MIT-WPU

F. Y. B. Tech.

Trimester-III

Applied Mathematics-II

Curve tracing

Curve Tracing

- Tracing of Curves
- Cartesian curves
- Polar curves
- Parametric curves

Introduction

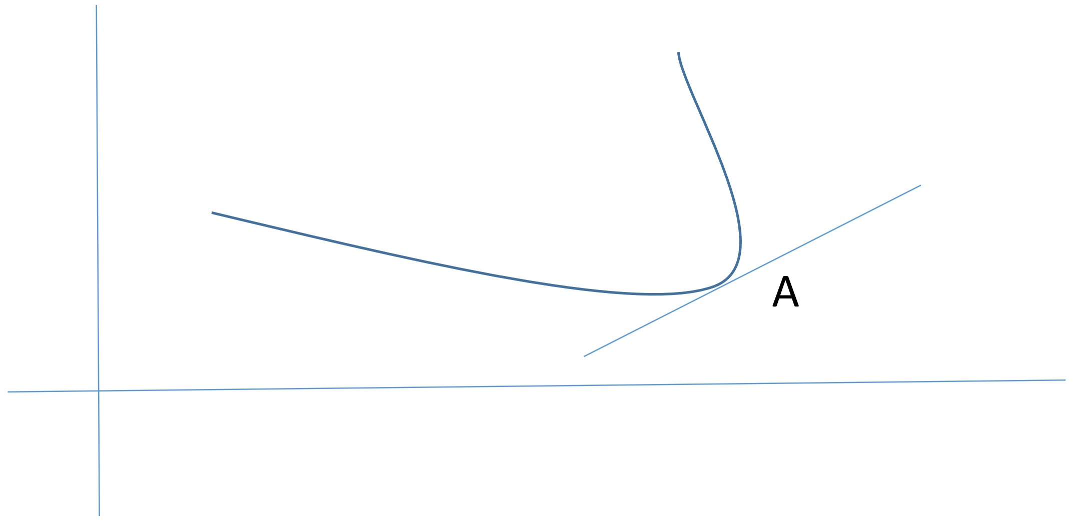
In this chapter, we shall deal with Tracing of Curves which means finding **approximate** shape of the curves using different features i.e. symmetry, intercepts, tangents, asymptotes, region of existence etc.

Curve tracing is useful in applications of integrations in finding area, mass, Centre of gravity, volume etc.

Concavity

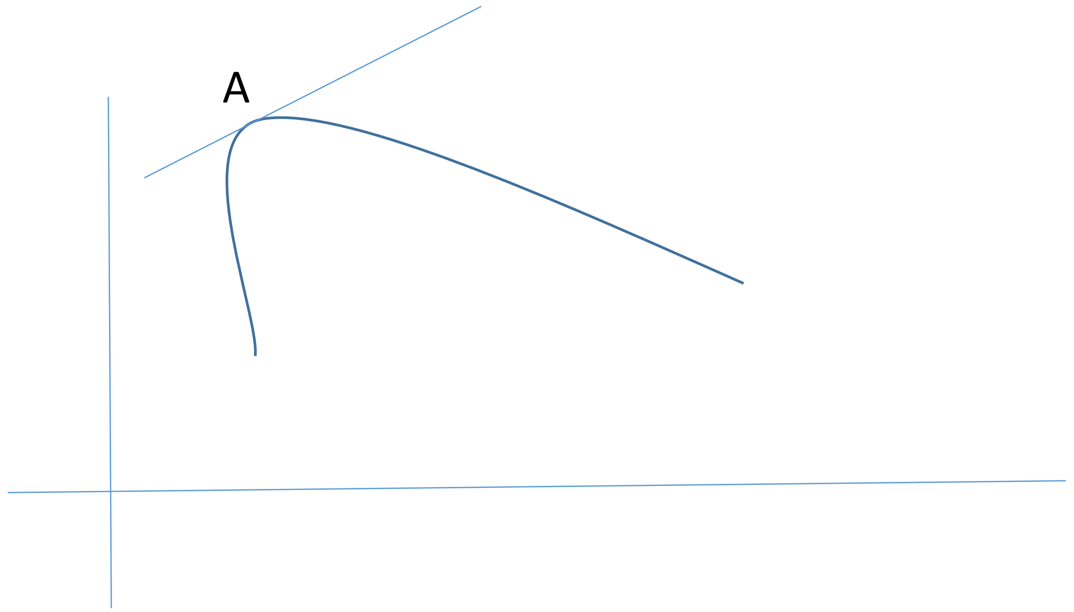
- **Concave Upward :-**

The curve is said to be concave upward at A if the portion of the curve on both sides of A lies above the tangent to the curve at A



