Analysis I, exercise 11

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Task 1

1. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point, i.e. $\exists \xi \in [0,1]$ such that $f(\xi) = \xi$.
2. Let $a, b \in \mathbb{R}$, $f : \mathbb{R} \to [a,b]$ be continuous. Assume that $\exists x_1, x_2 \in \mathbb{R} : \ f(x_1) = a$ and $f(x_2) = b$. Then prove that f is surjective.

Solution.

1. Consider the function $g(x) : [0,1] \mapsto [-1,1]$ defined by $g(x) = f(x) - x$.

   The interval is as claimed since for all $x \in [0,1]$, we have $g(x) \leq f(x) \leq 1$ and $g(x) \geq f(x) - 1 \geq -1$.

   By group task 1, we know that the identity $x \mapsto x$ is continuous. Furthermore, $x \mapsto f(x)$ is continuous by assumption, so by (3.3), it follows that their difference $x \mapsto f(x) - x = g(x)$ is also continuous.

   Note that $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$.

   If $f(0) = 0$, we can take $\xi = 0$. If $f(1) = 1$, we can take $\xi = 1$. Otherwise, we have $g(0) > 0$ and $g(1) < 0$. Since g is continuous on $[0,1]$, by the intermediate value theorem, g has a zero on $[0,1]$, so $\exists \xi \in [0,1]$ with $g(\xi) = 0$. For this ξ, we have $0 = g(\xi) = f(\xi) - \xi$, so ξ is a fixed point of f.

   Remark. This can be generalized (see Brouwer fixed-point theorem).

2. Assume first $x_1 < x_2$. Then the restriction $f : [x_1, x_2] \to [a,b]$ of f to the interval $[x_1, x_2]$ is still continuous. Let $y \in [a,b]$ be arbitrary.

   Since $f(x_1) = a$ and $f(x_2) = b$, by the intermediate value theorem, we have that $\exists x \in [x_1, x_2]$ with $f(x) = y$.

   Since this works for all $y \in [a,b]$, we conclude that f is surjective.

   Now assume $x_1 > x_2$. Analogously to above, we consider the continuous function $f : [x_2, x_1] \to [a,b]$. Let $y \in [a,b]$. Since $f(x_2) = b$ and $f(x_1) = a$, by the intermediate value theorem, there is some $x \in [x_2, x_1]$ with $f(x) = y$.

   Again, this holds for all $y \in [a,b]$, so f is surjective.

Task 2

Prove, using the $\varepsilon - \delta$ criterion, that

$$f : [0,1] \to \mathbb{R} : \; x \mapsto \sqrt{x}$$

is continuous. Further show that it is not Lipschitz continuous.

Solution. Let $x_0 \in (0,1]$ be arbitrary and let $\varepsilon > 0$. Define $\delta = \varepsilon \cdot \sqrt{x_0} > 0$ (note the restriction $x_0 > 0$).

We claim that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [0,1]$, we have $|f(x) - f(x_0)| < \varepsilon$. Indeed,

$$\begin{aligned} |f(x_0) - f(x)| &= \left|\sqrt{x_0} - \sqrt{x}\right| \\ &= \frac{\left|(\sqrt{x_0} - \sqrt{x})(\sqrt{x_0} + \sqrt{x})\right|}{\sqrt{x_0} + \sqrt{x}} \\ &= \frac{|x - x_0|}{\sqrt{x_0} + \sqrt{x}} < \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \\ &\leq \frac{\delta}{\sqrt{x_0}} = \varepsilon, \end{aligned}$$

as desired. Hence, f is continuous on $(0,1]$.

Now prove continuity at $x_0 = 0$. Let $\varepsilon > 0$. We choose $\delta = \varepsilon^2 > 0$. Then for all $x \in (-\delta, \delta) \cap [0,1]$, we have $x < \delta$ and hence

$$|f(x_0) - f(x)| = \left|\sqrt{x}\right| = \sqrt{x} < \sqrt{\delta} = \varepsilon,$$

as desired.

Assume now f is Lipschitz continuous with Lipschitz constant $L \in \mathbb{R}$. If $L \leq 1$, then simply choosing $u = \frac{1}{4}, w = 0$ would imply

$$\begin{aligned} \frac{1}{2} = \sqrt{\frac{1}{4}} - 0 &= ||f(u) - f(w)|| \\ &\leq L\,||u - w|| \leq |u - w| = \frac{1}{4}, \end{aligned}$$

a clear contradiction. Hence, $L > 1$. Choosing $u = \frac{1}{4L^2}, w = 0$ (this is why we had to ensure $L > 0$), we see that

$$\begin{aligned} \frac{1}{2L} = \sqrt{\frac{1}{4L^2}} - 0 &= ||f(u) - f(w)|| \\ &\leq L\,||u - w|| = L \cdot \frac{1}{4L^2} = \frac{1}{4L}, \end{aligned}$$

a clear contradiction.

