Analysis I, Exercise 9

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Task 1

Determine the behaviour of the followig power series for $x \in \mathbb{R}$ and justify your arguments:

1. $\sum\limits_{n=0}^{\infty} \frac{n(n-1)(x-3)^n}{2^n(2n+1)^2}$,
2. $\sum\limits_{n=1}^{\infty} \frac{(x-\sqrt{2})^{2n+1}}{2n}$,

Solution.

1. By the Cauchy-Hadamard theorem (see Group task 1), the radius of convergence is
$$r = \frac{1}{\limsup\limits_{n\to\infty} \left|\frac{n(n-1)}{2^n(2n+1)^2}\right|^{\frac{1}{n}}}.$$
We know that the polynomial part $\frac{n(n-1)}{(2n+1)^2}$ converges to $\frac{1}{4}$ as n approaches ∞ (since only the leading coefficients are relevant). Formally, we know that $\lim\limits_{n\to\infty} \frac{n(n-1)}{(2n+1)^2} = \lim\limits_{n\to\infty} \frac{1-\frac{1}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}$. Then $\frac{1}{n}$, $\frac{4}{n}$ and $\frac{1}{n^2}$ all converge to 0, so the entire limit is $\frac{1}{4}$.
By proposition (2.20b), we furthermore know that $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$ for any constant $c > 0$. Applying this to $c = \frac{1}{4}$ above, we see that the if we take the n-th root, the polynomial part will equal 1. Hence, only the exponential part, 2^n will remain, i.e. $\lim n \to \infty \left(\frac{n(n-1)}{2^n(2n+1)^2}\right)^{\frac{1}{n}} = \frac{1}{2} \lim\limits_{n\to\infty} \left(\frac{n(n-1)}{(2n+1)^2}\right)^{\frac{1}{n}} = \frac{1}{2}$.
Comparing this with the above expression, we see that we can replace lim sup by lim since the expression is convergent. Hence, the radius of convergence equals $r = \frac{1}{\frac{1}{2}} = 2$. Since the power series is centered at $x = 3$, the series converges for $x \in (1, 5)$.
2. By the Cauchy-Hadamard theorem, the radius of convegence is
$$r = \frac{1}{\limsup\limits_{n\to\infty} \left|\frac{1}{2n}\right|^{\frac{1}{n}}} = \frac{1}{\limsup\limits_{n\to\infty} \frac{1}{2^{\frac{1}{n}} \cdot n^{\frac{1}{n}}}}.$$
Again, by proposition (2.20a), we furthermore know that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$. Since $2^{\frac{1}{n}}$ converges to 1 as well by (2.20b), it follows that the radius of convergence is 1 (again, we replaced lim sup by lim, since the two are equivalent in case of convergence). Since the power series is centered at $x = \sqrt{2}$, the series converges only for $x \in (\sqrt{2}-1, \sqrt{2}+1)$.

Remark. Here, the interval is open, since the power series actually diverges by the alternating series test.

Task 2

Prove the followings:

1. $\frac{1}{(1-x)^2} = \sum\limits_{k=0}^{\infty}(1+k)x^k$,
2. $\frac{1}{(1-x)^3} = \sum\limits_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}x^k$,
3. Write the function $f(x) = \frac{3x^2}{5-2\sqrt[3]{x}}$ as follows:

$$f(x) = s(x) + t(x)x^{\frac{1}{3}} + r(x)x^{\frac{2}{3}}$$

such that $s(x)$, $t(x)$ and $r(x)$ are power series and then give the interval of convergence for each.

Solution.

