Analysis I, Christmas exercise

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Task 1: Infimum/Supremum

Show for bounded, nonempty sets A, B that

(a) $\sup\{a+b : a \in A,\ b \in B\} = \sup(A) + \sup(B)$,

(b) $\sup\{a-b : a \in A,\ b \in B\} = \sup(A) - \inf(B)$,

(c) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

(d) If additionally $A \subset B$ holds true, then one has $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

Solution.

(a) For any $a \in A, b \in B$, we have $a \leq \sup(A)$ and $b \leq \sup(B)$, so $a + b \leq \sup(A) + \sup(B)$.

Hence, $\sup A + \sup B$ is an upper bound for the set $\{a+b : a \in A,\ b \in B\}$. By (1.31),

$$\begin{aligned} &\sup(A) + \sup(B) = \sup\{a+b : a \in A,\ b \in B\} \\ \iff &\forall \varepsilon > 0 :\ \exists y \in \{a+b : a \in A,\ b \in B\} :\ y > \sup(A) + \sup(B) - \varepsilon. \end{aligned}$$

Let $\varepsilon > 0$. By (1.31), there is some $a \in A$ such that $a > \sup(A) - \frac{1}{2}\varepsilon$. Similarly, there is some $b \in B$ such that $b > \sup(B) - \frac{1}{2}\varepsilon$. Then $a + b > \sup(A) + \sup(B) - \varepsilon$, where $a + b \in \{a+b : a \in A,\ b \in B\}$, as desired.

(b) Let $B' = \{-b : b \in B\}$. Since $b \geq \inf(B)$ for any $b \in B$, we have $-b \leq -\inf(B)$, so $-\inf(B)$ is an upper bound for B'.

We claim that $\sup(B') = -\inf(B)$. Since we already now that this is an upper bound, by (1.31), it suffices to show that

$$\forall \varepsilon > 0 :\ \exists y \in B' : y > -\inf(B) - \varepsilon.$$

Let $\varepsilon > 0$. By (1.31), there is some $y \in B$ such that $y < \inf(B) + \varepsilon$. Equivalently, $-y > -\inf(B) - \varepsilon$. Since $-y \in B'$, the above inequality always holds for some $y \in B'$, so $\sup(B') = -\inf(B)$.

By part (a), $\sup\{a + b' : a \in A,\ b' \in B'\} = \sup(A) + \sup(B') = \sup(A) - \inf(B)$. Hence, we can conclude by noting that $\{a + b' : a \in A,\ b' \in B'\} = \{a - b : a \in A,\ b \in B\}$.

(c) Assume without loss of generality $\sup(A) \geq \sup(B)$ (otherwise swap A and B). Then $\max\{\sup(A), \sup(B)\} = \sup(A)$.

For any $a \in A$, we already have $a \leq \sup(A)$. For any $b \in B$, we have $b \leq \sup(B) \leq \sup(A)$, so $\sup(A)$ is an upper bound for $A \cup B$.

By (1.31), $\sup(A) = \sup(A \cup B)$ if and only if

$$\forall \varepsilon > 0 : \ \exists y \in A \cup B : \ y > \sup(A) - \varepsilon.$$

Let $\varepsilon > 0$. Since we already know $\sup(A)$ is the supremum of A, we can apply (1.31) to deduce that $y > \sup(A) - \varepsilon$ for some $y \in A$. Since $A \subseteq A \cup B$, $y \in A \cup B$ as well, so the above inequality always holds for some $y \in A \cup B$. Hence, $\sup(A) = \sup(A \cup B)$, as desired.

(d) For every $a \in A$, we have $a \in B$ and thus $a \leq \sup(B)$. Thus, $\sup(B)$ is an upper bound for A. However, $\sup(A)$ is by definition the least upper bound for A, so $\sup(A) \leq \sup(B)$.

For every $a \in A$, we analogously have $a \in B$ and thus $a \geq \inf(B)$. Thus, $\inf(B)$ is a lower bound for A. However, $\inf(A)$ is by definition the largest lower bound for B, so $\inf(B) \leq \inf(A)$.

Finally, since $\inf(A) \leq \sup(A)$ for nonempty A, we conclude that $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$, as desired.

Task 2: Complex Numbers

Prove that

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$$

for all $z, w \in \mathbb{C}$.

Solution. Using $|z|^2 = z\bar{z}$, we expand all parentheses to get

$$\begin{aligned}
|z+w|^2 + |z-w|^2 &= (z+w)\overline{(z+w)} + (z-w)\overline{(z-w)} \\
&= (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} + z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w} \\
&= 2z\bar{z} + 2w\bar{w} = 2\left(|z|^2 + |w|^2\right),
\end{aligned}$$

as desired.

Task 3: Recursive Sequences

Consider a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ given by

$$a_0 := 1, \text{ and for all } n > 1 : \ a_{n+1} := 1 + \frac{1}{a_n}.$$

Prove that $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$ and determine a.

First solution. Let $A = \frac{1+\sqrt{5}}{2}, B = \frac{1-\sqrt{5}}{2}$. By the quadratic formula, we have $A^2 = A+1$ and $B^2 = B+1$.

Lemma. *The equation $a_n = \frac{A^{n+2}-B^{n+2}}{A^{n+1}-B^{n+1}}$ holds for all $n \in \mathbb{N}$.*

Proof. We proceed by induction. For $n = 0$, we have

$$\frac{A^2 - B^2}{A - B} = A + B = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1,$$

which is true.

Now assume the statement holds for some $n \in \mathbb{N}$, i.e. $a_n = \frac{A^{n+2}-B^{n+2}}{A^{n+1}-B^{n+1}}$. Then by the recursion of $(a_n)_{n\in\mathbb{N}}$, we have

$$\begin{aligned} a_{n+1} = 1 + \frac{1}{a_n} &= 1 + \frac{A^{n+1} - B^{n+1}}{A^{n+2} - B^{n+2}} \\ = \frac{A^{n+1} + A^{n+2} - B^{n+1} - B^{n+2}}{A^{n+2} - B^{n+2}} &= \frac{A^{n+1}(1+A) - B^{n+1}(1+B)}{A^{n+2} - B^{n+2}} \\ \overset{(*)}{=} \frac{A^{n+1} \cdot A^2 - B^{n+1} \cdot B^2}{A^{n+2} - B^{n+2}} &= \frac{A^{n+3} - B^{n+3}}{A^{n+2} - B^{n+2}}, \end{aligned}$$

where the equality $(*)$ follows from the equations $A^2 = A + 1$ and $B^2 = B + 1$.

