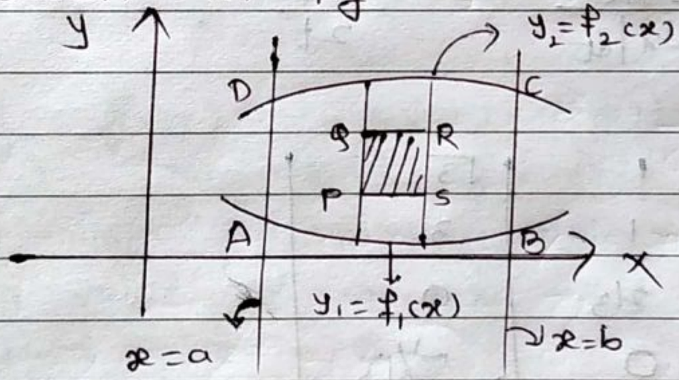
Area :- ~~Vet~~

(A) Area in cartesian co-ordinate system :-

Let R be the area enclosed by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x = a$, $x = b$ as shown in fig



Area of rectangle PQRS = $\delta x \delta y$

Area of vertical strip = $\lim_{\delta x \to 0} \sum_{y_1}^{y_2} \delta x \, \delta y$

$= \delta x \int_{y_1}^{y_2} dy$

Adding all such a strips from $x = a$ to $x = b$

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we get

$$\text{area } ABCD = \lim_{\delta x \to 0} \sum_a^b \delta x \int_{y_1}^{y_2} dy$$

$$= \int_a^b dx \int_{y_1}^{y_2} dy$$

$$= \int_{x=a}^{b} \int_{y_1 = f_1(x)}^{f_2(x)} dx\, dy \quad \text{—— (1)}$$

lly, if x having variable limits i.e.

$x_1 = \phi_1(y)$, $x_2 = \phi_2(y)$ & $y_1 = c$ $y_2 = d$

$$\text{Area} = \int_{y=c}^{d} \int_{x=\phi_1}^{\phi_2} dx\, dy \quad \text{—— (2)}$$

* Note → 1] The area bdd by the curve $y = f(x)$ the x-axis & the line $x = a$, $x = b$ is given by

$$\text{Area} = \int_{x=a}^{b} y\, dx = \int_a^b f(x)\, dx \text{ —— } \left(\because A = \int_{x=a}^{b} \int_{y=0}^{f(x)} dy\, dx \right)$$

2] The area bdd by the curve $x = \phi(y)$, y-axis & the lines $y = c$, $y = d$ is given by

$$\text{Area} = \int_{y=c}^{d} x\, dy = \int_c^d f(y)\, dy$$

$$\text{——} \left(\because A = \int_{y=c}^{d} \int_{x=0}^{f(y)} dx\, dy \right)$$

3] Area in polar form →

$$A = \iint_R dx\, dy \longrightarrow \text{cartesian form —— (1)}$$

put $x = r\cos\theta$ $y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$

Finding corresponding limits for r & θ

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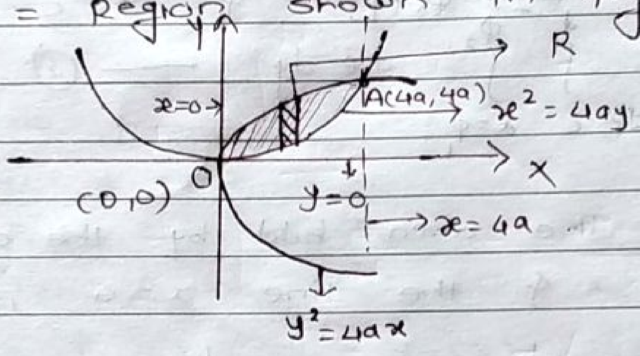
$$\therefore \text{ Area } = \int_{\theta=\alpha}^{\beta} \int_{r_1=f_1(\theta)}^{f_2(\theta)} r\,d\theta\,dr \text{ — (polar form)}$$

① eg. on cartesian form →

① Find area bdd by curves $y^2 = 4ax$ & $x^2 = 4ay$.

→ $\text{Area} = \iint_R dx\,dy$

R = Region shown in fig.



