Analysis I, Exercise 13

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Task 1

1. Prove that for every bounded domain $\mathbb{D} \subset \mathbb{C}$, there exists a sequence of polynomial functions $p_n : \mathbb{D} \to \mathbb{C}$ that converges uniformly to the exponential function on $\mathbb{D}$.
2. Does there also exist a sequence of polynomials on $\mathbb{C}$ that converges uniformly to the exponential function? Additionally, is there a sequence of continuous functions with this property?

Solution.

1. Let $p_n(x) = \sum\limits_{k=0}^{n} \frac{x^k}{k!}$ and $C > 0$. Then $I = [-C, C]$ is compact and the sequence $(p_n(x))_{n\in\mathbb{N}}$ converges pointwise to $\exp(x)$ by definition.

   We claim that the sequence even converges uniformly to $\exp(x)$ on I.

   Let $\varepsilon > 0$. Our goal is to find an $N \in \mathbb{N}$ such that $|\exp(x) - p_n(x)| < \varepsilon$ for all $n \geq N$ and $x \in I$.

   Define $a_n = \frac{C^k}{k!}$, so that $\exp(C) = \sum\limits_{k=0}^{\infty} a_k$. Note that $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{C^{k+1}/(k+1)!}{C^k/k!}\right| = \left|\frac{C}{k+1}\right| < 1$ for $k > |C|$, so $\left(\sum\limits_{k=0}^{n} a_k\right)_{n\in\mathbb{N}}$ converges absolutely to $\exp(C)$ by the quotient criterion.

   Hence, there is some $N \in \mathbb{N}$ such that $\sum\limits_{k=n+1}^{\infty} |a_k| < \varepsilon$ for all $n \geq N$. Let now $x \in I$. Then $|x| \leq C$ and for any $n \geq N$, we have by the triangle inequality,

   $$\begin{aligned} |\exp(x) - p_n(x)| &= \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right| \\ &\leq \sum_{k=n+1}^{\infty} \left| \frac{x^k}{k!} \right| = \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \\ &\leq \sum_{k=n+1}^{\infty} \frac{C^k}{k!} < \varepsilon. \end{aligned}$$

   Hence, $(p_n)_{n\in\mathbb{N}}$ converges uniformly to exp on I.

   Finally, we note that $\mathbb{D}$ is bounded. Thus, we can simply take some sufficiently large C with $\mathbb{D} \subseteq [-C, C]$ and we deduce that $(p_n)_{n\in\mathbb{N}}$ also converges uniformly to exp on the subdomain $\mathbb{D}$.

2. We claim there there is no sequence of polynomials on $\mathbb{C}$ that converges uniformly to the exponential function.

   Assume $(p_n)_{n\in\mathbb{N}}$ were a sequence of polynomials converging uniformly to exp over $\mathbb{C}$. Take $\varepsilon = 1$. By uniform convergence, there is some $N \in \mathbb{N}$ such that $|p_n(x) - \exp(x)| < 1$ for all $n \geq N$ and $x \in \mathbb{C}$.

   In particular, we have $\exp(x) - p_N(x) < 1$ for all $x \in \mathbb{R}$.

   Let $m = \deg(p_N)$ and $a_0, a_1, \dots, a_m \in \mathbb{C}$ be the coefficients of p_N with $p_N(x) = \sum_{k=0}^{m} a_k x^k$.

   Assume $x > 1$. Then

   $$\begin{aligned} p_N(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \\ &\leq a_0 x^m + a_1 x^m + \cdots + a_m x^m = (a_0 + a_1 + \cdots + a_m) x^m. \end{aligned}$$

   Hence, there is some constant $C = a_0 + a_1 + \cdots + a_m$ such that $p_N \leq C x^m$ for all $x > 1$.

   However, we also have

   $$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \frac{x^{m+1}}{(m+1)!}$$

   for all $x \in \mathbb{R}^+$. All in all, we obtain

   $$\frac{x^{m+1}}{(m+1)!} < \exp(x) < 1 + p_N(x) \leq C x^m$$

   for all $x > 1$, or, equivalently, $x < C(m+1)!$.

   Since the only restriction is $x > 1$, we can simply take x sufficiently large to arrive at a contradiction.

   However, there exist sequences of continuous functions converging to $\exp : \mathbb{C} \to \mathbb{C}$, for example the constant sequence $(\exp)_{n\in\mathbb{N}}$.

Task 2

1. Let $a, b, c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function fulfilling

   $$a + bx - cx^2 \leq f(x) \leq a + bx + cx^2,$$

   for all $x \in \mathbb{R}$. Show that f is differentiable at $x = 0$ and $f'(0) = b$.

2. Show that the function $f : \mathbb{R} \to \mathbb{R}$,

   $$x \mapsto \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   is differentiable in $x = 0$ and calculate its derivative at zero.

