

Successive differentiation is the process of differentiating a given function successively & result of such differentiation are called successive derivatives.

The higher order differential coefficients are of utmost importance in scientific & engineering applications.

* If y is function of x i.e. $y = f(x)$ then successive derivatives denoted by $f'(x), f''(x), \dots, f^{(n)}(x)$ or $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$ or $y', y'', y''', \dots, y^{(n)}$ or $y_1, y_2, y_3, \dots, y_n$

~~Note~~

* n^{th} derivative of some std functions:-

1) If $y = e^{ax}$

$$y_1 = ae^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

:

$$y_n = a^n e^{ax}$$

i.e. If $y = e^{ax}$ then $y_n = a^n e^{ax}$ — (1)

2) If $y = a^x$

$$y_1 = \log a (a^x)$$

$$y_2 = a^x (\log a)^2$$

$$y_n = a^n (\log a)^n$$

i.e. $\boxed{\text{If } y = a^x \text{ then } y_n = a^n (\log a)^n}$ — (2)

3) If $y = (ax+b)^m$, where m is any real number.

$$y = (ax+b)^m$$

$$y_1 = m(ax+b)^{m-1} a$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

$$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$$

$$\vdots \\ y_n = m(m-1)(m-2)\dots(m-(n-1))a^n(ax+b)^{m-n}$$

\therefore i.e. $\boxed{\text{If } y = (ax+b)^m, \text{ then } y_n = m(m-1)(m-2)\dots(m-n+1)a^n(ax+b)^{m-n}}$ — (3)

4) If $y = (ax+b)^m$, where m is any +ve integer.

$$\therefore y = (ax+b)^m$$

$$y_1 = -ma(ax+b)^{-m-1} = (-1)^m a$$

$$y_2 = -m(-m-1)a^2(ax+b)^{-m-2}$$

$$= (-1)^2 m(m+1)a^2(ax+b)^{-m-2}$$

$$y_3 = (-1)^3 m(m+1)(m+2)a^3(ax+b)^{-m-3}$$

$$\vdots \\ y_n = (-1)^3 m(m+1)(m+2)\dots(m+(n-1))a^n(ax+b)^{-m-n}$$

i.e. $\boxed{\text{If } y = \frac{1}{(ax+b)^m} \text{ where } m \text{ is +ve integer}} \\ \text{then } y_n = \frac{(-1)^n m(m+1)(m+2)\dots(m+(n-1))a^n}{(ax+b)^{m+n}}$ — (4)

Special case :-

① If $y = (ax+b)^m$, where m is +ve integer & $m > n$.
then $y_n = m(m-1)(m-2) \dots (m-n+1) a^n (ax+b)^{m-n}$.

This can be written as,

$$y_n = \frac{m(m-1)(m-2) \dots (m-n+1)(m-n)(m-n-1) \dots 3 \cdot 2 \cdot 1}{a^n (ax+b)^{m-n}}$$

$$\frac{(m-n)(m-n-1)(m-n-2) \dots 3 \cdot 2 \cdot 1}{(m-n)(m-n-1)(m-n-2) \dots 3 \cdot 2 \cdot 1}$$

$$\therefore \boxed{y_n = \frac{m! a^n (ax+b)^{m-n}}{(m-n)!}} \quad \text{--- } \textcircled{*}$$

i.e If $y = (ax+b)^m$ & m is +ve integer, $m > n$

$$\text{then } y_n = \frac{m! a^n (ax+b)^{m-n}}{(m-n)!}$$

② If $m=n$.

$$\text{then } y_n = \frac{n! a^n (ax+b)^{n-n}}{(n-n)!}$$

$$\text{i.e } \boxed{y_n = n! a^n}$$

③ If $m < n$. $\frac{d^n}{dx^n} (ax+b)^m = 0$

If $y = x^n$ then $y_n = n!$ but $y_{n-1} \neq (n-1)!$

To find y_{n-1} , in $\textcircled{*}$ put $m=n$, $n=n-1$, $a=1$, $b=0$.

$$y_{n-1} = \frac{n! a^{n-1} x^{n-(n-1)}}{(n-n+1)!}$$

$$\text{i.e. } y_{n-1} = \frac{n! x}{1} \text{ & if } \boxed{y = x^n \text{ then } y_{n-1} = n! x}$$

\Rightarrow If $y = \log(ax+b)$

$$y_1 = \frac{a}{ax+b}$$

case iv) If $y = (ax+b)^{-1}$ i.e. $y = \frac{1}{ax+b}$

In formula ④ put $m=1$

then $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

∴ i.e. $\boxed{\text{If } y = \frac{1}{ax+b} \text{ then } y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}}$ — (*)

