# Analysis I, Exercise 4

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## Task 1

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a strictly increasing function. Show that

1. $f : A \to f(A)$ is bijective;
2. $f^{-1}$ is strictly monotone.

Can we arrive at the same conclusion if $f$ is monotone?

**Solution.**

1. Assume $a_1 \neq a_2$ with $f(a_1) = f(a_2)$. Then by the totality of $<_A$, we either have $a_1 <_A a_2$ or $a_2 <_A a_1$.

   In the first case, $a_1 <_A a_2$ implies $f(a_1) <_{\mathbb{R}} f(a_2)$ since $f$ is strictly increasing. In particular, $f(a_1) \neq f(a_2)$, a contradiction to our assumption.

   In the second case, $a_2 <_A a_1$ implies $f(a_2) <_{\mathbb{R}} f(a_1)$ since $f$ is strictly increasing. In particular, $f(a_2) \neq f(a_1)$, a contradiction to our assumption.

   Hence, such $a_1, a_2$ cannot exists, so $f$ is injective.

   It remains to show that $f$ is surjective. However, this is obvious: If $y \in f(A)$, then $y = f(x)$ for some $x \in A$ by definition, so $y \in \operatorname{Im}(f)$. Since this holds for all $y \in f(A)$, we have $f(A) \subseteq \operatorname{Im}(f)$, so $f$ is surjective and thus, bijective.

2. Firstly $f^{-1} : f(A) \to A$ exists, since $f$ is bijective by part 1. We claim that $f^{-1}$ is strictly increasing and hence strictly monotone. Assume not, then there are $x, y \in f(A)$ with $x < y$, but $f^{-1}(x) \geq f^{-1}(y)$.

   Since $f$ is strictly increasing, it is in particular increasing, so $f^{-1}(x) \geq f^{-1}(y)$ implies $f(f^{-1}(x)) \geq f(f^{-1}(x))$ or $x \geq y$. This contradicts our assumption, so $f^{-1}$ must be strictly increasing.

If we only require $f$ to be monotone, then $f$ can also be constant, e.g. $f(a) = 0$ for all $a \in A$. However, for $|A| > 1$, this $f$ is not bijective.

## Task 2

Prove the following statements:

(a) For $p, q \in \mathbb{Q}$ with $p < q$, then there exists a $\xi \in \mathbb{R} \setminus \mathbb{Q}$ such that $p < \xi < q$.

**Solution.**

(a) Take $\xi = p + \frac{q-p}{\sqrt{2}}$. As shown in the lecture (see 1.39 in the notes), $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. By definition of the square-root, we also know that $\sqrt{2} \geq 0$. Assume that $0 \leq \sqrt{2} \leq 1$. Then $2 = \sqrt{2} \cdot \sqrt{2} \leq \sqrt{2} \cdot 1 \leq 1 \cdot 1 = 1$, a contradiction. Hence, $\sqrt{2} > 1$.

Since $q - p > 0$, we also have $\frac{q-p}{\sqrt{2}} > 0$, so $\xi = p + \frac{q-p}{\sqrt{2}} > p$.

Now since $\sqrt{2} > 1$, it follows that $\frac{1}{\sqrt{2}} < 1$, so $\frac{q-p}{\sqrt{2}} < q-p$. Then $\xi = p + \frac{q-p}{\sqrt{2}} < p+q-p = q$.

It remains to show that $\xi$ is irrational. Indeed, since $\sqrt{2} \notin \mathbb{Q}$, we also have $\frac{1}{\sqrt{2}} \notin \mathbb{Q}$ (otherwise one could just take the reciprocal of the fraction). Since $q - p \in \mathbb{Q}$ (since both $p, q$ are rational), we have $(q-p) \cdot \frac{1}{\sqrt{2}} \notin \mathbb{Q}$ by Lemma A.2. Then $\xi = p + \frac{q-p}{\sqrt{2}}$ with $p \in \mathbb{Q}$ and $\frac{q-p}{\sqrt{2}} \notin \mathbb{Q}$. By Lemma A.1 $\xi$, is irrational, as desired.