$y^2 = 4ax$, $x^2 = 4ay \Rightarrow \dfrac{x^2}{4a} = y$

$\Rightarrow \dfrac{x^4}{16a^2} = y^2$

$\Rightarrow \dfrac{x^4}{16a^2} = 4ax \quad \Rightarrow x^4 = 64a^3 x$

$\Rightarrow (x^4 - 64a^3x) = 0 \quad \Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0 \quad \text{or} \quad x^3 = 64a^3$

$x = 0 \quad \text{or} \quad x = 4a$

$x = 0 \Rightarrow y = 0$

$x = 4a \Rightarrow y = 4a$

$\therefore$ O = (0,0) , A (4a, 4a)

consider vertical strip →

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$\therefore$ limits of y is variable

limits : $y = \dfrac{x^2}{4a}$ to $y = +\sqrt{4ax}$ — ($\because$ in 1st qud)

$x = 0$ to $x = 4a$.

$$\therefore \text{Area} = \int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy\, dx$$

$$= \int_0^{4a} \left(y \Big|_{\frac{x^2}{4a}}^{\sqrt{4ax}} \right) dx = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[\frac{\sqrt{4a}\; x^{3/2}}{3/2} - \frac{1}{4a} \frac{x^3}{3} \right]_0^{4a}$$

$$= \sqrt{4a}\,(4a)^{3/2}\,\frac{2}{3} - \frac{1}{4a}\frac{(4a)^3}{3} - 0 - 0$$

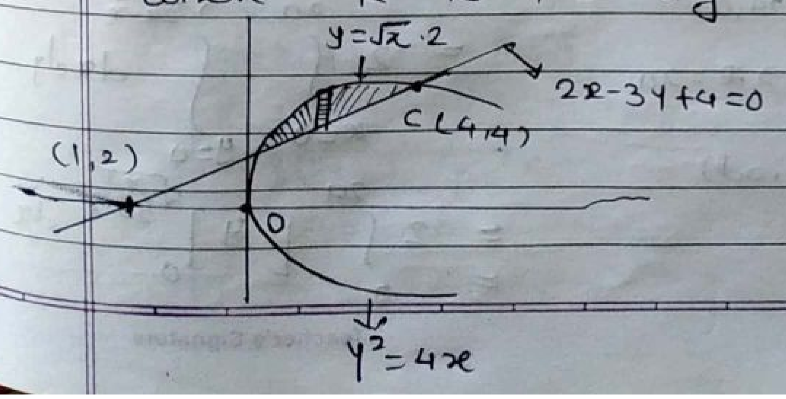
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\boxed{A = \frac{16a^2}{3}}$$

(2) Find the area betn the curves $y^2 = 4x$ & $2x - 3y + 4 = 0$.

→ Area $= \iint_R dx\, dy$

where R is the regn betn $y^2 = 4x$ & $2x-3y+4=0$



To find pt. of intersectn consider,

$y^2 = 4x$, $2x - 3y + 4 = 0$

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$y^2 = 2(2x)$

$y^2 = 2(-4+3y) \Rightarrow y^2 - 6y + 8 = 0$

$\Rightarrow y = 2 , \quad y = 4$

$\Rightarrow y = 2 , \quad x = 1$

$y = 4 , \quad x = 4$

$\therefore (1,2) \quad (4,4)$

consider, vertical strip -

limits of $y = \frac{2x+4}{3}$ to $y = 2\sqrt{x}$ &

moving strip from

$x = 1$ to $x = 4$

$$\therefore A = \int_1^4 \int_{y=\frac{2x+4}{3}}^{2\sqrt{x}} dy\,dx = \int_1^4 \left[2\sqrt{x} - \left(\frac{2x+4}{3}\right)\right] dx$$