5) $y = \log(ax+b)$

$$y_1 = \frac{a}{ax+b}$$

Take $(n-1)^{\text{th}}$ derivative of y_1 ,

$$\therefore \cancel{y_n = a \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}} \quad . \text{ by } \cancel{\text{(*)}} \quad y_n = \frac{a (-1)^{n-1} (n-1)! a^{n-1}}{(ax+b)^{n-1+1}}$$

i.e. $\boxed{\text{If } y = \log(ax+b)}$
then $y_n = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$

6) $y = \sin(bx+c)$

$$y_1 = b \cos(bx+c) = b \sin(bx+c + \pi/2)$$

$$y_2 = b^2 \cos(bx+c + \pi/2) = b^2 \sin(bx+c + \frac{2\pi}{2})$$

i.e. $\boxed{y_n = b^n \sin(bx+c + \frac{n\pi}{2})}$

i.e If $y = \sin(bx+c)$ then $y_n = b^n \sin(bx+c + \frac{n\pi}{2})$

2) similarly if $y = \cos(bx+c)$
then $y_n = b^n \cos(bx+c + \frac{n\pi}{2})$] \rightarrow prove

7) If $y = e^{ax} \sin(bx+c)$

then $y_n = r^n e^{ax} \sin(bx+c+n\theta)$

where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1} \frac{b}{a}$.

8) If $y = e^{ax} \cos(bx+c)$

then $y_n = r^n e^{ax} \cos(bx+c+n\theta)$

where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1} \frac{b}{a}$.

y	y_n
1) e^{ax}	$a^n e^{ax}$
2) a^x	$a^x (\log a)^n$
3) $(ax+b)^m$, m is any real number	$m(m-1)(m-2)\dots(m-n+1)a^n(ax+b)^{m-n}$
4) $\frac{1}{(ax+b)^m}$, m is +ve integer	$\frac{(-1)^n m(m+1)(m+2)\dots(m+n-1)a^n}{(ax+b)^{m+n}}$
5) $(ax+b)^m$, m is +ve integer $m > n$.	$\frac{m! a^n (ax+b)^{m-n}}{(m-n)!}$
6) $(ax+b)^m$, $m = n$.	$n! a^n$
7) $(ax+b)^m$, $m < n$	0

y	y_n
$\frac{1}{ax+b}$	$\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$
$\log(ax+b)$	$\frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$
$\sin(bx+c)$	$b^n \sin(bx+c + \frac{n\pi}{2})$
$\cos(bx+c)$	$b^n \cos(bx+c + \frac{n\pi}{2})$
$e^{ax} \sin(bx+c)$	$r^n e^{ax} \sin(bx+c + n\theta)$ where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1} \frac{b}{a}$

Ex. Q) Find n th derivative of

$$1) y = e^{5x}, 2) \sin 2x \cdot \cos 3x \quad 3) \cos^4 x \quad 4) a^{2x}$$

$$\rightarrow ① \quad y_n = 5^n e^{5x} \rightarrow \text{formula}$$

$$② \quad y = \sin 2x \cdot \cos 3x$$

$$= \frac{1}{2} [2 \cos 3x \sin 2x]$$

$$= \frac{1}{2} [\sin 5x - \sin x]$$

$$\therefore y_n = \frac{1}{2} \left[5^n \sin(5x + \frac{n\pi}{2}) - \sin(x + \frac{n\pi}{2}) \right]$$

$$③ \quad \cos^4 x = \left(\frac{1 + \cos 2x}{2} \right)^2$$

$$= \frac{1}{4} [1 + 2 \cos 2x + \cos^2 2x]$$

$$= \frac{1}{4} [1 + 2 \cos 2x + \frac{1 + \cos 4x}{2}]$$

$$= \frac{1}{8} [3 + 4 \cos 2x + \cos 4x]$$

$$\therefore y_n = \frac{1}{8} \left[4 \cos(2x + \frac{n\pi}{2}) + \cos(4x + \frac{n\pi}{2}) \right]$$

example: find 100th derivative of $y = \sin 2x$

$$y_{100} = 2^{100} \sin(2x + \frac{100\pi}{2}) \\ = 2^{100} \sin(2x + 50\pi)$$

$$\boxed{y_{100} = 2^{100} \sin 2x}$$

example ① If $y = (3x+2)^{99}$, then $y_{100} = ?$. $y_{99} = ?$.

example If $y = \log(3x+4)$ then $y_n = ?$.

$$y_n = \frac{(-1)^{n-1}(n-1)! 3^n}{(3x+4)^n}$$

example ② Find n^{th} derivative $\frac{1}{(x+1)(x+2)}$.

$$\therefore y = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$y = \frac{1}{x+1} - \frac{1}{x+2}$$

∴ By formula $y = \frac{1}{ax+b}$, $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

$$\boxed{y_n = \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}}$$

② $y = \frac{1}{1-5x+6x^2}$ then find y_n .

