Analysis I, Exercise 1

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October 2022

Task 1

Let $f : A \to B$ be a map, and let $X, Y \subseteq B$. Prove that

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$;
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$;
3. $f^{-1}(X) \setminus f^{-1}(Y) = f^{-1}(X \setminus Y)$.

Solution. We show that two sets P, Q are equal by showing $P \subseteq Q$ and $Q \subseteq P$.

1. Let $p \in f^{-1}(X \cup Y)$. Then $f(p) \in X \cup Y$, so $f(p) \in X$ or $f(p) \in Y$. In the first case, $p \in f^{-1}(X) \subseteq f^{-1}(X) \cup f^{-1}(Y)$. In the second case, $p \in f^{-1}(Y) \subseteq f^{-1}(X) \cup f^{-1}(Y)$. Thus in both cases, $p \in f^{-1}(X) \cup f^{-1}(Y)$. Since this holds for each $p \in f^{-1}(X \cup Y)$, this implies $f^{-1}(X \cup Y) \subseteq f^{-1}(X) \cup f^{-1}(Y)$.

   Now let $p \in f^{-1}(X) \cup f^{-1}(Y)$, so $p \in f^{-1}(X)$ or $f^{-1}(Y)$. In the first case, $f(p) \in X \subseteq X \cup Y$. In the second case, $f(p) \in Y \subseteq X \cup Y$. Thus, $f(p) \in X \cup Y \implies p \in f^{-1}(X \cup Y)$ for each $p \in f^{-1}(X) \cup f^{-1}(Y)$. Hence, $f^{-1}(X) \cup f^{-1}(Y) \subseteq f^{-1}(X) \cup f^{-1}(Y)$.

   Since both inclusions hold, the two sets are equal.

2. Let $p \in f^{-1}(X \cap Y)$, so $f(p) \in X \cap Y$. Thus, both $f(p) \in X$ and $f(p) \in Y$ hold. Hence, $p \in f^{-1}(X)$ and $p \in f^{-1}(Y)$, so $p \in f^{-1}(X) \cap f^{-1}(Y)$. Since this holds for each $p \in f^{-1}(X \cap Y)$, it follows that $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$.

   Now let $p \in f^{-1}(X) \cap f^{-1}(Y)$. Then $p \in f^{-1}(X)$ and $p \in f^{-1}(Y)$, so $f(p) \in X$ and $f(p) \in Y$. Finally, this yields $f(p) \in X \cap Y$, so $p \in f^{-1}(X \cap Y)$. Since this holds for each $p \in f^{-1}(X) \cap f^{-1}(Y)$, it follows that $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$.

   Since both inclusions hold, the two sets are equal.

3. Let $p \in f^{-1}(X) \setminus f^{-1}(Y)$. Then $p \in f^{-1}(X)$ and $p \notin f^{-1}(Y)$, so $f(p) \in X$ and $f(p) \notin Y$. Thus, $f(p) \in X \setminus Y$, so $p \in f^{-1}(X \setminus Y)$. Since this holds for each $p \in f^{-1}(X) \setminus f^{-1}(Y)$, we have $f^{-1}(X) \setminus f^{-1}(Y) \subseteq f^{-1}(X \setminus Y)$.

   Now let $p \in f^{-1}(X \setminus Y)$. Then $f(p) \in X \setminus Y$, so $f(p) \in X$ and $f(p) \notin Y$. Thus, $p \in f^{-1}(X)$ and $p \notin f^{-1}(Y)$. Hence, $p \in f^{-1}(X) \setminus f^{-1}(Y)$. Since this holds for each $p \in f^{-1}(X \setminus Y)$, it follows that $f^{-1}(X \setminus Y) \subseteq f^{-1}(X) \setminus f^{-1}(Y)$.

   Again both inclusions hold, so the two sets are equal.

