Triple Integration

It is extension of the concept of double integration. Triple integration represents volume or a physical quantity related to the volume.

$$I = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_{x=a}^{b} \left\{ \int_{y=f_1(x)}^{f_2(x)} \left[\int_{z=g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dy \right\} dx$$

Here we integrate 'z' first then y & finally 'x'. Note that order of integration depends on the distribution of limits

eg ①
$$I = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} f(x,y,z)\, dx\, dy\, dz$$

In this integral, we first integrate 'z', then 'x' & finally 'y'.

eg ②
$$I = \int_0^{\pi/2} \int_0^{x} \int_1^{y^2} f(x,y,z)\, dx\, dy\, dz$$

Here we first integrate 'z' first as 'x', 'y' are involved in limits. After that we integrate y & lastly 'x'

Problems on Direct evaluation

eg ① Evaluate: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

Sol^n:→ $I = \int_{z=-1}^{1}\int_{x=0}^{z}\left[\int_{y=x-z}^{x+z}(x+y+z)\,dy\right]dx\,dz$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy + \frac{y^2}{2} + yz\right]_{y=x-z}^{(x+z)} dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[x(x+z) + \frac{(x+z)^2}{2} + (x+z)z - x(x-z) - \frac{(x-z)^2}{2} - (x-z)z\right]dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[2xz + 2z^2 + \frac{(x+z)^2}{2} - \frac{(x-z)^2}{2}\right]dx\,dz$$

$$= \int_{z=-1}^{1}\left\{\int_{x=0}^{z}(2xz + 2z^2 + 2xz)\,dx\right\}dz$$

$$= \int_{z=-1}^{1}\left[x^2z + 2z^2x + x^2z\right]_{x=0}^{z} dz$$

$$= \int_{z=-1}^{1}\left(z^3 + 2z^3 + z^3 - 0\right)dz$$

$$= 4\int_{-1}^{1} z^3\,dz$$

$= 4 \times 0$ ($\because z^3$ is an odd func^n).

$= 0$

eg ② Show that $\int_0^1 dy \int_{y^2}^1 dx \int_0^{1-x} x\,dz = \frac{4}{35}$

Solⁿ:→ $LHS = \int_{y=0}^{1} \int_{x=y^2}^{1} \left[\int_{z=0}^{1-x} x\,dz \right] dx\,dy$

$= \int_{y=0}^{1} \int_{x=y^2}^{1} x\,[z]_{z=0}^{1-x}\,dx\,dy$

$= \int_{y=0}^{1} \int_{x=y^2}^{1} x(1-x)\,dx\,dy$

$= \int_{y=0}^{1} \left[\int_{x=y^2}^{1} (x-x^2)\,dx \right] dy$

$= \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=y^2}^{1} dy$

$= \int_{y=0}^{1} \left[\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3} \right] dy$

$= \left[\frac{1}{6} y - \frac{y^5}{10} + \frac{y^7}{21} \right]_0^1$

$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21}$

$= \frac{4}{35} = RHS.$

eg ③ Show that $\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$

Solⁿ :→ LHS $= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta}\left[\int_{z=0}^{\sqrt{a^2-r^2}} r\,dz\right] dr\,d\theta$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} r\,[z]_0^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left\{\int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\right\} d\theta$$

put $a^2-r^2=t$

$-2r\,dr=dt$

$$= \int_{\theta=0}^{\pi/2}\left[-\frac{1}{2}\frac{(a^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{a\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} -\frac{1}{3}\left[(a^2-a^2\cos^2\theta)^{3/2}-a^3\right] d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta-1)\,d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}\times 1-\frac{\pi}{2}\right]$$ (by Reduction f.)

$$= \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

= RHS

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eg ④ $\int_1^2 dx \int_2^3 dy \int_1^3 (x^2y+z)dz = \frac{47}{3}$

