Analysis I, Exercise 12

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Task 1

Prove or disprove:

1. There exists a continuous fuction $f : [0, 1] \to \mathbb{R}$ such that $f([0, 1]) = \mathbb{R}$.
2. There exists a continuous function $f : (0, 1) \to \mathbb{R}$ such that $f((0, 1)) = \mathbb{R}$.

Solution.

1. No, there is no such function, since continuous functions take compact sets to compact sets. $[0, 1]$ is closed and bounded (and hence compact), while $\mathbb{R}$ is not bounded (and hence not compact).
2. We claim that

$$f(x) = \begin{cases} \frac{1}{2x} - 1 & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2(x-1)} + 1 & \text{if } x > \frac{1}{2} \end{cases}$$

works. Firstly, since $2x \neq 0$ and $2(x-1) \neq 0$ for all $x \in (0, 1)$ it follows (for example by (3.3)) that $g(x) = \frac{1}{2x} - 1$ and $h(x) = \frac{1}{2(x-1)} + 1$ are both continuous on $(0, 1)$.

Since $f(x) = g(x)$ for $x \leq \frac{1}{2}$ and $f(x) = h(x)$ for $x > \frac{1}{2}$, it immediately follows that f is continuous on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

Since $g\left(\frac{1}{2}\right) = 0 = h\left(\frac{1}{2}\right)$, we deduce that f is also continuous at $x = \frac{1}{2}$.

Hence, f is continuous on all of $(0, 1)$.

Now note that $f\left(\left(0, \frac{1}{2}\right)\right) = g\left(\left(0, \frac{1}{2}\right)\right)$ contains every number in $[0, \infty)$. This is guaranteed by the intermediate value theorem and $\lim_{x \to 0} g(x) = \infty$, $g\left(\frac{1}{2}\right) = 0$. Similarly, $f\left(\left(\frac{1}{2}, 1\right)\right) = h\left(\left(\frac{1}{2}, 1\right)\right)$ contains every number in $(-\infty, 0]$. Hence, $f((0, 1))$ contains all of $\mathbb{R}$, as desired.

Remark. Functions like $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ work as well.

Task 2

(a) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Prove the following implications

$$f \text{ is Lipschitz continuous} \implies f \text{ is uniformly continuous} \implies \text{f is continuous.}$$

(b) Show that $f : (0, 1] \to \mathbb{R},\ x \mapsto \frac{1}{x^2}$ is continuous but not uniformly continuous.

(c) Show that $f : [0, 1] \to \mathbb{R},\ x \mapsto x^{\frac{1}{4}}$ is uniformly continuous but not Lipschitz continuous.

Solution.

(a) Assume f is Lipschitz continuous and let $L \in \mathbb{R}$ be such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in D$.

We now prove that f is uniformly continuous. Let $\varepsilon > 0$. We choose $\delta = \frac{\varepsilon}{L} > 0$. For any $x \in D$, take some $\bar{x} \in D$ with $|x - \bar{x}| < \delta$. Then by assumption,

$$|f(x) - f(\bar{x})| \leq L|x - \bar{x}| < L\delta = \varepsilon.$$

Hence, f is uniformly continuous.

Now assume f is uniformly continuous. Let $\varepsilon > 0$. Then there is some $\delta > 0$ such that $|x - \bar{x}| < \delta \implies |f(x) - f(\bar{x})| < \varepsilon$ for all $x, \bar{x} \in D$.

In particular, if we fix any $x \in D$, this shows continuity at x, since then $|f(x) - f(\bar{x})| < \varepsilon$ for all $\bar{x} \in (x - \delta, x + \delta) \cap D$. Thus, f is continuous at every $x \in D$, as desired.