$$\therefore y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$

$$\begin{aligned} \therefore y &= \frac{A}{2x-1} + \frac{B}{3x-1} \\ &= \frac{2}{2x-1} - \frac{3}{3x-1} \quad \text{formula} \end{aligned}$$

$$\therefore y_n = \frac{2(-1)^n n! 2^n}{(2x-1)^{n+1}} - \frac{3(-1)^n n! 3^n}{(3x-1)^{n+1}}$$

Note: If degree of numerator is greater than denominator the take actual division then use partial fraction method.

$$\text{eg. if } y = \frac{x^3}{(x-1)(x-2)}$$

Here degree of numerator is greater than denominator. Take actual division

$$\therefore y = \frac{x^3}{(x-1)(x-2)} = \frac{x^3}{x^2 - 3x + 2}$$

$$\begin{aligned} & \frac{x^3}{x^2 - 3x + 2} - \frac{x^3}{x^2 - 3x + 2} \\ &= \frac{x^3 - x^3 + 3x^2 - 2x}{x^2 - 3x + 2} \\ &= \frac{3x^2 - 2x}{x^2 - 3x + 2} \\ &= \frac{3x^2 - 9x + 6}{x^2 - 3x + 2} \\ &= \frac{7x - 6}{x^2 - 3x + 2} \end{aligned}$$

$$\therefore \textcircled{1} \quad x^3 = (x^2 - 3x + 2)(x+3)(x-6)$$

$$y = \frac{x^3}{(x-1)(x-2)} = x+3 + \frac{7x-6}{(x-1)(x-2)} \quad \text{--- } \textcircled{1}$$

$$\text{Now } \frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$= \frac{-1}{x-1} + \frac{8}{x-2}$$

$$\therefore y = x+3 - \frac{1}{x-1} + \frac{8}{x-2}$$

$$\therefore y_n = -\left[\frac{(-1)^n n!}{(x-1)^{n+1}} \right] + \frac{8(-1)^n n!}{(x-2)^{n+1}}$$

HW: find nth derivative of $\frac{x^3}{x-1}$

$$\text{eg. } y = \frac{1}{(x-1)^2(x-2)} \quad \text{find } y_n = ?$$

$$y = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{1}{x-2} - \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$$

or by formula

$$y_n = \frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{(-1)^n (n+1)!}{(x-1)^{n+2}}$$

$$\text{eg. find } y_n \text{ of } y = \frac{2x+3}{5x+7}$$

$$= \frac{2}{5} \left[\frac{5x + \frac{15}{2}}{5x+7} \right]$$

$$= \frac{2}{5} \left[\frac{5x+7 - 7 + \frac{15}{2}}{5x+7} \right]$$

$$= \frac{2}{5} \left[1 + \frac{1}{2(5x+7)} \right]$$

$$y = \frac{2}{5} + \frac{1}{5(5x+7)}$$

$$\boxed{y_n = \frac{1}{5} \frac{(-1)^n n! 5^n}{(5x+7)^{n+1}}}$$

* Find n th derivative of $y = \frac{1}{x^2 + a^2}$

$$\begin{aligned} \rightarrow y &= \frac{1}{x^2 + a^2} \\ &= \frac{1}{(x+ia)(x-ia)} \\ &= \frac{A}{x+ia} + \frac{B}{x-ia} \\ &= \frac{1}{2ia} \left[\frac{1}{x-ia} - \frac{1}{x+ia} \right] \\ y_n &= \frac{1}{2ia} \left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}} \right] \\ &= \frac{(-1)^n n!}{2ia} \left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}} \right] \end{aligned}$$

Put $x+ia = r [\cos\theta + i\sin\theta]$, $x-ia = r [\cos\theta - i\sin\theta]$

$$\therefore x = r\cos\theta, a = r\sin\theta \text{ & } \theta = \tan^{-1} \frac{a}{x}.$$

$$\begin{aligned} \therefore y_n &= \frac{(-1)^n n!}{2ia r^{n+1}} \left[\frac{1}{(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{(\cos\theta + i\sin\theta)^{n+1}} \right] \\ &= \frac{(-1)^n n!}{2ia r^{n+1}} \left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta} \right] \\ &= \frac{(-1)^n n!}{2ia r^{n+1}} [2i\sin(n+1)\theta] \\ &= \frac{(-1)^n n! \sin(n+1)\theta}{a \left(\frac{a^{n+1}}{\sin^{n+1}\theta} \right)} \quad \left[\because a = r\sin\theta \therefore r = \frac{a}{\sin\theta} \right] \end{aligned}$$

$$y_n = \frac{(-1)^n n! \sin(n+1)\theta \sin^{n+1}\theta}{a^{n+2}} \quad \text{where } \theta = \tan^{-1} \frac{a}{x}$$

$$\text{i.e. If } y = \frac{1}{x^2 + a^2} \text{ then } y_n = \frac{(-1)^n n! \sin(n+1)\theta \sin^{n+1}\theta}{a^{n+2}}$$

formula :- ~~If~~ $y = \tan^{-1} \frac{x}{a}$ then

$$y_1 = \frac{a}{x^2 + a^2}$$

$$y_1 = a \left(\frac{1}{x^2 + a^2} \right)$$

Differentiating $(n-1)$ times w.r.t x .