Sol^n:→

$$LHS = \int_1^2 dx \int_2^3 dy \int_1^3 (x^2y+z)dz$$

$$= \int_1^2 dx \int_2^3 \left[x^2yz + \frac{z^2}{2}\right]_{z=1}^{3} dy$$

$$= \int_1^2 dx \int_2^3 \left[x^2y(3-1) + \frac{3^2-1}{2}\right] dy$$

$$= \int_1^2 dx \int_2^3 (2x^2y+4) dy$$

$$= \int_1^2 \left[x^2y^2+4y\right]_2^3 dx$$

$$= \int_1^2 \left[x^2(3^2-2^2)+4(3-2)\right] dx$$

$$= \int_1^2 (5x^2+4) dx$$

$$= \left(5\frac{x^3}{3}+4x\right)_1^2$$

$$= 5\left(\frac{8}{3}-\frac{1}{3}\right)+4(2-1)$$

$$= \frac{35}{3}+4 = \frac{47}{3}$$

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Home Work

eg ① Show that $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz\,dy\,dx = \frac{a^6}{48}$

eg ② Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz\,dy\,dx$

eg ③ Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{a-\frac{r^2}{a}} r \, d\theta\,dr\,dz$

Note that spherical polar coordinates are given by:

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \Rightarrow \quad r^2 = x^2+y^2+z^2,\ \phi = \tan^{-1}(y/x),\ \cancel{\theta = \cos^{-1}} \quad \theta = \cos^{-1}\left(\frac{z}{r}\right) \text{ or } \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

We know that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ (Jacobian)

$\therefore \; dx\,dy\,dz = r^2\sin\theta \, dr\,d\theta\,d\phi$

Standard limits are given as follows:

① For complete sphere: $x^2+y^2+z^2=a^2$

$\theta \to 0$ to π, $\phi \to 0$ to 2π, $r \to 0$ to a

② For hemisphere: $x^2+y^2+z^2=a^2$ (upper half)

$\theta \to 0$ to $\pi/2$, $\phi \to 0$ to 2π, $r \to 0$ to a

③ For +ive octant of the sphere: $x^2+y^2+z^2=a^2$

$\theta \to 0$ to $\pi/2$, $\phi \to 0$ to $\pi/2$, $r \to 0$ to a.

2. Consider the Cylinderical polar coordinates
$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = \xi$
$\Rightarrow\ x^2 + y^2 = \rho^2,\ \phi = \tan^{-1}\left(\frac{y}{x}\right),\ z = \xi$

We know that $\frac{\partial(x, y, z)}{\partial(\rho, \theta, \xi)} = \rho$

$\therefore\ dx\,dy\,dz = \rho\,d\rho\,d\theta\,d\xi$

Standard limits for cylinder $x^2+y^2=a^2;\ z=h$
is given by $z: 0$ to h, $\rho: 0$ to a, $\phi: 0$ to 2π.

(3) Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

polar eqn of ellipsoid is given by

$$\left.\begin{aligned} x &= a r \sin\theta\cos\phi \\ y &= b r \sin\theta\sin\phi \\ z &= c r \cos\theta \end{aligned}\right\} \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Standard limits for complete ellipse is given by : $r \to 0$ to 1
$\theta \to 0$ to π
$\phi \to 0$ to 2π

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Type II : When limits are not given

eg ① Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ taken throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

Solⁿ:→ We polar coordinates for the sphere $x^2+y^2+z^2=a^2$ are given by

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \Rightarrow \begin{aligned} r^2 &= x^2+y^2+z^2 \\ dx\,dy\,dz &= r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

& limits : $r \to 0$ to a, $\theta \to 0$ to π, $\phi \to 0$ to 2π

$$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left\{\int_{r=0}^{a}\frac{r^2}{\sqrt{a^2-r^2}}\,dr\right\}\sin\theta\,d\theta\,d\phi$$

put $r = a\sin t \Rightarrow dr = a\cos t\,dt$

as $r\to 0$, $t\to 0$

as $r \to a$, $t \to \pi/2$

$$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left\{\int_{t=0}^{\pi/2}\frac{a^2\sin^2 t}{\sqrt{a^2-a^2\sin^2 t}}\cdot a\cos t\,dt\right\}\sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left\{\int_{t=0}^{\pi/2} a^2\sin^2 t\,dt\right\}\sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left\{\frac{1}{2}\times\frac{\pi}{2}\,a^2\right\}\sin\theta\,d\theta\,d\phi$$

$$= \frac{\pi a^2}{4}\int_{\phi=0}^{2\pi}\left\{\int_{\theta=0}^{\pi}\sin\theta\,d\theta\right\}d\phi = \frac{\pi a^2}{4}\int_{\phi=0}^{2\pi} 1\,d\phi = \frac{\pi^2 a^2}{2}$$

eg (2) Evaluate $\iiint_V \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$, over the volume V of sphere $x^2+y^2+z^2 = z$ in first octant.

