Analysis I, Exercise 10

David Schmitz

Task 1

An nth root of unity is a number $z \in \mathbb{C}$ such that $z^n = 1$. Let z be an nth root of unity. Prove that

$$\sum_{k=1}^{n} z^k = \begin{cases} n & \text{if } z = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution. For $z = 1$, all summands are simply 1, so the sum equals $\sum_{k=1}^{n} 1 = n$. Now assume $z \neq 1$. Note that this sum is just a finite geometric series. Analogously to (1.19), define $S = \sum_{k=1}^{n} z^k$. Then $zS = \sum_{k=1}^{n} z^{k+1} = \sum_{k=2}^{n+1} z^k$, so $zS - S = z^{n+1} - z = z^n \cdot z - z = z - z = 0$, since $z^n = 1$. Hence, $(z-1)S = 0$, which implies $S = 0$ (recall $z \neq 1$).

Remark. This can also be solved using Vieta's formula on a polynomial of the form $x^m - 1$, where m divides n.

Task 2

1. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence converging to $a \in \mathbb{C}$. Show that the sequence of the conjugates $(\overline{a_n})_{n\in\mathbb{N}}$ converges to $\bar{a}$.

2. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, for $|z| > 0$. Then

$$f(\bar{z}) = \overline{f(z)} \iff a_k \in \mathbb{R} \quad \forall k \in \mathbb{N}\,.$$

Solution.

1. Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ converges to a, there is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

   Let $z = x + \mathrm{i}y \in \mathbb{C}$ be arbitrary. Note that $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |\bar{z}|$.

   Setting $z = a_n - a$ for some $n \geq N$, we obtain that $|\overline{a_n - a}| = |a_n - a| < \varepsilon$. By the calculation rules for conjugation (see group task 2), we have $\overline{a_n - a} = \overline{a_n} - \bar{a}$.

   Hence, $|\overline{a_n} - \bar{a}| = |\overline{a_n} - \bar{a}| < \varepsilon$ for all $n \geq N$, so $(\overline{a_n})_{n\in\mathbb{N}}$ converges to $\bar{a}$, as desired.

2. Assume $a_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, so that $\overline{a_k} = a_k$. Let $z \in \mathbb{C}$. Applying the calculation rules for conjugation, we know that $\overline{z^2} = \bar{z}^2$. Repeating this, we find that by induction, $\overline{z^n} = \bar{z}^n$. Again, by the calculation rules for conjugation, we have

$$\begin{aligned}\overline{f(z)} &= \overline{\sum_{k=0}^{\infty} a_k z^k} = \sum_{k=0}^{\infty} \overline{a_k z^k} \\ &= \sum_{k=0}^{\infty} \overline{a_k}\,\overline{z^k} = \sum_{k=0}^{\infty} a_k \bar{z}^k = f(\bar{z}),\end{aligned}$$

as desired.

Assume now $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$. Observe that $f(\bar{z}) = \sum\limits_{k=0}^{\infty} a_k \bar{z}^k$ is a power series in $\bar{z}$. Similarly, by the calculation rules for conjugation,

$$\overline{f(z)} = \overline{\sum_{k=0}^{\infty} a_k z^k} = \sum_{k=0}^{\infty} \overline{a_k} \bar{z}^k$$

is a power series in $\bar{z}$. Since the two functions are equal for all $z \in \mathbb{C}$, the coefficients of the power series have to be equal as well, i.e. $a_k = \overline{a_k}$ for all $k \in \mathbb{N}$. This implies $a_k \in \mathbb{R}$, as desired.

Task 3

- Find, for $z = 1 + \mathrm{i}$, the following numbers:

$$z^n, \quad \tfrac{1}{z}, \quad \tfrac{1}{z^n}, \quad z^2 + 2z + 5 + \mathrm{i}.$$

- Give the set of solutions of the equation

$$x^2 + x + 1 = 0,$$

once when you assume that $x \in \mathbb{R}$, and then when you assume that $x \in \mathbb{C}$.

Solution.

- Note that $1 + \mathrm{i} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right) = \sqrt{2}(\cos(\pi/4) + \mathrm{i}\sin(\pi/4)) = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$.

  Hence, $z^n = \left(\sqrt{2}\right)^n \mathrm{e}^{\mathrm{i}n\frac{\pi}{4}}$. The value of the second factor depends only on the residue $n \pmod 8$. Hence,

$$z^n = \begin{cases} \sqrt{2}^n & \text{for } n \equiv 0 \pmod 8, \\ \sqrt{2}^{n-1}(1+\mathrm{i}) & \text{for } n \equiv 1 \pmod 8, \\ \sqrt{2}^n \mathrm{i} & \text{for } n \equiv 2 \pmod 8, \\ \sqrt{2}^{n-1}(-1+\mathrm{i}) & \text{for } n \equiv 3 \pmod 8, \\ -\sqrt{2}^n & \text{for } n \equiv 4 \pmod 8, \\ \sqrt{2}^{n-1}(-1-\mathrm{i}) & \text{for } n \equiv 5 \pmod 8, \\ -\sqrt{2}^n \mathrm{i} & \text{for } n \equiv 6 \pmod 8, \\ \sqrt{2}^{n-1}(1-\mathrm{i}) & \text{for } n \equiv 7 \pmod 8. \end{cases}$$

The values for $\frac{1}{z}$ and more generally, $\frac{1}{z^n}$ can simply be determined by looking up $-n$ into the above table, since $\frac{1}{z^n} = z^{-n}$ and our calculation also holds for negative n.

Specifically, $\frac{1}{z} = z^{-1} = \left(\sqrt{2}\right)^{-1} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} = \frac{1-\mathrm{i}}{2}$. Alternatively, we can simply observe that $\frac{1}{1+\mathrm{i}} = \frac{1-\mathrm{i}}{(1+\mathrm{i})(1-\mathrm{i})} = \frac{1-\mathrm{i}}{1-\mathrm{i}^2} = \frac{1-\mathrm{i}}{2}$ (or use the above table with $-1 \equiv 7 \pmod 8$).

Lastly, we compute $z^2 + 2z + 5\mathrm{i}$ as

$$\begin{aligned} z^2 + 2z + 5 + \mathrm{i} &= (1+\mathrm{i})^2 + 2(1+\mathrm{i}) + 5 + \mathrm{i} \\ &= 1 + 2\mathrm{i} + \mathrm{i}^2 + 2 + 2\mathrm{i} + 5 + \mathrm{i} = 8 + 5\mathrm{i} - 1 = 7 + 5\mathrm{i}. \end{aligned}$$

- Multiplying the equation by $x-1$, we see that $0 = (x-1)(x^2+x+1) = x^3 - 1$, so $x^3 = 1$. If $x \in \mathbb{R}$, this implies $x = 1$. However, plugging in $x = 1$ would mean $1 + 1 + 1 = 0$, a contradiction. Hence, the equation has no solution over $\mathbb{R}$.

  Over $\mathbb{C}$, we can either do the same thing (the solutions are the two primitive 3rd roots of unity) or use the quadratic formula to obtain

  $$x^2 + x + 1 = 0 \implies x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2}.$$

  Thus, over $\mathbb{C}$, the equation has the two solutions $x = \frac{-1+\mathrm{i}\sqrt{3}}{2}$ and $x = \frac{-1-\mathrm{i}\sqrt{3}}{2}$.