Theory of Matrices

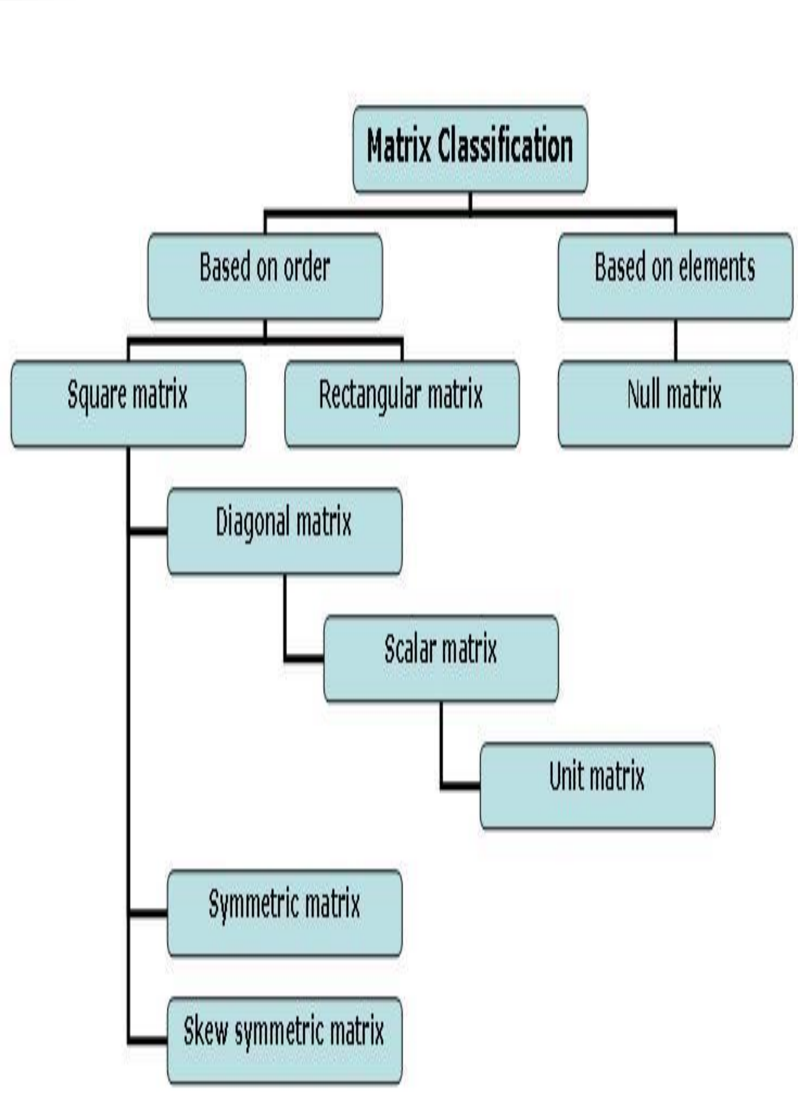
An arrangement of certain numbers in an array of m rows and n columns such as

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

m – no. of rows row in A, n – number of columns in A. Thus, matrix is denoted by $A_{m\times n}$.

$A = \{a_{ij}\}_{m\times n}$ where a_{ij} denotes an element belonging to i^{th} row and j^{th} $column$.

Types of Matrices



Name	*Definition*	*Example*
Rectangular	$n \times p$	$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \end{bmatrix}$
Column vector	$n \times 1$	$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$
Row vector	$1 \times p$	$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
Scalar	1×1	$[1]$
Square	$n \times n$	$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$
Symmetric	Square $a_{ij} = a_{ji}$	$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$
Diagonal	Square $a_{ij} = 0$ for $i \neq j$	$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$
Identity ($\mathbf{I}$)	Square $a_{ii} = 1$ $a_{ij} = 0$ for $i \neq j$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Upper/lower triangular	Square $a_{ij} = 0$ below/above the diagonal	$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$
Unit vector ($\mathbf{1}$)	$a_i = 1$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Unit matrix ($\mathbf{J}$)	$a_{ij} = 1$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
Null(zero) matrix ($\mathbf{0}$)	$a_{ij} = 0$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Operations of Matrices

➢ Addition and Subtraction of Matrices

Matrices can be added and subtracted if (and only if) they possess the same dimensions.

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} + \begin{bmatrix} j & m & p \\ k & n & q \\ l & o & r \end{bmatrix}$$

$$\begin{bmatrix} a+j & d+m & g+p \\ b+k & e+n & h+q \\ c+l & f+o & i+r \end{bmatrix}$$

Matrix Operations

- **Matrix Equality**
- Let *A* and *B* be two matrices.
  A=*B* if they have the **same number of rows and columns**, and every element at each position in *A* equals element at corresponding position in *B*.
- Matrices having equal corresponding entries.

- **Examples**

$$\begin{bmatrix} 5 & 0 \\ -\frac{4}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 0.75 \end{bmatrix} \qquad \begin{bmatrix} -2 & 6 \\ 0 & -3 \end{bmatrix} \neq \begin{bmatrix} -2 & 6 \\ 3 & -2 \end{bmatrix}$$

Operations of Matrices

Scaler x Multiplied to a Matrix

Scalar Multiplication

$$x\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax & bx \\ cx & dx \end{bmatrix}$$

Matrix Multiplication

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

2 x 4 4 x 3 2 x 3

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Important remarks

(Find example wherever possible)

- In general, $AB \neq BA$.
- If $AB = 0$ does not imply that $A = 0 \ or \ B = 0$.
- If $AB = AC$ does not imply that, $B = C$.
- Distributive Law holds for Matrix multiplication
  $$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC$$
- $A^2 = A \times A,\ A^3 = A^2 \times A \quad i.e\ (A^m)^n = A^{mn}$
- $A^m A^n = A^{m+n}$
- $(A')' = A$
- $(AB)' = B'A'$
- $(A + B)' = A' + B'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $|AB| = |A|.|B|$

Elementary Row and Column Operations

Operation description	Notation
Row operations	
1. Interchange rows i and j	R_i <--> R_j
2. Multiply row i by s, where $s \neq 0$	sR_i --> R_i
3. Add s times row i to row j	$sR_i + R_j$ --> R_j
Column operations	
1. Interchange columns i and j	C_i <--> C_j
2. Multiply column i by s, where $s \neq 0$	sC_i --> C_i
3. Add s times column i to column j	$sC_i + C_j$ --> C_j

Rank of a Matrix

The matrix is said to be of rank r if there is

1. At least one minor of the order r which is not equal to zero
2. Every minor of order (r+1) is equal to Zero.

Rank is denoted by $\rho(A)$.

$Ex. A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ Note that |A|=0 and also any minor of order 2×2 is 0. Hence, largest order of its non vanishing minor is 1.

Remark: $\rho(A_{m\times n}) \leq \min(m, n)$

Ways to find Rank

Echelon Form

Given Matrix is to be converted to Row Echelon form using **only elementary row operations.** Then, number of non-zero rows represents rank of Matrix.

Normal Form

Given matrix is converted to normal form(roughly identity matrix) **using both elementary row and column operations**. Then the order of (roughly) identity matrix is rank of matrix.

ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

Row Operation 1: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 5 & 9 \end{bmatrix}$ interchange the 1st row and the 3rd row $\begin{bmatrix} 5 & 9 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Row Operation 2: $\begin{bmatrix} 5 & 9 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ multiply the 1st row by 1/5 $\begin{bmatrix} 1 & \frac{9}{5} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Row Operation 3: $\begin{bmatrix} 1 & \frac{9}{5} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ interchange the 2nd row and the 3rd row $\begin{bmatrix} 1 & \frac{9}{5} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Example 1.

Row Operation 1:	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{matrix}$	add **-1** times the 2nd row to the 3rd row	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{matrix}$
Row Operation 2:	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{matrix}$	multiply the 3rd row by **1/3**	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$
Row Operation 3:	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$	add **-1** times the 3rd row to the 2nd row	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$
Row Operation 4:	$\begin{matrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$	add **-3** times the 2nd row to the 1st row	$\begin{matrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$

Example 2.

