Analysis I, Exercise 6

David Schmitz

Task 1

Decide which of the following statements are true or false and justify.

(a) Every bounded sequence converges.

(b) There exists a sequence that converges and diverges.

(c) Every divergent sequence is unbounded.

(d) For every convergent sequence there exists a maximum $\max_{n\in\mathbb{N}} a_n$.

(e) Every bounded sequence $(a_n)_{n\in\mathbb{N}}$ such that $\frac{a_{n+1}}{a_n} > 1$ is convergent.

(f) Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit a. If $a_n > 0$ for all $n \in \mathbb{N}$, then $a > 0$.

Solution.

(a) The statements is false, for example $a_n = (-1)^n$ is bounded (e.g. $|a_n| < 2$ for all $n \in \mathbb{N}$) and does not converge.

(b) The statement is false. If divergence is interpreted as ”non-convergence”, this is obvious (a sequence cannot converge and not converge at the same time). If it is interpreted as ”divergence to $\pm\infty$”, the statement is also false, since then every divergent series is unbounded and in particular cannot converge.

(c) The statement is true if divergence is interpreted as ”divergence to $\pm\infty$”, but false if interpreted as ”non-convergence” (since $(-1)^n$ would then be divergent and bounded).

Since divergence is usually understood in the second sense (in particular in our lecture), the statement is false.

(d) The statement is false, for example $a_n = 1 - \frac{1}{n}$ converges to 1, but the sequence has no maximum (only the supremum 1).

(e) The statement is true. Note that $\frac{a_{n+1}}{a_n} > 1$ implies $a_n \neq 0$ for all n and additionally, a_n and a_{n+1} have the same sign. Thus, all numbers in the sequence have the same sign. If $a_n > 0$ for all n (i.e. if all elements of the sequence are positive), then we can multiply the inequality by $a_n > 0$ to obtain $a_{n+1} > a_n$. Hence, the sequence is bounded and monotonic, so it must converge (by 2.27b in the lecture notes).

If, on the other hand, $a_n < 0$ for all $n \in \mathbb{N}$ (i.e. all terms of the sequence are negative), then we can multiply by $a_n < 0$ to obtain $a_{n+1} < a_n$. Again, the sequence is bounded and monotonic, so it must converge.

(f) The statement is false, for example $a_n = \frac{1}{n}$ converges to 0, but all terms a_n are positive.

Task 2: Sequential continuity of the distance function d

Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called sequentially continuous if for all $(a_n)_{n\in\mathbb{N}}$ sequences in X converging to a point $a \in X$ we have that the sequence $(f(a_n))_{n\in\mathbb{N}}$ of real numbers converges to the number $f(a)$.

This exercise is designed to let you prove that, given a metric space (X, d), the distance function $d : X \times X \to \mathbb{R}$ is a sequentially continuous function, through the following steps.

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that the numerical sequence $(d(x_n, x))_{n\in\mathbb{N}}$ converges to zero and the sequence $(d(x_n, y))_{n\in\mathbb{N}}$ converges to zero, for $x, y \in X$. Then $x = y$.

2. Suppose that there exists (now we know it is unique) $x \in X$ such that $(d(x_n, x))_{n\in\mathbb{N}}$ converges to zero. Show that for any fixed $y \in X$ the numerical sequence $(d(x_n, y))_{n\in\mathbb{N}}$ converges to the number $d(x, y)$.

3. Suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences such that $x_n \xrightarrow{d} x$ and $y_n \xrightarrow{d} y$. Prove that $(d(x_n, y_n))_{n\in\mathbb{N}} \to d(x, y)$.

4. (Application) Let (X, d) be a metric space. We define the function $d' : X \times X \to \mathbb{R}$ such that
$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad \forall x, y \in X.$$
Show that also d' is a distance in X, and that a sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$ if and only if the sequence of real numbers $(d'(x_n, x))_{n\in\mathbb{N}}$ converges to zero.

Solution.

1. Suppose $x \neq y$. Then by positive-definiteness, it follows that $d(x, y) = \varepsilon > 0$. Since $d(x_n, x)$ converges to zero, there is some $N_1 \in \mathbb{N}$ such that $d(x_n, x) < \frac{1}{2}\varepsilon$ for all $n \geq N$. Similarly, $d(x_n, y)$ converges to zero, so there is some $N_2 \in \mathbb{N}$ such that $d(x_n, y) < \frac{1}{2}\varepsilon$ for all $n \geq N_2$.

