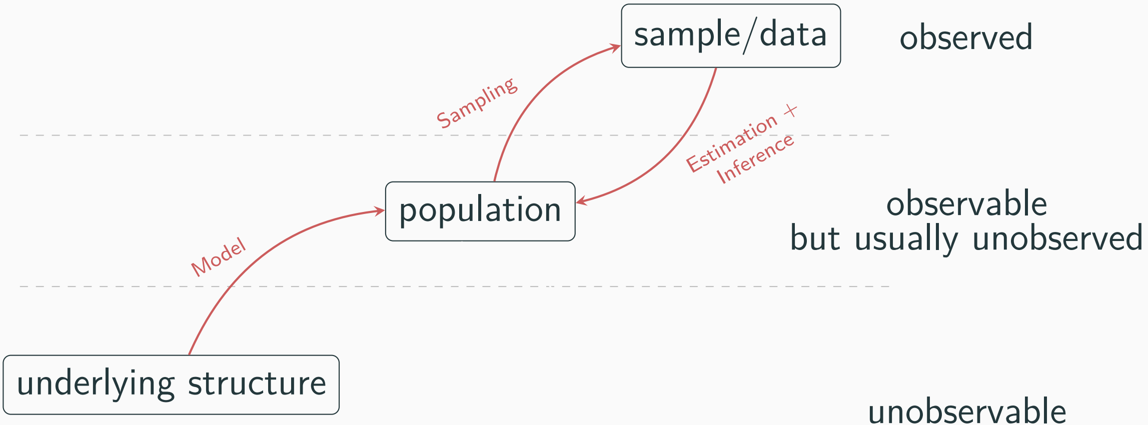

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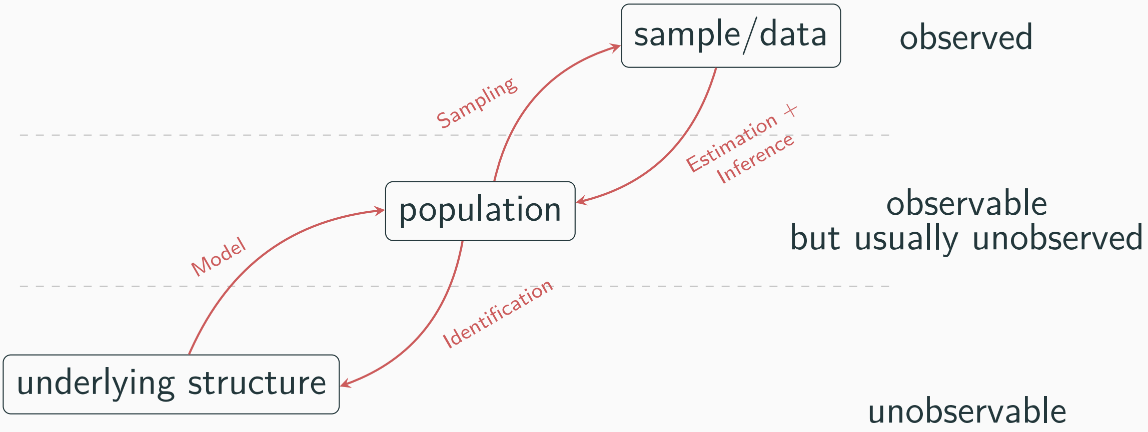

Terminology

Informal definition: Estimator/Estimate/Estimand

- The parameter to be estimated in a population is called the **estimand**

- **Estimand**: $\mathbb{E}[x]$:
  - e.g., mean salary of Goethe University alumna

  Proposed **estimator**: the average defined as $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

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- Instead describe it by **density** or **cumulative probabilities**

▸ Marathon finishing times

Distribution functions

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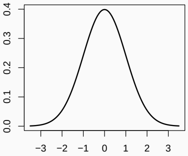

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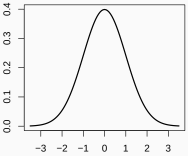

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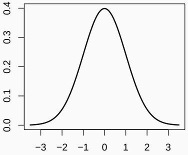

