# Analysis I, Exercise 5

David Schmitz

## Task 1

Investigate the convergence of the following sequences and determine the limit, if applicable.

(a) $a_n := \frac{1}{\sqrt{n}}$,

(b) $b_n := \frac{2^n+(-3)^n}{(-2)^n+3^n}$.

**Solution.**

(a) We claim that $\lim\limits_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon^2}$. Then for any $m \in \mathbb{N}$ with $m \geq N$, we have $\sqrt{m} \geq \sqrt{N} > \frac{1}{\varepsilon}$ (since the square-root is an increasing function) and thus $\frac{1}{\sqrt{m}} < \varepsilon$.

Hence, $a_m = \frac{1}{\sqrt{m}} < \varepsilon$ for all $m \geq N$, so the sequence converges to 0.

(b) Assume first $n = 2k$ is even ($k \in \mathbb{Z}$), then $(-2)^n = (-2)^{2k} = \left((-2)^2\right)^k = 4^k = 2^{2k} = 2^n$ and similarly, $(-3)^n = 3^n$. Thus, $b_n = \frac{2^{2k}+(-3)^{2k}}{(-2)^{2k}+3^{2k}} = \frac{4^k+9^k}{4^k+9^k} = 1$.

Now assume $n = 2k+1$ is odd. Then $(-2)^n = (-2)^{2k+1} = -2 \cdot (-2)^{2k} = -2 \cdot 4^k = -2^{2k+1} = -2^n$ and similarly, $(-3)^n = -3^n$.

Hence, $b_n = \frac{2^n+(-3)^n}{(-2)^n+3^n} = \frac{2^n-3^n}{-2^n+3^n} = -1$.

Therefore, $b_n = (-1)^n$. Assume this converges to some $x \in \mathbb{R}$. If we take $\varepsilon = 1$, then $|x - b_n| < 1$ would need to hold for all $n \geq N$ (where $N \in \mathbb{N}$ is fixed). However, there are both odd and even integers larger than $N$, so this would imply $|x-1| < 1$ and $|x+1| < 1$. This contradicts the triangle inequality, as $2 = |(x+1)+(1-x)| \leq |x+1| + |1-x| = |x+1| + |x-1| < 1+1 = 2$. Therefore, the sequence diverges.

## Task 2

1. Let $(\mathbb{F}, \leq_{\mathbb{F}})$ be an ordered field and $d$ a distance on $\mathbb{F}$. Prove that any convergent sequence in the metric space $(\mathbb{F}, d)$ is bounded, i.e. $d(a_n, 0_{\mathbb{F}}) < \infty$.

2. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two convergent sequences. Show that the sequence $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent.

3. Show that the sequence $(x^n)_{n\in\mathbb{N}}$ for $x \in \mathbb{Z}$ does not converge for $x \leq -1$.

**Solution.**

1. Take $\varepsilon = 1_{\mathbb{F}} > 0$ (see 1.26). Since $(a_n)_{n\in\mathbb{N}}$ converges to some $x \in \mathbb{F}$, there is some $N \in \mathbb{N}$ such that $d(a_n, x) < \varepsilon$ for all $n \geq N$.

   Let $M = \{d(a_0, x), d(a_1, x), \ldots, d(a_N, x), 1_{\mathbb{F}}\}$. Since $M$ is a finite set, it is bounded, i.e. $\exists C \in \mathbb{F}$ such that $m \leq C$ for all $m \in M$.

   We then claim that $d(a_n, x) \leq C$ for all $n$. If $n < N$, this is obvious, since $d(a_n, x) \in M$. If $n \geq N$, then by assumption, $d(a_n, x) \leq 1_{\mathbb{F}} \leq C$ (since $1_{\mathbb{F}} \in M$).

   Hence, $d(a_n, x) \leq C$ for all $n$ and the sequence is bounded, as desired.

2. Let the $a = \lim\limits_{n\to\infty} a_n, b = \lim\limits_{n\to\infty} b_n$.

   Let $\varepsilon > 0$ be arbitrary.

   By part 1, both sequences are bounded, so $|a_n| \leq C_1, |b_n| \leq C_2$ for some $C_1, C_2 \in \mathbb{R}_0^+$ and all $n \in \mathbb{N}$. Define $C = \max(C_1, C_2, a)$.

   By the convergence of $(a_n)$, there is some $N_1 \in \mathbb{N}$ with $|a_n - a| < \frac{\varepsilon}{2C}$ for all $n \geq N_1$. Analogously, there is some $N_2 \in \mathbb{N}$ with $|b_n - b| < \frac{\varepsilon}{2C}$ for all $n \geq N_2$.

   Now define $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| = |(a_n - a) b_n + a(b_n - b)| \\
&\overset{(*)}{\leq} |(a_n - a) b_n| + |a(b_n - b)| = |b_n|\,|a_n - a| + |a|\,|b_n - b| \\
&\leq C \cdot \frac{\varepsilon}{2C} + C \cdot \frac{\varepsilon}{2C} = \varepsilon,
\end{aligned}$$

   where we used the triangle inequality for $(*)$. Hence, $\lim\limits_{n\to\infty} a_n b_n = ab$. In particular, this sequence converges.