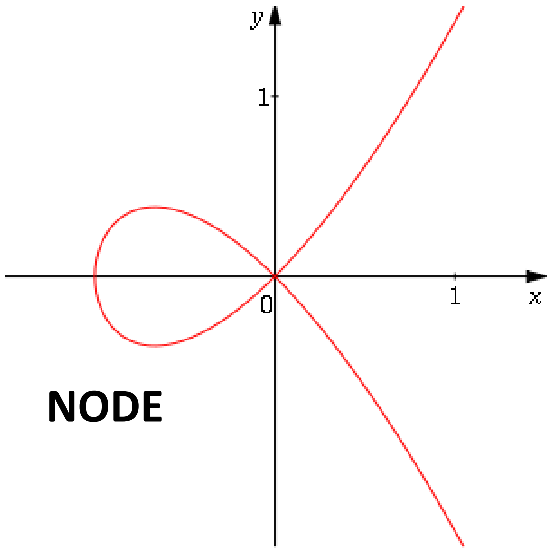
- **Concave Downward:-**

If the portion of the curve on both sides of A lies below the tangent to the curve at A.

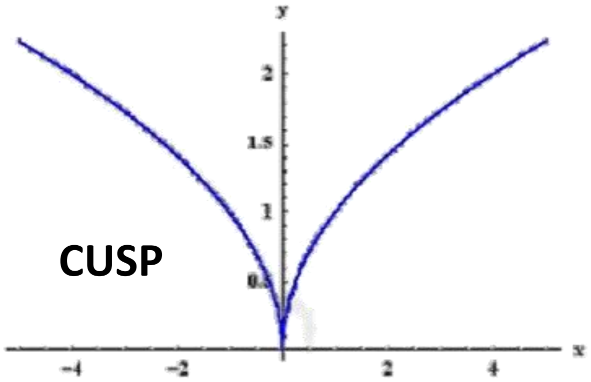


Singular Points:-

- **Double Point** : A point through which two branches of curve passes
- Multiple point : A point through which more than one branch passes
- Node : A double point is called as node if distinct branches have distinct tangents
- Cusp : A double point is called a cusp if two branches have a common tangent.