Hence, the statement also holds for $n + 1$, so by induction, we deduce that it is true for all $n \in \mathbb{N}$. ☐

Define now $b_n = A - \frac{B^{n+2}}{A^{n+1}}$ and $c_n = 1 - \frac{B^{n+1}}{A^{n+1}}$.

We claim that $\lim_{n\to\infty} b_n = A$, $\lim_{n\to\infty} c_n = 1$.

Let $\varepsilon > 0$. Note that $\sqrt{5} \in (2,3)$, so $|B| = \frac{1}{2}\left|\sqrt{5} - 1\right| < 1$. Similarly, $|A| > 1$. By (2.10), we know that $\left(\frac{1}{|A|^n}\right)_{n\in\mathbb{N}}$ converges to 0 since $\frac{1}{|A|} < 1$. Hence, there is some $N \in \mathbb{N}$ such that $\frac{1}{|A|^n} < \varepsilon$ for all $n \geq N$.

Therefore, for $n \geq N$, we have

$$|b_n - A| = \frac{|B|^{n+2}}{|A|^{n+1}} \leq \frac{1}{|A|^{n+1}} < \varepsilon$$

and

$$|c_n - 1| = \frac{|B|^{n+1}}{|A|^{n+1}} \leq \frac{1}{|A|^{n+1}} < \varepsilon.$$

Hence, $\lim_{n\to\infty} b_n = A$ and $\lim_{n\to\infty} c_n = 1$. Note that $c_n \neq 0$ for all n, so

$$\begin{aligned} A = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n} = \lim_{n\to\infty} \frac{b_n}{c_n} &= \lim_{n\to\infty} \frac{A - \frac{B^{n+2}}{A^{n+1}}}{1 - \frac{B^{n+1}}{A^{n+1}}} \\ &= \lim_{n\to\infty} \frac{A^{n+2} - B^{n+2}}{A^{n+1} - B^{n+1}} = \lim_{n\to\infty} a_n, \end{aligned}$$

so $(a_n)_{n\in\mathbb{N}}$ converges to $\frac{1+\sqrt{5}}{2}$.

Second solution. Again, let $A = \frac{1+\sqrt{5}}{2}, B = \frac{1-\sqrt{5}}{2}$. Observe that $A + B = 1$ and $A \cdot B = \frac{1-5}{4} = -1$.

Lemma. *For all $n \in \mathbb{N}$, we have $a_n \leq A$ if and only if $a_{n+1} \geq A$.*

Proof. Assume $a_n \leq A$. Then

$$\begin{aligned} a_{n+1} &= 1 + \frac{1}{a_n} \geq 1 + \frac{1}{A} \\ &= 1 - B = A. \end{aligned}$$

Assume now $a_n > A$. Then

$$\begin{aligned} a_{n+1} &= 1 + \frac{1}{a_n} < 1 + \frac{1}{A} \\ &= 1 - B = A. \end{aligned}$$

Hence, the two inequalities are equivalent. $\square$

Since $a_0 = 1 = \frac{1+1}{2} < \frac{1+\sqrt{5}}{2} = A$, it follows that $a_n \leq A$ if and only if n is even.

Lemma. *We have $a_n \leq a_{n+2}$ if and only if n is even.*

Proof. Applying the recursion twice shows

$$\begin{aligned} a_n \leq a_{n+2} &\iff a_n \leq 1 + \frac{1}{a_{n+1}} \\ &\iff a_n \leq 1 + \frac{1}{1 + \frac{1}{a_n}} = 1 + \frac{a_n}{a_n + 1} \\ &\iff a_n(a_n + 1) \leq a_n + 1 + a_n \\ &\iff a_n^2 - a_n - 1 \leq 0 \\ &\iff a_n^2 - (A + B)a_n + AB \leq 0 \\ &\iff (a_n - A)(a_n - B) \leq 0. \end{aligned}$$

Since $-B > 0$, the second factor is always positive, so this inequality is equivalent to $a_n - A \leq 0$ or $a_n \leq A$, which is true if and only if n is even by above lemma. $\square$

So far, we have shown that $(a_{2n})_{n\in\mathbb{N}}$ is an increasing sequence bounded above by A. Hence, it converges to some $A_1 \leq A$.

Similarly, we have shown that $(a_{2n+1})_{n\in\mathbb{N}}$ is a decreasing sequence bounded below by A. Hence, it converges to some $A_2 \geq A$.

It remains to show $A_1 = A_2$, which will show $A_1 = A = A_2$..

Lemma. *For $n \geq 1$, we have $\frac{3}{2} \leq a_n \leq 2$.*

Proof. We proceed by induction. Note that $a_1 = 2 \in [\frac{3}{2}, 2]$.

Now assume $a_n \in [\frac{3}{2}, 2]$. Then $a_{n+1} = 1 + \frac{1}{a_n} \geq 1 + \frac{1}{2} = \frac{3}{2}$ and $a_{n+1} = 1 + \frac{1}{a_n} \leq 1 + \frac{1}{\frac{3}{2}} \leq 2$, so $a_{n+1} \in [\frac{3}{2}, 2]$, as desired. $\square$

Note that for $n \geq 1$, we have $a_n - 1 = \frac{1}{a_{n-1}}$, so

$$\begin{aligned} \left|\frac{a_{n+1} - a_n}{a_n - a_{n-1}}\right| &= \left|\frac{1 + \frac{1}{a_n} - a_n}{a_n - \frac{1}{a_n - 1}}\right| = \left|\frac{a_n(a_n - 1) + a_n - 1 - a_n^2(a_n - 1)}{a_n^2(a_n - 1) - a_n}\right| \\ &= \left|\frac{-(a_n - 1)(a_n^2 - a_n - 1)}{a_n(a_n^2 - a_n - 1)}\right| = \left|\frac{-(a_n - 1)}{a_n}\right| = \left|\frac{1}{a_n} - 1\right| \\ &= 1 - \frac{1}{a_n} \leq 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

where the last line follows by the lemma. Thus, we can apply task 3c on exercise sheet 7 to deduce that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent. Thus, the two subsequences above have the same limit, so $A_1 = A_2 = A$ and the entire sequence converges to A.

Task 4: Convergence of sequences I

Is there a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = 0$ and $\limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$?

Solution. The answer is yes. Define the sequence

$$a_n := \begin{cases} \frac{1}{n} & \text{for odd } n, \\ \frac{1}{(n+1)^2} & \text{for even } n. \end{cases}$$

Then since $\frac{1}{(n+1)^2} \le \frac{1}{n+1} \le \frac{1}{n}$ and for all $n \in \mathbb{N}$, it follows that $0 \le a_n \le \frac{1}{n}$, so by the squeeze theorem, it follows that $\lim\limits_{n\to\infty} a_n = 0$.

