Analysis I, Exercise 7

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Task 1

In this task you will prove the remaining of the calculation rules corresponding of liminf and limsup from GW3:

a. First show for $\emptyset \neq A \subseteq \mathbb{R}, \sup A = -\inf\{-a : a \in A\}$ and then conclude the followings:

   I. $\limsup\limits_{n\to\infty}(-a_n) = -\liminf\limits_{n\to\infty}(a_n)$,

   II. $\liminf\limits_{n\to\infty}(-a_n) = -\limsup\limits_{n\to\infty}(a_n)$.

b. First show that for non-empty bounded sets $A, B \subseteq \mathbb{R}$ we have $\sup\{x + y : x \in A, y \in B\} \leq \sup A + \sup B$ and $\inf A + \inf B \leq \inf\{x + y : x \in A, y \in B\}$. Then suppose $\limsup\limits_{n\to\infty} a_n, \liminf\limits_{n\to\infty} a_n, \limsup\limits_{n\to\infty} b_n$, and $\liminf\limits_{n\to\infty} b_n$ are all finite. Then prove:

   a) $\limsup\limits_{n\to\infty}(a_n + b_n) \leq \limsup\limits_{n\to\infty}(a_n) + \limsup\limits_{n\to\infty}(b_n)$,

   b) $\liminf\limits_{n\to\infty}(a_n + b_n) \geq \liminf\limits_{n\to\infty}(a_n) + \liminf\limits_{n\to\infty}(b_n)$.

   Give an example where only $<$ or $>$ holds.

Solution.

a. Let $x = \sup A$. Since $a \leq x$ for all $a \in A$, it follows that $-a \geq -x$ for all $a \in A$. Hence, $-x$ is a lower bound for the set $\{-a : a \in A\}$. By (1.31), $x = \sup A$ implies that $\forall \varepsilon > 0 : \exists a \in A : a > x - \varepsilon$. Multiplying the last inequality by -1, we see that $\forall \varepsilon > 0 : \exists a \in A : -a < -x + \varepsilon$. Again, by (1.31), this implies $-x$ is the infimum of $\{-a : a \in A\}$ (since we already know that it's a lower bound).

   Hence, $\sup A = x = -(-x) = -\inf\{-a : a \in A\}$.

   Similarly, if $x = \inf A$, then $x \leq a$ for all $a \in A$, so $-x \geq -a$ and $-x$ is an upper bound for the set $\{-a : a \in A\}$. By (1.31), $x = \inf A$ implies $\forall \varepsilon > 0 : \exists a \in A : a < x + \varepsilon$. Multiplying the last inequality by -1, we see that $\forall \varepsilon > 0 : \exists a \in A : -a > -x - \varepsilon$. By (1.31), this implies $-x$ is the supremum of $\{-a : a \in A\}$ (since we already know that it's an upper bound).

   Hence, $\inf A = x = -(-x) = -\sup\{-a : a \in A\}$. We can now prove the statements regarding $\limsup$ and $\liminf$ of sequences.

   I. By definition, $\limsup\limits_{n\to\infty}(-a_n) = \lim\limits_{n\to\infty}(\sup\{-a_k : k \geq n\})$. By above discussion, $-\sup\{-a_k : k \geq n\} = \inf\{a_k : k \geq n\}$, so

   $$\limsup_{n\to\infty}(-a_n) = \lim_{n\to\infty}(-\inf\{a_k : k \geq n\}) = -\lim_{n\to\infty}(\inf\{a_k : k \geq n\}) = -\liminf_{n\to\infty}(a_k).$$

II. By definition, $\liminf_{n\to\infty}(-a_n) = \lim_{n\to\infty}(\inf\{-a_k : k \geq n\})$. By above discussion, $-\inf\{-a_k : k \geq n\} = \sup\{a_k : k \geq n\}$, so

$$\liminf_{n\to\infty}(-a_n) = \lim_{n\to\infty}(-\sup\{a_k : k \geq n\}) = -\lim_{n\to\infty}(\sup\{a_k : k \geq n\}) = -\limsup_{n\to\infty}(a_k).$$

b. Since $a \leq \sup A$ and $b \leq \sup B$ for all $a \in A$ and $b \in B$, it follows that $a+b \leq \sup A + \sup B$ for all $a \in A, b \in B$. Hence, $\sup A + \sup B$ is an upper bound for $\{x+y : x \in A, y \in B\}$. This already implies $\sup\{x+y : x \in A, y \in B\} \leq \sup A + \sup B$ (since the supremum cannot be bigger than an upper bound).