# Task 3

Here we want to complete the proof of the *Arithmetic and Geometric Inequality*. In G2 you proved that if this inequality holds for $n$, then it also holds for $2n$. Now if we show that when the inequality holds for $n+1$, then it also holds for $n$, we have completed the induction. Because we can move like this: $2 \to 4 \to 3 \to 6 \to 5 \to \ldots$. So we want to prove that if the *Arithmetic and Geometric Inequality* holds for $n+1$, then it also holds for $n$.

d. For $a_1, a_2, \ldots, a_n \in \mathbb{R}_{>0}$, show that $\frac{a_1+\ldots+a_n}{n} = \frac{a_1+\ldots+a_n+\frac{a_1+\ldots+a_n}{n}}{n+1}$.

e. Now using the above equality, prove that if the *Arithmetic and Geometric Inequality* holds for $n+1$, then it also holds for $n$.

**Solution.**

d. By distributivity,

$$\begin{aligned}
\frac{a_1 + \ldots + a_n + \frac{a_1+\ldots+a_n}{n}}{n+1} &= \frac{(a_1 + \ldots + a_n) \cdot 1 + (a_1 + \ldots + a_n) \cdot \frac{1}{n}}{n+1} \\
&= \frac{(a_1 + \ldots + a_n)\left(1 + \frac{1}{n}\right)}{n+1} \\
&= \frac{(a_1 + \ldots + a_n)\left(\frac{n+1}{n}\right)}{n+1} \\
&= \frac{(a_1 + \ldots + a_n)(n+1)}{(n+1)n} \\
&= \frac{a_1 + \ldots + a_n}{n},
\end{aligned}$$

as desired.

e. Assume we know AM-GM for $n+1$ variables, i.e.

$$\sqrt[n+1]{y_1 \cdot y_2 \cdot \ldots \cdot y_{n+1}} \leq \frac{y_1 + y_2 + \ldots + y_{n+1}}{n+1}$$

for all $y_1, y_2, \ldots, y_{n+1} \in \mathbb{R}_{>0}$. We want to show the same holds for $n$ variables. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}_{>0}$ be arbitrary.

Then we choose $y_1 = x_1, y_2 = x_2, \dots, y_n = x_n, y_{n+1} = \frac{x_1+x_2+\dots+x_n}{n}$. Using the equation from part d, we observe that

$$\begin{aligned}
\frac{x_1 + x_2 + \dots + x_n}{n} &= \frac{x_1 + \dots + x_n + \frac{x_1+\dots+x_n}{n}}{n+1} \\
&= \frac{y_1 + y_2 + \dots + y_{n+1}}{n+1} \\
&\overset{(*)}{\geq} \sqrt[n+1]{y_1 \cdot y_2 \cdot \dots \cdot y_{n+1}} \\
&= \sqrt[n+1]{x_1 x_2 \dots x_n} \cdot \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{\frac{1}{n+1}} \\
&= (x_1 x_2 \dots x_n)^{\frac{1}{n+1}} \cdot \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{\frac{1}{n+1}},
\end{aligned}$$

where we used AM-GM for $n+1$ variables in $(*)$.

We divide both sides by $\left(\frac{x_1+x_2+\dots+x_n}{n}\right)^{\frac{1}{n+1}}$ and then raise both sides to the power of $\frac{n+1}{n}$ (since $x \mapsto x^{\frac{n+1}{n}}$ is increasing, this does not change the ordering, see 1.46 in the lecture notes):

$$\begin{aligned}
&\left(\frac{x_1 + x_2 \dots + x_n}{n}\right)^{\frac{n}{n+1}} \geq (x_1 x_2 \dots x_n)^{\frac{1}{n+1}} \\
\implies &\frac{x_1 + x_2 + \dots + x_n}{n} = \left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)^{\frac{n}{n+1} \cdot \frac{n+1}{n}} \\
&\geq (x_1 x_2 \dots x_n)^{\frac{1}{n+1} \cdot \frac{n+1}{n}} = (x_1 x_2 \dots x_n)^{\frac{1}{n}} \\
&= \sqrt[n]{x_1 x_2 \dots x_n},
\end{aligned}$$

as desired.