1. Taking the Cauchy-product of two geometric series, we obtain

$$\frac{1}{(1-x)^2} = \frac{1}{1-x}\cdot\frac{1}{1-x} = \left(\sum_{i=0}^{\infty}x^i\right)\left(\sum_{j=0}^{\infty}x^j\right)$$
$$= \sum_{k=0}^{\infty}\sum_{i+j=k}x^i\cdot x^j = \sum_{k=0}^{\infty}\sum_{i+j=k}x^{i+j} = \sum_{k=0}^{\infty}\sum_{i+j=k}x^k.$$

For each k, there are exactly $(k+1)$ pairs (i,j) with $i+j=k$, namely $(0,k),(1,k-1),\dots,(k,0)$. Hence, the monomial x^k appears $k+1$ times, so $\frac{1}{(1-x)^2} = \sum\limits_{k=0}^{\infty}(k+1)x^k$.

2. We take the Cauchy-product with the previous term to obtain

$$\frac{1}{(1-x)^3} = \frac{1}{1-x}\cdot\frac{1}{(1-x)^2} = \left(\sum_{i=0}^{\infty}x^i\right)\left(\sum_{j=0}^{\infty}(1+j)x^j\right)$$
$$= \sum_{k=0}^{\infty}\sum_{i+j=k}(1+j)x^{i+j}.$$

Again, the pairs (i,j) with $i+j=k$ are $(0,k),(1,k-1),\dots,(k,0)$, so the coefficient in front of x^k equals $\sum\limits_{j=0}^{k}(1+j) = \sum\limits_{j=0}^{k}1 + \sum\limits_{j=0}^{k}j$. Note that $2\sum\limits_{j=0}^{k}j = \sum\limits_{j=0}^{k}j + \sum\limits_{j=0}^{k}(k-j) = \sum\limits_{j=0}^{k}k = k(k+1)$. Hence, $\sum\limits_{j=0}^{k}j = \frac{k(k+1)}{2}$.

Thus, $\sum\limits_{j=0}^{k}1 + \sum\limits_{j=0}^{j}k = k+1+\frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}$, so $\frac{1}{(1-x)^3} = \sum\limits_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}x^k$.

3. Observe that by the formula for the geometric series,

$$\frac{1}{5-2\sqrt[3]{x}} = \frac{\frac{1}{5}}{1-\left(\frac{2}{5}\sqrt[3]{x}\right)} = \frac{1}{5}\sum_{i=0}^{\infty}\left(\frac{2\sqrt[3]{x}}{5}\right)^i = \sum_{i=0}^{\infty}\frac{2^i}{5^{i+1}}x^{\frac{i}{3}},$$

so

$$\frac{3x^2}{5-2\sqrt[3]{x}} = \sum_{i=0}^{\infty}\frac{3\cdot 2^i}{5^{i+1}}x^{2+\frac{i}{3}}.$$

This is a power series in $x^{\frac{1}{3}}$. Separating out every third term, we see that

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{3\cdot 2^i}{5^{i+1}}x^{2+\frac{i}{3}} &= \sum_{i=0}^{\infty}\frac{3\cdot 2^{3i}}{5^{3i+1}}x^{2+\frac{3i}{3}} + \frac{3\cdot 2^{3i+1}}{5^{3i+2}}x^{2+\frac{3i+1}{3}} + \frac{3\cdot 2^{3i+2}}{5^{3i+3}}x^{2+\frac{3i+2}{3}} \\ &= \sum_{i=0}^{\infty}\frac{3\cdot 8^i}{5\cdot 125^i}x^{2+i} + x^{\frac{1}{3}}\sum_{i=0}^{\infty}\frac{6\cdot 8^i}{25\cdot 125^i}x^{2+i} + x^{\frac{2}{3}}\sum_{i=0}^{\infty}\frac{12\cdot 8^i}{125\cdot 125^i}x^{2+i}.\end{aligned}$$