Task 3

Consider the function $f : [0,1] \to [0,1]$ given by

$$f(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \cap [0,1], \text{ with } p, q \in \mathbb{N} \text{ coprime}, \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]. \end{cases}$$

Prove that f is continuous at $x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ and discontinuous at $x \in \mathbb{Q} \cap [0,1]$.

Solution. Note first that f is well-defined since the representation $x = \frac{p}{q}$ is unique if we force $\gcd(p, q) = 1$ (at least for $x \neq 0$, but we can simply choose $f(0) = 1$ or any other nonzero number, since this is not relevant in our proof).

Assume first $x_0 = \frac{p}{q} \in \mathbb{Q} \cap [0, \frac{1}{2}]$ with $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$. Define $(a_n)_{n\in\mathbb{N}}$ by $a_n = x_0 + \frac{\sqrt{2}}{2^{n+2}}$. Then the second summand is irrational, so $a_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}$. Observe that $(a_n)_{n\in\mathbb{N}}$ converges to x_0.

Note that $a_n \leq x_0 + \frac{\sqrt{2}}{2^2} \leq x_0 + \frac{1}{2} \leq 1$, for all n, so the sequences elements actually lie in $[0, 1]$. Thus, since all sequence elements are irrational, $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} 0 = 0 \neq \frac{1}{q} = f(x_0) = f\left(\lim\limits_{n\to\infty} a_n\right)$, so f is not continuous at x_0.

For $x_0 = (\frac{1}{2}, 1]$, we instead define $a_n = x_0 - \frac{\sqrt{2}}{2^{n+2}}$, which again converges to x_0 and all sequence elements are irrational, so analogously to above, we conclude that f is not continuous at x_0. This case distinction is necessary since using above sequence for $x_0 = 1$ results in a sequence for which all elements lie outside of $[0, 1]$.

Finally, we show continuity at $x_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ sufficiently large such that $N > \frac{1}{\varepsilon}$. Let also $M = \left\{\frac{p}{q} :\ p, q \in \{1, 2, \ldots, N\}\right\} \cap [0, 1]$, the set of rationals in $[0, 1]$ with denominator $\leq N$ when written in lowest terms.

The most important property of M is that it is finite since $|M| \leq N^2$ (there are N choices for q and for each of those, at most N for p). In addition, M is exactly the set of $x \in [0, 1]$ with $f(x) \geq \frac{1}{N}$. Thus, $f(x) < \frac{1}{N} < \varepsilon$ for all $x \notin M$.

Hence, we can simply choose $\delta = \min\{x_0 - m :\ m \in M\}$, which exists since M is finite. Also, $\delta > 0$ since otherwise $x_0 \in M$. However, this is impossible since x_0 is irrational and $M \subset \mathbb{Q}$.

Then for all $x \in (x_0 - \delta, x_0 + \delta)$, we have by definition of δ, $x \notin M$. Thus, since f is always nonnegative,

$$|f(x) - f(x_0)| = |f(x) - 0| = f(x) < \varepsilon,$$

so f is continuous at x_0, as desired.

Task 4

Using the intermediate value theorem, show the following:

1. $3x^2 - 4x = 3$ for $x \in \mathbb{R}$ is solvable in $[0, 2]$.
2. $x^3 = 2x^2 + 3x - 3$ for $x \in \mathbb{R}$ has at least three solutions in $[-2, 3]$.

Solution.

1. Let $f(x) = 3x^2 - 4x$. As a polynomial function, f is continuous. Since $f(0) = 0$ and $f(2) = 3 \cdot 4 - 8 = 4$, by the intermediate value theorem, there is some $x \in [0, 2]$ with $f(x) = 3$. For this x, we have $3 = f(x) = 3x^2 - 4x$, as desired.
2. Let $f(x) = x^3 - 2x^2 - 3x + 3$. We want to show that f has at least three roots in $[-2, 3]$.

   Note that

$$\begin{aligned} f(-2) = -7 < 0 < 3 = f(-1), \\ f(-1) = 3 > 0 > -1 = f(1), \\ f(1) = -1 < 0 < 3 = f(3). \end{aligned}$$

Furthermore, f is a polynomial and hence continuous. Thus, by the intermediate value theorem, it follows that f contains a root on each of the intervals $(-2,-1)$, $(-1,1)$ and $(1,3)$. Since these intervals are disjoint, it follows that f has at least three roots in $(-2,3)$, as desired.