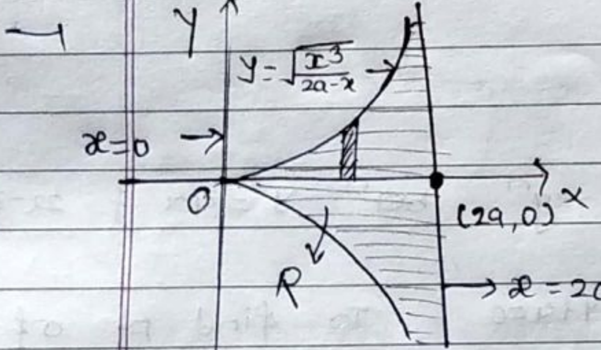
$$= \left[2 \cdot \frac{x^{3/2}}{3/2} - \left(\frac{x^2+4x}{3}\right)\right]_1^4$$

$$= \left(\frac{32}{3} - \frac{32}{3}\right) - \left(\frac{4}{5} - \frac{5}{3}\right)$$

$$A = \frac{1}{3}.$$

3] Find area bdd by the curves

$y^2(2a-x) = x^3$ & it's asymtote.



(Cissiod)

$x = 2a$ is asymtote.

$$\text{Area} = 2 \iint_R dx\,dy$$

$$= 2 \int_{x=0}^{2a} \int_{y=0}^{\sqrt{\frac{x^3}{2a-x}}} dx\,dy$$

$$= 2 \int_0^{2a} \left[y\right]_0^{\sqrt{\frac{x^3}{2a-x}}} dx$$

$=$

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$$= 2\int_0^{2a} \frac{x^{3/2}}{(2a-x)^{1/2}}\,dx$$

put $x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

limits :

x	0	$2a$
θ	0	$\pi/2$

$$\therefore A = 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{(2a-2a\sin^2\theta)^{1/2}}\,4a\sin\theta\cos\theta\,d\theta$$

$$= \frac{2\times 4a \times (2a)^{3/2}}{(2a)^{1/2}}\int_0^{\pi/2}\frac{(\sin^2\theta)^{3/2}}{\sqrt{1-\sin^2\theta}}\sin\theta\cos\theta\,d\theta$$

$$= 2\times 4a\,(2a)^{3/2-1/2}\int_0^{\pi/2}\frac{\sin^3\theta\sin\theta\cos\theta\,d\theta}{\cos\theta}$$

$$= 8a\times(2a)^1\int_0^{\pi/2}\sin^4\theta\,d\theta$$

by reduct^n formula $\int_0^{\pi/2}\sin^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{\pi}{2}$

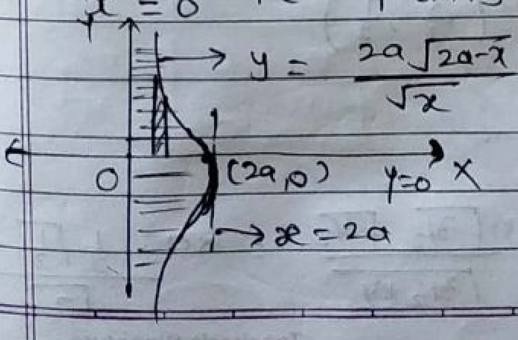
$n = \text{even}$

$$A = 16a^2\,\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\boxed{A = 3\pi a^2}$$

Q] Find area bet^n the curve $xy^2 = 4a^2(2a-x)$ & its asymptote

→ $x = 0$ i.e. y-axis is an asymptote.



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put $y=0 \Rightarrow 2a-x=0 \Rightarrow x=2a$

$\therefore$ x-intercept $(2a,0)$

By symmetry

$$A = 2\iint_R dx\,dy$$

$$= 2\int_{x=0}^{2a}\int_{y=0}^{2a\sqrt{\frac{2a-x}{x}}} dy\,dx$$

$$= 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$$

put $x=2at$

$\Rightarrow dx = 2a\,dt$

$\therefore$ ~~A =~~ limits:

x	0	$2a$
t	0	1

$$A = 2\int_0^1 2a\,(2a-2at)^{1/2}\,(2at)^{-1/2}\,2a\,dt$$

$$= 4a\int_0^1 (2a)^{1/2}(1-t)^{1/2}(2a)^{-1/2}\,t^{-1/2}\,2a\,dt$$

$$= 8a^2\int_0^1 (1-t)^{1/2}\,t^{-1/2}\,dt$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1}x^{n-1}\,dx$$