Thus, $f(x) = s(x) + t(x)x^{\frac{1}{3}} + r(x)x^{\frac{2}{3}}$ for $s(x) = \sum_{i=0}^{\infty} s_i x^i, t(x) = \sum_{i=0}^{\infty} t_i x^i, r(x) = \sum_{i=0}^{\infty} r_i x^i$, where $s_0 = s_1 = t_0 = t_1 = s_0 = s_1 = 0$ and for $i \geq 2$,

$$s_i = \tfrac{3\cdot 8^{i-2}}{5\cdot 125^{i-2}}, \quad t_i = \tfrac{6\cdot 8^{i-2}}{25\cdot 125^{i-2}}, \quad r_i = \tfrac{12\cdot 8^{i-2}}{125^{i-1}}.$$

Finally, we discuss the radius of convergence. For any fixed x, all coefficients s_i, t_i, r_i are simply constant multiples of $\left(\frac{8}{125}\right)^i$, i.e. $C\left(\frac{8}{125}\right)^i$ for some $C \in \mathbb{R}$ (dependent on x). Note $C \neq 0$ for $x \neq 0$.

Using the Cauchy-Hadamard theorem (see Group task 1, exercise 9), we obtain the radius of convergence as

$$r = \frac{1}{\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty}|C|^{\frac{1}{n}}\cdot\left|\left(\frac{8}{125}\right)^n\right|^{\frac{1}{n}}}.$$

Since $|C|^{\frac{1}{n}}$ converges to 1, it follows that the radius of convergence for all three series is simply $\frac{125}{8}$.

Remark. By using Newton's generalization of the binomial formula in the form $x^{\frac{1}{3}} = (1+(x-1))^{\frac{1}{3}} = \sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}(x-1)^n$, we can even write the expression as a single power series in x (though not centered at $x=0$). In addition, the radius of convergence gives insight into the behaviour of the function x: Since the power series is centered at 0 at f has a pole when $5-2\sqrt[3]{x}=0$ i.e. when $\sqrt[3]{x}=\frac{5}{2}$ or $x=\frac{125}{8}$, the radius of convergence cannot exceede this.

Task 3

1. Let $(a_n)_{n\in\mathbb{N}}$ be a monotone sequence and assume it has a subsequence that converges. Then prove $(a_n)_{n\in\mathbb{N}}$ has a limit and it is the same as the limit of its convergent subsequence.

2. Let $(a_n), (b_n) \subseteq \mathbb{R}$ be two sequences such that $(s_n) = \sum_{k=0}^{n} a_k$ and $(t_n) = \sum_{k=0}^{n} b_k$ are absolutely convergent. Let
$$\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad j \mapsto (\varphi_1(j), \varphi_2(j))$$
be bijective. Then prove that $\sum_{j=0}^{n} a_{\varphi_1(j)} b_{\varphi_2(j)}$ converges absolutely, and
$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{j=0}^{\infty} a_{\varphi_1(j)} b_{\varphi_2(j)}.$$

Solution.

1. Let $(a_n)_{n\in\mathbb{N}}$ be a monotone sequence. Assume that it is monotonically increasing and let $(c_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers, such that $(a_{c_n})_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$ converging to some $a \in \mathbb{R}$.

   Our goal $(a_n)_{n\in\mathbb{N}}$ also converges to a. Let $\varepsilon > 0$. Since $(a_{c_n})_{n\in\mathbb{N}}$ converges to a, there is some $N_1 \in \mathbb{N}$ such that $|a - a_{c_n}| < \varepsilon$ for all $n \geq N_1$. We claim that $|a - a_n| < \varepsilon$ holds even for $n \geq c_{N_1}$, which will imply that the whole sequence converges to a.

   Assume $n \geq c_{N_1}$. Then since $(c_n)_{n\in\mathbb{N}}$ is strictly increasing, there is some $N_2 \in \mathbb{N}, N_2 \geq N_1$ such that $c_{N_2} \geq n$. Since $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing, this implies $a_{c_{N_2}} \geq a_n \geq a_{c_{N_1}}$. Since $|a - a_i| < \varepsilon$ holds for both $i = c_{N_1}$ and $i = c_{N_2}$, it follows that $a_n - a \leq a_{c_{N_2}} - a \in (-\varepsilon, \varepsilon)$ and $a - a_n \leq a - a_{c_{N_1}} \in (-\varepsilon, \varepsilon)$, so $|a - a_n| < \varepsilon$ for all $n \geq c_{N_1}$. Hence, $(a_n)_{n\in\mathbb{N}}$ converges to a, as desired.