Normal Form

By performing elementary **row and column transformations**, any non-zero matrix A can be reduced to one of the following four forms, called the **Normal form**.

- $[I_r]$
- $[I_r \ \ 0]$
- $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$
- $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number ‘r‘so obtained is called the rank of the Matrix.

Example on Rank using Normal form

Ex. 2 : *Reduce the following matrix to the normal form and find the rank.* **(Dec. 96)**

$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

Sol. :

$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

By R_{12}

$$\sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

By $R_2 - 2R_1,\ R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

By $C_2 + 2C_1,\ C_3 - C_1,\ C_4 + 4C_1,\ C_5 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

By R_{23}

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

By $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix}$$

By $C_3 + C_2,\ C_4 - 3C_2,\ C_5 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix}$$

By $R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By $C_4 - 9C_3,\ C_5 + 4C_3$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ which is the required normal form of A.

$\therefore \quad \rho(A) = 3$

Example 2

Ex. 3 : *Reduce the following matrix to its normal form and hence find its rank where*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

Sol. :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

By $R_2 - 2R_1,\ R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & -5 & 0 \end{bmatrix}$$

By $C_2 - C_1,\ C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & -5 & 0 \end{bmatrix}$$

By $\frac{-1}{5}(C_2),\ \frac{1}{2}(C_3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

By $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

By $C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By $-1(R_3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim [I_3]$$

$$\therefore \rho(A) = 3$$

System of Linear Equations

- **A system of *m* linear equations in *n* variables** is a set of m equations, each of which is linear in the same n variables:

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_3 & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

Number of Solutions of a System of Linear Equations

For a system of linear equations in n variables, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has an infinite number of solutions (consistent system).
3. The system has no solution (inconsistent system).

Matrix form of System of Linear Equations

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

In above form,

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

is coefficient Matrix.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ vector of unknowns}$$

and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \text{ right hand side}$$

then the system can be written as

$$AX = B$$

The matrix $\left[A\middle|B\right]$ or $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the **augmented matrix**.

Solution of system of Linear Equations

AX=B
(m equations, n unknowns)

$\rho[A|B] = \rho(A)$
System AX=B is consistent

$\rho[A|B] \neq \rho(A)$
System AX=B is inconsistent

$\rho[A|B] = \rho(A)$< no. of variables
Infinitely many solutions

If $\rho[A|B] = \rho(A)$ = no. of variables
Unique Solution

Solution of System AX=B(Method)

1. Write the given system in matrix form(AX=B).
2. Consider the augmented matrix[A|B] from the given system.
3. Reduce the augmented matrix to the Echelon form.

Note : In [A|B], first part represents $\rho(A)$ and whole matrix represents $\rho(A|B)$.

4. Conclude the system has unique, infinite or no solution.
5. If consistent with $\rho(A|B) = \rho(A) =$ no. of variables then rewrite equations and find values.
6. If consistent with $\rho(A|B) = \rho(A) = r <$ no. of variables then put $n - r$ variables(free variables) as u, v, w etc. find values of other variables in terms of free variables.

System of linear equations having ***unique solution***

Ex. 1 : *Solve the system of equations by matrix method*

$$2x_1 + x_2 - x_3 + 3x_4 = 8$$
$$x_1 + x_2 + x_3 - x_4 + 2 = 0$$
$$3x_1 + 2x_2 - x_3 = 6$$
$$4x_2 + 3x_3 + 2x_4 + 8 = 0$$

Sol. : Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 0 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 6 \\ -8 \end{bmatrix}$$

i.e. $AX = B.$

Consider an augmented matrix.

$$(A, B) = \left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 8 \\ 1 & 1 & 1 & -1 & -2 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Step 1 : Get $a_{11} = 1$

Perform R_{12}
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 2 & 1 & -1 & 3 & 8 \\ 3 & 2 & -1 & 0 & 6 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Step 2 : Make a_{21}, a_{31} equal to zero. **(Hint : Use row transformation).**

Perform $R_2 - 2R_1$, $R_3 - 3R_1$
$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & -1 & -4 & 3 & 12 \\ 0 & 4 & 3 & 2 & -8 \end{array}\right]$$

Step 3 : Note that without making $a_{22} = 1$, we can get a_{32}, a_{42} equal to zero. For th
purpose, use again row transformation.

Perform $R_3 - R_2$, $R_4 + 4R_2$
$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -9 & 22 & 40 \end{array}\right]$$

Step 4 : Again keep a_{33} as it is and make $a_{43} = 0$.

Perform $R_4 - 9R_3$
$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -3 & 5 & 12 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 40 & 40 \end{array}\right]$$

Step 5 : This is reduced to echelon form.

$$\rho(A) = \rho(A, B) = 4 - 0 = 4 = \text{total number of variables}$$

Hence system possesses a unique solution and it is given as follows.

By R_4 $\quad 40x_4 = 40, \quad \therefore x_4 = 1$

By R_3 $\quad -x_3 - 2x_4 = 0, \quad \therefore x_3 = -2$

By R_2 $\quad -x_2 - 3x_3 + 5x_4 = 12, \quad \therefore x_2 = -3(-2) + 5(1) - 12 = -1$

By R_1 $\quad x_1 + x_2 + x_3 - x_4 = -2, \quad \therefore x_1 = -2 + 1 + 2 + 1 = 2$

Hence the solution set is $x_1 = 2$, $x_2 = -1$, $x_3 = -2$, $x_4 = 1$

Step 6 : Always check your answer (i.e. values of x_1, x_2, x_3, x_4) by substituting in one of the equation. i.e. $2x_1 + x_2 - x_3 + 3x_4 = 8 \;\therefore\; 2(2) - 1 + 2 + 3 = 8 \quad \therefore\; 8 = 8$

Hence the ...

Ex. 3 : *By considering the ranks of relevant matrices, examine for consistency the system of equations :*

$$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$ *and solve them if found consistent.* **(Dec. 2004, May 2010, Dec. 2014)**

Sol. : Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

i.e. $AX = B$

Consider an augmented matrix,

$$(A, B) = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Perform R_{12}
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Perform $R_2 - 2R_1,\ R_3 - 4R_1$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array}\right]$$

Perform $R_3 - 3R_2$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3 - 1 = 2, \quad \rho(A, B) = 3 - 1 = 2$

$\rho(A) = \rho(A, B) \quad \therefore$ System is consistent.

$\rho(A) = \rho(A, B) = 2 < 3$, number of variables, system possesses an infinite number of solutions given as follows.

By R_2 $\quad -5y - 3z = -2 \qquad$ Let $z = t \quad y = \dfrac{2 - 3t}{5}$

By R_1 $\quad x + 2y + z = 2 \qquad x = 2 - \dfrac{4 - 6t}{5} - t = \dfrac{10 - 4 + 6t - 5t}{5} = \dfrac{6 + t}{5}$

Hence solution set is $x = \dfrac{6 + t}{5}, \ y = \dfrac{2 - 3t}{5}, \ z = t.$

System of Linear Equation with ***Infinitely many solutions***

Note that different values of ‘t’ will yield different solutions

Ex. 2 : *Examine for consistency and solve, if consistent.*

$$x + y + z = 3$$
$$2x - y + 3z = 1$$
$$4x + y + 5z = 2$$
$$3x - 2y + z = 4$$

Sol. : The above system in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 4 & 1 & 5 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$(A, B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 1 \\ 4 & 1 & 5 & 2 \\ 3 & -2 & 1 & 4 \end{array}\right]$$

Perform $R_2 - 2R_1$, $R_3 - 4R_1$, $R_4 - 3R_1$ $\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -10 \\ 0 & -5 & -2 & -5 \end{array}\right]$

Perform $R_3 - R_2$, $-\frac{1}{5} R_4$ $\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & 1 & \frac{2}{5} & 1 \end{array}\right]$

Perform R_{24} $\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & \frac{2}{5} & 1 \\ 0 & 0 & 0 & -5 \\ 0 & -3 & 1 & -5 \end{array}\right]$

Perform $R_4 + 3R_2$ $\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & \frac{2}{5} & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & \frac{11}{5} & -2 \end{array}\right]$

Perform R_{34} $\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & \frac{2}{5} & 1 \\ 0 & 0 & \frac{11}{5} & -2 \\ 0 & 0 & 0 & -5 \end{array}\right]$

$\rho(A) = 4 - 1 = 3$, $\rho(A, B) = 4 - 0 = 4$ $\rho(A) \neq \rho(A, B)$

$\therefore$ System is inconsistent. $\therefore$ No solution exists.