Task 2

Let X, Y and Z sets and $f : X \to Y$ and $g : Y \to Z$ bijective mappings. Prove that $g \circ f : X \to Z$ is bijective and its inverse is given by

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Solution. Let $a, b \in X$ and assume $g(f(a)) = g(f(b))$. Since g is bijective (and thus in particular injective), this implies $f(a) = f(b)$. Since f is bijective, this implies $a = b$. Thus, $g \circ f$ is injective.

Let now $a \in Z$ be arbitrary. Since g is surjective, there is some $b \in Y$ with $g(b) = a$. Then since f is surjective, there is some $c \in X$ with $f(c) = b$. Observe that $g(f(c)) = g(b) = a$. Since $a \in Z$ was arbitrary, $g \circ f$ is surjective.

All in all, we deduce $g \circ f$ is bijective. For the second part, note that f^{-1}, g^{-1} are well-defined since f, g are bijective. Observe that $(f^{-1} \circ g^{-1}) \circ (g \circ f)(a) = f^{-1} \circ g^{-1} \circ g \circ f(a) = f^{-1}(g^{-1}(g(f(a)))) = f^{-1}(f(a)) = a$. Thus, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, as desired.

Task 3

Consider rational numbers as granted for the moment. Determine all $x \in \mathbb{R}$ such that the following inequalities hold

(a) $\left|\frac{x+4}{x-2}\right| < x$;

(b) $|x - a| + |x - b| \leq b - a$ for given $a \leq b$.

Solution.

(a) Note that the left side is not defined for $x = 2$, so $x \neq 2$.

- Case $x < -4$: Then $x + 4, x - 2 < 0$, so $\frac{x+4}{x-2}$ is positive and the inequality becomes $\frac{x+4}{x-2} < x$. Multiplying by $x - 2 < 0$, this becomes $x + 4 > x(x - 2) \iff 0 > (x - 4)(x + 1)$. Since both factors are negative $(x - 4 < x + 1 < -4 + 1 < 0)$, we get no solutions in this case.
- Case $-4 \leq x < 2$: Then $x - 2 < 0 < x + 4$, so $\frac{x+4}{x-2} < 0$ and the inequality becomes $-\frac{x+4}{x-2} < x$. Multiplying by $x - 2 < 0$, this becomes $-x - 4 > x(x - 2) \iff 0 > x^2 - x + 4 = \left(x - \frac{1}{2}\right)^2 + \frac{15}{4}$. Since squares are non-negative, this is false, so we get no solutions in this case.
- Case $x > 2$: Then $x + 4, x - 2 > 0$ and $\frac{x+4}{x-2}$ is positive, so the inequality becomes $\frac{x+4}{x-2} < x$. Multiplying by $x - 2 > 0$, this is equivalent to $x + 4 < x(x - 2) \iff 0 < x^2 - 3x - 4 = (x - 4)(x + 1)$. Since $x + 1 > 0$, this holds if and only if $x > 4$.

Hence, the solutions are all $x \in (4, \infty)$.

(b) Note that x must lie in one of the intervals $(-\infty, a), [a, b], (b, \infty)$.

- Case $x < a$: Then $|x - a| = a - x$ and $|x - b| = b - x$, so the inequality becomes

$$\begin{aligned} a + b - 2x &\leq b - a \\ \iff 2a &\leq 2x \\ \iff a &\leq x. \end{aligned}$$

Hence, there are no solutions in this case (the last inequality contradicts $x < a$).

- Case $a \leq x \leq b$: Then $|x-a| = x-a$ and $|x-b| = b-x$, so the inequality becomes

$$x - a + b - x \leq b - a,$$

which is always true.

- Case $b < x$: Then $|x-a| = x-a$ and $|x-b| = x-b$, so the inequality becomes

$$\begin{aligned} 2x - a - b &\leq b - a \\ \iff 2x &\leq 2b \\ \iff x &\leq b. \end{aligned}$$

Hence, there are no solutions in this case.

Finally, we deduce that x satisfies the inequality if and only if $x \in [a, b]$.