   Let now $M = \max(N_1, N_2)$.

   Then by the triangle inequality,
$$\varepsilon = d(x, y) \leq d(x, x_M) + d(x_M, y) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon,$$
   a clear contradiction. Hence, $x = y$.

2. Let $\varepsilon > 0$. Since $(d(x_n, x))_{n\in\mathbb{N}}$ converges to zero, there is some $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

   Thus, for $n \geq N$, it follows by the triangle inequality that
$$d(x_n, y) \leq d(x_n, x) + d(x, y) < \varepsilon + d(x, y),$$
   so $d(x_n, y) - d(x, y) < \varepsilon$. Similarly, we use the triangle inequality to deduce
$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \varepsilon + d(x_n, y),$$
   so $d(x, y) - d(x_n, y) < \varepsilon$.

   All in all, $|d(x, y) - d(x_n, y)| < \varepsilon$, so $(d(x_n, y))_{n\in\mathbb{N}}$ converges to $d(x, y)$, as desired.

3. We begin with a small lemma.

Lemma (Inverse triangle inequality for arbitrary metric spaces)**.** *Let $a, b, c \in X$. Then $|d(a,b) - d(a,c)| \leq d(b,c)$.*

Proof. By the ordinary triangle inequlity, $d(a,b) \leq d(a,c) + d(c,b)$, so $d(a,b) - d(a,c) \leq d(b,c)$. Similarly, $d(a,c) \leq d(a,b) + d(b,c)$, so $d(a,c) - d(a,b) \leq d(b,c)$.

Combining the two inequalities yields the claim. $\square$

Let $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ converges to x, by part 3, it follows that $(d(x_n, y))_{n\in\mathbb{N}}$ converges to $d(x,y)$. Thus, there is some $N_1 \in \mathbb{N}$ such that $|d(x_n, y) - d(x,y)| < \frac{1}{2}\varepsilon$ for all $n \geq N_1$.

Similarly, $(y_n)_{n\in\mathbb{N}}$ converges to y, so there is some $N_2 \in \mathbb{N}$ such that $|d(y_n, y)| < \frac{1}{2}\varepsilon$ for all $n \geq N_2$.

Define now $N = \max(N_1, N_2)$. Let $n \geq N$. Then, by the triangle inequality,

$$\begin{aligned}|d(x_n, y_n) - d(x,y)| &= |d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x,y)| \\ &\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x,y)|\,.\end{aligned}$$

Using the lemma with $a = x_n, b = y_n, c = y$ implies that $|d(x_n, y_n) - d(x_n, y)| \leq d(y_n, y)$. Thus,

$$|d(x_n, y_n) - d(x,y)| \leq d(y_n, y) + |d(x_n, y) - d(x,y)| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

so $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges to $d(x,y)$, as desired.

4. Note that the denominator $1 + d(x,y)$ is always ≥ 1 by positive-definiteness of d. Since the numerator $d(x,y)$ is non-negative as well, $d'(x,y)$ is non-negative too.

   Also, for $d'(x,y)$ is zero if and only if the numerator $d(x,y)$ is zero, i.e. if and only if $x = y$. Hence, d' is positive-definite.

   The symmetry of d' follows immediately from the symmetry of d:

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)} = \frac{d(y,x)}{1 + d(y,x)} = d'(y,x)$$

   for all $x, y \in X$.

   Finally, we show that d' satisfies the triangle inequality. This is simply algebraic manipulation. Let $x, y, z \in X$. Note that all denominators $d(x,y)+1, d(y,z)+1, d(z,x)+1$ are positive, so by clearing denominators,

$$\begin{gathered}d'(x,z) \leq d'(x,y) + d'(y,z) \\ \iff d(x,z)(1 + d(x,y))(1 + d(y,z)) \leq d(x,y)(1 + d(x,z))(1 + d(y,z)) \\ + d(y,z)(1 + d(x,z))(1 + d(x,y)) \\ \iff d(x,z) + d(x,z)d(x,y) + d(x,z)d(y,z) + d(x,y)d(y,z)d(z,x) \\ \leq d(x,y) + d(x,y)d(x,z) + 2d(x,y)d(y,z) \\ + 2d(x,y)d(y,z)d(z,x) + d(y,z) + d(y,z)d(x,z) \\ \iff d(x,z) \leq d(x,y) + d(y,z) \\ + 2[d(x,y)d(y,z) + d(x,y)d(y,z)d(z,x)].\end{gathered}$$

Here, the last inequality holds because d satisfies the triangle inequality.