Solⁿ:→ We know polar eqⁿ for sphere

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \Rightarrow \begin{aligned} &x^2+y^2+z^2 = r^2 \\ &dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

for given sphere $x^2+y^2+z^2 = z \Rightarrow r^2 = r\cos\theta$

$\Rightarrow r = \cos\theta$

∴ limits: $r \to 0$ to $\cos\theta$, $\theta \to 0$ to $\pi/2$, $\phi \to 0$ to $\pi/2$.

$$\therefore I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\cos\theta} \frac{r^2\cos^2\theta}{r^2}\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \cos^2\theta\sin\theta \left\{\int_{r=0}^{\cos\theta} r^2\,dr\right\} d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \cos^2\theta\sin\theta \left[\frac{r^3}{3}\right]_{r=0}^{\cos\theta} d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2} \left\{\int_{\theta=0}^{\pi/2} \frac{1}{3}\cos^5\theta\sin\theta\,d\theta\right\} d\phi$$

$$= \int_{\phi=0}^{\pi/2} \frac{1}{3}\left(\frac{1}{6}\right) d\phi$$

$$= \frac{1}{18}\int_{\phi=0}^{\pi/2} d\phi = \frac{\pi}{36}$$

eg (3) Evaluate the integral $\iiint_V \sqrt{x^2+y^2}\, dx\,dy\,dz$,

where V is $x^2+y^2=z^2$, $z>0$ & $z=0$, $z=1$.

Sol^n:→ Note that $z^2 = x^2+y^2$, $z>0$ is representing a right circular cone. Transforming to cylinderical polar co-ordinates:

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}\right\} \quad dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz \quad \& \quad x^2+y^2=\rho^2.$$

Hence limits: $z \to \rho$ to 1, $\phi \to 0$ to 2π, $\rho \to 0$ to 1.

$$\therefore \iiint_V \sqrt{x^2+y^2}\, dx\,dy\,dz = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi}\int_{z=\rho}^{1} \sqrt{\rho^2}\;\rho\, d\rho\, d\phi\, dz$$

$$= \int_{\rho=0}^{1} \rho^2 \int_{\phi=0}^{2\pi} \left\{ \int_{\rho}^{1} dz \right\} d\phi\, d\rho$$

$$= \int_{\rho=0}^{1} \rho^2 \int_{\phi=0}^{2\pi} (1-\rho)\, d\phi\, d\rho$$

$$= \int_{\rho=0}^{1} \rho^2(1-\rho)\, d\rho \cdot \int_{\phi=0}^{2\pi} d\phi$$

$$= \left[\frac{\rho^3}{3} - \frac{\rho^4}{4}\right]_0^1 \left[\phi\right]_0^{2\pi}$$

$$= \left(\frac{1}{3}-\frac{1}{4}\right)(2\pi)$$

$$= \frac{\pi}{6} \;//$$

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HW

eg ① Evaluate $\iiint_V \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$ over the volume of the sphere $x^2+y^2+z^2=2$.

$\left(\frac{8\sqrt{2}\pi}{9}\right)$

eg ② Evaluate $\iiint_V (x^2+y^2)\,dx\,dy\,dz$, where V is the volume of the cylinder $x^2+y^2=2z$; $z=2$.

$\left(\frac{16\pi}{3}\right)$

eg ③ Evaluate the integral

$\iiint_V \sqrt{1-x^2-\frac{y^2}{4}-z^2}\,dx\,dy\,dz$ over the volume of the ellipsoid $x^2+\frac{y^2}{4}+z^2=1$.

$\left(\frac{\pi^2}{2}\right)$.

Dirichlet's Theorem (for three variables)

① $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$

if $x+y+z\le 1$.