Similarly, $a \geq \inf A$ and $b \geq \inf B$ for all $a \in A$ and $b \in B$. Hence, $a+b \geq \inf A + \inf B$ for all $a \in A, b \in B$, so $\inf A + \inf B$ is a lower bound for $\{x+y : x \in A, y \in B\}$. As above, this already implies $\inf A + \inf B \leq \inf\{x+y : x \in A, y \in B\}$.

Now suppose the four limits (see problem statement) exist. This is equivalent to $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ being bounded (since a sequence has finite $\liminf$ iff it is bounded below and finite $\limsup$ iff it is bounded above).

a) By the definiton of limsup,

$$\begin{aligned}\limsup_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty}\sup\{a_k+b_k : k \geq n\} &\overset{(*)}{\leq} \lim_{n\to\infty}\sup\{a_i+b_j : i \geq n \wedge j \geq n\} \\ &\overset{(**)}{\leq} \lim_{n\to\infty}(\sup\{a_i : i \geq n\} + \sup\{b_j : j \geq n\}) \\ = \lim_{n\to\infty}\sup\{a_i : i \geq n\} + \lim_{n\to\infty}\sup\{b_j : j \geq n\} &= \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n).\end{aligned}$$

For $(*)$, we used the fact that $\{a_k+b_k : k \geq n\} \subseteq \{a_i+b_j : i \geq n \wedge j \geq n\}$ and $\sup X \leq \sup Y$ for $X \subseteq Y$. For $(**)$, we set $A = \{a_i : i \geq n\}, B = \{b_j : j \geq n\}$ and use the fact we proved above, i.e. $\sup\{a+b : a \in A, b \in B\} \leq \sup A + \sup B$. Then we take the limit $n \to \infty$ and use the fact that limits preserve weak inequalities.

b) This can be proven analogously to part a). However, we can also be clever and use I and II:

$$\begin{aligned}\liminf_{n\to\infty}(a_n+b_n) &\overset{\text{II}}{=} -\limsup_{n\to\infty}(-a_n-b_n) \\ &\overset{\text{a)}}{\geq} -\limsup_{n\to\infty}(-a_n) - \limsup_{n\to\infty}(-b_n) \overset{\text{I}}{=} \liminf_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n).\end{aligned}$$

We now give an example where both inequalities are strict. Let $a_n = (-1)^n, b_n = (-1)^{n+1}$. Then $a_n + b_n = 0$ for all n. However, $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$ and $\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} b_n = -1$. Thus, the two inequalities read $0 \leq 1+1$ and $0 \geq -1-1$. In both cases, the inequalities are strict.

Task 2: The Babylonian Method

Let $x_0 \in \mathbb{R}$ with $x_0 > 0$ and consider the following recursive sequence:

$$\begin{cases} a_0 = x_0, \\ a_1 = \frac{1}{2}\left(a_0 + \frac{x_0}{a_0}\right), \\ a_n = \frac{1}{2}\left(a_{n-1} + \frac{x_0}{a_{n-1}}\right). \end{cases}$$

Prove that a_n converges to $\sqrt{x_0}$. To do this, we use the following idea: First show that if the limit exists and it is positive, then it is the desired root (I). The prove that the limit actually exists and it is positive. To do so, first prove a_n is bounded (II) and then show it is monotone (III):

1. Assume that $a = \lim_{n\to\infty} a_n$ exists and $a > 0$, now show that $a = \sqrt{x_0}$.
2. Prove $\forall n \geq 1, a_n^2 \geq x_0$.
3. Prove $\forall n \geq 2, a_n \leq a_{n-1}$.

Solution. The suggested order is $2, 3, 1$.

1. The proof of 2 and 3 does not depend on this, so we assume these statements to be true.

   Since $a_n \geq \sqrt{x_0}$ for $n \geq 1$, we know that the sequence is bounded below. Also, it is monotonically decreasing (for $n \geq 2$) by part 3. Hence, by (2.27), it follows that the sequence converges to some a. Also, $\sqrt{x_0}$ is a lower bound of the sequence, so $a \geq \sqrt{x_0}$.

