Analysis I, Exercise 2

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Task 1

(a) Let $x_i \in \mathbb{Z}$ for $i \in \mathbb{N}$. Prove that

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|\,.$$

(b) Prove that $\sum\limits_{i=1}^{n} \frac{i-1}{i!} = 1 - \frac{1}{n!}$.
(For $n \in \mathbb{N}$ we define $n! = 1 \times 2 \times \ldots \times (n-1) \times n$.)

(c) For $n, k \in \mathbb{N}$ we define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and

$$\binom{n}{0} = 1 = \binom{n}{n}.$$

Now prove the following:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Solution.

(a) We proceed by induction. Observe that for $n = 0$, both sums are empty and thus $0 \leq 0$ is true. For $n = 1$, the inequality says $|x_1| \leq |x_1|$, which is again true. Now assume that the inequality holds for some $n \in \mathbb{N}$ and let $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Z}$. Then

$$\begin{aligned}
\left|\sum_{i=1}^{n+1} x_i\right| &= \left|\left(\sum_{i=1}^{n} x_i\right) + x_{n+1}\right| \\
&\overset{(1)}{\leq} \left|\sum_{i=1}^{n} x_i\right| + |x_{n+1}| \\
&\overset{(2)}{\leq} \left(\sum_{i=1}^{n} |x_i|\right) + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|\,,
\end{aligned}$$

where we used the triangle inequality for (1) and the induction hypothesis for (2). Thus, we have shown the statement is true for $n = 0$, $n = 1$ and the statement for n implies the statement for $n + 1$. Hence, we are done by induction.

(b) An easy way to see this is to note that

$$\sum_{i=1}^{n} \frac{i-1}{i!} = \sum_{i=1}^{n} \frac{i}{i!} - \frac{1}{i!} = \sum_{i=1}^{n} \frac{1}{(i-1)!} - \frac{1}{i!}.$$

Since this series is telescoping, it simplifies to $1 - \frac{1}{n!}$. For formality, we also provide an inductive proof.

For $n = 1$, the statement is true since $\sum_{i=1}^{1} \frac{i-1}{i!} = \frac{1-1}{1!} = 0 = 1 - \frac{1}{1!}$. Now assume the statement holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{i-1}{i!} &= \left(\sum_{i=1}^{n} \frac{i-1}{i!}\right) + \frac{n+1-1}{(n+1)!} \overset{(1)}{=} \left(1 - \frac{1}{n!}\right) + \frac{n}{(n+1)!} \\ &= 1 - \frac{n+1}{(n+1)!} + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!},\end{aligned}$$

where we used the induction hypothesis for (1). Hence, the statement also holds for $n+1$, so we are done by induction.

(c) This can be shown combinatorially, but since the binomial coefficients were defined algebraically in the exercise, we will give an algebraic proof. Let $n, k \in \mathbb{N}$ be arbitary. Observe that

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ = \frac{n! \cdot k}{k!(n-k+1)!} + \frac{n! \cdot (n-k+1)}{k!(n-k+1)!} &= \frac{n! \cdot (n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k},\end{aligned}$$

as desired.

Task 2

(a) Let $q \in \mathbb{N}$. If q^2 is even, then prove that q is even.

(b) Let $f : X \to Y$ be a map. Prove the equivalence of the following statements:

I. f is injective.

II. $f^{-1}(f(A)) = A$ for all $A \subset X$.

From now on, assume all the properties of real numbers that you know from school.

(c) Let x be a real number, and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\forall x \in \mathbb{R} : f(x) = 2x - 1$, then:

I. Show f is bijective.

II. Find f^{-1}.

(d) Determine all real numbers $a, b \in \mathbb{R}$ such that the map $f(x) = ax + b$ is bijective and then find the inverse of $f(x)$.

Solution.

(a) We show the contrapositive: if q is odd, then q^2 is odd. Let $q = 2k+1$ be odd with $k \in \mathbb{N}$. Then $q^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2k'+1$ for some $k' = 2k^2+2k \in \mathbb{N}$. Hence, q^2 is odd, as desired.