It remains to show $\limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$. Let $M > 0$ be arbitrary and define $N = 2\lceil M\rceil \in \mathbb{N}$, which is an even number. Hence, $\left|\frac{a_{N+1}}{a_N}\right| = \frac{a_{N+1}}{a_N} = \frac{\left(\frac{1}{N+1}\right)}{\left(\frac{1}{(N+1)^2}\right)} = N + 1 > N > M$.

Thus, $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)_{n\in\mathbb{N}}$ is not bounded above, so the lim sup of this sequence is ∞, as desired.

Task 5: Continuity

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $a, b \in \mathbb{R}$ it holds that $f(a + b) = f(a) + f(b)$. If f is continuous at $x = 0$, then prove that f is continuous in $\mathbb{R}$.

Solution. Note first that $f(0) = f(0 + 0) = f(0) + f(0)$, so $f(0) = 0$.

Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Since f is continuous at 0, there is some $\delta > 0$ such that $|f(x) - f(0)| < \varepsilon$ for all x with $|x - 0| < \delta$.

Since $f(0) = 0$, this means $|f(x)| < \varepsilon$ for all $x \in (-\delta, \delta)$.

Let now $x \in \mathbb{R}$ be such that $|x - x_0| < \delta$. Then by assumption that f is additive,

$$|f(x_0) - f(x)| = |f(x_0 - x)| < \varepsilon,$$

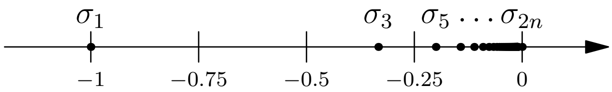
where the inequality follows from the fact that $x_0 - x \in (-\delta, \delta)$. Hence, f is also continuous at x_0. Since this holds for all $x_0 \in \mathbb{R}$, f is continuous in $\mathbb{R}$, as desired.

Remark. Actually, the graph of any nonlinear solutions to Cauchy's functional equation is dense in $\mathbb{R}^2$, so continuity at any point or boundedness in any interval of positive length implies $f(x) = cx$ for some $c \in \mathbb{R}$.

Task 6: Convergence of sequences II

For any sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ we define $\sigma_n := \frac{1}{n}\sum\limits_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$. Find a sequence $(a_n)_{n\in\mathbb{N}}$ such that $(a_n)_{n\in\mathbb{N}}$ diverges and $(\sigma_n)_{n\in\mathbb{N}}$ converges.

Solution. We claim $a_n = (-1)^n$ works. Obviously, this is a divergent sequence, so it remains to show that $(\sigma_n)_{n\in\mathbb{N}}$ converges.



Lemma. $\sum_{k=1}^{n} a_k$ *equals* -1 *for odd* n *and* 0 *for even* n.

Proof. We proceed by induction. For $n = 1$, we simply check that $\sum_{k=1}^{1} a_k = a_1 = -1$.

Now assume that the statement holds for some $n \in \mathbb{N}$.

- If n is odd, then by the induction hypothesis, $\sum_{k=1}^{n} a_k = -1$. Since $n+1$ is even, $(-1)^{n+1} = 1$, so
$$\sum_{k=1}^{n+1} a_k = a_{k+1} + \sum_{k=1}^{n} a_k = (-1)^{n+1} + (-1) = 1 - 1 = 0,$$
as desired.

- If n is even, then by the induction hypothesis, $\sum_{k=1}^{n} a_k = 0$. Since $n+1$ is odd, $(-1)^{n+1} = -1$, so
$$\sum_{k=1}^{n+1} a_k = a_{k+1} + \sum_{k=1}^{n} a_k = (-1)^{n+1} + 0 = -1 + 0 = -1,$$
as desired. $\square$

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ sufficiently large such that $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$, we have

$$|\sigma_n| = \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right| = \frac{1}{n} \left| \sum_{k=1}^{n} a_k \right| \leq \frac{1}{n},$$

where the last inequality follows because the sum is either -1 or 0.

Hence, $|\sigma_n - 0| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for all $n \geq N$, so $(\sigma_n)_{n\in\mathbb{N}}$ converges (to 0), as desired.

Task 7: Convergence of sequences III

Discuss which of the following sequences converge. If they do, name their limit.

(a) Let $a_n := n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$ for all $n \in \mathbb{N}_{>0}$,

(b) let $b_n := \frac{(3n^2-7n)(n-1)}{n^3+7n}$ for all $n \in \mathbb{N}_{>0}$,

(c) let $c_n := \frac{n}{e^n}$ for all $n \in \mathbb{N}$,

(d) let $d_n := \left(1 + \frac{1}{n^n}\right)^n$ for all $n \in \mathbb{N}$.

(e) Let $(e_n)_{n\in\mathbb{N}}$ be a real sequence for all $n \in \mathbb{N}$. Show that the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n := \cos(e_n)$ for all $n \in \mathbb{N}$ has a converging subsequence.

Solution.

(a) We claim that for $n \geq 1$, the inequality $a_n \geq 1$ holds. Indeed,

$$n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) \geq 1$$
$$\iff \sqrt{n^2+1} - \sqrt{n^2-1} \geq \frac{1}{n}.$$

Since both sides are nonnegative, squaring both sides will produce the equivalent inequality

$$n^2 + 1 + n^2 - 1 - 2\sqrt{n^4-1} \geq \frac{1}{n^2}$$
$$\iff 2n^2 - \frac{1}{n^2} \geq 2\sqrt{n^4-1}.$$

Again, both sides are nonnegative, so by squaring, we obtain the equivalent inequality

$$4n^4 - 4 + \frac{1}{n^4} \geq 4(n^4-1),$$

which is true.

We claim that $\lim_{n\to\infty} a_n = 1$. Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ sufficiently large such that $N > \max\left\{\frac{1}{\varepsilon}, (1+\varepsilon)^2\right\}$. We claim that $a_n < 1+\varepsilon$ for all $n \geq N$, i.e.

$$a_n = n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) < 1 + \varepsilon$$

Both sides are nonnegative, so by squaring, we obtain the equivalent inequality

$$n^2(n^2 + 1 + n^2 - 1 - 2\sqrt{n^4-1}) < (1+\varepsilon)^2$$
$$\iff 2n^4 - (1+\varepsilon)^2 < 2n^2\sqrt{n^4-1}.$$

Since $n^4 \geq N \geq (1+\varepsilon)^2$, both sides are nonnegative, so we can square to obtain the equivalent inequality

$$4n^8 - 4n^4(1+\varepsilon)^2 + (1+\varepsilon)^4 < 4n^4(n^4-1)$$
$$\iff (1+\varepsilon)^4 < 4n^4(2\varepsilon + \varepsilon^2).$$

Now since $N\varepsilon > 1$, we have $4n^4(2\varepsilon+\varepsilon^2) = 4n^2(2n^2\varepsilon + n^2\varepsilon^2) > 4n^2(2n+1) > 4n^2 \geq 4N^2 > N^2 > (1+\varepsilon)^4$, as desired.