   In particular, we have $a \leq a_n$ for all $n \geq 2$. Assume this were false, i.e. $a_N < a$ for some $N \geq 2$. Since the sequence is decreasing, this would imply $a_m \leq a_N$ for all $m \geq N$. However, this would mean almost all terms of the sequence are less than or equal to a_N, so the limit must also satisfy $a \leq a_N$, a contradiction to our assumption.

   We now show $a = \sqrt{x_0}$. Assume not, then $a - \sqrt{x_0} = \varepsilon > 0$.

   Choose $N \in \mathbb{N}$ sufficiently large such that $a_n - a < \varepsilon$ for all $n \geq N$. We can omit the absolute value since $a \leq a_n$ for $n \geq 2$.

   In particular, we have $a_N - a < \varepsilon$ and $a_{N+1} - a < \varepsilon$. However, by the recursive formula,

   $$\begin{aligned} a_{N+1} - a &= \frac{1}{2}\left(a_N + \frac{x_0}{a_N}\right) - a < \frac{1}{2}(a+\varepsilon) + \frac{x_0}{2a_N} - a \\ &= \frac{1}{2}(\varepsilon - a) + \frac{(a-\varepsilon)^2}{2a_N} = \frac{1}{2}(a-\varepsilon)\left(\frac{a-\varepsilon}{a_N} - 1\right) \\ &= \frac{1}{2}\sqrt{x_0}\left(\frac{\sqrt{x_0}}{a_N} - 1\right) \\ &< \frac{1}{2}\sqrt{x_0}\left(\frac{\sqrt{x_0}}{\sqrt{x_0}} - 1\right) = 0, \end{aligned}$$

   which contradicts the fact that $a_n \geq a$ for all $n \geq 2$. Thus, $a = \sqrt{x_0}$, as desired.

2. We first observe that all a_n are positive (this follows immediately from induction, since only positive terms appear in the recursion). Hence, $a_{n-1} > 0$ and $\frac{x_0}{a_{n-1}} > 0$ for all $n \geq 1$. Observe that, since squares are non-negative,

   $$0 \leq \frac{1}{2}\left(\sqrt{a_{n-1}} - \sqrt{\frac{x_0}{a_{n-1}}}\right)^2 = \frac{1}{2}\left(a_{n-1} + \frac{x_0}{a_{n-1}}\right) - \sqrt{a_{n-1}} \cdot \sqrt{\frac{x_0}{a_{n-1}}} = a_n - \sqrt{x_0},$$

   so $a_n \geq \sqrt{x_0}$ and $a_n^2 \geq x_0$, as desired.

3. This is simply algebraic manipulation. Note that

$$\begin{aligned} a_n \le a_{n-1} \iff \frac{1}{2}\left(a_{n-1} + \frac{x_0}{a_{n-1}}\right) \le a_{n-1} \\ \iff \frac{1}{2} \cdot \frac{x_0}{a_{n-1}} \le \frac{1}{2} a_{n-1} \iff \frac{x_0}{a_{n-1}} \le a_{n-1} \\ \iff x_0 \le a_{n-1}^2. \end{aligned}$$

By part 2, we already now that this holds for $n-1 \ge 1$ or, equivalently, $n \ge 2$. Hence, $a_n \le a_{n-1}$ holds for $n \ge 2$, as desired.

Task 3

(a) Prove: If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then for all $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \varepsilon$ for all $n \ge N_0$.

(b) Is the converse statement true? Give an argument or an example to support your claim.

(c) Assume there exists a $q \in (0,1)$ such that $|a_{n+1} - a_n| < q\,|a_n - a_{n-1}|$ for all $n \ge 2$. Prove that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(d) Show that the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_1 := 1, \quad a_{n+1} := \tfrac{1}{2+a_n}$$

is a Cauchy sequence and find its limit.

Solution.

(a) Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is some $N \in \mathbb{N}$ such that $d(a_m, a_n) = |a_m - a_n| < \varepsilon$ for all $m, n \ge N$. We claim that $N_0 = N$ works. Indeed, $|a_{n+1} - a_n| < \varepsilon$ follows from plugging $m = n+1 \ge N$ into the above inequality.

(b) The converse is not true. Consider the sequence $a_n := \sum\limits_{i=1}^{n} \frac{1}{i}$. Then $|a_{n+1} - a_n| = \frac{1}{n+1}$, which converges to 0. However, if the sequence was a Cauchy sequence, by (2.34b), it would have to converge, which is not true (see (2.44b)).