3. If $x = -1$, the sequence is just $(-1)^n$, which we already proved to not converge in Task 1, part (b).

   Now assume $x < -1$. We claim that $x^n$ is not bounded, which implies that the sequence does not converge by part 1. Assume otherwise, i.e. $|x^n| \leq C$ for some $C \in \mathbb{R}_0^+$. Since $x < -1$, we have $-x - 1 > 0$ and hence, there is some $n \in \mathbb{N}$ with $n > \frac{C}{-x-1}$. Using Bernoulli's inequality,

$$\begin{aligned}
x^{2n} &= \left(x^2\right)^n = \left((-x)^2\right)^n = (-x)^{2n} \\
&= (1 + (-x - 1))^{2n} \geq 1 + 2n(-x - 1) \geq n(-x - 1) \\
&> \frac{C}{-x-1} \cdot (-x - 1) = C.
\end{aligned}$$

   This contradicts the assumption that the sequence is bounded by $C$. Hence, $x^n$ must be unbounded.

# Task 3

Find two different distances $d_1, d_2$ on a set $X$ and a sequence $(a_n)_{n\in\mathbb{N}}$ that converges in the meaning of the first metric, but it does not converge in the meaning of the second metric.

In other words, find to metrics $d_1$ and $d_2$ and a sequence $(a_n)_{n\in\mathbb{N}}$ so that $(a_n)_{n\in\mathbb{N}}$ fulfils the definition of convergence on the metric space $(X, d_1)$ but not on the metric space $(X, d_2)$.

**Solution.** We take $X = \mathbb{R}$. Choose $d_1$ as the Euclidean distance, i.e. $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. As shown in the lecture notes (see 2.5), $(X, d_1)$ is a metric space. In addition, we have shown (see 2.8), that $a_n := \frac{1}{n}$ converges to 0 in this metric space.

For $d_2$, we take the discrete metric, i.e. $d_2(x, y) = 1$ for $x \neq y$ and $d_2(x, y) = 0$ for $x = y$. It's easy to see that $(X, d_2)$ is indeed a metric space. Positive definiteness is clear, since no negative distances appear $(1, 0 \geq 0)$ and $d_2(x, y) = 0$ if and only if $x = y$ by definition of $d_2$.

The symmetry follows from the fact that $d_2(x, y)$ only depends on whether $x = y$, which is symmetric (equivalent to $y = x$).

Let now $x, y, z \in \mathbb{R}$. We will now show $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$. Note that the left side is at most 1, so the inequality holds already if $d_2(x, y) = 1$ or $d_2(y, z) = 1$. Otherwise, $d_2(x, y) = 0 = d_2(y, z)$ implies $x = y = z$, so in particular, $x = z$ and $d_2(x, z) = 0$. Again, $0 \leq 0 + 0$ holds.

Finally, we can observe that $a_n$ does not converge in $(X, d_2)$. Assume it did converge to some $x \in \mathbb{R}$. Then for $\varepsilon = 1$, we need $d_2(x, a_n) < 1$ for all sufficiently large $n$. However, the only such possibility is $d_2(x, a_n) = 0$ and hence $x = a_n$. Thus, the only convergent sequences in $(X, d_2)$ are sequences which are eventually constant, but $a_n$ is not constant.

# Task 4

Fix $n \in \mathbb{N}$. Show in detail that $\mathbb{R}^n$ equipped with the Euclidean distance $d_2$ is a metric space, through the following steps.

1. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ define the map
$$\langle x, y \rangle : \mathbb{R}^n \times \mathbb{R}^n, \ \ \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$
which is called the Euclidean scalar product. Also, define $||x||^2 = \langle x, x \rangle$, which is called the Euclidean norm. Show that
$$d_2(x, y)^2 = \langle x - y, x - y \rangle = ||x - y||^2 .$$
Prove that $\langle x, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$.

2. For $\lambda \in \mathbb{R}$ define $\lambda x = (\lambda x_1, \dots, \lambda x_n)$ and show that for all $\lambda, \mu \in \mathbb{R}$, $x, y, z \in \mathbb{R}^n$ we have
$$\langle \lambda x, \mu y \rangle = \lambda\mu \langle x, y \rangle , \quad \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle , \quad \text{and } \langle x, y \rangle = \langle y, x \rangle .$$
Deduce from these properties that
$$||\lambda x + \mu y||^2 = \lambda^2 ||x||^2 + \mu^2 ||y||^2 + 2\lambda\mu \langle x, y \rangle . \tag{1}$$

3. By using $\langle z, z \rangle \geq 0$ and choosing a special vector $z \in \mathbb{R}^n$, prove the Cauchy-Schwarz inequality
$$\langle x, y \rangle \leq ||x|| \, ||y|| . \tag{2}$$

4. Use the Cauchy-Schwartz inequality (2) and (1) to prove the triangle inequality
$$||x - z|| \leq ||x - y|| + ||y - z|| , \quad \forall x, y, z \in \mathbb{R}^n.$$
Finally, show that this is equivalent to $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$.