Solution.

1. Define $g(x) = f(x) - a - bx$. Then by assumption,

$$g(x) = f(x) - a - bx \geq a + bx - cx^2 - a - bx = -cx^2,$$
$$g(x) = f(x) - a - bx \leq a + bx + cx^2 - a - bx = cx^2.$$

   In particular, for $x = 0$, we have $0 \leq g(x) \leq 0$, so $g(0) = 0$ and $f(0) = a$. In addition, for $x = 1$, we get $-c \leq g(1) \leq c$, so $c \geq 0$.

   With $c \geq 0$, it now follows that $-cx^2 \leq g(x) \leq cx^2 \implies |g(x)| \leq cx^2$ for all $x \in \mathbb{R}$.

   Our goal is to show that $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists and that it equals b.

   If $c = 0$, then $|g(x)| \leq 0$ for all x, so $g(x) = 0$ and $f(x) = a + bx$ for all x. Then we simply calculate

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{a + bx - a - b \cdot 0}{x} = \lim_{x\to 0} b = b,$$

   as desired.

   Now assume $c > 0$.

   Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{c} > 0$. Then for all $x \neq 0$ with $|x| < \delta$, we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{a + bx + g(x) - a}{x} = b + \frac{g(x)}{x}.$$

   Hence,

$$\begin{aligned} \left|\frac{f(x) - f(0)}{x - 0} - b\right| &= \left|b + \frac{g(x)}{x} - b\right| \\ &= \frac{|g(x)|}{|x|} \leq \frac{cx^2}{|x|} = \frac{c\,|x|^2}{|x|} \\ &= c\,|x| < c\delta = \varepsilon. \end{aligned}$$

   Therefore, $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = b$, so f is differentiable at $x = 0$ and $f'(0) = b$.

2. We claim that $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{f(x)}{x} = 0$, which will imply that f is differentiable at 0 and $f'(0) = 0$.

   Let $\varepsilon > 0$. Choose $\delta = \varepsilon$ and let $x \neq 0$ be such that $|x| < \delta$.

   - $x \in \mathbb{R} \setminus \mathbb{Q}$: In this case, we simply have $\left|\frac{f(x)}{x}\right| = \left|\frac{0}{x}\right| = 0 < \varepsilon$.
   - $x \in \mathbb{Q}$: In this case, we have $\left|\frac{f(x)}{x}\right| = \left|\frac{x^2}{x}\right| = |x| < \delta = \varepsilon$.

   Hence, we have $\frac{f(x)}{x} < \varepsilon$ for all such x, so this fraction approaches 0 as $x \to 0$. Thus, $\lim\limits_{x\to 0} \frac{f(x)}{x} = 0$ and $f'(0) = 0$.

Task 3

1. Show that the derivative of the logarithm is given by $\frac{1}{x}$.
2. Prove that for $a, b \in \mathbb{R}$ with $0 < a < b$, it holds that
$$1 - \frac{a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1.$$
3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f'(x) \neq 0$ for all $x \in \mathbb{R}$. How many roots can f have at most?
4. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

Solution.

1. Let $f(x) = \ln(x)$ and $a \in \mathbb{R}$ be arbitrary. We claim that f is continuous at $x = \exp(a)$, i.e.
$$\lim_{x \to \exp(a)} \frac{f(\exp(a)) - f(x)}{\exp(a) - x} = \frac{1}{\exp(a)}.$$
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers distinct from $\exp(a)$ and with limit $\exp(a)$. Our goal is to show that $\lim\limits_{n\to\infty} \frac{f(\exp(a))-f(a_n)}{\exp(a)-a_n} = \frac{1}{\exp(a)}$.

   Since $a_n > 0$ for all n, we can define $b_n = \ln(a_n)$. Then since $x \mapsto \ln(x)$ is continuous (see (3.9)), it follows that
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln(a_n) = \ln\left(\lim_{n\to\infty} a_n\right) = \ln(\exp(a)) = a.$$
Then
$$\lim_{n\to\infty} \frac{f(\exp(a)) - f(a_n)}{\exp(a) - a_n} = \lim_{n\to\infty} \frac{\ln(\exp(a)) - \ln(\exp(b_n))}{\exp(a) - \exp(b_n)}$$
$$= \lim_{n\to\infty} \frac{a - b_n}{\exp(a) - \exp(b_n)} = \left(\lim_{n\to\infty} \frac{\exp(a) - \exp(b_n)}{a - b_n}\right)^{-1}.$$
Since $\lim\limits_{n\to\infty} b_n = a$, it follows that the limit in parentheses equals $\exp'(a) = \exp(a)$. Hence,
$$\lim_{n\to\infty} \frac{f(\exp(a)) - f(a_n)}{\exp(a) - a_n} = \frac{1}{\exp(a)},$$
as desired.