Inconsistent system of Linear equations

Ex. 3 : *Investigate for what values of λ and μ, the system of simultaneous equations*

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

have (1) No solution. (2) A unique solution. (3) An infinite number of solution

Sol. : The above system in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

i.e. $AX = B$

$$(A, B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Perform $R_2 - R_1$, $R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

Perform $R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

Case 1 : For no solution : This will happen only when $\rho(A) \neq \rho(A, B)$

When $\lambda - 3 = 0$ and $\mu - 10 \neq 0$

i.e. $\lambda = 3$, $\mu \neq 10$ then $\rho(A) = 3 - 1 = 2$, $\rho(A, B) = 3 - 0 = 3$

$\rho(A) \neq \rho(A, B)$, system possesses no solution.

Hence for $\lambda = 3$, $\mu \neq 10$ then system possesses no solution.

Case 2 : For a unique solution : This will happen only when $\rho(A) = \rho(A, B) = 3 =$ number of variables.

i.e. When $\lambda - 3 \neq 0$ but μ can take any value then $\rho(A) = 3 - 0 = 3$,

$\rho(A, B) = 3 - 0 = 3.$

$\rho(A) = \rho(A, B) = 3 =$ number of variables, system possesses a unique solution.

Hence for $\lambda \neq 3$ and μ can take any value then system possesses a unique solution.

Case 3 : For an infinite number of solutions : This will happen only when

$\rho(A) = \rho(A, B) = 2 < 3$ (number of variables)

i.e. when $\lambda - 3 = 0$, $\mu - 10 = 0$

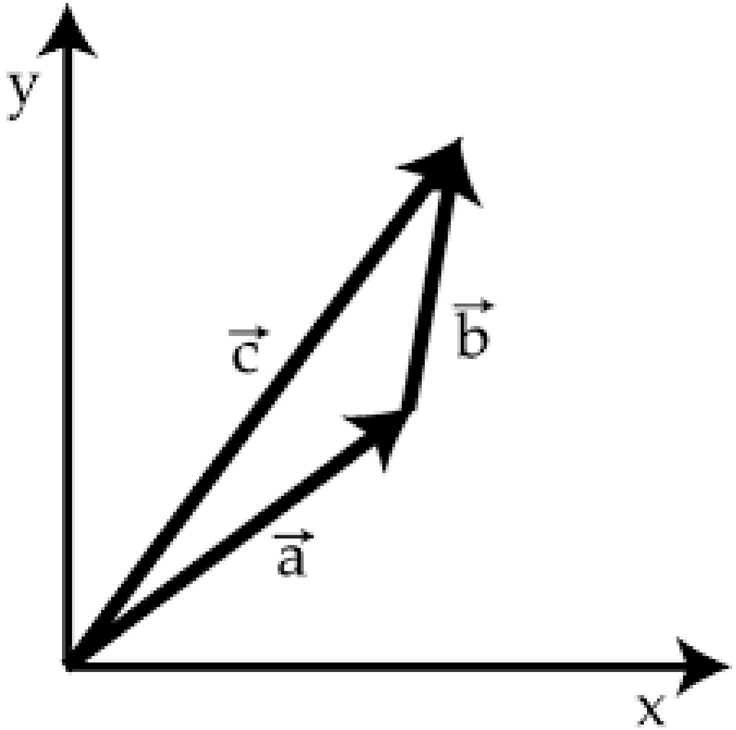
i.e. $\lambda = 3$, $\mu = 10$.

Then $\rho(A) = \rho(A, B) = 3 - 1 = 2 < 3$. System is consistent and possesses an infinite number of solutions.

Hence if $\lambda = 3$, $\mu = 10$ then system possesses an infinite number of solutions.

Follow the chart for solution of system of linear equations

Linear Dependence and Independence



By Triangle law,
$\vec{c} = \vec{a} + \vec{b}$

Set $\{\vec{a}, \vec{b}, \vec{c}\}$ is Linearly dependent.
Set $\{\vec{a}, \vec{b}\}$ is linearly Independent.

Can you think of the definition of Linearly dependent and Linearly Independent?

Definition

Linearly Independent

The system n vectors Set $x_1, x_2, x_3 \dots \dots . x_n$ is said to be Linearly Independent if every relation of the type

$$c_1x_1 + c_2x_2 + \cdots . . c_nx_n = 0$$

has a unique solution $c_1 = c_2 = \dots \dots . = c_n = 0$

Linearly Dependent

The system n vectors Set $x_1, x_2, x_3 \dots \dots . x_n$ is said to be Linearly dependent if every relation of the type

$$c_1x_1 + c_2x_2 + \cdots . . c_nx_n = 0$$

Has some $c_i \neq 0$.

Remarks

- $c_1x_1 + c_2x_2 + \cdots .. c_nx_n = 0$ is a linear equation with $c_1, c_2, c_3 \ldots \ldots c_n$ as unknowns.
- If vectors $x_1, x_2, x_3, \ldots \ldots x_n$ are linearly dependent then in equation $c_1x_1 + c_2x_2 + \cdots .. c_nx_n = 0$ by definition some $c_k \neq 0.$
- By simplification we get

$$c_kx_k = -c_1x_1 - c_2x_2 \ldots \ldots - c_nx_n$$

$$x_k = -\left(\frac{c_1}{c_k}\right)x_1 - \left(\frac{c_2}{c_k}\right)x_2 - \ \ldots \ldots \ldots . - \left(\frac{c_n}{c_k}\right)x_n$$

- ***If vectors are dependent then there exist a relation between the given vectors.***

Ex. 3 : *Define linear dependence and independence of vectors. Examine for linear dependence of vectors* $(1, 2, -1, 0)$, $(1, 3, 1, 2)$, $(4, 2, 1, 0)$, $(6, 1, 0, 1)$ *and find a relation between them if dependent.* **(May 2005, 2009)**

Sol. : Part 1 : For definition refer article 2.4.

Part 2 : Let the given vectors be

$x_1 = (1, 2, -1, 0)$, $x_2 = (1, 3, 1, 2)$, $x_3 = (4, 2, 1, 0)$, $x_4 = (6, 1, 0, 1)$

Consider the matrix equation $c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0$

$c_1(1, 2, -1, 0) + c_2(1, 3, 1, 2) + c_3(4, 2, 1, 0) + c_4(6, 1, 0, 1) = 0$

$$\begin{aligned} c_1 + c_2 + 4c_3 + 6c_4 &= 0 \\ 2c_1 + 3c_2 + 2c_3 + c_4 &= 0 \\ -c_1 + c_2 + c_3 &= 0 \\ 2c_2 + c_4 &= 0 \end{aligned}$$

In matrix form, $\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$(A, Z) = \left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 2 & 3 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Perform $R_2 - 2R_1$, $R_3 + R_1$ $\sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 0 & 1 & -6 & -11 & 0 \\ 0 & 2 & 5 & 6 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$

Perform $R_3 - 2R_2$, $R_4 - 2R_2$ $\sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 0 & 1 & -6 & -11 & 0 \\ 0 & 0 & 17 & 28 & 0 \\ 0 & 0 & 12 & 23 & 0 \end{array}\right]$

Perform $R_4 - \frac{12}{17} R_3$ $\sim \left[\begin{array}{cccc|c} 1 & 1 & 4 & 6 & 0 \\ 0 & 1 & -6 & -11 & 0 \\ 0 & 0 & 17 & 28 & 0 \\ 0 & 0 & 0 & \frac{55}{17} & 0 \end{array}\right]$

$$\rho(A) = \rho(A, Z) = 4 - 0 = 4$$

$\therefore$ System possesses a unique solution i.e. trivial solution. $c_1 = c_2 = c_3 = c_4 = 0$.