(b) We first prove continuity. Let $x \in (0, 1]$ and $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{8}|x|^4 \varepsilon, \frac{1}{2}x\right\} > 0$. Then for all $\bar{x}$ in the interval $(x - \delta, x + \delta) \cap (0, 1]$, we have $|x|, |\bar{x}| \leq 1$ and additionally $\bar{x} \geq x - \delta \geq x - \frac{1}{2}x$, so $|\bar{x}| \geq \frac{1}{2}|x|$. Hence, by the triangle inequality,

$$\begin{aligned}
|f(x) - f(\bar{x})| &= \left|\frac{1}{x^2} - \frac{1}{\bar{x}^2}\right| \\
= \left|\frac{1}{x} + \frac{1}{\bar{x}}\right|\left|\frac{1}{x} - \frac{1}{\bar{x}}\right| &= \frac{|x + \bar{x}|\,|x - \bar{x}|}{|x|^2|\bar{x}|^2} \\
\leq \frac{(|x| + |\bar{x}|)\,|x - \bar{x}|}{|x|^2|\bar{x}|^2} &\leq \frac{(1+1)\delta}{|x|^2|\bar{x}|^2}.
\end{aligned}$$

Using $|\bar{x}| \geq \frac{1}{2}|x|$, we can simplify further to

$$\begin{aligned}
|f(x) - f(\bar{x})| &\leq \frac{2\delta}{|x|^2|\bar{x}|^2} \\
\leq \frac{2\delta}{|x|^2\left(\frac{1}{2}|x|\right)^2} &= \frac{8\delta}{|x|^4} \\
\leq \frac{8\left(\frac{1}{8}|x|^4\varepsilon\right)}{|x|^4} &= \varepsilon,
\end{aligned}$$

as desired.

Assume now f were uniformly continuous. Then for $\varepsilon = 1$, there would have to be a $\delta > 0$ such that $|f(x) - f(\bar{x})| < 1$ for all $x, \bar{x} \in (0, 1]$ with $|x - \bar{x}| < \delta$.

Pick $n \in \mathbb{N}$ sufficiently large such that $n > \frac{1}{\delta}$. Then $0 < \frac{1}{n+1} < \frac{1}{n} < \delta$, so by taking $x = \frac{1}{n} \in (0, 1]$ and $\bar{x} = \frac{1}{n+1} \in (0, 1]$, we see that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \delta$. Thus, by assumption, we would have $|f(x) - f(\bar{x})| < 1$. However,

$$|f(x) - f(\bar{x})| = \left|\frac{1}{\left(\frac{1}{n}\right)^2} - \frac{1}{\left(\frac{1}{n+1}\right)^2}\right| = \left|n^2 - (n+1)^2\right| = 2n + 1 > 1,$$

a clear contradiction. Hence, f is not uniformly continuous.

(c) We first prove that f is continuous. Note that by (3.9) and since $\frac{1}{4} \in \mathbb{Q}$, we know that $f(x) = x^{\frac{1}{4}} = \mathrm{e}^{\ln\left(x^{\frac{1}{4}}\right)} = \mathrm{e}^{\left(\frac{1}{4}\ln(x)\right)}$. Also, by (3.9), we have that $x \mapsto \mathrm{e}^x$ and $x \mapsto \ln(x)$ are continuous. Finally, we have $x \mapsto \frac{1}{4}x$ is continuous. Since the composition of continuous functions is continuous by (3.3), we conclude that f, the composition of three such functions, is continuous.

Observe now that we only consider f on the *compact* domain $[0,1]$. By (3.24), the continuity of f is sufficient to imply uniform continuity on compact sets, so f is uniformly continuous on $[0,1]$.

Assume now f were Lipschitz continuous and take $L > 0$ such that $|f(x) - f(y)| \leq L\,|x-y|$ for all $x, y \in [0,1]$.

In particular, this has to hold if we choose $y = 0$, i.e. $\left|x^{\frac{1}{4}}\right| \leq L\,|x|$, or, since $x \in [0,1]$, $x^{\frac{1}{4}} \leq Lx$. Let $n \in \mathbb{N}$ be arbitrary. Choosing $x = \frac{1}{n^4} \in [0,1]$, we see that

$$x^{\frac{1}{4}} = \frac{1}{n} \leq Lx = \frac{L}{n^4},$$

i.e. $L \geq n^3$ for all positive integers n. However, no such L has this property, since $n^3 \geq n$ grows without bound. Hence, f is not Lipschitz continuous.

