Analysis I, Exercise 8

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TO BE REMOVED: Group work 1

$$\begin{tikzcd}
(a) \arrow[d] & \\
(c) \arrow[r, leftrightarrow] \arrow[d] & (g) \\
(d) \arrow[r, leftrightarrow] \arrow[d] \arrow[dr] & (h) \\
(f) \arrow[d] & (b) \\
(e) &
\end{tikzcd}$$

Task 1

Prove the convergence or divergence of the following series:

(a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$;

(b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}$ with $a \in \mathbb{Q}$.

Solution.

(a) Let $a_n = \frac{1}{2n+1} \geq 0$. Since $a_n \leq \frac{1}{n}$ and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, by the squeeze/sandwich theorem, $\lim\limits_{n\to\infty} a_n = 0$. In addition, for $n_1 \geq n_2$, we have $2n_1 + 1 \geq 2n_2 + 1$, so $\frac{1}{2n_1+1} \leq \frac{1}{2n_2+1}$, which implies that $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence. Thus, we can apply the Leibnitz criterion, (2.45), to deduce that the corresponding alternating series $\sum\limits_{n=1}^{\infty} (-1)^n a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ exists.

Remark. Since $\ln(1+x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$, this sum equals $-1 + \mathrm{Im}\left(\ln(1+\mathrm{i})\right) = -1 + \frac{\pi}{4} + 2\pi n$ for some $n \in \mathbb{Z}$. By merging consecutive terms, the sum can be bounded by $-1 + \frac{\pi^2}{6}$, so $n = 0$. Alternatively, the sum equals

$$-1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\Bigg|_{x=0}^{x=1} - 1$$
$$= -1 + \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n}\, \mathrm{d}x = -1 + \int_0^1 \sum_{n=0}^{\infty} \left(-x^2\right)^n \mathrm{d}x$$
$$= -1 + \int_0^1 \frac{1}{1+x^2}\, \mathrm{d}x = -1 + \arctan(1) - \arctan(0) = \frac{\pi}{4} - 1.$$

(b) If $\alpha = 0$, then the sum is $-1 + 1 - 1 + \dots$, which diverges. Similarly, if $\alpha < 0$, then $\left|\frac{(-1)^n}{n^\alpha}\right| = n^{-\alpha}$, which is an increasing function of n by (1.46). However, this means that the terms of the sequence are not even bounded, so the series must diverge.

Assume now $\alpha > 0$ and let $a_n = \frac{1}{n^\alpha}$. Analogously to above, n^α is a strictly increasing function of n and $\lim_{n\to\infty} n^\alpha = \infty$. Thus, $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing to 0. Hence, by the Leibnitz criterion, we deduce that $\frac{(-1)^n}{n^\alpha}$ exists.

Remark. Explicitly, this sum is $-\eta(\alpha) = (2^{1-\alpha} - 1)\zeta(\alpha)$ (note the pole at $\alpha = 1$), which converges even for every $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,(\alpha) > 0$.

Task 2

Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of real numbers. Prove that

1. If $\lim_{n\to\infty} a_n = a \in \mathbb{R}$, then all sub-sequences of $(a_n)_{n\in\mathbb{N}}$ converge to a.
2. The following definition is equivalent to ours:
$$\limsup_{n\to\infty} a_n = \inf_{k\geq 1}\left(\sup_{n\geq k} a_n\right) \text{ and } \liminf_{n\to\infty} a_n = \sup_{k\geq 1}\left(\inf_{n\geq k} a_n\right).$$
3. (Characterisation in the finite case) We have
$$\limsup_{n\to\infty} a_n = L \in \mathbb{R}$$
if and only if the following two properties are satisfied:
   a) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n \leq L + \varepsilon$ for all $n \geq N$;
   b) For every $k \in \mathbb{N}$, there exists $n_k > k, n_k \in \mathbb{N}$ such that $a_{n_k} \geq L - \varepsilon$.
4. The sequence $(a_n)_{n\in\mathbb{N}}$ always has a subsequence converging to the limes superior/inferior. More precisely, let $L = \limsup_{n\to\infty} a_n \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = L$.