   Therefore, $x \mapsto \ln(x)$ is continuous at $x = \exp(a)$ for every $a > 0$ with $f'(x) = \frac{1}{\exp(a)} = \frac{1}{x}$. Since $\exp(\mathbb{R}) = \mathbb{R}^+$, this proves that $x \mapsto \ln(x)$ is continuous at every point $x > 0$ with derivative $\ln'(x) = \frac{1}{x}$.
2. Define $f(x) = \ln(x) - \left(1 - \frac{1}{x}\right), g(x) = (x - 1) - \ln(x)$.

   If we let $x = \frac{b}{a} > 1$, then our problem is equivalent to showing $f(x), g(x) > 0$ for all $x > 1$.

Note that $f(1) = \ln(1) - \left(1 - \frac{1}{1}\right) = 0 = (1-1) - \ln(1) = g(1)$. Observe that f, g are differentiable on $\mathbb{R}^+$. Thus, by using the chain rule, it follows that for $x > 0$,

$$f'(x) = \frac{1}{x} - \frac{1}{x^2},$$
$$g'(x) = 1 - \frac{1}{x}.$$

Hence, $f'(1) = g'(1) = 0$ and for $x > 1$, we even have $\frac{1}{x} < 1$, so $g'(x) > 0$. Similarly, $x^2 > x$, so $\frac{1}{x^2} < \frac{1}{x}$ and $f(x) > 0$ for all $x > 1$.

Thus, f and g are strictly increasing on $(1, \infty)$ by (3.40) and monotonically increasing on $[1, \infty)$.

Therefore, $f(x) > f(1) = 0$ and $g(x) > g(1) = 0$ for all $x > 1$, as desired.

3. We claim that f can have at most one root. Assume f had two distinct roots a, b. Without loss of generality assume $a < b$.

   Then $f(a) = 0 = f(b)$, so by Rolle's theorem, there is some $\xi \in [a, b]$ with $f'(\xi) = 0$.

   However, this contradicts the assumption that f' has no roots.

   Hence, such f has at most one root. Indeed, one root is also attainable, for example by $f(x) = x$ with root $f(0) = 0$ and $f'(x) = 1 \neq 0$ for all x.

4. We claim that f is a solution if and only if $f(x) = C\exp(x)$ for some constant $C \in \mathbb{R}$.

   Firstly, since the derivative is linear, we have $(C\exp)' = C(\exp)' = C\exp$, so all such f are indeed solutions.

   Now assume f satisfies $f(x) = f'(x)$ for all x.

   Define $g(x) = f(x)\mathrm{e}^{-x}$. Then g is differentiable by (3.35) and $g'(x) = f'(x)\mathrm{e}^{-x} - f(x)\mathrm{e}^{-x} = (f'(x) - f(x))\,\mathrm{e}^{-x} = 0$.

   Hence, g is constant, so $g(x) = C$ for all x, or, equivalently, $g(x) = C\mathrm{e}^x$, as desired.

   Remark. The fact that g is constant can be shown as follows: assume $g(x) < g(y)$ for some $x, y \in \mathbb{R}$. Then by the mean value theorem, there is some $\xi \in [\min\{x, y\}, \max\{x, y\}]$ with $g'(\xi) = \frac{g(x)-g(y)}{x-y} \neq 0$, which contradicts $g'(\xi) = 0$.

Task 4

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $a, b \in \mathbb{R}$ with $a < b$ and $f'(a) < f'(b)$. Show that for every value $c \in \mathbb{R}$ with $f'(a) < c < f'(b)$, there exists a value $a < \xi < b$ such that $f'(\xi) = c$.

Solution. Even though this looks like the intermediate value theorem, we simply cannot apply it here, since it is possible that f is differentiable but not continuously differentiable.

Define $g : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) - cx$. Then $g'(x) = f'(x) - c$ and since g is continuous on the compact interval $[a, b]$, it follows by (3.24c) that g attains some minimum x_0, i.e. $g(x) \geq g(x_0)$ for some $x_0 \in [a, b]$ and all $x \in [a, b]$.

Since $g'(a) = f'(a) - c < 0 < f'(b) - c = g'(b)$, it follows that $g(a+\varepsilon) < g(a)$ and $g(b-\varepsilon) < g(b)$ for sufficiently small ε, so $x_0 \neq a$ and $x_0 \neq b$.

Hence, $x_0 \in (a, b)$. Thus, x_0 is an interior point of $[a, b]$ and g has a local minimum at x_0, so by (3.37), $0 = g'(x_0) = f'(x_0) - c$. Thus, we can take $\xi = x_0$.

Remark. This is known as Darboux's theorem.