   Now assume $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing. Then $(-a_n)_{n\in\mathbb{N}}$ is monotonically increasing and if $(a_n)_{n\in\mathbb{N}}$ has a subsequence converging to a, then $(-a_n)_{n\in\mathbb{N}}$ also has a subsequence (with the same set of indices) converging to $-a$. Since we already proved the statement for increasing sequences, we conclude that the whole sequence $(-a_n)_{n\in\mathbb{N}}$ converges to $-a$, so $(a_n)_{n\in\mathbb{N}}$ converges to a.

2. Let $i \in \mathbb{N}$. Define $M = \{\varphi_1(j) : j \leq i\} \cup \{\varphi_2(j) : j \leq i\}$ and let $N = \max(M)$.

   Then for all $j \leq i$, we know that $\varphi_1(j), \varphi_2(j) \leq N$, so if we expand $(|a_0| + |a_1| + \cdots + |a_N|)(|b_0| + |b_1| + \cdots + |b_N|)$, the term $\left|a_{\varphi_1(j)} b_{\varphi_2(j)}\right|$ will appear exactly once.

   Furthermore, since φ is bijective, it follows that two distinct values of j will result in a different pair of indices $(\varphi_1(j), \varphi_2(j))$. Thus, since the remaining terms are non-negative,
$$\begin{aligned}&(|a_0| + |a_1| + \cdots + |a_N|)(|b_0| + |b_1| + \cdots + |b_N|)\\ &\geq \left|a_{\varphi_1(0)} b_{\varphi_2(0)}\right| + \left|a_{\varphi_1(1)} b_{\varphi_2(1)}\right| + \cdots + \left|a_{\varphi_1(i)} b_{\varphi_2(i)}\right|.\end{aligned}$$

   Since $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are absolutely convergent, it follows that $\sum_{n=0}^{\infty} |a_n| = A < \infty$ and $\sum_{n=0}^{\infty} |b_n| = B < \infty$. Again, since the absolute values are non-negative, it follows that
$$(|a_0| + |a_1| + \cdots + |a_N|)(|b_0| + |b_1| + \cdots + |b_N|) \leq \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right) = AB.$$

Combining with the above inequality, we find that $\sum\limits_{j=0}^{i} \left|a_{\varphi_1(j)} b_{\varphi_2(j)}\right| \leq AB < \infty$. Since this holds for all i, it holds also if we take the limit $i \to \infty$. Hence, $\left(\sum\limits_{j=0}^{n} a_{\varphi_1(j)} b_{\varphi_2(j)}\right)_{n\in\mathbb{N}}$ converges absolutely.

Define $a = \sum\limits_{k=0}^{\infty} a_k, b = \sum\limits_{k=0}^{\infty} b_k$. It remains to show that $ab = \sum\limits_{j=0}^{\infty} a_{\varphi_1(k)} b_{\varphi_2(j)}$.