Hence, $a_n \geq 1$ for $n \geq 1$ and for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $a_n < 1 + \varepsilon$ for $n \geq N$. Thus, $\lim_{n\to\infty} a_n = 1$.

(b) Note that for $n \geq 1$, $b_n = \frac{(3n^2-7n)(n-1)}{n^3+7n} = \frac{3n^3-10n^2+7n}{n^3+7n} = \frac{3-\frac{10}{n}+\frac{7}{n^2}}{1+\frac{7}{n^2}}$. Since the numerator approaches 3 and the denominator approches 1, this limit equals 3, so $(b_n)_{n\in\mathbb{N}}$ converges to 3.

(c) Since $\mathrm{e} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots > \frac{1}{0!} + \frac{1}{1!} = 2$, we have $2^n < \mathrm{e}^n$ and hence $0 \leq c_n \leq \frac{n}{2^n}$.

Lemma. *For $n \geq 4$, we have $2^n \geq n^2$.*

Proof. We proceed by induction. For $n = 4$, we simply have $2^4 = 16 = 4^2$. Assume now $2^n \geq n^2$ holds for some $n \geq 4$. Then $2^{n+1} = 2 \cdot 2^n \geq 2n^2 = n^2 + n^2 \geq n^2 + 4n \geq n^2 + 2n + 1 = (n+1)^2$, so the statement also holds for $n + 1$ and by induction for all $n \in \mathbb{N}$. $\square$

Hence, for $n \geq 4$, we even have $0 \leq c_n \leq \frac{n}{n^2} = \frac{1}{n}$. Since both outer sequences converge to 0, by the squeeze theorem, we deduce that $\lim_{n\to\infty} c_n = 0$.

(d) Note that $1 + n^{-n} > 1$, so $d_n > 1$ for all $n \in \mathbb{N}$. We claim that $(d_n)_{n\in\mathbb{N}}$ converges to 1.

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ sufficiently large such that $N > \max\left\{1 + \frac{1}{\varepsilon}, 2\right\}$. Observe that for $n \geq N$, by the binomial theorem,

$$\begin{aligned} d_n = \left(1 + n^{-n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} n^{-kn} = 1 + \sum_{k=1}^{n} \binom{n}{k} n^{-kn} \\ &= 1 + \sum_{k=1}^{n} \frac{n(n-1)\dots(n-k+1)}{k!} n^{-kn} \leq 1 + \sum_{k=1}^{n} \frac{n^k}{k!} n^{-kn} \\ &\leq 1 + \sum_{k=1}^{n} n^k \cdot n^{-kn} = 1 + \sum_{k=1}^{n} n^{k(1-n)}. \end{aligned}$$

Since $n \geq 2$, we have $1-n \leq -1$ and so $n^{k(1-n)} \leq n^{-k}$. Hence, by the formula for geometric series,

$$\begin{aligned} d_n &\leq 1 + \sum_{k=1}^{n} n^{-k} < 1 + \sum_{k=1}^{\infty} n^{-k} \\ &= 1 + \frac{1/n}{1 - 1/n} = 1 + \frac{1}{n-1} \\ &\leq 1 + \frac{1}{N-1} \leq 1 + \frac{1}{1/\varepsilon} = 1 + \varepsilon. \end{aligned}$$

Since we already know $d_n \geq 1$ for all $n \geq 1$, this implies $\lim_{n\to\infty} d_n = 1$.

(e) Since $e_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, we have $\cos(e_n) \in [-1, 1]$, so $(\cos(e_n))_{n\in\mathbb{N}}$ is bounded (by 1). Hence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence.

Task 8: Convergence of series

Discuss which of the following series converge.

(a) $\sum\limits_{n=1}^{\infty} \frac{3}{2}\left(\frac{3}{4} - \frac{1}{n}\right)^n$

(b) $\sum\limits_{n=1}^{\infty} \frac{4n^2+37n+14n^{14}}{15n^{16}+3n^2}$

(c) $\sum\limits_{n=1}^{\infty} \frac{2n^3+7n+n^2}{5n^4+3n}$

(d) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

(e) Calculate the limit of $\sum_{n=3}^{\infty} \frac{4^{n-1}}{5^{n+1}}$ explicitly.

Solution.

(a) Note that for $n \geq 2$, we have $0 < \frac{1}{n} < \frac{3}{4}$, so $0 < \frac{3}{4} - \frac{1}{n} < \frac{3}{4}$. Hence, $\left(\frac{3}{4} - \frac{1}{n}\right)^n < \left(\frac{3}{4}\right)^n$ for $n \geq 2$, so

$$\begin{aligned}\sum_{n=1}^{\infty} \left|\frac{3}{2}\left(\frac{3}{4} - \frac{1}{n}\right)^n\right| &= \left|\frac{3}{2}\left(\frac{3}{4} - 1\right)^1\right| + \sum_{n=2}^{\infty} \left|\frac{3}{2}\left(\frac{3}{4} - \frac{1}{n}\right)^n\right| \\ &< \frac{3}{8} + \sum_{n=2}^{\infty} \left|\frac{3}{2} \cdot \left(\frac{3}{4}\right)^n\right| = \frac{3}{8} + \frac{3}{2} \cdot \left(\frac{3}{4}\right)^2 \sum_{n=0}^{\infty} 34^n \\ &= \frac{3}{8} + \frac{9}{32} \cdot \frac{1}{1 - \frac{3}{4}} < \infty.\end{aligned}$$

Hence, the series converges (even absolutely).

(b) Note that all terms of the series are positive, so it suffices to bound the sum from above to show convergence.

Note that for $n \geq 1$, $15n^{16} + 3n^2 \geq 15n^{16}$ and $4n^2 + 37n + 14n^{14} \leq 150n^2 + 150n + 150n^{14} \leq 150n^{14} + 150n^{14} + 150n^{14} = 450n^{14}$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4n^2 + 37n + 14n^{14}}{15n^{16} + 3n^2} &\leq \sum_{n=1}^{\infty} \frac{450n^{14}}{15n^{16}} \\ &= 30 \sum_{n=1}^{\infty} n^2 < \infty.\end{aligned}$$

The last expression is finite and equal to $5\pi^2$, see for example (2.54b).