② $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\, h^{l+m+n}$

if $x+y+z\le h$.

③ $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = a^l b^m c^n \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$

where V is volume of tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Problems On Dirichlet's Theorem:

eg ① Show that $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = a^l b^m c^n$

$\times \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$, where V is the volume of the tetrahedron $x=0, y=0, z=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol^n:→ $I = \iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz$

put $x = au$, $y = bv$, $z = cw$

$\Rightarrow dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$

$\therefore I = \iiint_V (au)^{l-1} (bv)^{m-1} (cw)^{n-1}\, a\,du\, b\,dv\, c\,dw$

$= a^l b^m c^n \iiint_V u^{l-1} v^{m-1} w^{n-1}\, du\,dv\,dw$

Limits: $x=0 \Rightarrow u=0$, $y=0 \Rightarrow v=0$, $z=0 \Rightarrow w=0$

& $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow u+v+w=1$

Hence the volume inscribed is $u+v+w \leq 1$.

Therefor by Dirichlet's theorem

$I = a^l b^m c^n \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$

Hence proved.

eg ② Evaluate $\iiint_V xyz\,dx\,dy\,dz$ over the first octant of the sphere $x^2+y^2+z^2=a^2$

Solⁿ:→ Consider $I = \iiint_V xyz\,dx\,dy\,dz$

given sphere $x^2+y^2+z^2=a^2$

putting $x^2 = a^2u$, $y^2 = a^2v$, $z^2 = a^2w$

$\rightarrow x\,dx = \frac{a^2}{2}du$, $y\,dy = \frac{a^2}{2}dv$, $z\,dz = \frac{a^2}{2}dw$

$$\therefore I = \iiint_V \frac{a^2}{2}du \cdot \frac{a^2}{2}dv \cdot \frac{a^2}{2}dw$$

$$= \frac{a^6}{8}\iiint_V du\,dv\,dw$$

Also, eqⁿ of sphere : $x^2+y^2+z^2=a^2$

$\Rightarrow u+v+w=1$

Hence over the volume of first octant of given sphere $u+v+w \le 1$. By Dirichlet's condⁿ

$$I = \frac{a^6}{8}\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)}$$

$$= \frac{a^6}{8}\frac{1}{\Gamma 4}$$

$$= \frac{a^6}{8}\frac{1}{3!}$$

$$= \frac{a^6}{48} //.$$

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Q(3) Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Solⁿ: → Consider $I=\iiint_V (xy+yz+zx)\,dx\,dy\,dz$

& given sphere $x^2+y^2+z^2=a^2$.

Putting $x^2=a^2u,\ y^2=a^2v,\ z^2=a^2w$

$\Rightarrow x=a\sqrt{u},\ y=a\sqrt{v},\ z=a\sqrt{w}$

$\Rightarrow dx=\frac{a}{2\sqrt{u}}du,\ dy=\frac{a}{2\sqrt{v}}dv,\ dz=\frac{a}{2\sqrt{w}}dw$

∴ eqⁿ of sphere $x^2+y^2+z^2=a^2 \Rightarrow a^2u+a^2v+a^2w=a^2$

$\Rightarrow u+v+w=1$

Hence volume of +ive octant is $u+v+w\le 1$.

Also

$$I=\iiint_V \left(a\sqrt{u}\cdot a\sqrt{v}+a\sqrt{v}\cdot a\sqrt{w}+a\sqrt{w}\,a\sqrt{u}\right)\cdot\frac{a\,du}{2\sqrt{u}}\,\frac{a\,dv}{2\sqrt{v}}\cdot\frac{a\,dw}{2\sqrt{w}}$$

$$=\frac{a^5}{8}\iiint_V\left(\frac{1}{\sqrt{w}}+\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{v}}\right)du\,dv\,dw$$

$$=\frac{a^5}{8}\left[\iiint_V w^{-1/2}du\,dv\,dw+\iiint_V u^{-1/2}du\,dv\,dw+\iiint_V v^{-1/2}du\,dv\,dw\right]$$

$$=\frac{a^5}{8}\,3\times\frac{\Gamma 1\,\Gamma 1\,\Gamma 1/2}{\Gamma(1+1+1/2+1)}$$ (By Dirichlet's condⁿ)

$$=\frac{3a^5}{8}\,\frac{\sqrt{\pi}}{\Gamma 7/2}=\frac{3a^5}{8}\,\frac{\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}}=\frac{a^5}{5}\ //.$$

eg (4) Evaluate $\iiint_V x^2y^2z^2\,dx\,dy\,dz$ taken throught the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$.