   A similar statement holds true for the $\liminf a_n$. Conclude from this point a different proof of the Bolzano-Weierstrass theorem.

5. We have $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ if and only if for each sub-sequence of $(a_n)_{n\in\mathbb{N}}$ there exists a sub-sub-sequence convergent to a. This means that there exists some infinite subset $J \subseteq \mathbb{N}$ such that $\lim_{\substack{n\to\infty \\ n\in J}} a_n = a$.

6. If $\limsup_{n\to\infty} a_n$ is finite, then $\mathrm{Cl}[(a_n)]$, the set of cluster points of $(a_n)_{n\in J}$, is bounded above and non-empty, and furthermore

$$\limsup_{n\to\infty} a_n = \max \mathrm{Cl}[(a_n)].$$

If $\liminf_{n\to\infty} a_n$ is finite, then $\mathrm{Cl}[(a_n)]$ is bounded below, non-empty, and

$$\liminf_{n\to\infty} a_n = \min \mathrm{Cl}[(a_n)].$$

Show just the first equality, the second one is very similar.

Solution.

1. Assume $\lim_{n\to\infty} a_n = a$. Let $\varphi : \mathbb{N} \to I \subseteq \mathbb{N}$ be a strictly increasing function, then $\left(a_{\varphi(n)}\right)_{n\in\mathbb{N}} = (a_n)_{n\in I}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   We now claim that $\varphi(n) \geq n$ for all n. Note that $\varphi(0) \geq 0$ since the image of φ is a subset of $\mathbb{N}$. Now assume $\varphi(n) \geq n$. Then since φ is strictly increasing, we have $\varphi(n+1) > \varphi(n) \geq n$. Hence, $\varphi(n+1) \geq n+1$, so the claim follows from induction.

   Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ converges to a, there is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. For $n \geq N$, we also have $\varphi(n) \geq n \geq N$, so $\left|a_{\varphi(n)} - a\right| < \varepsilon$. Thus, $(a_{\varphi(n)})_{n\in\mathbb{N}}$ also converges to a, as desired.

2. By our definition, $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup\{a_k : k \geq n\})$. Observe that the sequence $b_k := \sup\{a_n : n \geq k\}$ is monotonically decreasing, since if $k_1 \leq k_2$, then $\{a_k : k \geq k_1\} \supseteq \{a_k : k \geq k_2\}$, so $b_{k_1} = \sup\{a_k : k \geq k_1\} \geq \sup\{a_k : k \geq k_2\} = b_{k_2}$.

   Comparing with our definition, we furthermore have $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

   Since $(b_k)_{k\in\mathbb{N}}$ is monotonically decreasing, it either diverges to $-\infty$ or converges to some $b \in \mathbb{R}$.

   In the first case, $(b_k)_{k\in\mathbb{N}}$ cannot be bounded below, so $\lim_{k\to\infty} b_k = -\infty$. Similarly, $\inf_{k\geq 1}(b_k)$ must equal $-\infty$, since if this were a finite real number, then $(b_k)_{k\in\mathbb{N}}$ would be bounded below.

   Hence, $\limsup_{n\to\infty} a_n = -\infty = \inf_{k\geq 1}(b_k) = \inf_{k\geq 1}\left(\sup_{n\geq k} a_n\right)$, as desired.

   Now assume $\lim_{k\to\infty} b_k = b \in \mathbb{R}$.

   Lemma. *$b \leq b_k$ for all $k \geq 1$.*

   Proof. Assume not, i.e. $b_{k'} < b$ for some $k' \geq 1$. Then since $(b_k)_{k\in\mathbb{N}}$ is monotonically decreasing, it would follow that $b_k \leq b_{k'}$ for all $k \geq k'$. If we take $\varepsilon = b - b_{k'} > 0$, by convergence, there would need to be an $N \in \mathbb{N}$ such that $|b_k - b| < \varepsilon$ for all $k \geq N$. However, if we let $M = \max(N, k')$, then it follows that $\varepsilon > |b - b_M| \geq b - b_M \geq b - b_{k'} = \varepsilon$, a contradiction. Hence, such a k' cannot exist. $\square$

Thus, we see that b is a lower bound for b_k. By (1.31), we only need to show $\forall \varepsilon > 0 : \exists k \in \mathbb{N} : b_k < b + \varepsilon$ to conclude $b = \inf_{k \geq 1} (b_k)$.