(c) Note that for $n \geq 1$, $5n^4 + 3n \leq 5n^4 + 3n^4 = 8n^4$ and $2n^3 + 7n + n^2 \geq 2n^3$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2n^3 + 7n + n^2}{5n^4 + 3n} &\geq \sum_{n=1}^{\infty} \frac{2n^3}{8n^4} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.\end{aligned}$$

Since the harmonic series diverges, our original series must also diverge.

(d) Note that $n + 1 > n$, so $0 \leq \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ for all $n \geq 1$. Since the outer two sequences converge to 0 for $n \to \infty$, by the squeeze theorem, it follows that $\lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0$.

We now show that $\left(\frac{\sqrt{n}}{n+1}\right)_{n\in\mathbb{N}}$ is a decreasing sequence. We first note that all terms of the sequence are positive. Let $a, b \in \mathbb{N}$ with $a \leq b$. By squaring (both sides are nonnegative) we obtain

$$\begin{aligned}
&\frac{\sqrt{a}}{a+1} \geq \frac{\sqrt{b}}{b+1} \\
\iff &\frac{a}{(a+1)^2} \geq \frac{b}{(b+1)^2} \\
\iff &a(b+1)^2 \geq b(a+1)^2 \\
\iff &ab^2 + 2ab + a \geq a^2 b + 2ab + b \\
\iff &(b-a)(ab-1) \geq 0.
\end{aligned}$$

Since both $ab \geq 1$ and $b - a \geq 0$, this always holds, so the sequence is indeed decreasing. Hence, we can apply the alternating series test to deduce that

$$\sum_{kn=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$$

converges to a finite number.

(e) By shifting indices and applying the formula for geometric series,

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{4^{n-1}}{5^{n+1}} &= \sum_{n=3}^{\infty} \frac{4^2}{5^4} \cdot \frac{4^{n-3}}{5^{n-3}} \\
= \frac{16}{625} \sum_{n=3}^{\infty} \left(\frac{4}{5}\right)^{n-3} &= \frac{16}{625} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^{n} \\
= \frac{16}{625} \cdot \frac{1}{1 - \frac{4}{5}} &= \frac{16}{125}.
\end{aligned}$$

Task 9: Radius of convergence

(a) Determine all $x \in \mathbb{R}$ such that $\sum\limits_{n=0}^{\infty} \frac{4n}{3^n} x^n$ converges.

(b) Determine the radius $r > 0$ such that the series $\sum\limits_{n=0}^{\infty} (n+1)2^n x^n$ converges for all $|x| < r$ and calculate it explicitly for these values.

Solution.

(a) By the quotient criterion, we know that the series converges if $\limsup\limits_{n\to\infty} \left| \frac{4(n+1)\cdot 3^{-n-1}\cdot x^{n+1}}{4n\cdot 3^{-n}\cdot x^n} \right| = \limsup\limits_{n\to\infty} \frac{(n+1)|x|}{3n} < 1$. Now since $\lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} 1 + \frac{1}{n} = 1 + \lim\limits_{n\to\infty} \frac{1}{n} = 1$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{(n+1)\,|x|}{3n} &= \frac{|x|}{3} \limsup_{n\to\infty} \frac{n+1}{n} \\
&\overset{(*)}{=} \frac{|x|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{3}.
\end{aligned}$$

In $(*)$, we used that lim and lim sup are equivalent for converging sequences.

Thus, we already know the series converges if $|x| < 3$ and diverges if $|x| > 3$. It remains to inspect convergence for $|x| = 3$. In this case $\left|\frac{4n}{3^n} x^n\right| = \frac{4n}{3^n} |x|^n = \frac{4n}{3^n} \cdot 3^n = 4n$, which is not even bounded, so the series diverges.

All in all, the series conveges if and only if $|x| < 3$.

(b) Analogously to above, we can use the quotient criterion to deduce convergence if

$$\limsup_{n\to\infty} \left|\frac{(n+1)2^{n+1}x^{n+1}}{n2^n x^n}\right| = \limsup_{n\to\infty} \frac{2(n+1)|x|}{n} < 1.$$

As above, $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$, so we can replace lim sup by lim to obtain

$$\limsup_{n\to\infty} \frac{2(n+1)|x|}{n} = 2|x| \lim_{n\to\infty} \frac{n+1}{n} = 2|x|.$$

Thus, the series converges if $2|x| < 1$ and diverges if $2|x| > 1$, so the radius of convergence is $r = \frac{1}{2}$.

Finally, we calculate the value of the series for $|x| < \frac{1}{2}$. In task 2, exercise sheet 9, we showed that $\frac{1}{(1-X)^2} = \sum\limits_{k=0}^{\infty}(1+k)X^k$ if $|X| < 1$.

Since $|2x| < 1$, we can set $X = 2x$ to deduce that

$$\sum_{n=0}^{\infty}(n+1)2^n x^n = \sum_{n=0}^{\infty}(n+1)(2x)^n = \frac{1}{(1-2x)^2}.$$

Remark. Although it is not part of the problem, one can use similar reasoning as in part (a) to show that the series diverges for $|x| = \frac{1}{2}$, so we find that the series converges if and only if $|x| < \frac{1}{2}$.

Task 10: Induction

Show that for all $n \in \mathbb{N}$, we have

(a) $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$, and

(b) $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Solution.

(a) We proceed by induction. Note that for $n = 1$,

$$\sum_{k=1}^{n} k^3 = 1^3 = 1 = \frac{1^2 \cdot 2^2}{4} = \frac{1^2(1+1)^2}{4}.$$

Now assume that the statement holds for some $n \in \mathbb{N}$, i.e. $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. Observe that

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= (n+1)^3 + \sum_{k=1}^{n} k^3 \\ &= (n+1)^3 + \frac{n^2(n+1)^2}{4} \\ &= (n+1)^2 \left(n + 1 + \frac{n^2}{4}\right) \\ &= (n+1)^2 \left(\frac{n^2+4n+4}{4}\right) = \frac{(n+1)^2(n+2)^2}{4},\end{aligned}$$

so the statement also holds for $n+1$. Hence, the formula holds true for all $n \in \mathbb{N}$ by induction.

(b) Again, we proceed by induction. Note that for $n = 1$,

$$\sum_{k=1}^{n} k = 1 = \frac{1 \cdot 2}{2}.$$

Now assume that the statement holds for some $n \in \mathbb{N}$, i.e. $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$. Observe that

$$\begin{aligned}\sum_{k=1}^{n+1} k &= (n+1) + \sum_{k=1}^{n} k \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= (n+1) \left(1 + \frac{n}{2}\right) \\ &= (n+1)^2 \left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2},\end{aligned}$$

so the statement also holds for $n+1$. Hence, the formula holds true for all $n \in \mathbb{N}$ by induction.