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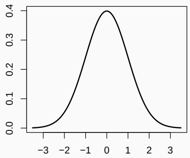

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Cumulative distribution function (cdf)

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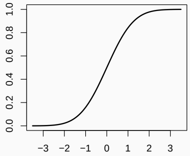

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$$F_y(a) = \Pr(y \leq a) = \int_{-\infty}^{a} f_y(u)\mathrm{d}u.$$

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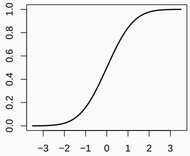

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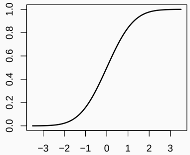

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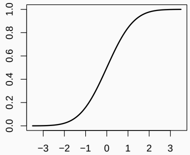

Sometimes it is more convenient to visualize distributions as:

$$F_y(a) = \Pr(y \leq a) = \int_{-\infty}^{a} f_y(u)\mathrm{d}u.$$

Ex.: Probability of finishing a marathon within 4 to 6 hrs

- $F_y(6)$ is the probability of finishing within 6 hrs
- $F_y(4)$ is the probability of finishing within 4 hrs
- $F_y(6) - F_y(4)$ is the prob of finishing within 4-6 hrs

- $f_y(\cdot)$ and $F_y(\cdot)$ carry the same information about y's distribution

Moments (1)

We often need summaries of distributions and don't want to describe the whole $f_y(\cdot)$ or $\Pr(\cdot)$.

Mean:

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  - Discrete:

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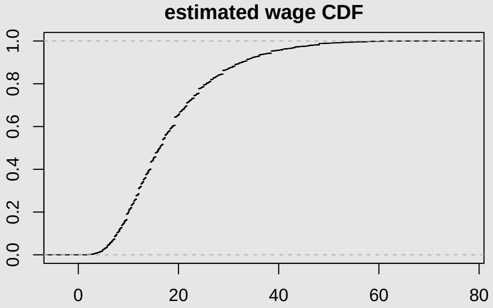
  - Continuous:

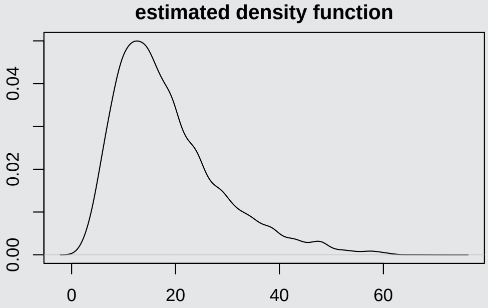
$$\mathbb{E}[y] = \int_{-\infty}^{+\infty} t f_y(t) \mathrm{d}t$$

- often but not *always* a great summary measure

Example of a skewed distribution

Ex.: Hourly wages of 61k people (2004 Current Population Survey)





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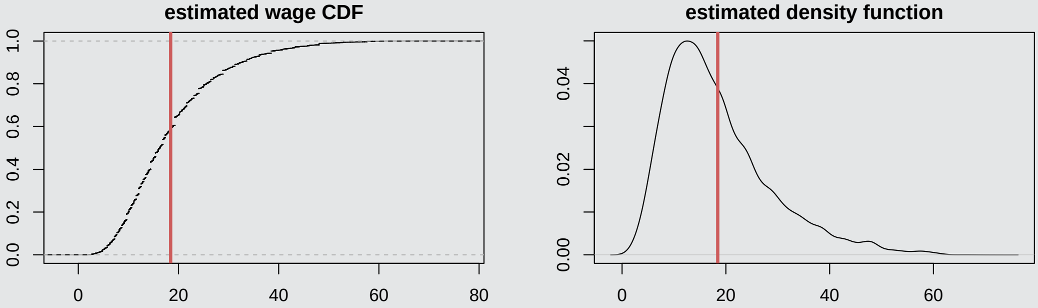

$\mathbb{E}(\text{wage}) = 18.44\$/\text{hr}$

Moments (2)

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- alternatively we often use the standard deviation:

$$\sigma_y = \sqrt{\mathrm{var}(y)}$$

Example of a skewed distribution (i.e. with outliers)