Now we show that as metrics, d and d' are equivalent.

Assume that $(x_n)_{n\in\mathbb{N}}$ converges to x, i.e. $(d(x_n,x))_{n\in\mathbb{N}}$ converges to zero. Note that for all $x,y\in X$, we have the inequality $1+d(x,y)\geq 1$ and so $d'(x,y)=\frac{d(x,y)}{1+d(x,y)}\leq d(x,y)$. Hence, $0\leq d'(x_n,x)\leq d(x_n,x)$ for all $n\in\mathbb{N}$, so by squeeze/sandwich theorem, $(d'(x_n,x))_{n\in\mathbb{N}}$ also converges to zero.

Now assume that $(d'(x_n,x))_{n\in\mathbb{N}}$ converges to zero. Let $\varepsilon>0$ and define $\tilde{\varepsilon}=\frac{\varepsilon}{1+\varepsilon}>0$. By convergence, there is some $N\in\mathbb{N}$ such that $d'(x_n,x)<\tilde{\varepsilon}$ for all $n\geq N$.

Substituting this into the defintion of d' and multiplying by $(1+d(x_n,x))(1+\varepsilon)>0$ yields

$$\begin{aligned}&\frac{d(x_n,x)}{1+d(x_n,x)}<\frac{\varepsilon}{1+\varepsilon}\\ \iff\ & d(x_n,x)(1+\varepsilon)<(1+d(x_n,x))\varepsilon\\ \iff\ & d(x_n,x)<\varepsilon.\end{aligned}$$

Hence, $(d(x_n,x))_{n\in\mathbb{N}}$ converges to zero, or, equivalently, $(x_n)_{n\in\mathbb{N}}$ converges to x, as desired.

Task 3

Prove the following statements.

(a) The following sequence converges

$$a_n:=\frac{(2n^2-3n)(n^3+1)}{(n+2)(n^2+n^4)}.$$

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(a_{n+1}-a_n)_{n\in\mathbb{N}}$ converges to a, then $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ converges to a.

(c) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $a_n>0$ for $n\in\mathbb{N}$. If $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ converges to a, then $\left(\sqrt[n]{a_n}\right)_{n\in\mathbb{N}}$ converges to a.

(d) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be given by

$$a_n=\frac{n}{\sqrt[n]{n!}}\text{ and }b_n=\frac{1}{n^2}\sqrt[n]{\frac{(3n)!}{n!}}.$$

Then $\lim\limits_{n\to\infty}a_n=\mathrm{e}$ and $\lim\limits_{n\to\infty}b_n=\frac{27}{\mathrm{e}^2}$.

Solution.

(a) Write

$$a_n=\frac{2n^5-3n^4+2n^2-3n}{n^5+2n^4+n^3+2n^2}=\frac{2n^4-3n^3+2n-3}{n^4+2n^3+n^2+2n}=2+\frac{-7n^3-2n^2-2n-3}{n^4+2n^3+n^2+2n}=2-b_n,$$

where $b_n:=\frac{7n^3+2n^2+2n+3}{n^4+2n^3+n^2+2n}$. Observe that $n^4+2n^3+n^2+2n\geq n^4$ and $7n^3+2n^2+2n+3\leq 7n^3+2n^3+2n^3+3n^3=14n^3$, so $b_n\leq\frac{14n^3}{n^4}=\frac{14}{n}$. Hence, $0\leq b_n\leq c_n$, where $c_n=\frac{14}{n}$. Since both the zero sequence and $(c_n)_{n\in\mathbb{N}}$ converge to zero, by the squeeze/sandwich theorem, $(b_n)_{n\in\mathbb{N}}$ also converges to zero, so $(a_n)_{n\in\mathbb{N}}$ converges to 2.

(b) Write $b_n := a_{n+1} - a_n$. We claim that $\sum_{i=1}^{n} b_i = a_{n+1} - a_1$. For $n = 1$, this follows from the definition of b_n. Now assume that $\sum_{i=1}^{n} b_i = a_{n+1} - a_1$. Then $\sum_{i=1}^{n+1} b_i = b_{n+1} + a_{n+1} - a_1 = a_{n+2} - a_{n+1} + a_{n+1} - a_1 = a_{n+2} - a_1$. Hence, our claim follows by induction.

