Analysis I, Exercise 3

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Task 1

Assume the axiom of choice. Let A, B be non-empty sets, and $f : A \to B$ be a function. The function f is said to have a

(i) *Left-hand side inverse*, if there exists a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$. In this case, the function g is called *the left-hand side inverse of f.*

(ii) *Right-hand side inverse*, if there exists a function $h : B \to A$ such that $f(h(b)) = b$ for all $b \in B$. In this case, the function h is called *the right-hand side invese of f.*

Prove the following:

1. The function f is injective iff f has a left-hand side inverse.
2. The function f is surjective iff f has a right-hand side inverse.
3. There exists an injective map from A to B iff there exists a surjective map from B to A.
4. If A is a countable set and there exists a surjective map $f : A \to B$, then the set B is finite or countable.

Solution.

1. Assume f has a left-hand side inverse g and let $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$. Applying g to both sides, we find that $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$, so $a_1 = a_2$ and f is injective.

   Assume now that f is injective. Then for each $b \in B$, we have $\left|f^{-1}(\{b\})\right| \leq 1$, for if $f(a_1) = f(a_2)$ with $a_1, a_2 \in A$, then $a_1 = a_2$ by the injectivity of f. Choose some $a_0 \in A$ arbitrarily. We then define the function

   $$g : B \to A, \quad b \mapsto \begin{cases} a_0 \text{ if } f^{-1}(\{b\}) = \{\}, \\ a_b \text{ if } f^{-1}(\{b\}) = \{a_b\}. \end{cases}$$

   g is well-defined since $f^{-1}(\{b\})$ has at most one element. We claim that g is a left-hand side inverse. Indeed, let $a \in A$ be arbitrary. Then $f(a) \in B$ and since $f^{-1}(\{f(a)\}) = \{a\}$ (since f is injective), it follows that $g(f(a)) = a$, as desired.

2. Assume f has a right-hand side inverse h. Let $b \in B$. Observe that $h(b) \in A$ with $f(h(b)) = b$. Since $b \in B$ is arbitrary, f is surjective.

   Assume now f is surjective. Let $X = \{f^{-1}(\{b\}) \mid b \in B\}$. We know that $f^{-1}(\{b\}) \subseteq A$ is non-empty by the surjectivity of f, so $X \subseteq \mathcal{P}(A)$ is a set of non-empty sets. Thus, by the axiom of choice, there is some choice function $h_1 : X \to A$ such that $\forall Y \in X : h_1(Y) \in X$.

Denote by $h_2 : B \to X,\ b \mapsto f^{-1}(\{b\})$ and let $h = h_1 \circ h_2 : B \to A$. We claim that h is a right-hand side inverse.

Let $b \in B$ be arbitrary and let $a^* = h_1(h_2(b))$. We want to show $b = f(h(b)) = f(h_1(h_2(b))) = f(a^*)$. By the definition of h_1, $a^* \in h_2(b) = f^{-1}(\{b\})$. Thus, $f(a^*) = b$, as desired.

3. Assume there exists an injective map $f : A \to B$. By 1., f has some left-hand side inverse $g : B \to A$. Since $g(f(a)) = a$ for all $a \in A$, it follows that g must be surjective. In particular, a surjection $B \to A$ exists.

   Assume there exists a surjective map $f : B \to A$. By 2., f has some right-hand side inverse $h : A \to B$. Let $a_1, a_2 \in A$ with $h(a_1) = h(a_2)$. Then $a_1 = f(h(a_1)) = f(h(a_2)) = a_2$. Therefore, h is injective. In particular, an injection $A \to B$ exists.

4. Let A be countable and $f : A \to B$ be surjective. Since A is countable, there is injective function $g : A \to \mathbb{N}$. By 3., there exists an injection $\tilde{f} : B \to A$. It's easy to see that $\tilde{g} = g \circ \tilde{f} : B \to \mathbb{N}$ is injective: if $b_1, b_2 \in B$ with $g(\tilde{f}(b_1)) = g(\tilde{f}(b_2))$, then by the injectivity of g, $f(b_1) = f(b_2)$. Again, $b_1 = b_2$ by the injectivity of f, so $\tilde{g}$ is injective. By definition, this implies B is countable.

Task 2

Conclude from the axioms of a field $\mathbb{F}$ that for all $x \in \mathbb{F}$ the additive inverse $-x$ is uniquely determined and that $-(-x) = x$ holds.

Solution. Let $x \in \mathbb{F}$ be arbitrary and assume $a, b \in \mathbb{F}$ are both additive inverses of x, i.e. $x + a = 0 = x + b$. Adding a to both sides yields $a + (x + b) = a + (x + a)$. Using the associativity of addition, this is equivalent to $(a + x) + b = (a + x) + a$. By the commutativity of addition, $0 = x + a = a + x$, so $0 + b = 0 + a$. Since 0 is the neutral element with respect to addition, this implies $b = a$. Hence, additive inverses are unique.

For each x, the additive inverse is written as $-x$. Hence, $x + (-x) = 0$ for all x. By the commutativity of addition, $(-x) + x = 0$ also holds, so x must be the additive inverse of $-x$. However, this is nothing but $-(-x)$. Thus, $-(-x) = x$, as desired.

Task 3

Prove that for all $x, y \in \mathbb{R}$

- $\max(x, y) = \frac{1}{2}(x + y + |x - y|)$, and
- $\min(x, y) = \frac{1}{2}(x + y - |x - y|)$.