Let $\varepsilon > 0$ be arbitrary. Since $(b_k)_{k \in \mathbb{N}}$ converges to b, there exists an $N \in \mathbb{N}$ such that $|b - b_k| < \varepsilon$ for all $k \geq N$. By the lemma, we know $|b - b_N| = b_N - b$. Hence, $b_N - b < \varepsilon \iff b_N < b + \varepsilon$, so taking $k = N$ above suffices. Thus, $b = \inf_{k \geq 1} (b_k)$. Finally, we deduce $\limsup_{n \to \infty} a_n = \lim_{k \to \infty} b_k = b = \inf_{k \geq 1} b_k = \inf_{k \geq 1} \left(\sup_{n \geq k} a_n \right)$, as desired.

To show the other equality, we could work analogously. However, we can also apply the previous result. For each $A \subseteq \mathbb{R}$, we abbreviate $-A = \{-a : a \in A\}$. Then $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$ (see exercise sheet 7, task 1). Then, it follows that

$$\begin{aligned} \liminf_{n \to \infty} a_n &\overset{\text{def}}{=} \lim_{n \to \infty} \inf\{a_k : k \geq n\} = -\left(\lim_{n \to \infty} \left(-\inf\{a_k : k \geq n\} \right) \right) \\ &= -\lim_{n \to \infty} \sup\{-a_k : k \geq n\} \overset{\text{def}}{=} -\limsup_{n \to \infty} (-a_n) \end{aligned}$$

Now using the case we have already shown above, we know that this equals

$$\begin{aligned} -\inf_{k \geq 1} \left(\sup_{n \geq k} (-a_k) \right) &= -\inf_{k \geq 1} \left(-\inf_{n \geq k} (a_k) \right) = -\left(-\sup_{k \geq 1} \left(\inf_{n \geq k} a_k \right) \right) \\ &= \sup_{k \geq 1} \left(\inf_{n \geq k} a_n \right), \end{aligned}$$

as desired.

3. Assume $\limsup_{n \to \infty} = L \in \mathbb{R}$.

   By the definition of the $\limsup$, $L = \lim_{k \to \infty} b_n$, where $b_n = \sup_{n \geq k} a_n$. Thus, for each $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|L - b_k| < \varepsilon$ for all $k \geq N$.

   In part 2, we already proved that $(b_k)_{k \in \mathbb{N}}$ is monotonically decreasing and (see the lemma) deduced $L \leq b_k$ for all $k \geq 1$. Thus, $|L - b_N| = b_N - L < \varepsilon$. Recall $b_N = \sup_{n \geq N} a_n$, which implies $b_N \geq a_n$ for all $n \geq N$. Combining with the previous inequality, this implies $a_n \leq b_N < L + \varepsilon$ for all $n \geq N$. This proves a).

   Recall $b_k \geq L$ for all $k \geq 1$. In particular, $L \leq b_{k+1} = \sup_{n \geq k+1} a_n = \sup_{n > k} a_n$. By (1.31), this implies $\forall \varepsilon > 0 : \exists k' > k : a_{k'} > \sup_{n > k} a_n - \varepsilon \geq L - \varepsilon$. Upon taking $n_k = k'$, we conclude that b) holds.

   It remains to prove the converse. Assume both a) and b) hold for some real L. By taking $\varepsilon = 1$ in a), we see that $a_n \leq L + 1$ for sufficiently large n, so the sequence must be bounded above and hence $\limsup_{n \to \infty} a_n < \infty$. Assume $\limsup_{n \to \infty} a_n = -\infty$. Taking $\varepsilon = 1$ for b), we see that for all $k \in \mathbb{N}$, there is an $n_k > k$ with $a_{n_k} \geq L - 1$. In particular, the number of terms a_n with $a_n \geq L - 1$ is infinite, so the $\limsup$ is at least $L - 1$, which is finite. Hence, we can let $\limsup_{n \to \infty} a_n = L' \in \mathbb{R}$. Our goal is to show $L = L'$. Using the other direction of the statement we already proved, we deduce that a) and b) also hold for L'.