$\therefore$ x_1, x_2, x_3, x_4 are linearly independent.

Hence there does not exist any relation among them.

Linearly Independent set of Vectors

Ex. 1 : *Examine for linear dependence or independence of vectors (2, –1, 3, 2), (1, 3, 4, 2) and (3, –5, 2, 2). Find a relation between them if dependent.* **(May 2004, 2009, 2014)**

Sol. : Step 1 : Let the given vectors be $x_1 = (2, -1, 3, 2)$, $x_2 = (1, 3, 4, 2)$ and $x_3 = (3, -5, 2, 2)$. Consider the matrix equation $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$

$$c_1 (2, -1, 3, 2) + c_2 (1, 3, 4, 2) + c_3 (3, -5, 2, 2) = (0, 0, 0, 0).$$

i.e.
$$\begin{aligned} 2c_1 + c_2 + 3c_3 &= 0 \\ -c_1 + 3c_2 - 5c_3 &= 0 \\ 3c_1 + 4c_2 + 2c_3 &= 0 \\ 2c_1 + 2c_2 + 2c_3 &= 0 \end{aligned}$$

which is a homogeneous system of equations.

Step 2 : In matrix form,

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A, Z) = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ -1 & 3 & -5 & 0 \\ 3 & 4 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

Perform $R_1 + R_2$, $R_3 + 3R_2$, $R_4 + 2R_2$
$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ -1 & 3 & -5 & 0 \\ 0 & 13 & -13 & 0 \\ 0 & 8 & -8 & 0 \end{array}\right]$$

Perform $R_2 + R_1$, $\frac{1}{13} R_3$, $\frac{1}{8} R_4$
$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Perform $R_2 - 7R_3$, $R_4 - R_3$
$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Perform R_{23}
$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = \rho(A, Z) = 4 - 2 = 2 < 3$$

$\therefore$ System possesses a non-trivial solution.

By R_2 $\quad c_2 - c_3 = 0, \quad c_3 = t, \quad c_2 = t$

By R_1 $\quad c_1 + 4c_2 - 2c_3 = 0, \quad c_1 = -2t$

Note that we got all c_1, c_2, c_3 non-zero, therefore, x_1, x_2, x_3 are linearly dependent.

Step 3 : From the matrix equation $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$

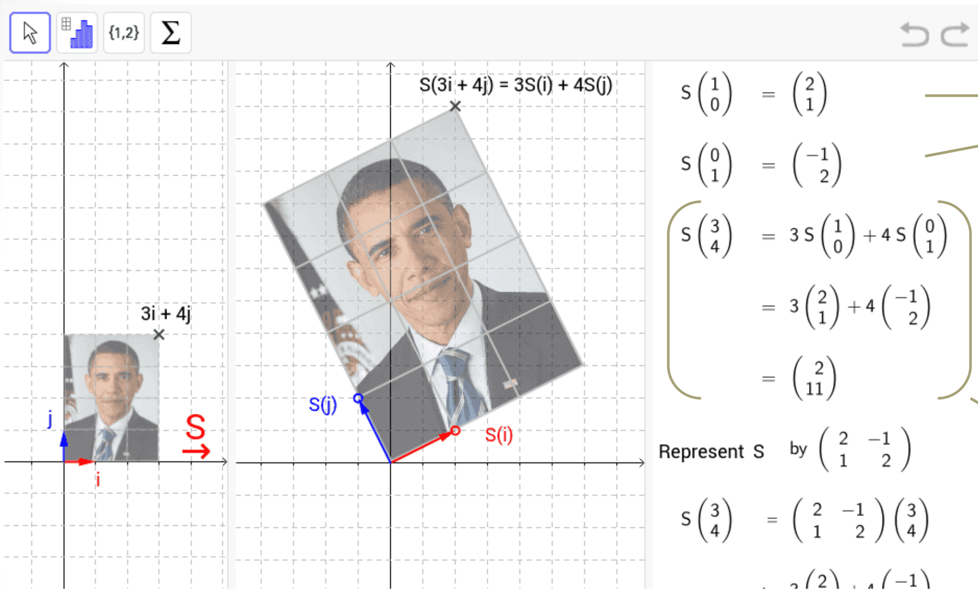
i.e. $-2t\, x_1 + t\, x_2 + t\, x_3 = 0$ or $2x_1 = x_2 + x_3$ which is the required relation among them.

Step 4 : Final check : $\quad 2x_1 = x_2 + x_3$

$$2(2, -1, 3, 2) = (1, 3, 4, 2) + (3, -5, 2, 2)$$
$$(4, -2, 6, 4) = (4, -2, 6, 4)$$

Linearly Dependent. Hence, relation between vectors is found.

Linear Transformation



Matrix S is applied to vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

Matrix S transforms each and every vector on the plane

A Linear Transformation is a map from 'n' dimensional space to itself generally represented by

$$y_1 = a_{11}x_1 + a_{12}x_2 \ldots. + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 \ldots. + a_{2n}x_n$$
$$.$$
$$.$$
$$y_n = a_{n1}x_1 + a_{n2}x_2 \ldots. + a_{nn}x_n$$

Matrix Representation of Linear Transformation

- Matrix representation of the above Linear Transformation is given by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- Every Linear Transformation is the form Y = AX (**Expressing Y in terms of X**)
- Properties of Linear Transformation depends on matrix ‘A’
- If $|A| = 0$, then matrix A is called singular and transformation is called Singular transformation.
- If $|A| \neq 0$, then matrix A is called non-singular and transformation is called non-singular or regular transformation.

Ex. 1 : *Given the transformation* $Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Find the co-ordinates* $(x_1\ x_2\ x_3)$ *in X corresponding to (1, 2, –1) in Y.* **(May 2004, May 2010)**

Sol. : $Y = AX$ i.e. $AX = Y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

This is non-homogeneous system of equations with $m = n = 3$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -4 + 4 - 1$$

$$|A| = -1 \neq 0$$

System possesses a unique solution $X = A^{-1}Y$.

$$A^{-1} = \frac{1}{|A|}\text{ adj. } A$$

$$\text{adj } A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix} \quad \therefore A^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$$X = A^{-1}Y = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$. Hence (–1, 3, 0) corresponds to (1, 2, –1) in Y.

Ex. 2 : *Express each of the transformations* $\left.\begin{array}{l} x_1 = 3y_1 + 2y_2 \\ x_2 = -y_1 + 4y_2 \end{array}\right\}$ and $\left.\begin{array}{l} y_1 = z_1 + 2z_2 \\ y_2 = 3z_1 \end{array}\right\}$ *in the matrix form and find the composite transformation which expresses* x_1, x_2 *in terms of* z_1, z_2.

Sol. : The transformation $\left.\begin{array}{l} x_1 = 3y_1 + 2y_2 \\ x_2 = -y_1 + 4y_2 \end{array}\right\}$ in matrix form can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$X = AY$$

Also the transformation $\left.\begin{array}{l} y_1 = z_1 + 2z_2 \\ y_2 = 3z_1 \end{array}\right\}$ in matrix form can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$Y = BZ$$

$\therefore$ Required composite transformation is

$$X = AY = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\therefore$ $\left.\begin{array}{l} x_1 = 9z_1 + 6z_2 \\ x_2 = 11z_1 - 2z_2 \end{array}\right\}$ is the required transformation.