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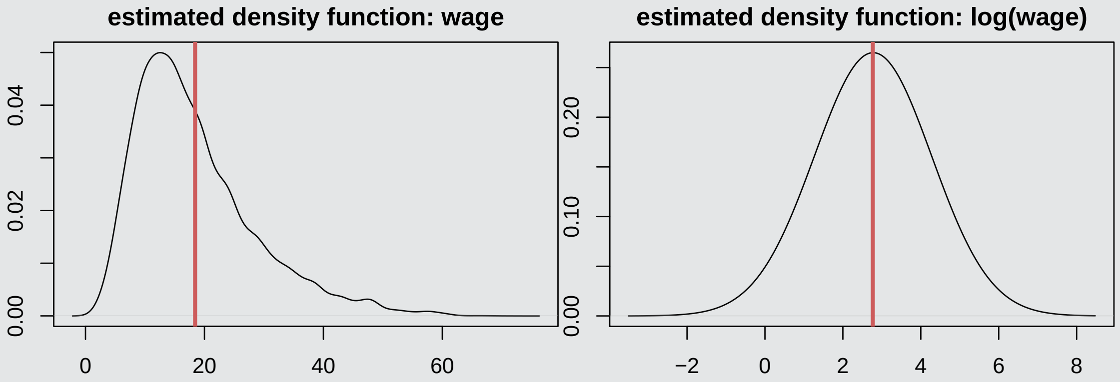

log(wage) is precisely described by mean and variance. The distribution of wage is asymmetric and more

Joint distributions and covariance

Covariance: The covariance between x and y is

$$\mathrm{cov}(x, y) = \mathbb{E}[(x - \mu_x)(y - \mu_y)] = \sigma_{xy}.$$

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- its units are (units of x) $\times$ (units of y)

Correlation coefficient:

- **unitless** measure of **linear** association defined in terms of covariance and variances:

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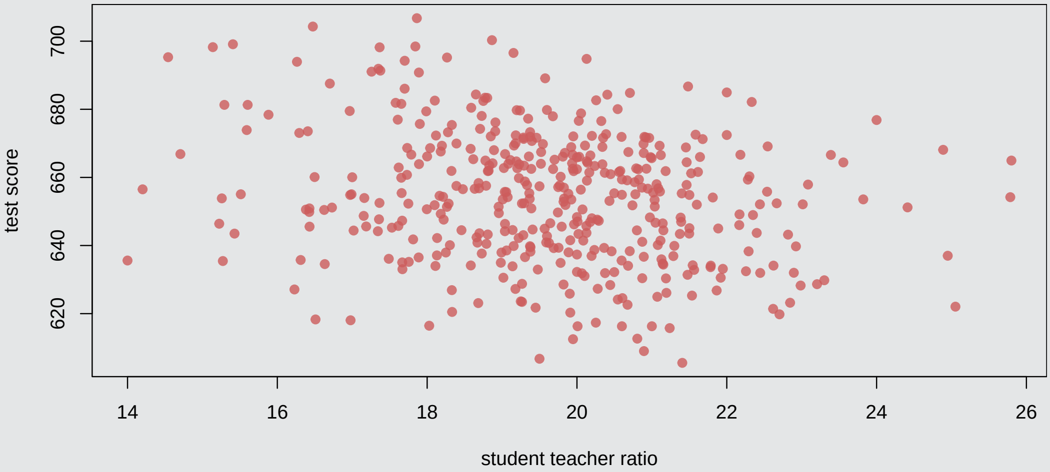
Properties:

- $-1 \leq \operatorname{corr}(x, y) \leq 1$.
- $\operatorname{corr}(x, y) = 1$ means perfect positive linear association
- $\operatorname{corr}(x, y) = -1$ means perfect negative linear association
- $\operatorname{corr}(x, y) = 0$ means no linear association

Ex.: Districts' avg. test score and class size data (CASchools)



$\operatorname{corr}(\text{test score}, \text{student teacher ratio}) = -0.23$ and $\operatorname{cov}(\text{test score}, \text{student teacher ratio}) = -8.16$

Recap: Random variables

- Tool to mathematically describe things that are random/uncertain

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Note: Sample moments vs population moments

- note difference between the **mean** (a property of the RV, also called "**population moment**") and the **average** (also called **sample moment**), which refers to the estimator and is itself a random variable.