(b) Firstly, we show $f^{-1}(f(A)) \supseteq A$ for all $A \subseteq X$ (with no constraint on f). Let $a \in A$. Then $f(a) \in f(A)$, so $a \in f^{-1}(f(A))$. Since this holds for each $a \in A$, we know $A \subseteq f^{-1}(f(A))$.

Now assume that f is injective, but there exists an A with $f^{-1}(f(A)) \neq A$. By above consideration, we must have $A \subsetneq f^{-1}(f(A))$, so $\exists b \in f^{-1}(f(A))$ and $b \notin A$. Now $b \in f^{-1}(f(A)) \iff f(b) \in f(A) \iff \exists a \in A : f(b) = f(a)$. However, since f is injective, $f(b) = f(a) \implies b = a \in A$, a contradiction.

Finally, we show that if $f^{-1}(f(A)) = A$ for all $A \subset X$, then f is injective. Indeed, let $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. By taking $A = \{a_1\}$, we see that $\{a_1\} = A = f^{-1}(f(A)) = f^{-1}(f(a_2)) \ni a_2$, so $a_1 = a_2$, as desired.

(c) If $x, y \in \mathbb{R}$ with $f(x) = f(y)$, then $2x-1 = 2y-1 \iff 2x = 2y \iff x = y$, so f is injective. Let now $y \in \mathbb{R}$ be arbitrary. Then $\frac{y+1}{2} \in \mathbb{R}$ and $f\left(\frac{y+1}{2}\right) = 2 \cdot \frac{y+1}{2} - 1 = y$. Since y was arbitrary, we deduce that f is surjective (and hence bijective).

To find f^{-1}, we use above observation that $f\left(\frac{y+1}{2}\right) = y$ for all y, so $f^{-1} : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x+1}{2}$.

(d) We claim that the set of such (a, b) is precisely $\{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$. Firstly, we show that this is necessary.

Assume $a = 0$. Then $f(0) = b = f(1)$, so f is not injective (and not bijective). Now we show that if $a \neq 0$, then f is bijective.

Assume $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2) \iff ax_1 + b = ax_2 + b \iff 0 = a(x_1 - x_2)$. Since $a \neq 0$, we can divide to get $x_1 = x_2$. Hence, f is injective. To show surjectivity, observe that for any $y \in \mathbb{R}$, we have $\frac{y-b}{a} \in \mathbb{R}$ (since $a \neq 0$) and $f\left(\frac{y-b}{a}\right) = a \cdot \frac{y-b}{a} + b = y$. Since y was arbitrary, f is surjective. Furthermore, above calculation shows f^{-1} is determined by $x \mapsto \frac{x-b}{a}$.

Task 3

Let G be a set. For all $i \in \mathbb{N}$, let $M_i \subset G$. We define the sets:

1. $\liminf\limits_{i\in\mathbb{N}} M_i = \bigcup\limits_{j\in\mathbb{N}} \left(\bigcap\limits_{i\in[m\in\mathbb{N}:m\geq j]} M_i \right)$ (called lim inferior),

2. $\limsup\limits_{i\in\mathbb{N}} M_i = \bigcap\limits_{j\in\mathbb{N}} \left(\bigcup\limits_{i\in[m\in\mathbb{N}:m\geq j]} M_i \right)$ (called lim superior).

Show the following statements, for all $x \in G$:

a. $x \in \liminf\limits_{i\in\mathbb{N}} M_i \iff \exists k \in \mathbb{N} : \forall n \geq k : x \in M_n$ (i.e. $\liminf\limits_{i\in\mathbb{N}}$ contains precisely those elements of G that are in all except finitely many of the M_i).

b. $x \in \limsup\limits_{i\in\mathbb{N}} M_i \iff \forall k \in \mathbb{N} : \exists n \geq k : x \in M_n$ (i.e. $\limsup\limits_{i\in\mathbb{N}}$ contains precisely those elements of G that are contained in infinitely many of the M_i).

c. Conclude that $\liminf_{i\in\mathbb{N}} M_i \subseteq \limsup_{i\in\mathbb{N}} M_i$.