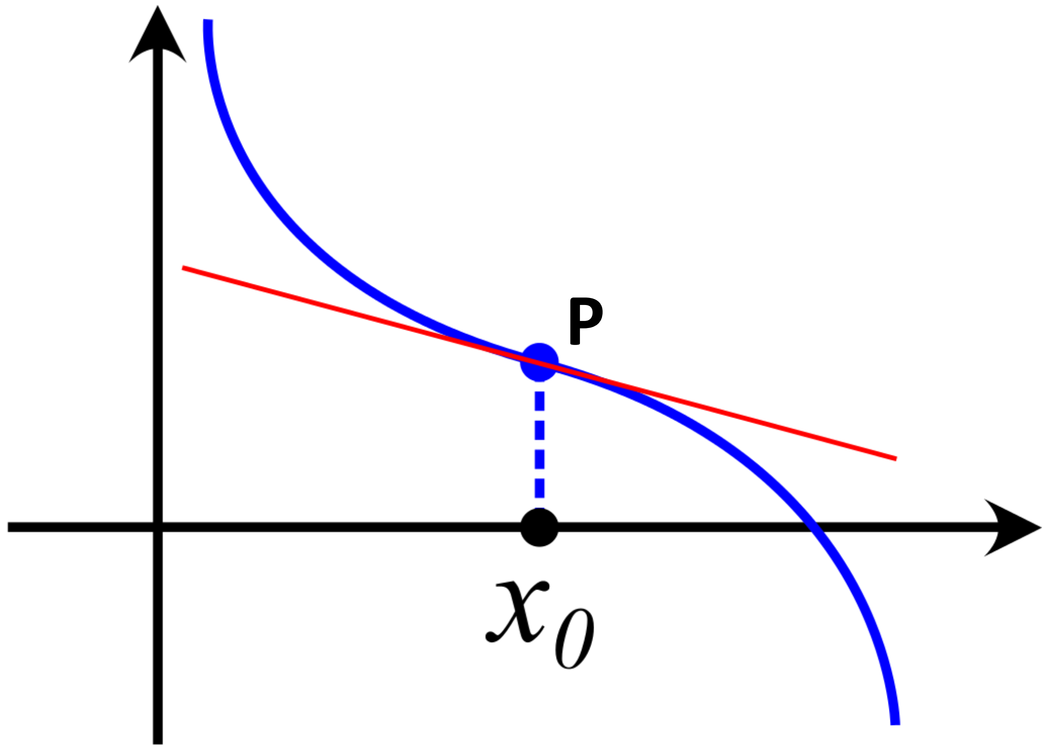
NODE



CUSP

- <u>Point of Inflexion :</u>

A curve has inflexion at P if it changes from concavity. upward to concavity downwards, or vice versa, as a point moving along the curve passes through P

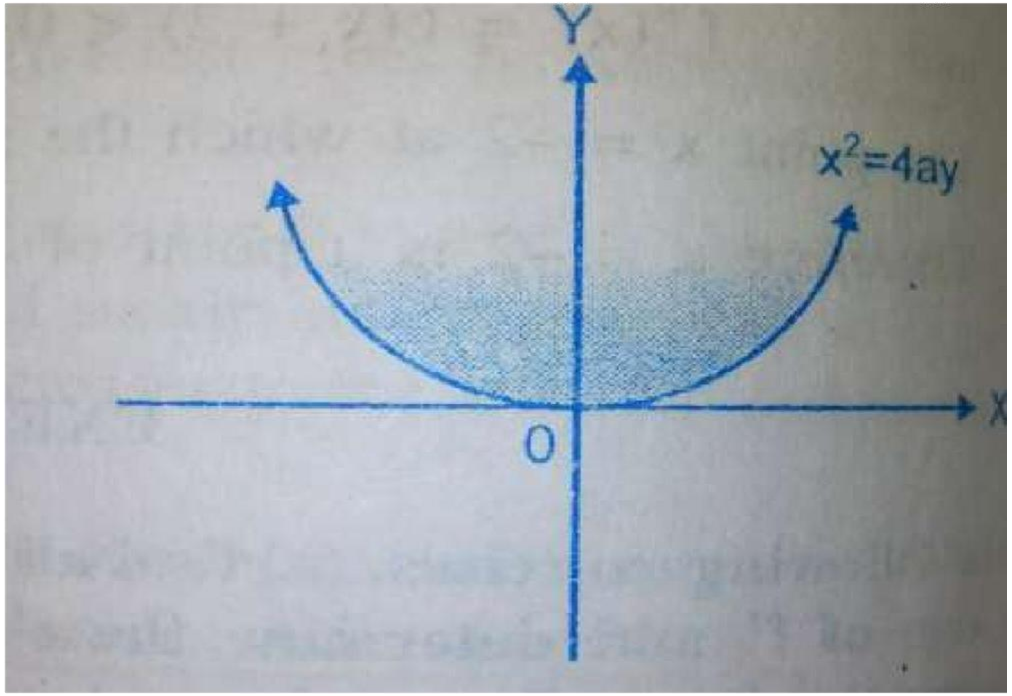


Rules for Tracing of Cartesian Curves

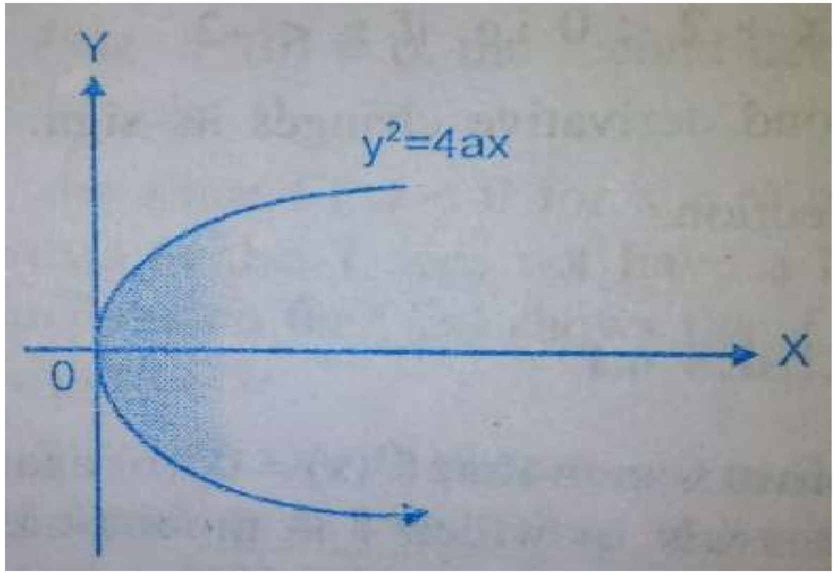
Symmetry: Find out whether the curve is symmetric about any line or a point. The various kinds of symmetry arising from the form of the equation are as follows:

i) symmetric about the y-axis

If the equation of the curve remain unchanged when x is replace by –x and the curve is an even function of x. (OR Even power of x)