Remark. If we include $0 \in \mathbb{N}$, then both formulae still hold, since then the right expressions evaluate to 0 and the sums on the left side are empty.

Task 11: Set theory

In this task, we study the cardinality of real numbers. It will be enough to consider the interval $[0, 1]$, since it has the same cardinality as $\mathbb{R}$. To characterize all real numbers in this interval, we consider the sets

$$\begin{aligned}A &:= \{(a_n)_{n\in\mathbb{N}} \subseteq \{0,1\} : \ \forall n \in \mathbb{N} : \ \exists k \geq n : \ a_k \neq 1\}, \\ B &:= \left\{a \in \mathbb{R} : \ \exists (a_n)_{n\in\mathbb{N}} \in A : \ a = \sum_{n=1}^{\infty} a_n 2^{-n}\right\} \cup \{1\}.\end{aligned}$$

We start by proving that all rational numbers are included in B and that B is complete.

(a) Show that for any $q \in \mathbb{Q} \cap [0, 1)$, there exists $(a_n)_{n\in\mathbb{N}} \in A$ such that $q = \sum\limits_{n=0}^{\infty} a_n 2^{-n}$.

(b) Show that for any $a \in B \setminus \{1\}$, there exists exactly one $(a_n)_{n\in\mathbb{N}} \in A$ such that $a = \sum\limits_{n=1}^{\infty} a_n 2^{-n}$.

(c) Show that B is complete, i.e. for any $C \subseteq B$, we have $\inf C, \sup C \in B$.

For simplicity, we want to drop the assumption '$\forall n \in \mathbb{N} : \ \exists k \geq n : \ a_k \neq 1$' and identify $[0, 1]$ by $\{0, 1\}$-valued sequences.

(d) Show that A and $C := \{(c_n)_{n\in\mathbb{N}} \subseteq \{0, 1\}\}$ have the same cardinality, i.e. find a bijection $f : A \to C$.

(e) Conclude that C and $[0, 1]$ have the same cardinality.

Now, we want to use the power set of natural numbers as a set with the same cardinality as $[0, 1]$ and prove that the natural numbers and the real numbers do not have the same cardinality.

(f) Let D be an arbitrary countable set. Show that the power set $\mathcal{P}(D)$ does not have the same cardinality as D.

(g) Show that C and $\mathcal{P}(D)$ have the same cardinality.

(h) Conclude that $[0, 1]$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.

One can even generalize the result from f).

(i) Let A be an arbitrary set, show that A and $\mathcal{P}(A)$ do not have the same cardinality.

Solution.

(a) Let $q \in \mathbb{Q} \cap [0, 1)$.

We recursively define two sequences $(q_n)_{q\in\mathbb{N}}, (a_n)_{n\in\mathbb{Z}^+}$ by

$$q_0 = q, \quad \forall n \in \mathbb{N} : \ q_{n+1} = q_n - a_{n+1} 2^{-(n+1)},$$
$$\forall n \in \mathbb{N} : \ a_{n+1} = \begin{cases} 0 & \text{if } q_n < 2^{-(n+1)} \\ 1 & \text{if } q_n \geq 2^{-(n+1)}. \end{cases}$$

Note that we do not define a_0.

Lemma. *For all $n \in \mathbb{N}$, we have $q_n \in [0, 2^{-n})$ and $q_0 = q_n + \sum\limits_{k=1}^{n} a_k 2^{-k}$.*

Proof. We proceed by induction. For $n = 0$, we have $q_0 = q \in [0, 1)$ by assumption.

In addition, $q_0 = q_0 + \sum\limits_{k=1}^{0} a_k 2^{-k}$ holds since the sum is empty.

Now assume both statements hold for some $n \in \mathbb{N}$, i.e. $q_n \in [0, 2^{-n})$ and $q_0 = q_n + \sum\limits_{k=1}^{n} a_k 2^{-k}$.

Then by definition, $q_{n+1} = q_n - a_{n+1}2^{-(n+1)}$, or, equivalently, $q_n = q_{n+1} + a_{n+1}2^{-(n+1)}$. Hence,

$$\begin{aligned} q_0 = q_n + \sum_{k=1}^{n} a_k 2^{-k} &= q_{n+1} + a_{n+1}2^{-(n+1)} + \sum_{k=1}^{n} a_k 2^{-k} \\ &= q_{n+1} + \sum_{k=1}^{n+1} a_k 2^{-k}. \end{aligned}$$

Finally, we use $q_n \in [0, 2^{-n})$.

- Case $q_n \in [0, 2^{-(n+1)})$: In this case, we have $q_n < 2^{-(n+1)}$, so by definition, $a_{n+1} = 0$ and hence $q_{n+1} = q_n \in [0, 2^{-(n+1)})$.
- Case $q_n \in [2^{-(n+1)}, 2^{-n})$: In this case, we have $q_n \geq 2^{-(n+1)}$, so by definition, $a_{n+1} = 1$ and hence $q_{n+1} = q_n - 2^{-(n+1)} \in [0, 2^{-(n+1)})$.

Thus, we see that $q_{n+1} \in [0, 2^{-(n+1)})$ holds in both cases.

Hence, the lemma follows by induction. □

Note that $q_n \in [0, 2^{-n})$ implies $\lim\limits_{n\to\infty} q_n = 0$. Thus, taking the limit of both sides in the equation $q_0 = q_n + \sum\limits_{k=1}^{n} a_k 2^{-k}$, we have $q_0 = \sum\limits_{k=1}^{\infty} a_k 2^{-k}$. We claim that the left sides lies in B. To do this, we simply have to exclude the possibility that there is some $N \in \mathbb{N}$ with $a_k = 1$ for all $k \geq N$.

Assume not, i.e. $a_k = 1$ for all $k \geq N$. Then since $q_0 = q_N + \sum\limits_{k=1}^{N} a_k 2^{-k}$, it would follow that

$$\begin{aligned} q_N = q_0 - \sum_{k=1}^{N} a_k 2^{-k} &= \sum_{k=1}^{\infty} a_k 2^{-k} - \sum_{k=1}^{N} a_k 2^{-k} \\ &= \sum_{k=N+1}^{\infty} a_k 2^{-k}. \end{aligned}$$

Since $a_k = 1$ for all $k \geq N$, we can simplify this using the formula for geometric series to obtain

$$\begin{aligned} q_N = \sum_{k=N+1}^{\infty} 2^{-k} &= 2^{-(N+1)} \sum_{k=0}^{\infty} 2^{-k} \\ &= 2^{-(N+1)} \cdot \frac{1}{1-\frac{1}{2}} = 2^{-N}. \end{aligned}$$

However, this contradicts $q_N \in [0, 2^{-N})$, so the sequence $(a_k)_{k\in\mathbb{N}}$ does not terminate in 1s eventually.