$$\therefore y_n = a \left[\frac{(-1)^{n-1} (n-1)! \sin n\theta \sin^n \theta}{a^{n+1}} \right] \text{ where } \theta = \tan^{-1} \frac{a}{x}$$

∴ If $y = \tan^{-1} \frac{x}{a}$ then $y_n = \frac{(-1)^{n-1} (n-1)! \sin n\theta \sin^n \theta}{a^n}$
where $\theta = \tan^{-1} \frac{a}{x}$

———— ***

eg. ① If $y = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$ then find y_n .

$$\rightarrow y = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right) \text{ put } x = \tan \phi$$

$$= \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \phi} - 1}{\tan \phi} \right)$$

$$= \tan^{-1} \left(\frac{\sec \phi - 1}{\tan \phi} \right) = \tan^{-1} \left(\frac{1 - \cos \phi}{\sin \phi} \right)$$

$$= \tan^{-1} \left(\frac{2 \sin^2 \phi / 2}{2 \sin \phi / 2 \cos \phi / 2} \right)$$

$$= \tan^{-1} (\tan \phi / 2)$$

$$= \phi / 2$$

$$y = \frac{\tan^{-1} x}{2} \therefore \text{by } ***$$

$$\therefore y_n = \frac{1}{2} (-1)^{n-1} (n-1)! \sin n\theta \sin^n \theta \text{ where } \theta = \tan^{-1} \frac{1}{x}.$$

2) find n^{th} derivative of $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

$$\therefore y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$$

$$= \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

$$\text{put } x = \tan \phi$$

$$\therefore y = \cos^{-1}\left(\frac{\tan^2 \phi - 1}{\tan^2 \phi + 1}\right)$$

$$= \cos^{-1}(-\cos 2\phi)$$

$$= \cos^{-1}(\cos(\pi + 2\phi))$$

$$= \pi + 2\phi$$

$$\therefore y = \pi + 2\tan^{-1}x.$$

$$\therefore y_n = 2(-1)^{n-1}(n-1)! \sin^n \theta \sin n\theta, \text{ where } \theta = \tan^{-1} \frac{1}{x}$$

HW:- * find n^{th} derivative of

$$\textcircled{1} \quad y = \tan^{-1}\left(\frac{2x}{1+x^2}\right), \textcircled{2} \quad \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

$$\textcircled{3} \quad y = \sin^{-1}\left(\frac{2x}{1+x^2}\right).$$

* Leibnitz's theorem:-

This is useful to find n^{th} derivative of product of two functions.

Statement: If $y = uv$ where u & v are functions of x then n^{th} derivative

$$y_n = {}^n C_0 u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + {}^n C_n u v_n$$

where suffixes denote the order of the derivatives. & ${}^n C_0, {}^n C_1, {}^n C_2, \dots, {}^n C_n$ are Binomial coefficients.

Ex.

Note:- The choice of u & v is important. That function should be chosen as v whose successive derivative vanish earlier & that function should be taken as u whose n^{th} derivative is known.

Eg. find n^{th} derivative of

$$\textcircled{1} \quad y = x^2 e^{ax} \quad \textcircled{2} \quad x^3 \cos x \quad \textcircled{3} \quad e^x (2x+3)^2$$

$$\textcircled{1} \quad y = x^2 e^{ax}$$

$$u = e^{ax}, \quad v = x^2$$

$$u_n = a^n e^{ax}, \quad v_1 = 2x$$

$$u_{n-1} = a^{n-1} e^{ax}, \quad v_2 = 2$$

$$u_{n-2} = a^{n-2} e^{ax}, \quad v_3 = 0$$

$$\therefore y_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + u v_n$$

$$\therefore y_n = a^n e^{ax} (x^2) + n(a^{n-1} e^{ax}) 2x + \frac{n(n-1)}{2!} (a^{n-2} e^{ax})(2)$$