Conditional distributions

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Probability of completing a marathon given that one is older than 50

(Discrete, as we're not looking at finishing times ($\Omega = \mathbb{R}^+$) but at finishing: $\Omega = \{\text{no}, \text{yes}\}$)

Bayes rule yields conditional distributions

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For two events A and B with $\Pr(B) > 0$, we have that:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

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Ex.: Completing a marathon (discrete)

$$\Pr(\text{completing}|\text{age} > 50) = \frac{\Pr(\text{completing,age}>50)}{\Pr(\text{age}>50)}$$

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$$\Pr(\text{completing}|\text{age} > 50) = \frac{\Pr(\text{completing}, \text{age}>50)}{\Pr(\text{age}>50)}$$

In the **continuous case**:

$$f_{y|x}(a|b) = \frac{f_{y,x}(a,b)}{f_x(b)},$$

- $f_x(\cdot)$ is the density of x
- $f_{x,y}(\cdot)$ is the joint density of x and y
- $f_{y|x}(\cdot)$ is the conditional density of y given x

The conditional expectation/mean of y given x ...

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Ex.: Marathon (discrete)
The probability of finishing a marathon given that one is older than 50.

If y is discrete with mass points at $\{y_1, y_2, \ldots\}$:

$$\mathbb{E}[y|x = b] = \sum_{j=1}^{\infty} y_j \Pr(y = y_j|x = b).$$

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For simplicity, treat this data as if it's the population, not sample. Ignore that means are estimated sample means.

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- $\mathbb{E}[\log(\text{wage})] = 2.77$

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Implication for discrete x

we can compute $\mathbb{E}[y]$ from $\mathbb{E}[y|x]$ through casework:

$$\mathbb{E}[y] = p_1\,\mathbb{E}[y|x = v_1] + p_2\,\mathbb{E}[y|x = v_2] + p_3\,\mathbb{E}[y|x = v_3] + \ldots$$

Law of iterated expectations

Ex.: Hourly wages of 61k people (2004 Current Population Survey)

Consider again hourly wages

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Observe that:

- $\Pr[\text{gender} = \text{male}] = 0.56$ and $\Pr[\text{gender} = \text{female}] = 0.44$

Law of iterated expectations

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L.I.E. implies:

$$\begin{aligned}\mathbb{E}[\log(\text{wage})] = \; & \mathbb{E}\left[\log(\text{wage})\,|\text{gender}{=}\text{male}\right] \times \Pr(\text{gender}{=}\text{male}) \\ & + \mathbb{E}\left[\log(\text{wage})\,|\text{gender}{=}\text{female}\right] \times \Pr(\text{gender}{=}\text{female}) \\ = & 2.86 \times 0.56 + 2.65 \times 0.44 \approx 2.77.\end{aligned}$$

Law of iterated expectations

Ex.: LIE in a proof

To be proven: $\mathbb{E}[e|x] = 0 \Rightarrow \mathbb{E}[ex] = 0$

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&= \mathbb{E}[x \cdot 0] \\
&= \mathbb{E}[0] \\
&= 0
\end{aligned}
$$

Conditional variance

Just ike the mean, other moments have conditional counterparts

Conditional variance

$$\mathrm{var}(y|x=b) \equiv \sigma_y^2(x=b) = \mathbb{E}[(y-\mathbb{E}[y|x=b])^2|x=b]$$

Recap: Conditional distributions

We are often interested in describing the relationship between 2+ RVs

- Conditional distributions describe how y behaves for given values of x
- Conditional distributions can be obtained via Bayes rule.
- Conditional distributions can be described by conditional moments (e.g., conditional mean, variance)

Regression lines (OLS) are econometric models for conditional means

A larger part of what we study in this class are how to estimate conditional moments (means).

References

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Appendix

Marathon finishing times