   Assume now $L \neq L'$.

- Case $L < L'$: Let $\varepsilon = \frac{1}{3}(L' - L) > 0$. Then since a) holds for L, there is some $N \in \mathbb{N}$ such that $a_n \leq L + \varepsilon$ for all $n \geq N$. Since b) holds for L', there is some $n_N > N$ such that $a_{n_N} \geq L' - \varepsilon$. However, $n_N > N$ implies $a_{n_N} \leq L + \varepsilon$. This is a contradiction, since
$$L' - \varepsilon \leq a_{n_N} \leq L + \varepsilon$$
would imply $3\varepsilon = L' - L \leq 2\varepsilon$.
- Case $L' < L$: Let $\varepsilon = \frac{1}{3}(L - L') > 0$. Then since a) holds for L', there is some $N \in \mathbb{N}$ such that $a_n \leq L' + \varepsilon$ for all $n \geq N$. Since b) holds for L, there is some $n_N > N$ such that $a_{n_N} \geq L - \varepsilon$. However, $n_N > N$ implies $a_{n_N} \leq L' + \varepsilon$. This is a contradiction, since
$$L - \varepsilon \leq a_{n_N} \leq L' + \varepsilon$$
would imply $3\varepsilon = L - L' \leq 2\varepsilon$.

Since both cases lead to a contradiction, we have $L = L'$, as desired.

4. I believe the axiom of choice is needed for this, but I could be mistaken.

Anyway, let $L = \limsup\limits_{n\to\infty} a_n$. By definition, this equals $\lim\limits_{k\to\infty} b_k$, where $b_k = \sup\limits_{n\geq k} a_n$ is defined as above. Let $\varepsilon > 0$ and $M \in \mathbb{N}$ be arbitrary. Since the sequence $(b_k)_{k\in\mathbb{N}}$ converges to L, there is some $N_1 \in \mathbb{N}$ such that $|L - b_k| < \varepsilon$ for all $n \geq N_1$. In part 2, we proved (see the lemma) that $b_k \geq L$ for all $k \geq 1$, so $|L - b_k| = b_k - L < \varepsilon$.

Define now $N = \max(N_1, M)$. Then $|L - b_N| < \varepsilon$. By definition, $b_N = \sup\limits_{n\geq N} a_n$ is a supremum of a set, so by (1.31), we have $\forall \tilde{\varepsilon} > 0 : \exists k' \geq N : a_{k'} > b_N - \tilde{\varepsilon}$. In particular, this holds for $\tilde{\varepsilon} = \varepsilon$, so there is some $k' \geq N$ with $a_{k'} > b_N - \varepsilon \geq L - \varepsilon$.

Also, b_N must be an upper bound for $\{a_n : n \geq N\}$ (since it is the supremum of this set), so $a_{k'} \leq b_N$ holds. Combining with the above inequality $b_N - L < \varepsilon$, we deduce that $a_{k'} - L < \varepsilon$.

All in all, it follows that $|a_{k'} - L| < \varepsilon$ for some $k' \in \mathbb{N}, k' \geq M$, where $\varepsilon > 0$ and $M \in \mathbb{N}$ are arbitrary.

We now recursively define the following sequence of positive integers: For c_0, we take $c_0 \geq 1$ with $|L - a_{c_0}| < 1$, whose existence is guaranteed by the above discussion by taking $\varepsilon = M = 1$. Assume we now know c_n. Then we take $c_{n+1} \geq c_n + 1$ such that $\left|L - a_{c_{n+1}}\right| < 2^{-n-1}$, whose existence is guaranteed by the above discussion by taking $\varepsilon = 2^{-n-1}$ and $M = c_n + 1$.

Then, we see that $(c_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers (since $c_{n+1} \geq c_n$), so $(a_{c_n})_{n\in\mathbb{N}}$ is a subsequene of $(a_n)_{n\in\mathbb{N}}$.

We now claim that this subsequence converges to L, which will finish the proof. Let $\varepsilon > 0$ be arbitrary. Since $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$ by (2.10), there is some $N \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ for all $n \geq N$.