Orthogonal Transformation

- Transformation $Y = AX$ is said to be Orthogonal if A is Orthogonal matrix.
- Matrix A is called orthogonal if $A'A = AA' = I$ where A' is transpose of A,.

Properties of Orthogonal Matrix :

1. $A^{-1} = A'$
2. $|A| = \pm 1$
3. A is Orthogonal then A^{-1} is also Orthogonal.

Ex. 3 : *Show that* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ *is orthogonal.*

Sol. : We have by definition A is orthogonal if

$$AA' = A'A = I$$

$$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AA' = I \Rightarrow$ A is orthogonal

Ex. 5 : *Determine the values of a, b, c when* $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ *is orthogonal.*

(May 05, 09; Dec. 2010)

Sol. : If A is orthogonal then it requires

$$AA' = I$$

$$AA' = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

$$= \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix}$$

But $\quad AA' = I$

$\therefore \quad 4b^2 + c^2 = 1, \qquad 2b^2 - c^2 = 0$

$\therefore \quad 6b^2 = 1 \qquad b^2 = \frac{1}{6} \qquad \therefore \ b = \pm \frac{1}{\sqrt{6}}$

$c^2 = \frac{1}{3} \qquad \therefore \ c = \pm \frac{1}{\sqrt{3}}$

Also $\quad a^2 + b^2 + c^2 = 1, \qquad a^2 = \frac{1}{2}, \qquad a = \pm \frac{1}{\sqrt{2}}$

$\therefore \quad a = \pm \frac{1}{\sqrt{2}}, \qquad b = \pm \frac{1}{\sqrt{6}}, \quad c = \pm \frac{1}{\sqrt{3}}$

Eigen Values and Eigen Vectors

Linearly Transformed
(Using some matrix 'A')

- Observe blue and red vectors before and after the transformation.

Blue vector has same direction even after the transformation where as red vector changes its direction.

Eigen Value and Eigen Vectors

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathbf{x} = \lambda\mathbf{x}$; such an **x** is called an *eigenvector corresponding to* λ.

Eigenvector of Matrix **A**

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Eigenvalue of Matrix **A**

Remember
Blue vector?

Graphically Eigen vector is the one that doesn't change its direction
And Eigen values are the extent by which Eigen vector changes its length.

Let x be an eigenvector of the matrix **A.** Then there must exist an eigenvalue λ such that $\mathbf{Ax = \lambda x}$ or, equivalently,

$$\mathbf{Ax - \lambda x = 0} \quad \text{or}$$

$$\mathbf{(A - \lambda I)x = 0}$$

If we define a new matrix $\mathbf{B = A - \lambda I}$, then

$$\mathbf{Bx = 0}$$

If **B** has an inverse then $\mathbf{x = B^{-1}0 = 0}$. But an eigenvector cannot be zero.

Thus, it follows that x will be an eigenvector of **A** if and only if **B** does not have an inverse, or equivalently $\mathbf{det(B)=0}$, or

$$\mathbf{det(A - \lambda I) = 0}$$

This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

Characteristic Equation

- Characteristic Equation is also called characteristic Polynomial

because $|A - \lambda I| = 0$ is a polynomial of degree equal to order of square matrix A.

- For 2×2 matrix, characteristic polynomial is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\lambda^2 - S_1\lambda + |A| = 0$$

Where S_1 = sum of diagonal matrix = Trace (A)

- For 3×3 matrix, characteristic polynomial is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where S_1 = sum of diagonal matrix =Trace(A)

S_2 = sum of minors of order two of diagonal elements

$$S_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Method

(To Eigen values and vectors for a given matrix 'A')

1. Write the characteristic polynomial for 'A' i.e $|A - \lambda I| = 0.$ (use the shortcuts for 2×2 and 3×3 matrix)
2. Roots of the Characteristic polynomial are Eigen Values. (say λ_1, $\lambda_2, \lambda_3 \ldots\ldots$)
3. For each Eigen Value λ_i, we can find Eigen vectors by solving the system of linear equation

$$[A - \lambda_i I]X = 0$$

Brain Teaser

Can same Eigen vector can correspond to two different Eigen Values?

If no, then why? If yes, can you find it?

HINT: USE THE BASIC DEFINITION OF EIGEN VECTOR

Ex. Find the eigen value and eigen vectors for the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solⁿ: Characteristic polynomial is $\lambda^2 - S_1\lambda + |A| = 0$

$S_1 = -1 + 1$
$= 0$

$|A| = -1$

So, $\lambda^2 - 1 = 0$

$(\lambda+1)(\lambda-1) = 0 \Rightarrow \lambda = 1, -1$

Eigen values of A are $\lambda = 1, -1$.

Eigen vector corresponding to $\lambda = 1$

$[A - (1)I]\,X = [0]$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x = 0$

Note that y is free variable

Let $y = t$

$x = 0$

Eigen vectors corresponding to $\lambda = 1$ is $\begin{bmatrix} 0 \\ t \end{bmatrix}$

$t = 1$ gives particular eigen vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = -1$.

$[A - (-1)I]\,X = [0]$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) X = [0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2y = 0$

Note that x is free

Put $x = r$
$y = 0$

Eigen vectors correspond to $\lambda = -1$ is $\begin{bmatrix} r \\ 0 \end{bmatrix}$

Put $r = 1$, gives particula eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Can you quickly check Eigen values for $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$?

Any conclusion?

Ex) Find the eigen vector corresponding to largest eigen value for the matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Solution: Characteristic polynomial is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = 4 + 3 + 1$
$= 8$

$S_2 = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix}$

$= -5 - 0 + 22$

$S_2 = 17$

$S_3 = \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix} = 4(-5) - 2(-1) - 2(-14)$
$= 10$

So, $\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$(\lambda-1)(\lambda^2 - 7\lambda + 10) = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-5) = 0$

Eigen values are $\lambda = 1, 2, 5$.

Largest Eigen value is $\lambda = 5$. Eigen vector corresponding to $\lambda = 5$ is

$[A - 5I]X = [0]$

$\left(\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) X = [0]$

$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Augmented Matrix

$\left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ -5 & -2 & 2 & 0 \\ -2 & 4 & -4 & 0 \end{array}\right]$

$R_2 - 5R_1$, $R_3 - 2R_1$

$\left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$-\frac{1}{12}R_2$

$\sim \left[\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Rewritting equation

$\left.\begin{array}{r} -x + 2y - 2z = 0 \\ y - z = 0 \end{array}\right\}$ z-free variable

Put $z = t$, so, $y = t$

$x = 0$

Eigen vector is $\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$.

As a homework, Try finding Eigen vectors for $\lambda = 1$ & 2?

c) Find the eigen vector corresponding to smallest eigen value for $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Sol^n: Chacteristic polynomial is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = 0$ $\quad S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$= -3$

$|A| = 0 - (1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= 2$

So, we get $\lambda^3 - 3\lambda - 2 = 0$.

Note that -1 is the root. So,

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & \downarrow & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \Rightarrow \begin{array}{l} (\lambda+1)(\lambda^2 - \lambda - 2) = 0 \\ (\lambda+1)(\lambda-2)(\lambda+1) = 0 \end{array}$$

Eigen values are $\lambda = -1, -1, 2$.

Eigen vector corresponding to $\lambda = -1$.

$[A - (-1)I]X = [0]$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 - R_1$, $R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A|0) = 1 < 3$

2 free variables i.e $(3-1=2)$

$x + y + z = 0$

Put $y = r$ $\quad z = s$

So, $x = -r - s$

Eigen vector is $\begin{bmatrix} -r-s \\ r \\ s \end{bmatrix}$

Put $r = s = 1$ we get particul[ar] vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Interesting facts about this particular example:

1. Repeated Eigen value.
2. Two variables
3. Matrix symmetric

$$\begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two vectors (-1,1,0) and (-1,0,1) are linearly independent.

What does it mean graphically?