Let $\varepsilon > 0$ and define $\tilde{\varepsilon} = \min\left(\frac{\varepsilon}{8A}, \frac{\varepsilon}{8B}\right) > 0$ (see remark). By convergence, there exists some $N_1 \in \mathbb{N}$ such that $\left|\sum\limits_{k=0}^{n} a_k - a\right| = \left|\sum\limits_{k=n+1}^{\infty} a_k\right| < \tilde{\varepsilon}$ for all $n \geq N_1$. Similarly, there exists some $N_2 \in \mathbb{N}$ such that $\left|\sum\limits_{k=0}^{n} b_k - b\right| = \left|\sum\limits_{k=n+1}^{\infty} b_k\right| < \tilde{\varepsilon}$ for all $n \geq N_2$. Analogously, since the series also converge absolutely, there is some $N_3 \in \mathbb{N}$ such that $\left|A - \sum\limits_{k=0}^{n} |a_k|\right| = \sum\limits_{k=n+1}^{\infty} |a_k| < \tilde{\varepsilon}$ for all $n \geq N_3$ and some $N_4 \in \mathbb{N}$ such that $\left|B - \sum\limits_{k=0}^{n} |b_k|\right| = \sum\limits_{k=n+1}^{\infty} |b_k| < \tilde{\varepsilon}$ for all $n \geq N_4$.

Denote $N = \max(N_1, N_2, N_3, N_4)$.

Let now $M = \max\{\varphi^{-1}(i,j) : 0 \leq i,j \leq N\}$, which is well-defined since φ is bijective. Then

$$\left|\sum_{j=0}^{M} a_{\varphi_1(j)} b_{\varphi_2(j)} - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| = \left|\sum_{\substack{j\in\{0,1,\ldots,M\},\\ \varphi(j)\notin\{(k_1,k_2):0\leq k_1,k_2\leq N\}}} a_{\varphi_1(j)} b_{\varphi_2(j)}\right|$$

since all terms $a_{k_1} b_{k_2}$ with $0 \leq k_1, k_2 \leq N$ appear as $a_{\varphi_1(j)} b_{\varphi_2(j)}$ for some $j \in \{0, 1, \ldots, M\}$ (by the definition of M). We now bound this sum as follows. Using the triangle inequality, we obtain

$$\begin{aligned}
\left|\sum_{\substack{j\in\{0,1,\ldots,M\},\\ \varphi(j)\notin\{(k_1,k_2):0\leq k_1,k_2\leq N\}}} a_{\varphi_1(j)} b_{\varphi_2(j)}\right| &\leq \sum_{\substack{j\in\{0,1,\ldots,M\},\\ \varphi(j)\notin\{(k_1,k_2):0\leq k_1,k_2\leq N\}}} \left|a_{\varphi_1(j)} b_{\varphi_2(j)}\right| \\
\leq \sum_{\substack{j\in\mathbb{N},\\ \varphi(j)\notin\{(k_1,k_2):0\leq k_1,k_2\leq N\}}} \left|a_{\varphi_1(j)} b_{\varphi_2(j)}\right| &= \sum_{\substack{k_1,k_2\in\mathbb{N},\\ k_1>N\vee k_2>N}} |a_{k_1} b_{k_2}| \\
&\leq \sum_{\substack{k_1,k_2\in\mathbb{N},\\ k_1>N}} |a_{k_1} b_{k_2}| + \sum_{\substack{k_1,k_2\in\mathbb{N},\\ k_2>N}} |a_{k_1} b_{k_2}| \\
= (|a_N + 1| + |a_{N+2}| + \ldots)(|b_0| + |b_1| + \ldots) &+ (|b_N + 1| + |b_{N+2}| + \ldots)(|a_0| + |a_1| + \ldots) \\
&< \tilde{\varepsilon} B + \tilde{\varepsilon} A.
\end{aligned}$$

All of these inequalities follow since absolute values are non-negative and each sum contains all the terms that the one preceding it contains (and maybe more). For the last inequality, we used that $N \geq N_3, N_4$.

By choice of $\tilde{\varepsilon}$, the final term is at most $\frac{1}{4}\varepsilon$. Finally, we turn our attention to the desired value of our product, ab. Again, by the triangle inequality,

$$\begin{aligned}\left|ab - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| &= \left|ab - a\left(\sum_{k=0}^{N} b_k\right) + a\left(\sum_{k=0}^{N} b_k\right) - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| \\ &\leq |a|\left|b - \sum_{k=0}^{N} b_k\right| + |b|\left|a - \sum_{k=0}^{N} a_k\right|.\end{aligned}$$

Since $N \geq N_1, N_2$, it follows that the factors $\left|a - \sum\limits_{k=0}^{N} a_k\right|$ and $\left|b - \sum\limits_{k=0}^{N} b_k\right|$ are at most $\tilde{\varepsilon}$.

