Error Functions

Def^n: Error function of x is defined as $\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$ and is denoted by $\mathrm{erf}(x)$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$$

Complementary error function: It is defined as $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$ and is denoted by $\mathrm{erfc}(x)$.

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$$

* **Properties**

① $\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du$ .... (by definition)

Put $u^2 = t \rightarrow u = t^{1/2}$

$du = \frac{1}{2}t^{-1/2}\,dt$

$u = 0$ then $t = 0$, $u = \infty$ then $t = \infty$

$$= \frac{\cancel{2}}{\sqrt{\pi}}\int_0^\infty e^{-t}\frac{t^{-1/2}}{\cancel{2}}\,dt = \frac{1}{\sqrt{\pi}}\cdot(\sqrt{\pi}) \quad \text{... (by gamma function)}$$

$$= 1.$$

② $\mathrm{erf}(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-u^2}\,du = 0$ ... (by properties of definite integral).

③ $\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$

LHS $= \mathrm{erf}(x) + \mathrm{erfc}(x)$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\left[\int_0^\infty e^{-u^2}\,du\right] \text{ — (by properties of definite integral)}$$

$$= \mathrm{erf}(\infty) = 1.$$

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④ erf(x) is an odd function

Proof :- $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$\text{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-u^2} du$ ... Put $u = -y$, $du = -dy$, $u = 0$ then $y = 0$, $u = -x$ then $y = x$

$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} (-dy) = -\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy = -\text{erf}(x)$

Examples ① S.T $\frac{d}{dx} \text{erf}(ax) = \frac{2a e^{-a^2x^2}}{\sqrt{\pi}}$

$\text{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$

diff w. r to x,

$\frac{d}{dx} \text{erf}(ax) = \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du\right)$ ... (x-parameter using DUIS rule - II)

$= \frac{2}{\sqrt{\pi}} \left[\int_0^{ax} \frac{d}{dx}\left(e^{-u^2}\right) du + \left(\frac{d}{dx}(ax)\right) e^{-(ax)^2} - \left\{\frac{d}{dx}(0)\right\} e^{-0} \right]$

$= \frac{2}{\sqrt{\pi}} \left[0 + a e^{-a^2x^2} - 0 \right] = \frac{2a e^{-a^2x^2}}{\sqrt{\pi}}$.

② S.T $\int_0^t \text{erf}(ax)\, dx = t \cdot \text{erf}(at) + \frac{1}{a\sqrt{\pi}} e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}}$.

$\int_0^t \text{erf}(ax)\, dx = \int_0^t \text{erf}(ax) \cdot 1\, dx$

$= \left[\text{erf}(ax) \cdot x\right]_0^t - \int_0^t \frac{d}{dx}\left(\text{erf}(ax)\right) \cdot x\, dx$

$= \text{erf}(at) \cdot t - \int \frac{2a e^{-a^2x^2}}{\sqrt{\pi}} \cdot x\, dx$ ... (from ex. ①)

$= t\, \text{erf}(at) + \frac{1}{a\sqrt{\pi}} \int_0^t e^{-a^2x^2} (-2a^2x\, dx)$

$= t\, \text{erf}(at) + \frac{1}{a\sqrt{\pi}} \left(e^{-a^2t^2} - 1\right) = \text{RHS}.$

③ S.T $\quad erfc(-x) + erfc(x) = 2$

We know that $\quad erf(x) + erfc(x) = 1$

So, $\quad erf(-x) + erfc(-x) = 1$

$$-erf(x) + erfc(-x) = 1$$

$$erfc(-x) = 1 + erf(x)$$

LHS $= erfc(-x) + erfc(x)$

$= 1 + erf(x) + erfc(x) \quad = 1 + 1 = 2.$

④ S.T $\quad \int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \left[erf(b) - erf(a) \right]$

We know that $\quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$$erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx$$

$$1 = \frac{2}{\sqrt{\pi}} \left[\int_0^a e^{-x^2} dx + \int_a^b e^{-x^2} dx + \int_b^\infty e^{-x^2} dx \right]$$

$$1 = \cancel{\frac{2}{\sqrt{\pi}}} \left[erf(a) + \frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx + erfc(b) \right]$$

$$\frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx = \left(1 - erfc(b)\right) - erf(a)$$

but $erf(b) + erfc(b) = 1.$

$$\frac{2}{\sqrt{\pi}} \int_a^b e^{-x^2} dx = erf(b) - erf(a)$$

$$\Longrightarrow \quad \int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \left(erf(b) - erf(a) \right)$$

Ex.) S.T $\int_0^\infty e^{-x^2-2bx}\,dx = \frac{\sqrt{\pi}}{2}\, e^{b^2}\left(1-erf(b)\right)$.

$$I = \int_0^\infty e^{-x^2-2bx}\,dx$$

$$= \int_0^\infty e^{-x^2-2bx-b^2+b^2}\,dx = \int_0^\infty e^{-(x+b)^2} \cdot e^{b^2}\,dx$$

$$= e^{b^2}\int_0^\infty e^{-(x+b)^2}\,dx$$

$x+b=t$ $\quad$ $x=0$ then $t=b$

$dx=dt$ $\quad$ $x=\infty$ then $t=\infty$

$$= e^{b^2}\int_b^\infty e^{-t^2}\,dt$$

$$= e^{b^2}\, erfc(b)$$

$$= e^{b^2}\left(1-erf(b)\right).$$

H.W

① S.T $\frac{d}{dt}\left(erf\sqrt{t}\right) = \frac{e^{-t}}{\sqrt{\pi t}}$ and hence evaluate $\int_0^\infty e^{-t} erf(\sqrt{t})\,dt$

② Find $\frac{d}{dx}\, erfc(ax^n)$

③ P.T $\frac{1}{x}\frac{d}{da}\left(erfc(ax)\right) = -\frac{1}{a}\frac{d}{dx}\, erf(ax)$