Eigenvalue Properties

- Eigenvalues of $\mathbf{A}$ and $\mathbf{A}^T$ are equal
- Singular matrix has at least one zero eigenvalue
- Eigenvalues of $\mathbf{A}^{-1}$: $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$
- Eigenvalues of diagonal and triangular matrices are equal to the diagonal elements
- Trace

$$Tr(\mathbf{A}) = \sum_{j=1}^{n} \lambda_j$$

- Determinant

$$|\mathbf{A}| = \prod_{j=1}^{n} \lambda_j$$

Cayley-Hamilton Theorem

Statement: **Every square matrix satisfies its own characteristic equation.**

We know that characteristic equation of a square matrix is

$$|A - \lambda I| = 0$$

i.e. $a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

C-H theorem says that A satisfies above equation, which means if we replace λ by matrix A, we will get null matrix on RHS.

i.e. $a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = [0]$

Where I is identity matrix of size n

e.g. 1 Verify cayley-Hamilton theorem for matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

sol^n :- ch. eq^n is $\lambda^2 - 4\lambda - 5 = 0$

To verify the theorem, we should show

$$A^2 - 4A - 5I = [0]$$

$$L.H.S = A^2 - 4A - 5I$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-9 & 16-16 \\ 8-8 & 17-17 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **Applications of Cayley-Hamilton Theorem:**

1. Nth power of any square matrix can be expressed as linear combination of lower powers of A.
2. To find inverse of non-singular matrix A.

Ex. 1 : *Verify Cayley-Hamilton theorem for* $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ *and use it to find* A^4 *and* A^{-1}.

Sol. : Step 1 : The characteristic equation of A is $|A - \lambda I| = 0 \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0.$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 1 + 3 + 1 = 5$$

$$S_2 = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 3 + 1 + 5 = 9$$

$$|A| = 1(3) + 1(-2) = 1$$

Hence the characteristic equation of A is $\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$

Step 2 : By using Cayley-Hamilton theorem, we have $A^3 - 5A^2 + 9A - I = 0$ ... (1)

Step 3 : To verify Cayley-Hamilton theorem, we first obtain A^2 and A^3 as follows :

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

By using equation (1),

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley-Hamilton theorem is verified.

Step 4 : To find A^4 we multiply equation (1) by A.

$$A^4 - 5A^3 + 9A^2 - AI = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$= 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

Step 5 : To find A^{-1} we multiply equation (1) by A^{-1}.

$$A^2 - 5A + 9I - A^{-1}I = 0$$

$$A^{-1} = A^2 - 5A + 9I$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

e. g. **2 Verify Cayley-Hamilton theorem for** $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ **and use it to find the matrix**

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$$

Sol. : Step 1 : The characteristic equation of A is $|A-\lambda I|=0$ $\begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$

$$\lambda^3-S_1\lambda^2+S_2\lambda-|A|=0$$

where $S_1=2+1+2=5$

$$S_2=\begin{vmatrix}1&0\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\0&1\end{vmatrix}=2+3+2=7$$

$$|A|=2(2)-1(0)+1(-1)=3$$

Hence the characteristic equation of A is

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

Step 2 : By Cayley-Hamilton theorem, 'A' satisfies its own characteristic equation

i.e. we have $A^3-5A^2+7A-3I=0$ ... (1)

Step 3 : Verification : To verify

$$A^3-5A^2+7A-3I=0$$

Let $A^2=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$

$$A^3=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}=\begin{bmatrix}14&13&13\\0&1&0\\13&13&14\end{bmatrix}$$

$$A^3-5A^2+7A-3I=\begin{bmatrix}14&13&13\\0&1&0\\13&13&14\end{bmatrix}-5\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+7\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Hence Cayley–Hamilton theorem is verified.

Step 4 : Now the given expression can be written as

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5(A^3 - 5A^2 + 7A - 3) + A(A^3 - 5A^2 + 7A - 3) + A^2 + A + I$$

$$= A^2 + A + I \text{ [By using equation (1) } \because A^3 - 5A^2 + 7A - 3 = 0]$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Application of Matrices in 2D & 3D Transformations

Transformations

What are they?

- changing something to something else via rules
- mathematics: mapping between values in a range set and domain set (function/relation)
- geometric: translate, rotate, scale, shear,…

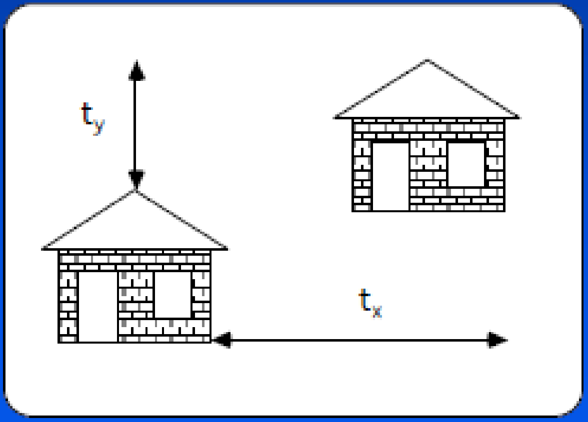
Why are they important to graphics?

- moving objects on screen / in space
- specifying the camera's view of a 3D scene
- mapping from model space to world space to camera space to screen space
- specifying parent/child relationships
- …

Application in 2D

Translation

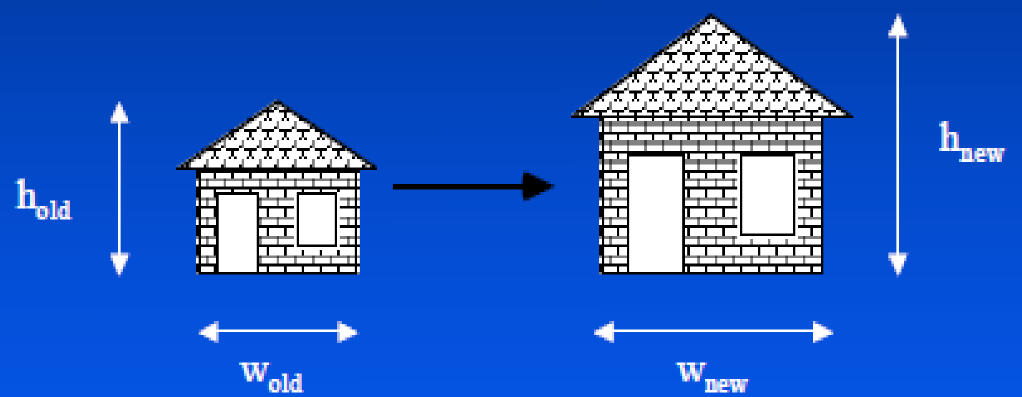
Moving an object is called a translation. We translate a point by adding to the x and y coordinates, respectively, the amount the point should be shifted in the x and y directions. <u>We translate an object by translating each vertex in the object.</u>



$$x_{new} = x_{old} + t_x;\ y_{new} = y_{old} + t_y$$

Scaling

Changing the size of an object is called a scale. We scale an object by scaling the x and y coordinates of each vertex in the object.



$s_x = w_{new}/w_{old}$ $\qquad$ $s_y = h_{new}/h_{old}$

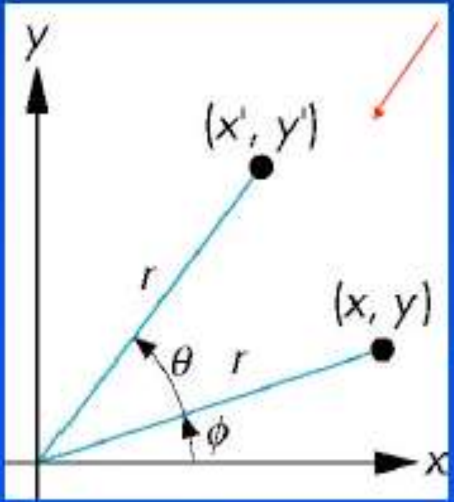
$x_{new} = s_x x_{old}$ $\qquad$ $y_{new} = s_y y_{old}$

Rotation about the origin

Consider rotation about the origin by Θ degrees

- radius stays the same, angle increases by Θ

$$x' = r\cos(\phi + \theta)$$
$$y' = r\sin(\phi + \theta)$$



$$x = r\cos\phi$$
$$y = r\sin\phi$$

Rotation about the origin (cont.)