# Task 4

Here we want to continue task 3 from the group work.

d. (Proposition) For $0 \leq i, j \leq n$, let $a_{i,j} \in \mathbb{R}$, prove that:

$$\sum_{i=0}^{n} \sum_{j=0}^{i} a_{i,j} = \sum_{j=0}^{n} \sum_{i=j}^{n} a_{i,j}.$$

(You can understand the above equality better if you try to write down the left hand side for some small values of $n$ and then try to construct the summation like the right hand side.)

e. (Main step) Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $n$. Show that there is a polynomial $g : \mathbb{R} \to \mathbb{R}$ with degree $n$ such that $\forall x \in \mathbb{R},\ f(x) = g(x - \bar{x})$.

f. (Conclusion) Use the above steps to prove the main statement.

**Solution.**

d. Both sides are simply the sum of all terms $a_{i,j}$ with $0 \le j \le i \le n$. For formality, we provide a proof by induction.

If $n = 0$, then both sides simply equate to $a_{0,0} = a_{0,0}$, which is true.

Assume now the identity holds for some $n$. Let $a_{i,j} \in \mathbb{R}$ for $0 \le i, j \le n+1$. Then by the induction hypothesis,

$$\sum_{i=0}^{n} \sum_{j=0}^{i} a_{i,j} = \sum_{j=0}^{n} \sum_{i=j}^{n} a_{i,j} =: S_1.$$

Now we add $S_2 := a_{n+1,0} + a_{n+1,1} + \ldots + a_{n+1,n+1}$ to both sides. Observe that this sum can also be written as

$$S_2 = \sum_{j=0}^{n+1} a_{n+1,j} = \sum_{i=n+1}^{n+1} \sum_{j=0}^{n+1} a_{i,j} = \sum_{i=n+1}^{n+1} \sum_{j=0}^{i} a_{i,j}.$$

This follows because we restrict $i$ to only be $n+1$.

Hence,

$$S_1 + S_2 = \sum_{i=0}^{n} \sum_{j=0}^{i} a_{i,j} + \sum_{i=n+1}^{n+1} \sum_{j=0}^{i} a_{i,j} = \sum_{i=0}^{n+1} \sum_{j=0}^{i} a_{i,j},$$

where we could combine the outer sums because the inner sums are the same expression.

For the other half, we can write $S_2$ as

$$S_2 = a_{n+1,n+1} + \sum_{j=0}^{n} a_{n+1,j} = a_{n+1,n+1} + \sum_{j=0}^{n} \sum_{i=n+1}^{n+1} a_{i,j}.$$

Analogously to before, we find

$$\begin{aligned} S_1 + S_2 &= a_{n+1,n+1} + \sum_{j=0}^{n} \sum_{i=j}^{n} a_{i,j} + \sum_{j=0}^{n} \sum_{i=n+1}^{n+1} a_{i,j} \\ &= a_{n+1,n+1} + \sum_{j=0}^{n} \left( \sum_{i=j}^{n} a_{i,j} + \sum_{i=n+1}^{n+1} a_{i,j} \right) = a_{n+1,n+1} + \sum_{j=0}^{n} \sum_{i=j}^{n+1} a_{i,j}. \end{aligned}$$

To combine this into one sum, simply write $a_{n+1,n+1} = \sum_{j=n+1}^{n+1} a_{n+1,j} = \sum_{j=n+1}^{n+1} \sum_{i=j}^{n+1} a_{i,j}$. Again, this works because the only summand is $i = j = n+1$. Continuing the above chain of equalities, we find that

$$S_1 + S_2 = \sum_{j=n+1}^{n+1} \sum_{i=j}^{n+1} a_{i,j} + \sum_{j=0}^{n} \sum_{i=j}^{n+1} a_{i,j} = \sum_{j=0}^{n+1} \sum_{i=j}^{n+1} a_{i,j}.$$