(c) Let $|a_2 - a_1| = C$. We know $C > 0$, since otherwise $|a_3 - a_2| < C = 0$, which is impossible. We claim that $|a_{n+1} - a_n| \le q^{n-1}C$. For $n = 1$, this follows from the definition of C. If $|a_{n+1} - a_n| \le q^{n-1}C$ holds for some $n \in \mathbb{N}$, then it follows that $|a_{n+2} - a_{n+1}| < q\,|a_{n+1} - a_n| \le q \cdot q^{n-1}C = q^n C$. Hence, our claim follows by induction.

Let $\varepsilon > 0$. By (2.10), the sequence $b_n = q^n$ converges to 0, so there is some $N \in \mathbb{N}$ such that $q^n < \frac{\varepsilon(1-q)q}{C}$ for $n \ge N$. Here, we can omit the absolute value since q^n is always positive.

Let $m, n \geq N$. Without loss of generality let $m \geq n$. Then by the triangle inequality and our claim,

$$\begin{aligned}
|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_{m-2} \pm \cdots - a_n| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq q^{n-1}C + q^n C + \cdots + q^{m-2}C = q^{n-1}C(1 + q + q^2 + \cdots + q^{m-n-1}) \\
&< q^{n-1}C \cdot \frac{1}{1-q} = q^n \cdot \frac{C}{q(1-q)} \\
&< \frac{\varepsilon(1-q)q}{C} \cdot \frac{C}{q(1-q)} = \varepsilon.
\end{aligned}$$

Here, we replaced the finite geometric sequence by the corresponding infinite geometric series (which is possible since $q \in (0,1)$) and applied the formula for geometric series. In the last line, we used the fact that $q^n < \frac{\varepsilon(1-q)q}{C}$ for $n \geq N$.

All in all, we see that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, as desired.

(d) The sequence converges to $\sqrt{2} - 1$. Note that

$$\begin{aligned}
|a_{n+1} - a_n| &= \left|\frac{1}{2+a_n} - a_n\right| = \left|\frac{1 - a_n(2+a_n)}{2+a_n}\right| = \left|\frac{1 - 2a_n - a_n^2}{2+a_n}\right| \\
&= \left|\frac{1 - \frac{2}{2+a_{n-1}} - \frac{1}{(2+a_{n-1})^2}}{2 + \frac{1}{2+a_{n-1}}}\right| = \left|\frac{(2+a_{n-1})^2 - 2(2+a_{n-1}) - 1}{2(2+a_{n-1})^2 + 2 + a_{n-1}}\right| \\
&= \left|\frac{a_{n-1}^2 + 2a_{n-1} - 1}{2a_{n-1}^2 + 9a_{n-1} + 10}\right|
\end{aligned}$$

and

$$|a_n - a_{n-1}| = \left|\frac{1 - 2a_{n-1} - a_{n-1}^2}{2+a_{n-1}}\right| = \left|\frac{a_{n-1}^2 + 2a_{n-1} - 1}{2+a_{n-1}}\right|.$$

Hence,

$$\frac{|a_{n+1} - a_n|}{|a_n - a_{n-1}|} = \left|\frac{a_{n-1}+2}{2a_{n-1}^2 + 9a_{n-1} + 10}\right| = \left|\frac{a_{n-1}+2}{(a_{n-1}+2)(2a_{n-1}+5)}\right| = \left|\frac{1}{2a_{n-1}+5}\right|.$$

We claim that $a_n > 0$ for all $n \geq 1$. For $n = 1$, this holds since $a_1 = 1 > 0$. If $a_n > 0$ for some n, then we also have $2 + a_n > 0$ and hence $a_{n+1} = \frac{1}{2+a_n} > 0$. By induction, our claim follows.