Then for $n \geq N$, we have $|L - a_{c_n}| < 2^{-n} < \varepsilon$ by the recursion, so $(a_{c_n})_{n\in\mathbb{N}}$ converges to L, as desired.

We now deduce the Bolzano-Weierstrass theorem. Suppose $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence. Then $\limsup\limits_{n\to\infty} a_n = L < \infty$ is a finite real number, so by above discussion, there is some subsequence of $(a_n)_{n\in\mathbb{N}}$ converging to L. In particular, a convergent subsequence exists, which implies the Bolzano-Weierstrass theorem.

5. Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence converging to $a \in \mathbb{R}$.

   Then by part 1, every subsequence also converges to a. In particular, for every subsequence there is a sub-subsequence converging to a (e.g. the subsequence itself, since any sub-subsequence is still a subsequence, but actually any sub-subsequence works).

   We now show the reverse direction. Assume every subsequence has a sub-subsequence converging to a. The, we claim that $(a_n)_{n\in\mathbb{N}}$ also converges to a. Assume not. Then by (2.24), it would follow that for some $\varepsilon > 0$, the set $M = \{n \in \mathbb{N} : |a_n - a| \geq \varepsilon\}$ is infinite. In particular, $(a_n)_{n\in M}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$. By assumption, there must be a sub-subsequence for this subsequence which converges to a. However, this is obviously impossible, since all terms of the subsequence (and thus also of the sub-subsequence) have distance $\geq \varepsilon$ from a.

   Thus, M must be finite, which implies that $(a_n)_{n\in\mathbb{N}}$ converges to a.

6. Assume $\limsup\limits_{n\to\infty} a_n = L$ is finite. By part 5, there is a subsequence of $(a_n)_{n\in\mathbb{N}}$ converging to L, so $L \in \mathrm{Cl}[(a_n)]$. In particular, the set of cluster points is non-empty.

   We now show that every cluster point L' of $(a_n)_{n\in\mathbb{N}}$ satisfies $L' \leq L$, which will imply that the set of these cluster points is bounded above (by L) and that $\limsup\limits_{n\to\infty} = L = \max \mathrm{Cl}[(a_n)]$. Assume $L' > L$ is a cluster point. By definition, this means there is some subsequence $(a_n)_{n\in M}$ converging to L'.

   Let $\varepsilon = L' - L > 0$. By convergence, there is some $N \in \mathbb{N}$ such that $|a_n - L'| < \frac{1}{2}\varepsilon$ for all $n \geq N, n \in M$. In particular, the set $M' = \{n \in \mathbb{N} : |a_n - L'| < \frac{1}{2}\varepsilon\}$ is infinite (it contains $M \cap [N, \infty)$).

   Hence, for all $k \in \mathbb{N}$, there is some $n_k \in \mathbb{N}, n_k \geq k$ with $n_k \in M'$, or, equivalently, $|a_{n_k} - L'| < \frac{1}{2}\varepsilon$. This implies $\frac{1}{2}(L' - L) > |a_{n_k} - L'| \geq L' - a_{n_k} \iff a_{n_k} > \frac{1}{2}(L + L')$.

   Thus, for each $k \in \mathbb{N}$, $\sup\limits_{n\geq k} a_n > \frac{1}{2}(L + L')$, since the supremum must be at least as large as the a_{n_k} in the previous paragraph. Taking the infimum of these sets for all $k \geq 1$, the strict inequality becomes weak, i.e. $\inf\limits_{k\geq 1}\left(\sup\limits_{n\geq k} a_n\right) \geq \frac{1}{2}(L + L')$. However, by part 2, we know that the term on the left equals $\limsup\limits_{n\to\infty} a_n = L$, so $L \geq \frac{1}{2}(L + L')$, which contradicts $L < L'$. Hence, such an L' cannot exist and L is indeed the largest cluster point.

   The equality for lim inf can be shown analogously or the same trick of flipping signs and exchanging sup and inf as in part 2 can be used.