Thus, $(a_k)_{k\in\mathbb{N}} \in A$, so $q = q_0 = \sum\limits_{n=1}^{\infty} a_n 2^{-n} \in B$, as desired.

(b) Assume for some $a \in B \setminus \{1\}$, there are two different sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in A$ with $\sum\limits_{n=1}^{\infty} a_n 2^{-n} = a = \sum\limits_{n=1}^{\infty} b_n 2^{-n}$. Since the two sequences are different, there is some smallest index $i \in \mathbb{N}$ with $a_i \neq b_i$. Without loss of generality assume $a_i = 1, b_i = 0$ (otherwise interchange $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$).

Then $a_j = b_j$ for all $j < i$, so

$$\begin{aligned} 0 = a - a &= \sum_{n=1}^{\infty} a_n 2^{-n} - \sum_{n=1}^{\infty} b_n 2^{-n} = \sum_{n=1}^{\infty} (a_n - b_n) 2^{-n} \\ &= (a_i - b_i)2^{-i} + \sum_{n=i+1}^{\infty} (a_n - b_n) \\ &= 2^{-i} + \sum_{n=i+1}^{\infty} (a_n - b_n) 2^{-i}. \end{aligned}$$

Since $a_n - b_n \geq -1$, we have, by the formula for the geometric series,

$$-2^{-i} = \sum_{n=i+1}^{\infty} (a_n - b_n) 2^{-i} \geq \sum_{n=i+1}^{\infty} -2^{-n} = -2^{-i}.$$

Comparing both ends of the inequality, we see that this must be an equality, so $a_n - b_n = -1$ must hold for all $n \geq i+1$. This is of course only possible if $b_n = 1$ for all $n \geq i+1$, which contradicts the assumption that sequences in A do not terminate in only 1s.

Hence, there cannot be two different sequences representing the same number.

(c) We essentially already proved this in part (a). If $\sup(C) = 1$ or $\inf(C) = 1$, we are already done since $1 \in B$. Otherwise, we have $\sup(C) \in [0,1)$ or $\inf(C) \in [0,1)$. Following the proof in part (a), we did not use the condition that $q \in \mathbb{Q}$.

Hence, the proof works for all $q \in [0,1)$, so B is complete.

(d) We partition C into sequences with infinitely many 0s and 1s and sequence which are eventually constant 0s or 1s. Define

$$\begin{aligned} M_0 &= \{(c_n)_{n\in\mathbb{N}} \in C :\ \exists k \in \mathbb{N} :\ \forall n \geq k :\ a_k = 0\}, \\ M_1 &= \{(c_n)_{n\in\mathbb{N}} \in C :\ \exists k \in \mathbb{N} :\ \forall n \geq k :\ a_k = 1\}, \\ M_{\text{mix}} &= \{(c_n)_{n\in\mathbb{N}} \in C :\ \forall k \in \mathbb{N} :\ \exists n_0, n_1 \geq k :\ c_{n_0} = 0, c_{n_1} = 1\}. \end{aligned}$$

Thus, $C = M_0 \cup M_1 \cup M_{\text{mix}}$ and $A = M_0 \cup M_{\text{mix}}$.

We claim that $f : A \to C$ defined by

$$f((a_n)_{n\in\mathbb{N}}) = \begin{cases} (a_n)_{n\in\mathbb{N}} & \text{if } (a_n)_{n\in\mathbb{N}} \in M_{\text{mix}}, \\ (a_{n+1})_{n\in\mathbb{N}} & \text{if } (a_n)_{n\in\mathbb{N}} \in M_0, a_0 = 0, \\ (1 - a_{n+1})_{n\in\mathbb{N}} & \text{if } (a_n)_{n\in\mathbb{N}} \in M_0, a_0 = 1 \end{cases}$$

is a bijection.

We first prove surjectivity. Let $(c_n)_{n\in\mathbb{N}} \in C$.

- If $(c_n)_{n\in\mathbb{N}} \in M_{\text{mix}}$, then $(c_n)_{n\in\mathbb{N}} \in A$ and $f((c_n)_{n\in\mathbb{N}}) = (c_n)_{n\in\mathbb{N}}$.
- If $(c_n)_{n\in\mathbb{N}} \in M_0$, then we define the sequence $(a_n)_{n\in\mathbb{N}}$ by $a_0 = 0, a_{n+1} = c_n$. Since $(c_n)_{n\in\mathbb{N}}$ terminates in a sequence of only 0s, $(a_n)_{n\in\mathbb{N}}$ does as well and $(a_n)_{n\in\mathbb{N}} \in A$. By the definition of f, we have $f((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{n\in\mathbb{N}} = (c_n)_{n\in\mathbb{N}}$.

- If $(c_n)_{n\in\mathbb{N}} \in M_1$, then we define the sequence $(a_n)_{n\in\mathbb{N}}$ by $a_0 = 1, a_{n+1} = 1 - c_n$. Since $(c_n)_{n\in\mathbb{N}}$ terminates in a sequence of only 1s, $(a_n)_{n\in\mathbb{N}}$ terminates in a sequence of 0s, so $(a_n)_{n\in\mathbb{N}} \in A$.
  By the definition of f, we have $f((a_n)_{n\in\mathbb{N}}) = (1 - a_{n+1})_{n\in\mathbb{N}} = (1 - (1 - c_n))_{n\in\mathbb{N}} = (c_n)_{n\in\mathbb{N}}$.

Thus, we see that $(c_n)_{n\in\mathbb{N}} \in \operatorname{Im}(f)$ holds in all cases, so f is surjective.

Now we prove that f is injective. Assume $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in A$ are such that $f((a_n)_{n\in\mathbb{N}}) = f((b_n)_{n\in\mathbb{N}})$. We distinguish four different cases.