Observe that $2a_{n-1} + 5 \geq 5$, so $\frac{|a_{n+1}-a_n|}{|a_n - a_{n-1}|} \leq \frac{1}{5}$. Hence, setting $q = \frac{1}{5}$ in part (c) implies that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. In particular, the sequence converges. It remains to find its limit. Since any convergent sequence is bounded, there is some $C > 0$ such that $a_n < C$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is some $N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$. In particular, we have $|a_{N+1} - a_n| < \varepsilon$. Plugging this into our recursion yields

$$\begin{aligned}
\varepsilon &> \left|\frac{1}{2+a_n} - a_n\right| = \left|\frac{1 - 2a_n - a_n^2}{2+a_n}\right| \\
\iff |2 + a_n|\,\varepsilon &> \left|a_n^2 + 2a_n - 1\right| = \left|a_n + 1 + \sqrt{2}\right|\left|a_n + 1 - \sqrt{2}\right|
\end{aligned}$$

Since $a_n > 0$, we know $2 + a_n, a_n + 1 + \sqrt{2}$ are both positive, so

$$(2+C)\varepsilon > (2+a_n)\varepsilon > (a_n+1+\sqrt{2}) \left|a_n + 1 - \sqrt{2}\right| > (1+\sqrt{2}) \left|a_n + 1 - \sqrt{2}\right| > \left|a_n + 1 - \sqrt{2}\right|.$$

So far, we have shown that for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $\left|a_n - (\sqrt{2} - 1)\right| < (2+C)\varepsilon$. Since the constant $2+C$ does not depend on n, this means that a_n must converge to $\sqrt{2} - 1$.

Remark. Alternatively, the problem can be attacked using the theory of Möbius transformations.

Task 4

1. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in $\mathbb{R}$, prove:
   a) $(c_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ is also Cauchy.
   b) $(c_n)_{n\in\mathbb{N}} = (a_n b_n)_{n\in\mathbb{N}}$ is also Cauchy.
2. If $(c_n)_{n\in J}$ with $J \subseteq \mathbb{N}$ is a sequence in $\mathbb{R}$ and it converges to some limit c, then every bijective re-ordering of (c_n) also converges to c.

Solution.

1. a) Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences, there is some $N \in \mathbb{N}$ such that $|a_m - a_n| < \frac{1}{2}\varepsilon$ and $|b_m - b_n| < \frac{1}{2}\varepsilon$ for all $m, n \geq N$. Then by the triangle inequality, for all $m, n \geq N$, we have

   $$|c_m - c_n| = |a_m + b_m - a_n - b_n| \leq |a_m - a_n| + |b_m - b_n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

   Hence, $(c_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence.

   b) By (2.34), both sequences converge, so in particular, both sequences are bounded. Let $C > 0$ be such that $|a_n| < C$ and $|b_n| < C$ for all n.
   Let $\varepsilon > 0$. Define $\tilde{\varepsilon} = \frac{\varepsilon}{2C}$. Take $N \in \mathbb{N}$ sufficiently large such that $|a_m - a_n| < \tilde{\varepsilon}$ and $|b_m - b_n| < \tilde{\varepsilon}$ for all $m, n \geq N$, which exists because both sequences are Cauchy sequences.
   Then by the triangle inequality, for all $m, n \geq N$, we have

   $$\begin{aligned} |a_m b_m - a_n b_n| &= |a_m b_m - a_m b_n + a_m b_n - a_n b_n| \leq |a_m b_m - a_m b_n| + |a_m b_n - a_n b_n| \\ &= |a_m| \, |b_m - b_n| + |b_n| \, |a_m - a_n| < C\tilde{\varepsilon} + C\tilde{\varepsilon} = \varepsilon, \end{aligned}$$

   so $(a_n b_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence.

Remark. Since we already know that in $\mathbb{R}$, convergent sequences are exactly the Cauchy sequences, the problem also follows from the fact that $(a_n + b_n)_{n\in\mathbb{N}}$ and $(a_n b_n)_{n\in\mathbb{N}}$ converge if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge.

2. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the bijective re-ordering of the indices. Let $\varepsilon > 0$. By convergence, there is some $N \in \mathbb{N}$ such that $|a_n - c| < \varepsilon$ for all $n \geq N$, i.e. the set $M = \{n \in \mathbb{N} : |a_n - c| \geq \varepsilon\}$ is finite. Consider now the set $\sigma^{-1}(M) = \{\sigma^{-1}(n) \mid n \in M\}$. Since it is finite, it is in particular bounded. Take $N' \in \mathbb{N}$ sufficiently large such that $n < N'$ for all $n \in \sigma^{-1}(M)$. Then for $n \geq N'$, we have $n \notin \sigma^{-1}(M)$, so $\sigma(n) \notin M$, which is equivalent to $\left|a_{\sigma(n)} - c\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the sequence $(a_{\sigma(n)})_{n\in\mathbb{N}}$ also converges to c, as desired.