By assumption, $(b_n)_{n\in\mathbb{N}}$ converges to a. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ sufficiently large such that $|b_n - a| < \frac{1}{2}\varepsilon$ for all $n \geq N$.

Then for $n \geq N$, by the triangle inequality,

$$\begin{aligned}
\left|\frac{a_{n+1}}{n+1} - a\right| &= \left|\frac{a_1 + \sum\limits_{i=1}^{n} b_i}{n+1} - a\right| = \left|\frac{a_1 - a}{n+1} + \sum_{i=1}^{n} \frac{b_i - a}{n+1}\right| \\
&\leq \left|\frac{a_1 - a}{n+1}\right| + \sum_{i=1}^{n} \left|\frac{b_i - a}{n+1}\right| = \frac{|a_1 - a|}{n+1} + \sum_{i=1}^{N-1} \frac{|b_i - a|}{n+1} + \sum_{i=N}^{n} \frac{|b_i - a|}{n+1} \\
&< \frac{1}{n+1}\left(|a_1 - a| + \sum_{i=1}^{N-1} |b_i - a|\right) + \sum_{i=N}^{n} \frac{\frac{1}{2}\varepsilon}{n+1} = \frac{R}{n+1} + \frac{\varepsilon(n+1-N)}{2(n+1)},
\end{aligned}$$

where $R = |a_1 - a| + \sum_{i=1}^{N-1} |b_i - a|$ is some real number independent of n.

Denote now $R' = R - \frac{1}{2}\varepsilon N$, so that $\left|\frac{a_{n+1}}{n+1} - a\right| < \frac{R'}{n+1} + \frac{\varepsilon(n+1)}{2(n+1)} = \frac{R'}{n+1} + \frac{1}{2}\varepsilon$.

If we now let $N' \in \mathbb{N}$ be larger than $\max(\frac{2R'}{\varepsilon}, N)$, then for $n \geq N'$, we even have

$$\left|\frac{a_{n+1}}{n+1} - a\right| < \frac{R'}{n+1} + \frac{1}{2}\varepsilon \leq \frac{R'}{\left(\frac{2R'}{\varepsilon}\right)} + \frac{1}{2}\varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that $\left(\frac{a_n}{n}\right)_{n\in\mathbb{N}}$ converges to a, as desired.

Remark. This is also a nice consequence of the Stolz-Cesàro-theorem.

(c) Write $b_n = \frac{a_{n+1}}{a_n}$. We claim that $\prod_{i=1}^{n} b_i = \frac{a_{n+1}}{a_1}$. For $n = 1$, this follows from the definition of b_n. Now assume that $\prod_{i=1}^{n} b_i = \frac{a_{n+1}}{a_1}$ holds for some $n \in \mathbb{N}$. Then $\prod_{i=1}^{n+1} b_i = b_{n+1} \cdot \frac{a_{n+1}}{a_1} = \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+1}}{a_1} = \frac{a_{n+2}}{a_1}$. Hence, our claim follows by induction.

By assumption, $(b_n)_{n\in\mathbb{N}}$ converges to a. Let $\varepsilon \in (0, a)$ be arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that $|a - b_n| < \varepsilon$ for all $n \geq N$.

Denote $P = a_1 \prod_{i=1}^{N-1} b_i$, a real number independent of n. Since $a_1 > 0$ and the b_i are positive too (ratios of positive numbers), we know that $P > 0$. Then for all $n \geq N$,

$$\begin{aligned}
a_{n+1} = a_1 \prod_{i=1}^{n} b_i = a_1 \prod_{i=1}^{N-1} b_i \prod_{i=N}^{n} b_i = P \prod_{i=N}^{n} b_i \\
< P \prod_{i=N}^{n} (a + \varepsilon) = P(a + \varepsilon)^{n+1-N}.
\end{aligned}$$

Hence, $\sqrt[n+1]{a_{n+1}} < P^{\frac{1}{n+1}}(a+\varepsilon)^{\frac{n+1-N}{n+1}} = \left(\frac{P}{(a+\varepsilon)^N}\right)^{\frac{1}{n+1}}(a+\varepsilon)$.