Solution. We consider the cases $x \geq y$ and $x < y$ separately.

- Assume first $x \geq y$. Then $\max(x, y) = x$ and $x - y \geq 0$, so $|x - y| = x - y$. Thus, $\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + x - y) = x = \max(x, y)$ holds.

  If $x < y$, then $\max(x, y) = y$ and $x - y < 0$, so $|x - y| = y - x$. Thus, $\frac{1}{2}(x + y + |x - y|) = \frac{1}{2}(x + y + y - x) = y = \max(x, y)$ holds.

- This can be proven analogously. A nicer solution might be to note that for all $x, y \in \mathbb{R}$,
$$\begin{aligned}\max(x,y)+\min(x,y) = x+y &= \frac{1}{2}(2x+2y-|x-y|+|x-y|)\\ &= \frac{1}{2}(x+y+|x-y|)+\frac{1}{2}(x+y-|x-y|).\end{aligned}$$
Since we already now that the first summands are equal, the second summands have to be equal as well.

Task 4

Show that

1. $$\left(1+\frac{1}{n}\right)^n \le \sum_{k=0}^{n}\frac{1}{k!} \le 3, \quad \text{for all } n \ge 1,$$
2. $$\left(\frac{n}{3}\right)^n \le \frac{1}{3}n! \le \frac{1}{3}n^n, \quad \text{for all } n \ge 1.$$

Solution.

1. By the binomial formula,
$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\cdot 1^{n-k}\cdot\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!\cdot n^k}\\ &= \sum_{k=0}^{n}\frac{n(n-1)\dots(n-k+1)}{k!\cdot n^k} \le \sum_{k=0}^{n}\frac{1}{k!}.\end{aligned}$$
Here, we used the inequality $n(n-1)\dots(n-k+1) \le n^k$, which follows from multiplying $n \le n, n-1 \le n, \dots, n-k+1 \le n$. To show other direction, we first note that $k! \ge 2^k$ for $k \ge 4$. It trivially holds for $k = 4$, as $24 \ge 16$, and for any $k \ge 4$, if $k! \ge 2^k$, then $(k+1)k! \ge 5k! \ge 2k! \ge 2\cdot 2^k = 2^{k+1}$, so the result follows from induction.
Using this inequality, we observe that for $n \ge 4$,
$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k!} &= 1+1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{k!} = \frac{8}{3}+\sum_{k=4}^{n}\frac{1}{k!}\\ &\le \frac{8}{3}+\sum_{k=4}^{n}\frac{1}{2^k}\end{aligned}$$
Let $S = \sum\limits_{k=4}^{n} 2^{-k}$. Then (or by the formula for geometric series)
$$\begin{aligned}S &= 2S - S = \sum_{k=4}^{n}2^{-k+1} - \sum_{k=4}^{n}2^{-k}\\ &= \sum_{k=3}^{n-1}2^{-k} - \sum_{k=4}^{n}2^{-k} = \left(2^{-3}+\sum_{k=4}^{n-1}2^{-k}\right) - \left(2^{-n}+\sum_{k=4}^{n-1}2^{-k}\right)\\ &= 2^{-3} - 2^{-n} \le 2^{-3}.\end{aligned}$$

Finally, we obtain

$$\sum_{k=0}^{n} \frac{1}{k!} \leq \frac{8}{3} + \sum_{k=4}^{n} 2^{-k} \leq \frac{8}{3} + \frac{1}{8} = \frac{67}{24} < 3.$$

The cases $n \geq 3$ can either be checked by hand or simply note that $\sum\limits_{k=0}^{n} \frac{1}{k!}$ is an increasing function of n.

A more efficient way to prove the upper bound is to note that

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{k!} &= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \leq 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 2 + \sum_{k=2}^{n} \left(\frac{k}{k(k-1)} - \frac{k-1}{k(k-1)} \right) \\ &= 2 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 2 + \left(\sum_{k=1}^{n-1} \frac{1}{k} \right) - \left(\sum_{k=2}^{n} \frac{1}{k} \right) = 2 + 1 - \frac{1}{n} \leq 3.\end{aligned}$$

2. The upper bound is obvious and follows from

$$\frac{1}{3} n! = \frac{1}{3} \prod_{k=1}^{n} k \leq \frac{1}{3} \prod_{k=1}^{n} n = \frac{1}{3} n^n.$$

Proving the lower bound is surprisingly subtle when neither Stirling's results nor derivatives and logarithms are useable. We proceed by induction.

For $n = 1$, the inequality is satisfied, as $\frac{1}{3} \leq \frac{1}{3} \cdot 1!$. Now assume that $\left(\frac{n}{3}\right)^n \leq \frac{1}{3} n!$ for some $n \geq 1$. Multiplying this by $\left(1 + \frac{1}{n}\right)^n \leq 3$ (see above), it follows that

$$\begin{aligned} n! &\geq \left(1 + \frac{1}{n}\right)^n \left(\frac{n}{3}\right)^n \\ &= \left(\frac{n+1}{3}\right)^n. \end{aligned}$$

Multiplying this by $\frac{n+1}{3}$, we obtain $\frac{1}{3}(n+1)! \geq \left(\frac{n+1}{3}\right)^{n+1}$, as desired.

Note. The approximation $\frac{67}{24} \approx 2.79167$ for $\mathrm{e} \approx 2.71828$ has a relative error of around 2.7%.