**Solution.**

1. Let $x, y \in \mathbb{R}^n$. The Euclidean distance $d_2$ is defined as

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

Define $z = x - y \in \mathbb{R}^n$, so $z_i = x_i - y_i$. Squaring the above, we obtain

$$\begin{aligned} d_2(x,y)^2 &= \sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n} z_i^2 \\ &= \langle z, z \rangle = \langle x - y, x - y \rangle \stackrel{\text{def}}{=} \|x - y\|^2 . \end{aligned}$$

Finally, let $n = (0, 0, \ldots, 0) \in \mathbb{R}^n$ be the zero vector. Then $\langle x, x \rangle = \langle x - n, x - n \rangle = d_2(x, n)^2$. Since squares are non-negative, this is $\geq 0$, as desired.

2. By distributivity,

$$\begin{aligned} \langle \lambda x, \mu y \rangle &= \sum_{i=1}^{n} (\lambda x_i)(\mu y_i) \\ &= \sum_{i=1}^{n} \lambda \mu x_i y_i = \lambda \mu \sum_{i=1}^{n} x_i y_i \\ &= \lambda \mu \langle x, y \rangle . \end{aligned} \tag{*}$$

Similarly, we use distributivity to deduce

$$\begin{aligned} \langle x + z, y \rangle &= \sum_{i=1}^{n} (x_i + z_i) y_i = \sum_{i=1}^{n} (x_i y_i + z_i y_i) \\ &= \left( \sum_{i=1}^{n} x_i y_i \right) + \left( \sum_{i=1}^{n} z_i y_i \right) \\ &= \langle x, y \rangle + \langle z, y \rangle . \end{aligned} \tag{**}$$

Finally, by the commutativity of multiplication,

$$\begin{aligned} \langle x, y \rangle &= \sum_{i=1}^{n} x_i y_i \\ &= \sum_{i=1}^{n} y_i x_i = \langle y, x \rangle , \end{aligned} \tag{***}$$

as desired. To show (1), we also need

$$\begin{aligned} \langle x, y + z \rangle &\stackrel{(***)}{=} \langle y + z, x \rangle \stackrel{(**)}{=} \langle y, x \rangle + \langle z, x \rangle \\ &\stackrel{(***)}{=} \langle x, y \rangle + \langle x, z \rangle . \end{aligned} \tag{****}$$

We can now prove that the Euclidean norm is bilinear:

$$
\begin{aligned}
||\lambda x + \mu y||^2 &\overset{\text{def}}{=} \langle \lambda x + \mu y, \lambda x + \mu y \rangle \overset{(**)}{=} \langle \lambda x, \lambda x + \mu y \rangle + \langle \mu y, \lambda x + \mu y \rangle \\
&\overset{(****)}{=} \langle \lambda x, \lambda x \rangle + \langle \lambda x, \mu y \rangle + \langle \mu y, \lambda x \rangle + \langle \mu y, \mu y \rangle \\
&\overset{(*)}{=} \lambda^2 \langle x, x \rangle + \lambda\mu \langle x, y \rangle + \mu\lambda \langle y, x \rangle + \mu^2 \langle y, y \rangle \\
&\overset{(***)}{=} \lambda^2 \langle x, x \rangle + 2\lambda\mu \langle x, y \rangle + \mu^2 \langle y, y \rangle \\
&\overset{\text{def}}{=} \lambda^2 ||x||^2 + \mu^2 ||y||^2 + 2\lambda\mu \langle x, y \rangle .
\end{aligned}
$$

3. If $x$ is the zero vector, then $\langle x, y \rangle = \sum_{i=1}^{n} 0 \cdot y_i = 0$ and $||x|| = 0$. Hence, the inequality becomes $0 \leq 0$.
   Similarly, if $y$ is the zero vector, the inequality becomes $0 \leq 0$, which is true.
   Now assume $||x||, ||y|| > 0$.
   We choose $\lambda = ||y||, \mu = ||x||$ in (1). Denote $z = ||y||\, x - ||x||\, y$. Then

$$
\begin{aligned}
&0 \leq ||z||^2 = ||(||y||\, x - ||x||\, y)|| \overset{(1)}{=} ||y||^2 ||x||^2 + ||x||^2 ||y||^2 - 2\, ||y||\, ||x|| \langle x, y \rangle \\
&\quad = 2\, ||x||\, ||y|| \left( ||x||\, ||y|| - \langle x, y \rangle \right) .
\end{aligned}
$$

   Since $2\, ||x||\, ||y|| > 0$ by assumption, it follows that $0 \leq ||x||\, ||y|| - \langle x, y \rangle$, as desired.

4. Note that

$$
\begin{aligned}
||x - z||^2 = ||(x - y) + (y - z)||^2 &\overset{(1)}{=} ||x - y||^2 + ||y - z||^2 + 2 \langle x - y, y - z \rangle \\
&\overset{(2)}{\leq} ||x - y||^2 + ||y - z||^2 + 2\, ||x - y||\, ||y - z|| \\
&= \left( ||x - y|| + ||y - z|| \right)^2 .
\end{aligned}
$$

   If we take the square root, this becomes the triangle inequality.