$$y_n = a^n e^{ax} \left[x^2 + \frac{2xn}{a} + \frac{n(n-1)}{a^2} \right]$$

④ $y = x^3 \cos x$

$$u = \cos x, \quad v = x^3$$

$$u_n = \cos\left(x + \frac{n\pi}{2}\right), \quad v_1 = 3x^2$$

$$u_{n-1} = \cos\left(x + \frac{(n-1)\pi}{2}\right), \quad v_2 = 6x$$

$$u_{n-2} = \cos\left(x + \frac{(n-2)\pi}{2}\right), \quad v_3 = 6.$$

$$\therefore y_n = u_n v_1 + n u_{n-1} v_2 + \frac{n(n-1)}{2!} u_{n-2} v_3 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3$$

$$y_n = \cos\left(x + \frac{n\pi}{2}\right)x^3 + n \cos\left(x + \frac{(n-1)\pi}{2}\right)3x^2 \\ + \frac{n(n-1)}{2!} \cos\left(x + \frac{(n-2)\pi}{2}\right)(6x) + \frac{n(n-1)(n-2)}{3!} \cos\left(x + \frac{(n-3)\pi}{2}\right)$$

(6).

$$\therefore y_n = x^3 \cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 \cos\left(x + \frac{(n-1)\pi}{2}\right) + 3n(n-1)x \cos\left(x + \frac{(n-2)\pi}{2}\right) \\ + n(n-1)(n-2) \cos\left(x + \frac{(n-3)\pi}{2}\right).$$

e.g. ③ If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ then show that

$$y_2(1-x^2) - 3xy_1 - y = 0 \text{ & hence prove that}$$

$$(1-x^2)y_{n+2} - (2n+3)x y_{n+1} - (n+1)^2 y_n = 0.$$

$$\therefore y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2} y = \sin^{-1}x$$

$$\therefore (1-x^2)y^2 = (\sin^{-1}x)^2 \quad \dots \textcircled{1}$$

Differentiating both sides w.r.t x .

$$\therefore (1-x^2)2yy_1 - 2xy^2 = 2\sin^{-1}x \left(\frac{1}{\sqrt{1-x^2}} \right)$$

$$\therefore (1-x^2)y_1 - xy = 1.$$

$$\therefore (1-x^2)y_1 - xy = 0 \quad \dots \textcircled{2}$$

Differentiating $\textcircled{2}$ w.r.t x .

$$\therefore (1-x^2)y_2 - 2xy_1 - xy_1 - y_{11} = 0$$

$$\therefore \boxed{(1-x^2)y_2 - 3xy_1 - y_{11} = 0} \quad \dots \textcircled{3}$$

Differentiate $\textcircled{3}$ n times by Leibnitz thm.

$$(1-x^2)y_{n+2} - n2xy_{n+1} + \frac{n(n-1)}{2!}(-2)y_n$$

$$- 3[xy_{n+1} + ny_n] - y_n = 0.$$

$$\therefore (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n^2 - n + 3n + 1)y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n^2 + 2n + 1)y_n = 0.$$

$$\therefore \boxed{(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0} \quad \text{is}$$

required result.

② If $y = e^{as\sin^{-1}x}$ then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0.$$

$$y = e^{ax \sin^{-1} x}$$

$$y_1 = \frac{ae^{\sin^{-1} x}}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$\therefore y_1 \sqrt{1-x^2} = ay$$

squaring both sides

$$\therefore y_1^2(1-x^2) = a^2 y^2$$

Differentiating once again

$$\therefore 2y_1 y_2 (1-x^2) - 2xy_1^2 = 2a^2 y y_1$$

$$\therefore y_2 (1-x^2) - xy_1 = a^2 y$$

Differentiating n times by using Leibnitz thm.

$$\therefore \left[(1-x^2)y_{n+2} - 2nx y_{n+1} + \frac{n(n-1)}{2} y_{n-2} \right] \\ - [y_{n+1}x + ny_n] = a^2 y_n = 0$$

$$\boxed{(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2) y_n = 0}$$

is required result.

H.W : - ① If $y = \cos(m \log x)$, show that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (m^2+n^2) y_n = 0$$

② If $x = \sin t$, $y = \sin at$ then show that

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2-a^2) y_n = 0$$

③ Find 4th derivative of $x^2 \sin 3x$.

④ If $x = \tan(\log y)$, prove that

$$(1+x^2)y_{n+1} + (2nx+1)y_n + n(n-1)y_{n-1} = 0$$

Taylor's and Maclaurin's theorems:-

» Maclaurin's thm:-

Let $f(x)$ be a function of x which can be expanded in ascending powers of x (i.e power series in x)

If $f(x)$ is term by term differentiable then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)$$

is called Maclaurin's series.

* Expansion of some standard functions by using Maclaurin's theorem.