Task 3

We call a function one periodic if it fulfills the following property: Let $s : \mathbb{R} \to \mathbb{R}$ be a continuous, nonconstant and bounded function such that

$$s(x) = s(x+1) \tag{1}$$

for all $x \in \mathbb{R}$. The property (1) is called 1-periodicity. Furthermore, let $f, g : (0,\infty) \to \mathbb{R}$ be given by

$$f(x) := s\left(\frac{1}{x}\right) \text{ and } g(x) := xs\left(\frac{1}{x}\right).$$

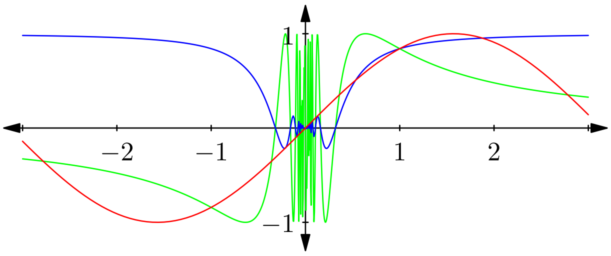
1. Draw examples of s, f and g.
2. Are f and g continuous on $(0,\infty)$? Calculate the limit $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ if they exist.
3. Do there exist $\alpha, \beta \in \mathbb{R}$ such that the functions $f^*, g^* : [0,\infty) \to \mathbb{R}$ given by

$$f^*(x) := \begin{cases} s\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ \alpha & \text{if } x = 0, \end{cases}$$
$$g^*(x) := \begin{cases} xs\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ \beta & \text{if } x = 0 \end{cases}$$

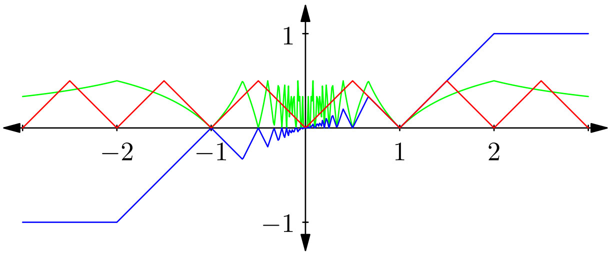
are continuous?

Solution.

(a) We first consider the case when $s(x) = \sin(2\pi x)$. Then by (2.76b), we know that $\sin(x + 2\pi) = \sin(x)$ (so this s is 1-periodic and obviously also nonconstant). Furthermore, by (3.7), we know that it is continuous. In this case, $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = x\sin\left(\frac{1}{x}\right)$. For a graph of these functions, see below (we don't just plot $(0,\infty)$, but the entire domain).



For a second example, consider the function $s(x) = \left|x - 1 - \left\lfloor x - \frac{1}{2}\right\rfloor\right|$. This function is 1-periodic, which can be proven by a quick case distinction (see figure below for the shape of the graph). Again, we also plot $f(x)$ and $g(x)$.



(b) We claim that f and g are continuous. Note that $x \mapsto \frac{1}{x}$ is continuous (on $(0,\infty)$) and $x \mapsto s(x)$ is also continuous by assumption.

Since the composition of continuous functions is continuous by (3.3), we deduce that f must be continuous as well.

Similarly, $x \mapsto x$ is continuous and we know that f is continuous. By (3.3), this implies that their pointwise product, $x \mapsto xf(x) = g(x)$ is also continuous.

We claim that $\lim\limits_{x\to 0} g(x) = 0$. By assumption, s is bounded, so choose $L > 0$ sufficiently large such that $|s(x)| \le L$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{L} > 0$. Then for all $x \in (0,\delta)$, we have

$$|g(x)| = \left|xs\left(\frac{1}{x}\right)\right| = |x|\left|s\left(\frac{1}{x}\right)\right| \le |x|\,L < \delta L = \varepsilon.$$

Hence, we know that $g(x)$ approaches 0 as $x \to 0$, so $\lim\limits_{x\to 0} g(x) = 0$.

We now show that $\lim\limits_{x\to 0} f(x)$ does not exist.

Since s is nonconstant, there are $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$. By periodicity of s, we can assume $p, q \in (0,1]$. Consider the sequences $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}}$ defined by $p_n = \frac{1}{p+n}, q_n = \frac{1}{q+n}$.

Then $\lim_{n\to\infty} p+n = \infty = \lim_{n\to\infty} q+n$, so $\lim_{n\to\infty} p_n = 0 = \lim_{n\to\infty} q_n$.