- Case $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in M_{\text{mix}}$: Then by the definition of f, we have $f((a_n)_{n\in\mathbb{N}}) = (a_n)_{n\in\mathbb{N}}$ and $f((b_n)_{n\in\mathbb{N}}) = (b_n)_{n\in\mathbb{N}}$. Hence, it follows that $(a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}$.
- Case $(a_n)_{n\in\mathbb{N}} \in M_0, (b_n)_{n\in\mathbb{N}} \in M_{\text{mix}}$: Then $f((a_n)_{n\in\mathbb{N}}$ terminates in only 0s or only 1s (depending on whether $a_0 = 0$ or $a_0 = 1$). In any case, we definitely have $f((a_n)_{n\in\mathbb{N}}) \notin M_{\text{mix}}$. However, since $(b_n)_{n\in\mathbb{N}} \in M_{\text{mix}}$, it follows by the definition of f that $f((b_n)_{n\in\mathbb{N}}) = (b_n)_{n\in\mathbb{N}} \in M_{\text{mix}}$, so $f((a_n)_{n\in\mathbb{N}}) \neq f((b_n)_{n\in\mathbb{N}})$, a contradiction to our assumption.
- Case $(a_n)_{n\in\mathbb{N}} \in M_{\text{mix}}, (b_n)_{n\in\mathbb{N}} \in M_0$: Interchange $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ and apply the previous case.
- Case $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in M_0$:
  - Subcase $a_0 = 0$: By the definition of f, we know that $f((a_n)_{n\in\mathbb{N}}) = f((b_n)_{n\in\mathbb{N}})$ terminates with only 0s, so we must have $b_0 = 0$. Then, again by the definition of f, we have $(a_{n+1})_{n\in\mathbb{N}} = f((a_n)_{n\in\mathbb{N}}) = f((b_n)_{n\in\mathbb{N}}) = (b_{n+1})_{n\in\mathbb{N}}$, so $a_i = b_i$ for all $i \geq 1$. Since $a_0 = 0 = b_0$ as well, we have $(a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}$.
  - Subcase $a_0 = 1$: By the definition of f, we know that $f((a_n)_{n\in\mathbb{N}}) = f((b_n)_{n\in\mathbb{N}})$ terminates with only 1s, so we must have $b_0 = 1$. Then, again by the definition of f, we have $(1 - a_{n+1})_{n\in\mathbb{N}} = f((a_n)_{n\in\mathbb{N}}) = f((b_n)_{n\in\mathbb{N}}) = (1 - b_{n+1})_{n\in\mathbb{N}}$, so $a_i = b_i$ for all $i \geq 1$. Since $a_0 = 1 = b_0$ as well, we have $(a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}$.

Hence, $(a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}$ holds in all cases, so f is injective. Summarizing, we have that f is bijective, so $|A| = |C|$, as desired.

(e) We know that $|A| = |B|$, since $(a_n)_{n\in\mathbb{Z}^+} \mapsto \sum\limits_{n=1}^{\infty} a_n 2^{-n} \in B$ is a bijection by part (b).

Furthermore, by part (d), we have $|A| = |C|$. Finally, in part (c), we showed that $B = [0, 1]$, so $|C|$ and $[0, 1]$ have the same cardinality.

(f) This follows from part (i) by taking $A = D$. Alternatively, we also provide a more visual variant of Cantor's diagonal argument.

If D is finite, then $|\mathcal{P}(D)| = 2^{|D|} > |D|$ always holds. Assume now D is countably infinite. By definition, this means that there is some sequence $(d_n)_{n\in\mathbb{N}}$ which covers each element of D, i.e. $\forall d \in D: \ \exists n \in \mathbb{N}: \ d = d_n$.

Assume now $f : D \to \mathcal{P}(D)$ is a bijection.

Define a_{ij} as 0 if $d_i \notin f(d_j)$ and 1 if $d_i \in f(d_j)$. Then we can represent f as a matrix

$$\begin{array}{c|c|c|c|c} a_{ij} & 0 & 1 & 2 & \dots \\ \hline 0 & 1 & 0 & 1 & \dots \\ \hline 1 & 1 & 1 & 1 & \dots \\ \hline 2 & 0 & 1 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

Then, we can simply define a set S by flipping the entries on the diagonal $(1 - a_{ii})$. Since this new set is different from all sets in the list in at least one place, we know that S does not appear in the list, a contradiction to our assumption that f is bijective.

Remark. This is not a formal proof, only the visual intuition. The formal proof is exactly the same as in part (i).

(g) Since D is countable, there is a sequence $(d_n)_{n\in\mathbb{N}}$ of all elements of D, i.e. $\forall d \in D: \exists n \in \mathbb{N}: d = d_n$.

We then claim that $f : C \to \mathcal{P}(D),\ (c_n)_{n\in\mathbb{N}} \mapsto \{d_n \in D : c_n = 1\}$ is bijective, which will be sufficient.

Firstly, assume $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}} \in C$ are two different sequences. Then, there must exist an $i \in \mathbb{N}$ such that $p_i \neq q_i$. Without loss of generality, we can assume $p_i = 1, q_i = 0$ (otherwise interchange $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$).

By definition of f, we have $d_i \in f((p_n)_{n\in\mathbb{N}})$ since $p_i = 1$. Similarly, we have $d_i \notin f((q_n)_{n\in\mathbb{N}})$ since $q_i = 0$. Hence, $f((p_n)_{n\in\mathbb{N}}) \neq f((q_n)_{n\in\mathbb{N}})$, so f is injective.

Let now $S \in \mathcal{P}(D)$ be arbitrary and define $(s_n)_{n\in\mathbb{N}}$ by $s_n = 1$ if $d_n \in S$ and 0 otherwise. Then observe that for any $i \in \mathbb{N}$, we have

$$d_i \in S \iff s_i = 1 \iff d_i \in f((s_n)_{n\in\mathbb{N}}),$$

so $S = f((s_n)_{n\in\mathbb{N}})$. Since this works for any $S \in \mathcal{P}(D)$, it follows that f is surjective.

Finally, we have that f is bijective and thus, C and $\mathcal{P}(D)$ have the same cardinality.

(h) By part (e), we know that C and $[0, 1]$ have the same cardinality. Furthermore, by part (g), we know that C and $\mathcal{P}(D)$ have the same cardinality if D is a countably infinite set. Since $\mathbb{N}$ is a countably infinite set, this holds true in particular for $D = \mathbb{N}$, so C and $\mathcal{P}(\mathbb{N})$ have the same cardinality.

All in all, $|[0, 1]| = |C| = |\mathcal{P}(\mathbb{N})|$, as desired.

(i) This is Cantor's theorem. Assume $f : A \to \mathcal{P}(A)$ is a bijection. Then define $S = \{a \in A : a \notin f(a)\}$, which is possible since $f(a) \subseteq A$ for all $a \in A$.

Note that S is a subset of A. Since f is bijective, there is some $s \in A$ with $f(s) = S$. We distinguish two cases.

- Case $s \in S$: Then $s \in S = f(s)$. However, by definition of the set S, we have $s \notin f(s)$, a clear contradiction.
- Case $s \notin S$: Then $s \notin S = f(s)$. Thus, by definition of the set S, we have $s \in S$. Again, this is a contradiction.

Since both cases lead to a contradiction, there can be no bijection between A and $\mathcal{P}(A)$, so the two sets have different cardinalities.