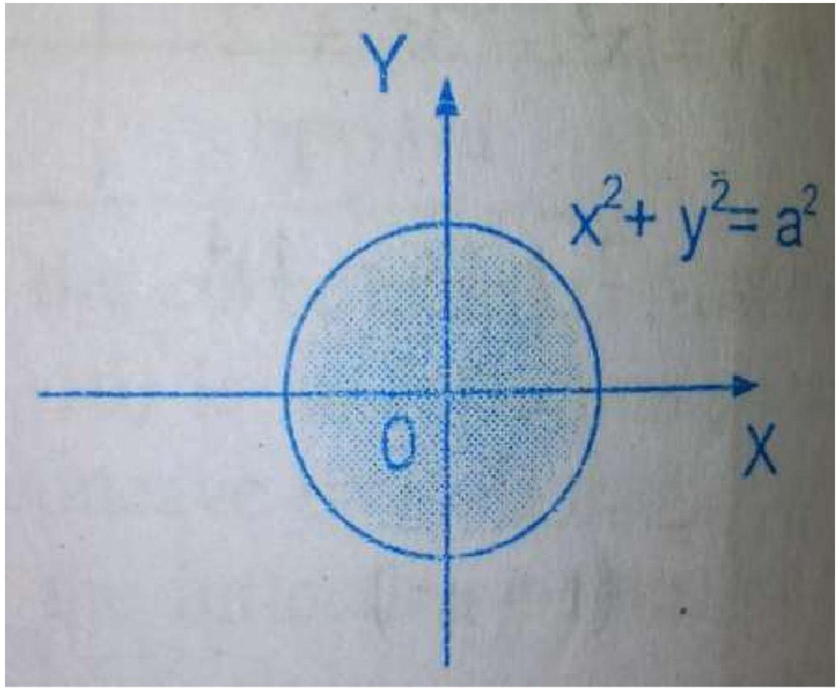


ii) <u>symmetric about x-axis</u>



If the equation of the curve remains unchanged when y is replaced by –y and the curve is an even function of y.(OR even power of y)

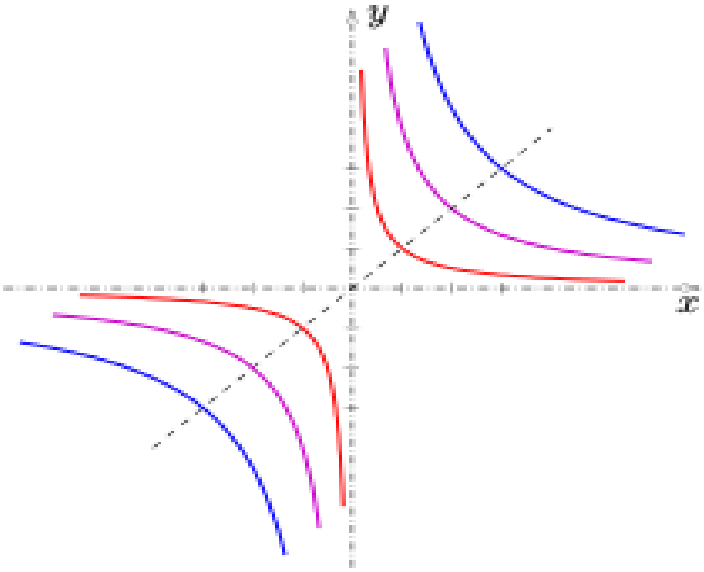
<u>iii) symmetric about both x and y axes</u>



If the equations of the curve is such that the powers of x and y both are even everywhere then the curve is symmetrical about both the axes. for example, the circle..