From the double angle formulas:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Thus,

$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

Transformations as matrices

Scale:

$x_{new} = s_x x_{old}$

$y_{new} = s_y y_{old}$

$$\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x \cdot x \\ s_y \cdot y \end{pmatrix}$$

Rotation:

$x_2 = x_1 \cos\theta - y_1 \sin\theta$

$y_2 = x_1 \sin\theta + y_1 \cos\theta$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

Translation:

$x_{new} = x_{old} + t_x$

$y_{new} = y_{old} + t_y$

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}$$

Reflection about x-axis

Example 1 (A reflection). Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Take a generic point $\mathrm{x} = (x, y)$ in the plane, and write it as the column vector $\mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the matrix product $A\mathrm{x}$ is

$$A\mathrm{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Thus, the matrix A transforms the point (x, y) to the point $T(x, y) = (x, -y)$. You'll recognize this right away as a reflection across the x-axis.

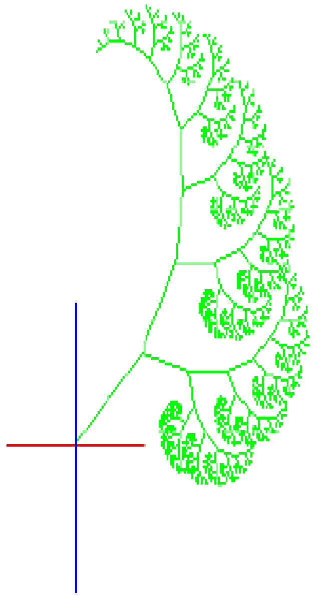


Figure 1: Basic leaf

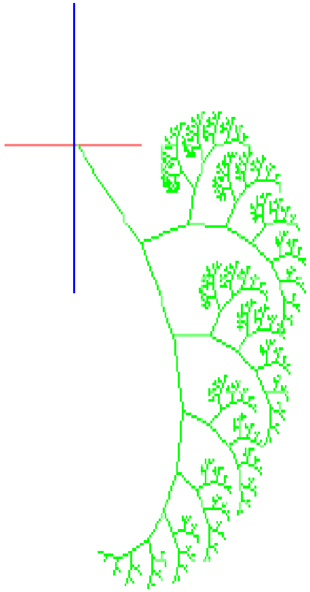
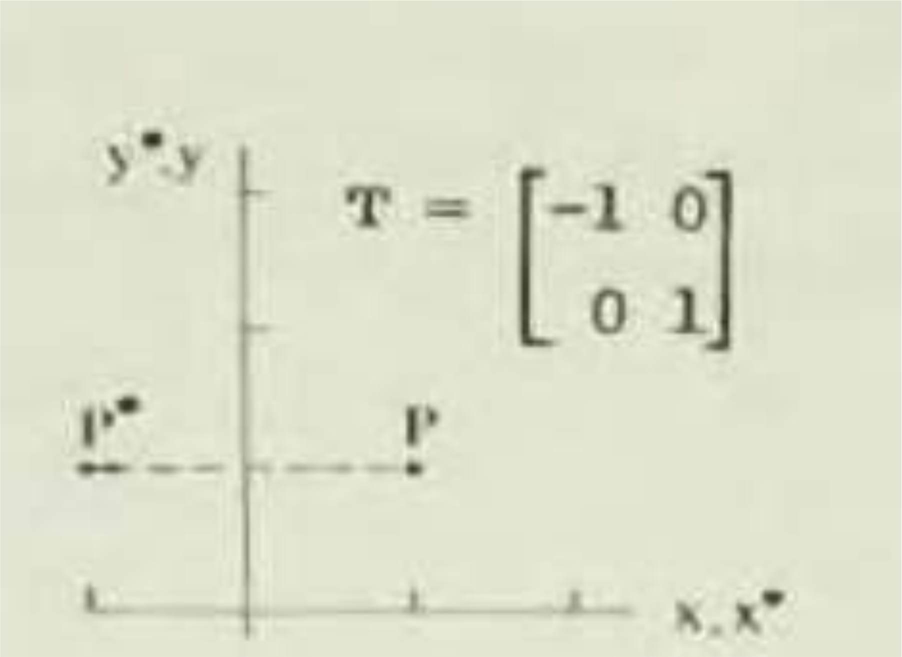


Figure 2: Reflected across x-axis

Reflection about Y-axis



Rotation in 3D:

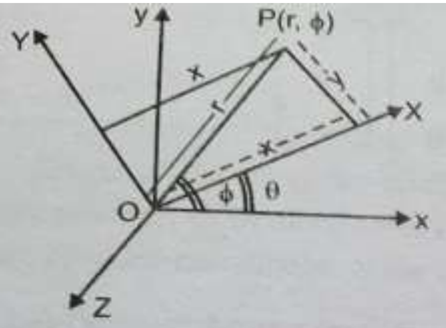


Fig. 3.3

Consider a point P whose coordinates w.r.t. ox, oy, oz are (x, y, z) and w.r.t. OX, OY, OZ are (X, Y, Z). Consider rotation taking place about the z-axis of the model coordinate system as shown in Fig. 3.3. It is clear that z coordinates in both systems will remain same.

Let (r, ϕ) be polar coordinates of the point P in xoy plane.

$$\therefore \quad x = r\cos\phi, \quad y = r\sin\phi \qquad \text{... (4)}$$

If angle of rotation of the axes is θ, then

$$X = r\cos(\phi - \theta), \quad Y = r\sin(\phi - \theta)$$

$$\therefore \quad X = r[\cos\phi\cos\theta + \sin\phi\sin\theta] \qquad \text{... (5)}$$

$$Y = r[\sin\phi\cos\theta - \cos\phi\sin\theta]$$

$$\therefore \quad X = (r\cos\phi)\cos\theta + (r\sin\phi)\sin\theta \qquad \text{... (6)}$$

$$Y = (r\sin\phi)\cos\theta - (r\cos\phi)\sin\theta$$

$$\therefore \quad X = x\cos\theta + y\sin\theta \qquad \text{... (7)}$$

$$Y = -x\sin\theta + y\cos\theta$$

$$Z = z \qquad \text{[As rotation is about z-axis]}$$

Relation (7) can be put in matrix form as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \dots (8)$$

Similarly, for rotation about x-axis, we shall obtain

$$\begin{aligned} X &= x \qquad \dots (9) \\ Y &= y\cos\theta + z\sin\theta \\ Z &= -y\sin\theta + z\cos\theta \end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \ldots (10)$$

And rotation about y-axis will give

$$\begin{aligned} Y &= y \\ X &= -z\sin\theta + x\cos\theta \\ Z &= z\cos\theta + x\sin\theta \end{aligned} \qquad \ldots (11)$$

or, in matrix form,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \ldots (12)$$

NOTE

- Let P(x,y,z) be coordinates of given point. If origin is shifted to (u,v,w). Let Z-axis be the axis of rotation. Then co-ordinates of P (X,Y,Z) in NEW COORDINATE SYSTEM is:

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta & 0 & -u \\ -sin\theta & cos\theta & 0 & -v \\ 0 & 0 & 1 & -w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- If axis of rotation is changed then only rotation matrix (first three columns) in above expression will change. Everything else remains same.