By (2.49), the "infinite" version of the triangle inequality, we also have

$$|a| = \left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k| = A$$

and analogously $|b| \leq B$. All in all, we find that

$$\left|ab - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| \leq A\tilde{\varepsilon} + B\tilde{\varepsilon} \leq \frac{1}{4}\varepsilon.$$

Combining this with the inequality

$$\left|\sum_{j=0}^{M} a_{\varphi_1(j)} b_{\varphi_2(j)} - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| \leq \frac{1}{4}\varepsilon$$

from above, we can deduce (again with the triangle inequality) that

$$\begin{aligned}&\left|ab - \sum_{j=0}^{M} a_{\varphi_1(j)} b_{\varphi_2(j)}\right| \\ &= \left|ab - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right) + \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right) - \sum_{j=0}^{M} a_{\varphi_1(j)} b_{\varphi_2(j)}\right| \\ &\leq \left|ab - \left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right)\right| + \left|\left(\sum_{k=0}^{N} a_k\right)\left(\sum_{k=0}^{N} b_k\right) - \sum_{j=0}^{M} a_{\varphi_1(j)} b_{\varphi_2(j)}\right| \\ &\leq \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon < \varepsilon.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we deduce that

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = ab = \sum_{j=0}^{\infty} a_{\varphi_1(j)} b_{\varphi_2(j)},$$

as desired.

Remark. In above solution, we use expressions of the form $\frac{\varepsilon}{8A}$ and $\frac{\varepsilon}{8B}$. If $A = 0$ or $B = 0$, these are not defined. However, since $A = 0$ implies that $a_i = 0$ for all i, this is only possible if one of the two sequences is identically 0, in which case the statement is trivially true.

Task 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 1, f(1) = \mathrm{e}$ and $f(x + y) = f(x)f(y)$. Prove that $f(x) = \mathrm{e}^x$ for all $x \in \mathbb{Q}$.

Solution. Actually, the assumption $f(0) = 1$ is extraneous and follows from $\mathrm{e} = f(0 + 1) = f(0)f(1) = f(0) \cdot \mathrm{e}$.

Lemma. *Let $n \in \mathbb{Z}$. Then for all $x \in \mathbb{R}$, we have $f(nx) = f(x)^n$.*

Proof. We proceed by induction. For $n = 0$, we have $f(0 \cdot x) = f(0) = 1 = f(x)^0$, which is always true.

Now assume $f(nx) = f(x)^n$ for some $n \in \mathbb{N}$. Then $f((n + 1)x) = f(nx + x) = f(nx)f(x) = f(x)^n f(x) = f(x)^{n+1}$, so by induction, the claim holds for all $n \in \mathbb{N}$.

Assume now $n < 0$. Then we can apply the lemma for $-n$ (since we already proved it for positive values) to deduce $1 = f(0) = f(nx + (-nx)) = f(nx)f(-nx) = f(nx)f(x)^{-n}$. Multiplying the above equation by $f(x)^n$ implies $f(x)^n = f(nx)$, so this holds for all $n \in \mathbb{Z}$, as desired. $\square$

Applying the lemma to $x = 1$ shows $f(n) = f(1)^n = \mathrm{e}^n$ for all $n \in \mathbb{Z}$. Let now $q \in \mathbb{Q}$. Write q as $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Applying the lemma to $x = q, n = b$, we see that $f(q)^b = f(q \cdot b) = f(a) = \mathrm{e}^a$. Taking both sides to the power of $\frac{1}{b}$ implies $f(q) = \mathrm{e}^{\frac{a}{b}} = \mathrm{e}^q$, as desired.