1) $f(x) = \sin x$; $f(0) = 0$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x, \quad f''(0) = 0$$

$$f'''(x) = -\cos x, \quad f'''(0) = -1$$

$$f^{(iv)}(x) = \sin x, \quad f^{(iv)}(0) = 0 \quad \dots \dots \dots$$

Substituting this value in Maclaurin's series then we get series for $\sin x$.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \dots$$

$$\therefore \sin x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \dots$$

∴ Expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots$$

Similarly we can find expansion of $\cos x$

$$2) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$3) \tan x = x + \frac{x^3}{3!} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$$

$$4) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$5) \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$6) \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$7) \tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

$$8) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$9) \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

Differentiate ⑧ w.r.t x then

$$10) \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

Differentiate ⑨ w.r.t x then

$$11) \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$12) (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

is called Binomial expansion.

This all are standard formulae

example :-

1) Expand $\sqrt{1 + \sin x}$ upto x^6 .

$$y = \sqrt{1 + \sin x}$$

$$= \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \sqrt{(\cos \frac{x}{2} + \sin \frac{x}{2})^2}$$

$$= \cos \frac{x}{2} + \sin \frac{x}{2}$$

$$= \left(1 - \frac{1}{2!} \left(\frac{x}{2} \right)^2 + \frac{1}{4!} \left(\frac{x}{2} \right)^4 \dots \right) + \left(\left(\frac{x}{2} \right) - \frac{1}{3!} \left(\frac{x}{2} \right)^3 + \frac{1}{5!} \left(\frac{x}{2} \right)^5 \right)$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080}$$

2) Expand $\log(1+x+x^2+x^3)$ upto a term in x^8

$$f(x) = \log(1+x+x^2+x^3)$$

$$= \log[(1+x)+x^2(1+x)]$$

$$= \log[(1+x)(1+x^2)]$$

$$= \log(1+x) + \log(1+x^2)$$

we have std formula for $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\therefore \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots \right) + \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} \right)$$

$$= x - \frac{x^2}{2} + x^2 + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^4}{2} + \frac{x^5}{5} + \frac{x^6}{3} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{4} - \frac{x^8}{8} + \dots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{1}{5}x^5 + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \dots$$

3) Expand $(1+x)^n$ in ascending powers of x ,
expansion being correct upto fifth power.

$$\rightarrow y = (1+x)^n$$

$$\therefore \log y = x \log(1+x)$$

$$= x \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right)$$

$$= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots$$

$$= z \text{ (say)}$$

$$\therefore \cancel{y = e^z}$$

$$\therefore y = e^z$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots \right) + \frac{1}{2} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \dots \right)^2$$

$$+ \frac{1}{6} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \dots \right)$$

neglect x^6 & higher powers of x , we get

$$\therefore y = 1 + x^2 - \frac{1}{2}x^3 + \frac{x^4}{3} - \frac{x^5}{4} + \frac{1}{2}(x^4 - x^5) + \dots$$

$$y = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \dots$$

example :- prove that

4) $\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots}$

$$\text{Let } y = \tan^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \text{We know } \frac{1}{1+x} = 1-x+x^2-x^3+x^4-\dots$$

$$\frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

Integrating both side w.r.t x

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

i.e $\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots}$. This is

standard formula.

If similarly we can find expansion of

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots$$

$$\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \dots \right]$$

example ① :

$$\text{Prove that } \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right)$$

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\text{put } x = \tan \theta$$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 \theta} - 1}{\tan \theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec \theta - 1}{\tan \theta}\right)$$

$$= \tan^{-1}\left(\frac{1 - \cos \theta}{\sin \theta}\right)$$

$$= \tan^{-1}\left(\frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}\right) = \tan^{-1}(\tan \theta/2) = \frac{\theta}{2}$$

$$= \frac{1}{2} \tan^{-1} x$$

$$= \frac{1}{2} \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right] \text{ By standard expansion of } \tan^{-1} x$$

(2) Expand $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$ in ascending powers of x .

$$y = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

$$\text{put } x = \tan \theta$$

$$\therefore y = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right)$$

$$y = \sin^{-1} (\sin 2\theta)$$

$$= 2\theta$$

$$= 2 \tan^{-1} x$$

$$\boxed{y = 2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots \right)}$$

(3) HW: Expand $\cos^{-1} \left(\frac{x-x^{-1}}{x+x^{-1}} \right)$ in ascending powers of x .