Examples:

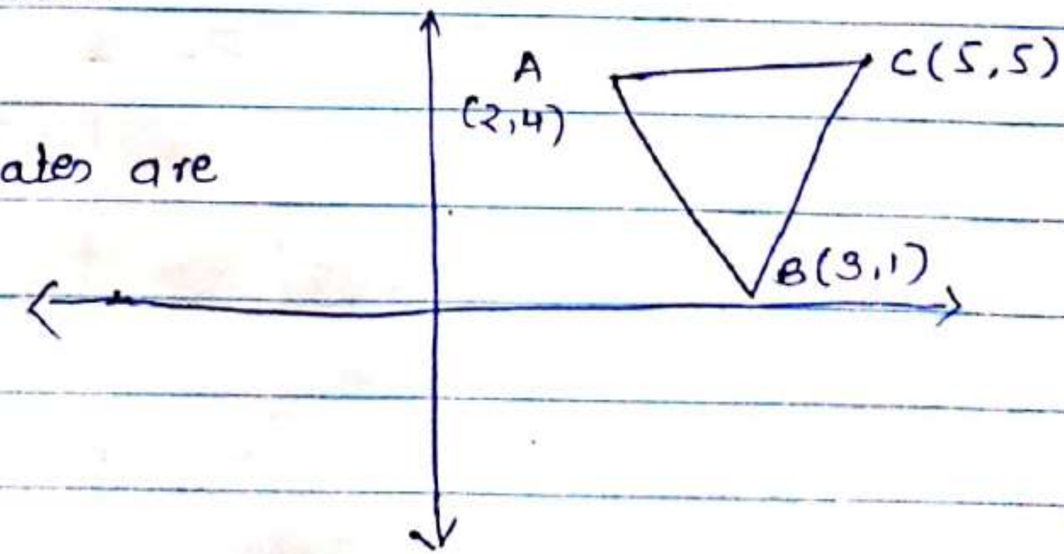
① * Find the reflection of the vectors $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ along X - axis

→ We know that, Reflection along X-axis is given by the matrix $\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\therefore \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

2 Translate the triangle 4 units left & 5 units down.



→ New coordinates are

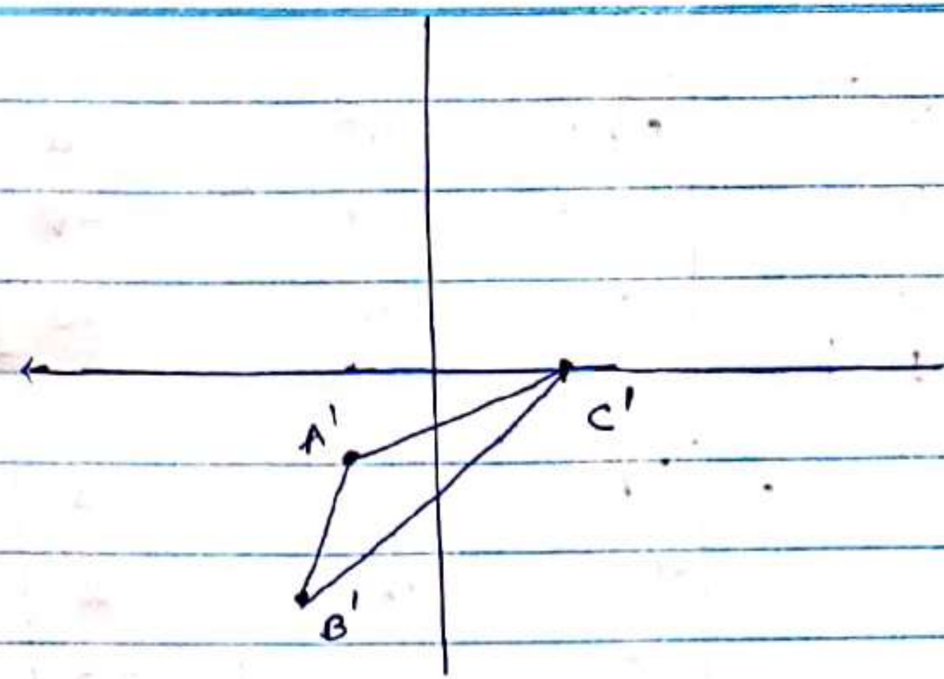
$A'(2-4, 4-5)$

$= A'(-2, -1)$

$B'(3-4, 1-5) = B'(-1, -4)$

$C'(5-4, 5-5) = C'(1, 0)$

Sol^n -



3) Rotate the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ by an angle 45° anticlockwise & find resulting vector.

$\rightarrow$ We know that rotation matrix is given by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Her $\theta = 45^\circ$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

4) coordinates of point P are (50, 50, 50). Origin is shifted to the point (5, -2, 3). Rotation is about z-axis through 45°. Find the coordinates of P in new coordinate system.

→ As the rotation is about z-axis & translation is (u, v, w), rotation & translation together in matrix form is given by

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & -u \\ -\sin\theta & \cos\theta & 0 & -v \\ 0 & 0 & 1 & -w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & -5 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & -5 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 50 \\ 50 \\ 50 \\ 1 \end{bmatrix}$$

$$X = \frac{100}{\sqrt{2}} - 5 = 65.71$$

$$Y = 0 + 2 = 2$$

$$Z = 50 - 3 = 47$$

5) centre of arc of the circle is (10, 10, 10)
Origin is (0, 0, 0). Rotation is about X-axis through an angle 60°. Find the centre of arc of the circle in new coordinate system

→ As the rotation is about x-axis, thus

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\theta = 60^\circ$, $x = 10$, $y = 10$, $z = 10$.

$$\therefore \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

$\therefore$ $X = 10$, $Y = 13.66$, $Z = -3.66$.

Exercises:

① verify cayley-Hamilton theorem for the following matrix A & use it to find A^{-1}.

a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

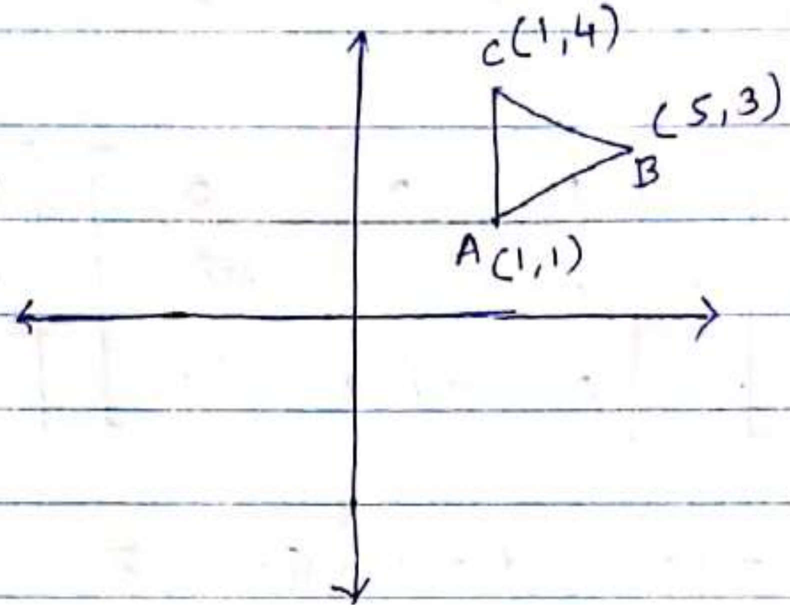
b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

2) Find the characteristic eq^n for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

& use it to find the simplified expression for $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$

Ans: $919A + 942I$

3)



a) Rotate triangle 90^{0} clockwise

b) Rotate triangle 90^{0} counterclockwise

c) Take its reflection around X-axis

d) Take its reflection around Y-axis

e) Translate the triangle 6 units right & 5 units down.

4) Centre of the arc of the circle in a given coordinate system is (100, 100, 100). origin is shifted to the point (−10, −5, 2). Rotation is carried out about Y-axis through an angle of 30^{0}. Find the centre of the arc of the circle in new coordinate system.

Ans. (46·66, 105, 134·66)

5) Centre of arc of the circle in a given coordinate system is (100, 100, 100). Origin is (0,0,0) Axis of rotation is Y-axis. Angle of rotation is 30°. Find the centre of arc of the circle in new coordinate system.