$\Rightarrow m-1 = 1/2 \qquad n-1 = -1/2$

$\Rightarrow m = 3/2, \qquad n = 1/2$

$\therefore A = 8a^2\,\beta\left(\frac{3}{2},\frac{1}{2}\right)$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\therefore A = 8a^2\,\frac{\Gamma 3/2\,\Gamma 1/2}{\Gamma(3/2+1/2)} = 8a^2\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\sqrt{\pi}}{\Gamma 2}$$

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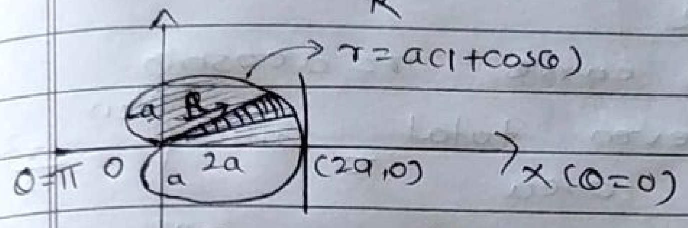
$$= \frac{8a^2 \; \frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{1}$$

$$\boxed{A = 4a^2\pi}$$

* Area in polar co-ordinate :-

1. Find area of cardiode $r = a(1+\cos\theta)$

$$\text{Area} = \iint_R dx\,dy = \iint_R r\,dr\,d\theta$$



By symmetry

$$A = 2\iint_R r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$= 2\int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2}\,d\theta$$

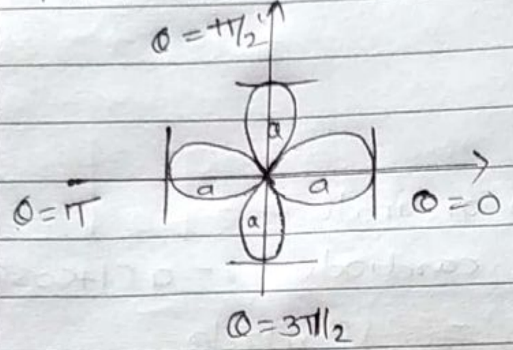
$$= \frac{2a^2}{2}\int_0^{\pi}(1+\cos^2\theta+2\cos\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos2\theta}{2}\right)d\theta$$

$$= a^2\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin2\theta}{4}\right]_0^{\pi}$$

$$= a^2\left[\frac{3\pi}{2}+0\right] = \frac{3\pi a^2}{2}$$

(2) Find total area of the curve $r = a\cos 2\theta$



Four leaved rose

Note : For the curve $r = a\cos n\theta$

(1) If n is even total area $= 2n \int_{-\pi/2n}^{\pi/2n} \frac{r^2}{2}\, d\theta$.

(2) If n is odd total area $= n \int_{-\pi/2n}^{\pi/2n} \frac{r^2}{2}\, d\theta$.

$\therefore$ Required area $= 2\cdot 2 \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\, d\theta$

$= 2 \int_{-\pi/4}^{\pi/4} \underline{a^2} \cos^2 2\theta \, d\theta$

$= 2 \times 2\underline{a^2} \int_0^{\pi/4} \cos^2 2\theta \, d\theta$

put $2\theta = u$

$\Rightarrow d\theta = du/2$

limits

θ	0	$\pi/4$
u	0	$\pi/2$

$\therefore A = 4a^2 \int_0^{\pi/2} \cos^2 u \, \frac{du}{2}$

$$= 2a^2 \int_0^{\pi/2} \cos^2 u \, du$$

Reductⁿ formula $= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{\pi}{2}$ (n = even)

$$\therefore \int_0^{\pi/2} \cos^2 u \, du = \frac{1}{2} \cdot \frac{\pi}{2}.$$

$$\therefore A = 2a^2 \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\boxed{A = \frac{\pi a^2}{2}}$$

H.W. ① Find area bdd by following curves.

① $y^2 = x$, $x^2 = -8y$ —— (Ans : $8/3$)

② $y = x^2 - 6x + 3$ & stright line —— (Ans : $32/3$)

③ $x^2 + y^2 = a^2$, $x + y = a$ in 1st quad —— (Ans : $(\pi - 2)a^2/4$]

④ $x^2 = 4y$ & $x - 2y + 4 = 0$ —— (Ans : 9)

5] $y^2 = 4 - x$ & $y^2 = x$ —— (Ans : $16\sqrt{2}/3$]