(4) Expand $\cos^{-1} (4x^3 - 3x)$ in ascending powers of x .
(Hint put $x = \cos \theta$)

(5) Expand $\sin^{-1} (3x - 4x^3)$ in ascending powers of x .
(Hint put $x = \sin \theta$)

⑥ Expand $\sec x$ by Maclaurin's theorem as far as the terms upto x^4 .

$$\rightarrow f(x) = \sec x, f(0) = 1$$

$$f'(x) = \sec x \cdot \tan x, f'(0) = 0$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x, f''(0) = 1$$

$$f'''(x) = 3 \sec^2 x \tan x + \sec x \tan^3 x + 2 \tan x \sec^3 x \\ \therefore f'''(0) = 0$$

$$f''(x) = 3 [\sec^5 x + \tan^2 x \cdot 3 \sec^3 x] + \sec^3 x \cdot 3 \tan^2 x$$

$$+ \tan^4 x \sec x + 2 [\tan^2 x \cdot 3 \sec^3 x + \sec^5 x]$$

$$\therefore f''(0) = 5 \dots$$

Put the values of $f(0), f'(0), f''(0), f'''(0) \dots$
in Maclaurin's theorem.

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \frac{x^5}{5!} f^{(5)}(0) \\ = 1 + x(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(5) + \dots$$

$$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{4!} + \dots$$

H.W: Expand by Maclaurin's theorem.

- ① ~~$\log(1+x)$~~ $e^x \sec x$
- ② $e^x \sin x$
- ③ $\log \sec x$

Taylor's Theorem:-

Statement:- Let $f(x+h)$ be a function, which can be expanded in powers of h

i.e
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \dots$$

Deductions: 1) Replace x by h & h by x then we get

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots + \frac{x^n}{n!} f^{(n)}(h) + \dots$$

is expansion in powers of x .

2) we can write $x = x-a+a$ then

$$f(x) = f(x-a+a)$$

put $y = x-a$ & $h = a$ then

~~$f(x) = f(y+h) = f(h) + yf'(h) + \frac{y^2}{2!} f''(h) + \frac{y^3}{3!} f'''(h) + \dots + \frac{y^n}{n!} f^{(n)}(h) + \dots$~~

i.e
$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots$$

is expansion in powers of $x-a$

3) In Deduction ① put $h=0$ then we get
Maclaurin's series

i.e
$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

eg. ① Using Taylor's theorem, express ~~$(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$~~ in powers of x .

Let $f(x+h) = (x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$, where $h = -2$.

by Taylor's thm,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{(iv)}(h) + \dots$$

$$\text{Now } f(h) = h^4 - 3h^3 + 4h^2 + 5, \quad f(-2) = 61$$

$$f'(h) = 4h^3 - 9h^2 + 8h, \quad f'(-2) = -84$$

$$f''(h) = 12h^2 - 18h + 8, \quad f''(-2) = 92$$

$$f'''(h) = 24h - 18, \quad f'''(-2) = -66$$

$$f^{(iv)}(h) = 24, \quad f^{(iv)}(-2) = 24.$$

$$\therefore f(x-2) = 61 - 84x + 92 \frac{x^2}{2!} - 66 \frac{x^3}{3!} + 24 \frac{x^4}{4!}$$

$$\therefore f(x-2) = 61 - 84x + 46x^2 - 11x^3 + x^4$$

eg. ② Expand $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$

$$\leftarrow f(x) = 2x^3 + 7x^2 + x - 6$$

By Taylor's thm

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(3) + \dots$$

$$\therefore f(x) = 2x^3 + 7x^2 + x - 6, \quad f(2) = 40$$

$$f'(x) = 6x^2 + 14x + 1, \quad f'(2) = 53$$

①

$$f''(x) = 12x + 14, \quad f''(2) = 38$$

$$f'''(x) = 12, \quad f'''(2) = 12$$

$$f^{(4)}(x) = 0, \quad f^{(4)}(2) = 0$$

put all values in ①

$$\therefore f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12) + 0$$

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Q-3) Using Taylors thm find expansion of $\tan(x + \frac{\pi}{4})$ in ascending powers of x upto terms in x^4 & find approximately the value of $\tan(43^\circ)$

$$\rightarrow f(x+h) = \tan(x + \frac{\pi}{4}), \text{ then } h = \frac{\pi}{4}.$$

$$f(h) = \tan(h), \quad f(\frac{\pi}{4}) = 1$$

$$f'(h) = \sec^2(h) = 1 + \tan^2 h, \quad f'(\frac{\pi}{4}) = 2$$

$$f''(h) = 2 \tanh \sec^2 h = 2 \tanh(1 + \tan^2 h), \quad f''(\frac{\pi}{4}) = 4 \\ = 2[\tanh + \tan^3 h]$$

$$\underline{f'''(h) = 2[}$$

$$f'''(h) = 2[\sec^2 h + 3\tan^2 h \sec^2 h]$$

$$= 2[1 + \tan^2 h + 3\tan^2 h(1 + \tan^2 h)]$$

$$= 2[1 + 4\tan^2 h + 3\tan^4 h]$$

$$\therefore f'''(\frac{\pi}{4}) = 16$$

$$f''(h) = 2 [8 \tan^2 h \sec^2 h + 12 \tan^3 h \sec^2 h]$$

$$\therefore f''\left(\frac{\pi}{4}\right) = 80$$

Substitute all values in Taylor's thm.