Task 3

Inverstigate the (absolute) convergence or divergence of the following series:

(a) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1-n^2}{n^2(n+1)}$;

(b) $\sum\limits_{n=1}^{\infty} \frac{\left(3^{n+1}\right)^2}{17 \cdot 2^{3n}}$.

Solution.

(a) Note that $(-1)^{n-1}\frac{1-n^2}{n^2(n+1)} = (-1)^n\frac{(n-1)(n+1)}{n^2(n+1)} = (-1)^n\frac{n-1}{n^2}$. Let now $a_n = \frac{n-1}{n^2}$. Then $0 \le a_n \le \frac{n}{n^2} = \frac{1}{n}$. By the squeeze/sandwich theorem, $\lim\limits_{n\to\infty} a_n = 0$.

We claim that $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing for $n \ge 2$. Let $m \ge n \ge 2$. Then

$$a_m \le a_n \iff (m-1)n^2 \le (n-1)m^2 \iff 0 \le m^2n - m^2 - mn^2 + n^2 = (m-n)(mn-m-n).$$

Since $m \ge n$, the first factor is non-negative. It remains to show that $mn - m - n \ge 0$. Since $m, n \ge 2$, we know $m-1, n-1 \ge 1$, so $(m-1)(n-1) \ge 1$, which is equivalent to $mn - m - n \ge 0$, as desired.

Hence, $(a_n)_{n\in\mathbb{N}}$ converges to 0 and is monotonically decreasing (at least for $n \ge 2$). Thus, by the Leibnitz criterion, it follows that $\sum\limits_{n=1}^{\infty}(-1)^n a_n = \sum\limits_{n=1}(-1)^{n-1}\frac{1-n^2}{n^2(n+1)}$ is finite.

We now show that the sequence does not converge absolutely, i.e. $\sum\limits_{n=1}^{\infty} a_n = \infty$. Indeed, for $n \ge 2$, we have $n - 1 \ge \frac{1}{2}n$, so $a_n = \frac{n-1}{n^2} \ge \frac{\frac{1}{2}n}{n^2} = \frac{1}{2}\cdot\frac{1}{n}$. Hence, $\sum\limits_{n=1}^{\infty} a_n \ge \sum\limits_{n=2}^{\infty} a_n \ge \frac{1}{2}\sum\limits_{n=2}^{\infty}\frac{1}{n}$. However, by (2.44b), we now that this diverges to ∞. Hence, the original sequence does not converge absolutely.

(b) Note that

$$\frac{\left(3^{n+1}\right)^2}{17\cdot 2^{3n}} = \frac{3^{2n+2}}{17\cdot 8^n} = \frac{9\cdot 9^n}{17\cdot 8^n} = \frac{9}{17}\cdot\left(\frac{9}{8}\right)^n.$$

Since $\frac{9}{8} > 1$, this is an increasing function of n (see (2.10)). Hence, the terms of the series are not bounded, which implies that it must diverge (to $+\infty$).

Task 4: Application of series

Check that the definition

$$0{,}\overline{9} := \sum_{i=1}^{\infty}\frac{9}{10^i},$$

is meaningful (i.e. that $0{,}\overline{9} \in \mathbb{R}$), and prove that $0{,}\overline{9} = 1$.

Solution. By (2.35c), it suffices to give a sequence of rational numbers which converge to $0{,}\overline{9}$. For this, let $a_n = 9\cdot 10^{-n-1}$. Then $a_0 = 0{,}9, a_1 = 0{,}09, a_2 = 0{,}009$, etc. Hence, we can look at the corresponing series $s_k = \sum\limits_{n=0}^{k} a_n$. Since $\left|\frac{9}{10}\right| < 1$, by (2.44a), this geometric series converges, so $0{,}\overline{9} \in \mathbb{R}$. In addition, applying the formula for geometric series, we see that $0{,}\overline{9} = \sum\limits_{n=0}^{\infty} 9\cdot 10^{-n-1} = \frac{9}{10}\sum\limits_{n=0}^{\infty}\left(\frac{9}{10}\right)^n = \frac{9}{10}\cdot\frac{1}{\left(1-\frac{1}{10}\right)} = \frac{9}{10}\cdot\frac{10}{9} = 1$, as desired.