Solution.

a. " $\Longleftarrow$ ": Let $x \in G$ and assume $\exists k \in \mathbb{N} : \forall n \geq k : x \in M_n$. Then $x \in M_i$ for all $i \geq k$, so x is also in the intersection of these sets, i.e.

$$x \in \bigcap_{i\in[m\in\mathbb{N}:m\geq k]} M_i.$$

Observe that this set appears in definition 1 by taking $j = k$. Hence,

$$x \in \bigcap_{i\in[m\in\mathbb{N}:m\geq k]} M_i \subseteq \bigcup_{j\in\mathbb{N}} \left(\bigcap_{i\in[m\in\mathbb{N}:m\geq j]} M_i \right) = \liminf_{i\in\mathbb{N}} M_i.$$

" $\Longrightarrow$ ": Now we show that if $x \in \liminf_{i\in\mathbb{N}} M_i$, then such a k must exist. Since $\liminf_{i\in\mathbb{N}} M_i$ is a union of sets, x must be an element of one of them, so $\exists k' \in \mathbb{N}$ with

$$x \in \bigcap_{i\in[m\in\mathbb{N}:m\geq k']} M_i.$$

This is now an intersection of sets, so x must be an element of each of them, i.e. $x \in M_i$ for all $i \geq k'$. We now see that the statement $\forall n \geq k : x \in M_n$ is satisfied for $k = k'$. Hence, the two statements are equivalent.

b. " $\Longrightarrow$ ": Let $x \in \limsup_{i\in\mathbb{N}} M_i$. Assume now $\forall k \in \mathbb{N} : \exists n \geq k : x \in M_n$ is false. This then means $\exists k \in \mathbb{N} : \nexists n \geq k : x \in M_n$ is true, which can also be written as $\exists k \in \mathbb{N} : \forall n \geq k : x \notin M_n$, i.e. $x \notin M_n$ for any $n \in \mathbb{N}, n \geq k$. However, then

$$x \notin \bigcup_{i\in[m\in\mathbb{N}:m\geq k]} M_i \supseteq \bigcap_{j\in\mathbb{N}} \left(\bigcup_{i\in[m\in\mathbb{N}:m\geq j]} M_i \right) = \limsup_{i\in\mathbb{N}} M_i.$$

This is implies $x \notin \limsup_{i\in\mathbb{N}} M_i$, a contradiction to our assumption, so $\forall k \in \mathbb{N} : \exists n \geq k : x \in M_n$ must be true.

" $\Longleftarrow$ ": Assume $\forall k \in \mathbb{N} : \exists n \geq k : x \in M_n$. Then for all k, we can find some integer $f(k) \in \mathbb{N}$ (depending on k) and with $f(k) \geq k$ such that $x \in M_{f(k)}$. Observe that for all $k \in \mathbb{N}$,

$$x \in M_{f(k)} \overset{(1)}{\subseteq} \bigcup_{i\in[m\in\mathbb{N}:m\geq k]} M_i,$$

where (1) follows from $f(k) \geq k$. Since x lies in each of these sets, it also lies in their intersection, which is exactly $\limsup_{i\in\mathbb{N}} M_i$ (just replace k by j).

c. Assume for the sake of contradicton that $x \in \liminf_{i\in\mathbb{N}} M_i$, but $x \notin \limsup_{i\in\mathbb{N}} M_i$. By part a, this means

$$\exists k \in \mathbb{N} : \forall n \geq k : x \in M_n$$

is true. In particular, we can choose some $k_1 \in \mathbb{N}$ with $\forall n \geq k_1 : x \in M_n$. By part b, the statement $\forall k \in \mathbb{N} : \exists n \geq k : x \in M_n$ is false, so the negation

$$\exists k \in \mathbb{N} : \forall n \geq k : x \notin M_n$$

must be true. In particular, we can choose some $k_2 \in \mathbb{N}$ with $\forall n \geq k_2 : x \notin M_n$.

However, this now leads to a contradiction. Let $k_3 = \max(k_1, k_2)$. Then since $k_3 \geq k_1$, we have $x \in M_{k_3}$. Since $k_3 \geq k_2$, we also have $x \notin M_{k_3}$, an obvious contradiction. Therefore, such an x cannot exists.