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f''''(h) + \dots$$

$$\therefore f(x+h) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{x^3}{3!} f'''\left(\frac{\pi}{4}\right) + \frac{x^4}{4!} f''''\left(\frac{\pi}{4}\right) + \dots$$

$$\therefore \tan(x+\frac{\pi}{4}) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots \quad \text{--- } ①$$

$$\text{Now } \tan(43^\circ) = \tan(x+h) = \tan(-2^\circ + 45^\circ) \\ = \tan\left(\frac{-2\pi}{180} + \frac{\pi}{4}\right)$$

$$\therefore x = -2^\circ = -\frac{2\pi}{180} = -0.0349, \quad h = \frac{\pi}{4}$$

Putting $x = -0.0349$ in ①

$$\therefore \tan(43^\circ) = 1 - 2(0.0349) + 2(0.0349)^2 - \frac{8}{3}(0.0349)^3$$

$$+ \frac{10}{3}(0.0349)^4 + \dots$$

$$= 0.9326 \text{ approximately}$$

Q.4) Expand e^x in powers of $x-2$

$$\therefore f(x) = e^x; \quad \text{Now } x = x-2+2$$

By Taylor's thm

$$\therefore f(x) = f((x-2)+2) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \dots \quad \text{--- } ①$$

$$\therefore f(x) = e^x \quad \therefore f(2) = e^2$$

$$f'(x) = e^x, \quad f'(2) = e^2$$

$$f''(x) = e^x, \quad f''(2) = e^2$$

$$f'''(x) = e^x, \quad f'''(2) = e^2$$

$$\begin{aligned}\therefore f(x) &= e^2 + (x-2)e^2 + \frac{(x-2)^2}{2!}e^2 + \frac{(x-2)^3}{3!}e^2 + \dots \\ &= \cancel{e^2} \boxed{\quad} \\ &= e^2 \left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \dots \right).\end{aligned}$$

Q.5) Use Taylor's thm to find $\sqrt{25.15}$

Here $f(x+h) = \sqrt{x+h}$, where $x=25$, $h=0.15$.

$$\Rightarrow f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}, \quad f'''(x) = \frac{9}{8}x^{-\frac{5}{2}}, \quad f^{(iv)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$$

\therefore By Taylors thm,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(x) + \dots$$

$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} = \frac{h^2}{8(\sqrt{x})^3} + \frac{h^3}{16(\sqrt{x})^5} = \frac{5h^4}{128(\sqrt{x})^7} + \dots$$

a. put $x=25$, $h=0.15$.

$$\sqrt{25.15} = 5 + \frac{0.15}{2\sqrt{25}} = \frac{(0.15)^2}{8\times 125} + \frac{(0.15)^3}{16\times 3125} = \frac{5(0.15)^4}{128\times 78125} + \dots$$

$$= 5 + 0.015 - 0.000225 + 0.000000675 - \dots$$

≈ 5.01478 approximately

Q. 6) Show that $\frac{1}{2}[f(x) - f(2a-x)] = (x-a)f'(a)$

$$+ \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^v(a) + \dots$$

→ Let $f(x) = f(a+(x-a))$

By Taylors thm,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$$+ \frac{(x-a)^4}{4!} f^{IV}(a) + \frac{(x-a)^5}{5!} f^v(a) \quad \textcircled{1}$$

Let $f(2a-x) = f(a+(a-x))$

∴ By Taylors thm,

$$f(2a-x) = f(a) + (a-x)f'(a) + \frac{(a-x)^2}{2!} f''(a) + \frac{(a-x)^3}{3!} f'''(a) + \dots$$

subtract ② from ①

$$+ \frac{(a-x)^4}{4!} f^{IV}(a) + \frac{(a-x)^5}{5!} f^v(a) \quad \textcircled{2}$$

~~add ① & ②~~

$$\therefore f(x) - f(2a-x) = 2f(a)$$

$$f(x) - f(2a-x) = 2 \left[(x-a)f'(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^v(a) \right] + \dots$$

$$\therefore \frac{1}{2} [f(x) - f(2a-x)] = (x-a)f'(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^v(a)$$

$$+ \dots$$

- HW:-
- ① Expand $3x^3 - 2x^2 + x - 4$ in powers of $(x+2)$
 - ② Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of $x-3$
 - ③ Using Taylors thm express $7 + (x+2) + 3(x+2)^3 + (x+2)^4$ in powers of x .
 - ④ Using Taylors thm obtain $\sqrt{9 \cdot 12}$

eg prove that $\log \sin(x+h) = \log \sin x + h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x + \frac{h^3}{3} \operatorname{cosec}^3 x \cot x + \dots$