Similarly to above, we could combine the outer sums because the inner sums were the same expression. Finally, we deduce that

$$\sum_{i=0}^{n+1} \sum_{j=0}^{i} a_{i,j} = S_1 + S_2 = \sum_{j=0}^{n+1} \sum_{i=j}^{n+1} a_{i,j},$$

which is the desired statement for $n+1$. Hence, we are done by induction.

e. This simply follows from shifting the variable and expanding all brackets. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ with $a_i \in \mathbb{R}$ for $0 \leq i \leq n$ and $a_n \neq 0$. Using the binomial formula,

$$\begin{aligned} f(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i((x-\bar{x})+\bar{x})^i &= \sum_{i=0}^{n} a_i \sum_{j=0}^{i} \binom{i}{j}(x-\bar{x})^j \bar{x}^{i-j} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\left(\binom{i}{j} a_i \bar{x}^{i-j}\right)(x-\bar{x})^j. \end{aligned}$$

Define $a_{i,j} := a_i\binom{i}{j}\bar{x}^{i-j}(x-\bar{x})^j \in \mathbb{R}$ (or if we don't pick $x$ first, then $a_{i,j} \in \mathbb{R}[x]$) for $0 \leq i, j \leq n$. If $j > i$, the binomial coefficient is either not defined or can be regarded as zero. Also, the exponent $i - j$ might be a problem if $\bar{x} = 0$, but those pairs $(i, j)$ are not relevant, as $j$ ranges between 0 and $i$. Then by part d, we can rearrange to

$$\begin{aligned} f(x) &= \sum_{i=0}^{n}\sum_{j=0}^{i} a_{i,j} \\ &= \sum_{j=0}^{n}\sum_{i=j}^{n} a_{i,j} = \sum_{j=0}^{n}\sum_{i=j}^{n}\left(a_i\binom{i}{j}\bar{x}^{i-j}\right)(x-\bar{x})^j \\ &= \sum_{i=0}^{n}(x-\bar{x})^i b_i, \end{aligned}$$

where $b_i := \sum\limits_{j=0}^{i} a_i\binom{i}{j}\bar{x}^{i-j}$ are constants not dependent on $x$ (and thus not dependent on $x - \bar{x}$). Thus, $g(x) = \sum\limits_{i=0}^{n} b_i x^i$ is a polynomial of degree at most $n$. From above, we also see that $f(x) = g(x - \bar{x})$.

It remains to show that $g$ has degree exactly $n$. In above expansion, we see that $x - \bar{x}$ appears with exponent $j$. For $j = n$, since $i$ ranges between $j$ and $n$, the only summand is $i = j = n$. Hence, the coefficient of $x^n$ in $g(x)$ is simply $a_n\binom{n}{n}\bar{x}^0 = a_n\binom{n}{n} = a_n \neq 0$. This coefficient is non-zero, since otherwise $f$ would have degree $< n$.

f. We know $f(x) = g(x - \bar{x}) = \sum\limits_{i=0}^{n} b_i(x-\bar{x})^i$ for some constants $b_i$ by part e. Plugging in $x = \bar{x}$, we see that $0 = f(\bar{x}) = g(0) = \sum\limits_{i=0}^{n} b_i \cdot 0^i = b_0$, so $b_0 = 0$. However, then

$$\begin{aligned} f(x) = b_0 + \sum_{i=1}^{n} b_i(x-\bar{x})^i &= \sum_{i=1}^{n} b_i(x-\bar{x})^i \\ &= (x-\bar{x})\sum_{i=0}^{n-1} b_{i+1}(x-\bar{x})^i. \end{aligned}$$

Here, we took out a factor of $x-\bar{x}$ from the sum and shifted indices. Define the polynomial $h(x) = \sum\limits_{i=0}^{n-1} b_{i+1}x^i$ of degree (exactly) $n - 1$ (since $b_n \neq 0$). Hence, $f(x) = (x - \bar{x})h(x)$ for a polynomial $h$ of degree $n - 1$, as desired.