Analogously, for $n \geq N$,

$$\begin{aligned} a_{n+1} = a_1 \prod_{i=1}^{n} b_i = a_1 \prod_{i=1}^{N-1} b_i \prod_{i=N}^{n} b_i = P \prod_{i=N}^{n} b_i \\ > P \prod_{i=N}^{n} (a-\varepsilon) = P(a-\varepsilon)^{n+1-N}. \end{aligned}$$

so $\sqrt[n+1]{a_{n+1}} > P^{\frac{1}{n+1}}(a-\varepsilon)^{\frac{n+1-N}{n+1}} = \left(\frac{P}{(a-\varepsilon)^N}\right)^{\frac{1}{n+1}}(a-\varepsilon)$.

Thus, letting $p_n = \left(\frac{P}{(a-\varepsilon)^N}\right)^{\frac{1}{n+1}}(a-\varepsilon)$ and $q_n = \left(\frac{P}{(a+\varepsilon)^N}\right)^{\frac{1}{n+1}}(a-\varepsilon)$ (note that the denominators are positive since $\varepsilon < a$), we see that

$$p_n < \sqrt[n]{a_n} < q_n$$

for all $n \geq N+1$.

By (2.20b), we know that $\lim\limits_{n\to\infty} p_n = a-\varepsilon$ and $\lim\limits_{n\to\infty} q_n = a+\varepsilon$.

Since $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are both convergent (and thus bounded), the union of their sets $\{p_n \mid n \in \mathbb{N}\} \cup \{q_n \mid n \in \mathbb{N}\}$ is bounded too. Since for $n \geq N+1$, each term $\sqrt[n]{a_n}$ lies inside of (p_n, q_n), we know that $\{a_n \mid n \geq N+1\}$ is bounded. Hence, $\{a_n \mid n \in \mathbb{N}\}$ is bounded as well, since we only added finitely many terms.

Thus, $(\sqrt[n]{a_n})_{n\in\mathbb{N}}$ has a (finite) cluster point (for instance, $\limsup\limits_{n\to\infty} \sqrt[n]{a_n}$). Let A be such a cluster point.

Then we can consider the subsequence $(\sqrt[n]{a_n})_{n\in M}$ converging to A, where $M \subseteq \mathbb{N}$ is infinite. Restricting our attention to the integers $n \in M \cap [N+1, \infty)$, the sequences $p_n < \sqrt[n]{a_n} < q_n$ are three convergent sequences with limits $a-\varepsilon, A, a+\varepsilon$ respectively.

Since limits preserve weak inequalities, this implies $a-\varepsilon \leq A \leq a+\varepsilon$ for all such cluster points A. However, $\varepsilon > 0$ was arbitrary in the beginning. Hence, the only such A is a.

Finally, we see that $(\sqrt[n]{a_n})_{n\in\mathbb{N}}$ has exactly one cluster point in $A = a$ and that it is bounded. Hence, it must converge to a, as desired.

(d) Define $c_n = \frac{n^n}{n!}$. Then $\frac{c_{n+1}}{c_n} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \frac{(n+1)^n/n!}{n^n/n!} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n$, which converges to e by definition. Using part (c), this implies $(\sqrt[n]{c_n})_{n\in\mathbb{N}}$ converges to e. However, $\sqrt[n]{c_n} = \frac{n}{\sqrt[n]{n!}} = a_n$, as desired.

Then

$$\begin{aligned} \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt[n]{(3n)!}}{n^2 \sqrt[n]{n!}} = \lim_{n\to\infty} \frac{\sqrt[n]{(3n)!}}{n^3 \sqrt[n]{n!/n^n}} = \lim_{n\to\infty} \frac{\frac{3^3}{3^3} \cdot \sqrt[n]{(3n)!/n^{3n}}}{\sqrt[n]{n!/n^n}} \\ = \lim_{n\to\infty} \frac{3^3 \sqrt[n]{(3n)!/(3n)^{3n}}}{\sqrt[n]{n!/n^n}} = \lim_{n\to\infty} \frac{27 a_{3n}^{-3}}{a_n^{-1}}. \end{aligned}$$

By the calculation rules for limits (see 2.13) and since $(a_n)_{n\in\mathbb{N}}$ converges to e, this equals $\frac{27/\mathrm{e}^3}{\mathrm{e}} = \frac{27}{\mathrm{e}^2}$, as desired.