Sol^n:→ Consider $I=\iiint_V x^2y^2z^2\,dx\,dy\,dz$.

put $x^2=a^2u,\ y^2=b^2v,\ z^2=c^2w$

$\Rightarrow x\,dx=\frac{a^2}{2}du,\ y\,dy=\frac{b^2}{2}dv,\ z\,dz=\frac{c^2}{2}dw$

∴ eq^n of ellipsoid reduces to $u+v+w\le 1$.

And $I=\iiint_V a^2u\cdot b^2v\, c^2w\ \frac{a^2du}{2}\ \frac{b^2dv}{2}\cdot\frac{c^2}{2}dw$

$=\frac{a^4b^4c^4}{8}\iiint_V uvw\,du\,dv\,dw$

By Dirichlet's theorem, we first evaluate integral for first octant of ellipsoid,

$=\frac{a^4b^4c^4}{8}\ \frac{\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)}$

$=\frac{a^4b^4c^4}{8\times(6!)}$

Therefore for the complete ellipsoid

$I=8\times\frac{a^4b^4c^4}{8\times(6!)}$

$I=\frac{a^4b^4c^4}{720}$ //.

HW:

eg ① Evaluate $\iiint x^2 y^3 z^4 \, dx\,dy\,dz$ over the volume of tetrahedron bounded by $x=0$, $y=0$, $z=0$, $x+\frac{y}{2}+\frac{z}{3}=1$.

eg ② Evaluate $\iiint_V (x+y+z) \, dx\,dy\,dz$ over the tetrahedron formed by $x=0$, $y=0$, $z=0$, $x+y+z=1$.

eg ③ Evaluate $\iiint_V x y^2 z \, dx\,dy\,dz$ over the volume of the sphere $x^2+y^2+z^2=a^2$.

eg ④ Evaluate $\iiint_V (xy^2+yz^2) \, dx\,dy\,dz$ over the volume of the sphere $x^2+y^2+z^2=9$ in first octant.

eg ⑤ Evaluate $\iiint_V xyz \, dx\,dy\,dz$ taken over the volume of the ellipsoid $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$.

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Relation between co-ordinate systems :-

1) Cartesian Co-ordinate system :- (x, y, z)

2) Spherical Polar : (r, θ, ϕ) where
$0 < r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and,

$x = r \sin\theta \cos\phi$, $r = \sqrt{x^2+y^2+z^2}$
$y = r \sin\theta \sin\phi$, $\theta = \cos^{-1}\left(\frac{z}{r}\right)$
$z = r \cos\theta$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

3) Cylindrical Co-ordinate system :- (ρ, ϕ, z)
$0 < \rho < \infty$, $0 \le \phi \le 2\pi$, $-\infty < z < \infty$

$x = \rho \cos\phi$, $\rho = \sqrt{x^2+y^2}$
$y = \rho \sin\phi$, $\phi = \tan^{-1}(y/x)$
$z = z$, $z = z$

Hans

Ex Find the spherical polar & Cylindrical co-ordinates of a point (1,1,1).

$\Rightarrow$ Here $(x, y, z) = (1, 1, 1)$

a) spherical polar (r, θ, ϕ).

$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+1} = \sqrt{3} \quad (\because r > 0)$

$\theta = \cos^{-1}(y/r) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.74^\circ$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(1/1) = \tan^{-1}(1) = 45^\circ$.

$\therefore (r, \theta, \phi) = (\sqrt{3}, 54.74^\circ, 45^\circ)$.

b) Cylindrical Co-ordinates (ρ, ϕ, z).

$\rho = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2} \quad (\because \rho > 0)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = 45^\circ$

$z = 1$

$\therefore (\rho, \phi, z) = (\sqrt{2}, 45^\circ, 1)$.