Assume now $\lim_{x\to 0} f(x)$ would exist. Then

$$\lim_{n\to\infty} f(p_n) = \lim_{x\to 0} f(x) = \lim_{n\to\infty} f(q_n).$$

However, s is 1-periodic, so $f(p_n) = s\left(\frac{1}{\left(\frac{1}{p+n}\right)}\right) = s(p+n) = s(p)$ and $f(q_n) = s\left(\frac{1}{\left(\frac{1}{q+n}\right)}\right) = s(q+n) = s(q)$. Thus,

$$\begin{aligned}\lim_{n\to\infty} f(p_n) &= \lim_{n\to\infty} s(p) = s(p)\\ &\neq s(q) = \lim_{n\to\infty} s(q) = \lim_{n\to\infty} f(q_n),\end{aligned}$$

which contradicts above equality. Hence, $\lim_{x\to 0} f(x)$ does not exist.

(c) Note first that f^* and g^* are equal to f and g on $(0,\infty)$. Since f, g are continuous by part (a), it suffices to consider continuity at $x = 0$.

Choose $\beta = 0$. Since $\lim_{x\to 0} g^*(x) = \lim_{x\to 0} g(x) = 0 = g^*(0)$, by the sequential definition of continuity, it follows that g^* is continuous at 0. Hence, g^* is continuous on all of $[0,\infty)$.

Assume there was some α for which f^* was continuous.

Then again, by sequential continuity at $x = 0$, we would have

$$\alpha = f^*(0) = \lim_{x\to 0} f^*(x) = \lim_{x\to 0} f(x).$$

However, in part (b), we have proven that this limit does not exist. Hence, f^* cannot be continuous.

Remark. Actually, the condition that s is bounded is extraneous. Since the restriction of s to the compact interval $[0,1]$ is still continuous, we deduce that $s([0,1])$ is compact and hence bounded. However, by the periodicity of s, it follows that $s([0,1]) = s(\mathbb{R})$, so s is bounded.

Task 4

(a) Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R})$ is a finite set.

(b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0)\cdot f(1) < 0$ and $f(1) + f(0) \neq 0$. Show that there exists some $\xi \in [0,1]$ such that $f(\xi) = f(0) + f(1)$.

Solution.

(a) If $f(x) = c$ is constant, then $f(\mathbb{R}) = \{c\}$ is finite. We claim that no other f work. Assume f is not constant, then $f(x_1) = y_1 \neq f(x_2) = y_2$ for some $x_1 \neq x_2 \in \mathbb{R}$.

- Case $x_1 < x_2$: Since f is continuous, we can apply the intermediate value theorem on the interval $[x_1, x_2]$ to deduce that all numbers in $[y_1, y_2]$ (if $y_1 < y_2$) or all numbers in $[y_2, y_1]$ (if $y_2 < y_1$) are images of f. Hence, $f(\mathbb{R})$ would contain an interval of positive length. However, any interval of positive length contains infinitely many numbers (for example, since the rationals are dense in $\mathbb{R}$). Hence, this is not possible.

- Case $x_1 > x_2$: Analogously to above, f is continuous, so by applying the intermediate value theorem on the interval $[x_2, x_1]$, we see that all numbers in $[y_1, y_2]$ or $[y_2, y_1]$ are contained in $f(\mathbb{R})$.

  These intervals contain infinitely many numbers again, so $f(\mathbb{R})$ would be infinite.

All in all, we see that only constant f work.

(b) Note that we have $f(0) \neq 0$ and $f(1) \neq 0$, since otherwise $f(0) \cdot f(1) = 0$.

- Case $f(0) < 0$: Then since $f(0) \cdot f(1) < 0$, it follows that $f(1) > 0$. Hence, $f(0) < f(0) + f(1) < f(1)$, or, equivalently, $f(0) + f(1) \in [f(0), f(1)]$. Since f is continuous on $[0, 1]$, we can apply the intermediate value theorem to find a $\xi \in [0, 1]$ for every $y \in [f(0), f(1)]$ with $f(\xi) = y$.

  Choosing $y = f(0) + f(1)$, we find the desired ξ.
- Case $f(0) > 0$: Analogously to above, we find that $f(1) < 0$ since $f(0) \cdot f(1) < 0$. Thus, $f(1) < f(0) + f(1) < f(0)$, or, equivalently, $f(0) + f(1) \in [f(1), f(0)]$. Since f is continuous on $[0, 1]$, we can apply the intermediate value theorem to find a $\xi \in [0, 1]$ for every $y \in [f(1), f(0)]$ with $f(\xi) = y$.

  Choosing $y = f(0) + f(1)$, we find the desired ξ.