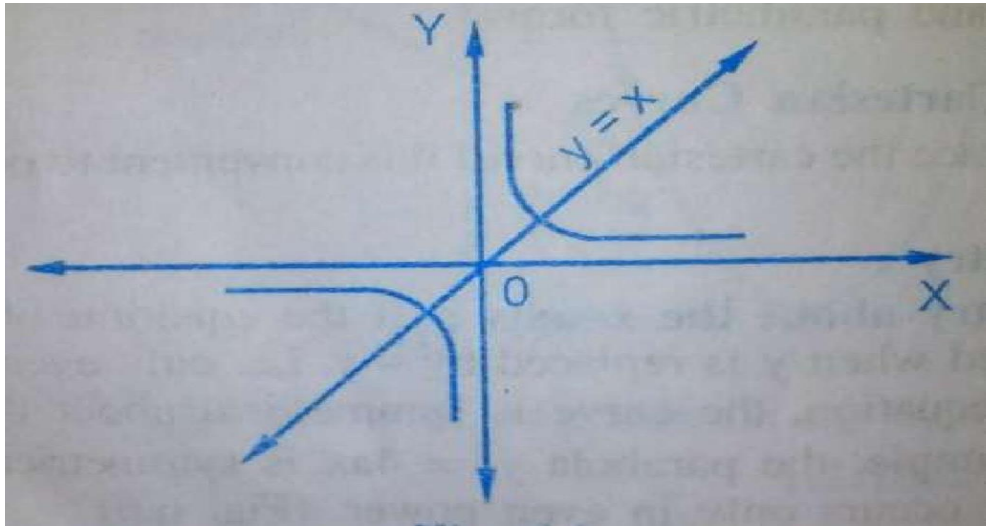
iv)<u>symmetry in the opposite quadrants</u>

If the equation of the curve remains unchanged when x is replaced by –x and y is replaced by –y simultaneously, the curve is symmetrical in opposite quadrant. for example, the hyperbola…,$xy = c^2$



v) <u>symmetrical about the line y = x</u>

If the equation of the curve remains unchanged we replace x by y and y by x simultaneously then the curve is symmetric about the line y=x



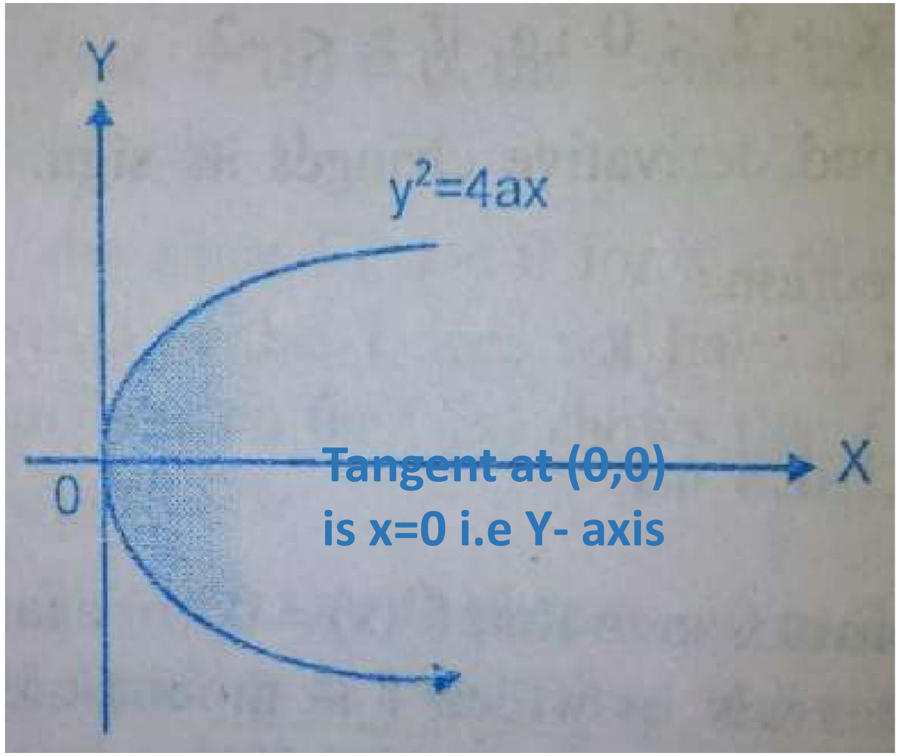
Origin:

(A)Curve through the origin:-

If the equations of the curve does not contain any constant term, the curve passes through the origin. Thus it will pass through the origin if the equation is satisfied by (0,0)

(B)Tangents at the origin:-

The equations of the tangents to the curve at the origin is obtained by equating the lowest degree terms taken togeather in the given equation to zero, provided the curve passes through the origin.



(C) To find Intersection with coordinate axes:-

Put y = 0 for x-axis and x=0 for y-axis.

Tangent at any other point:-

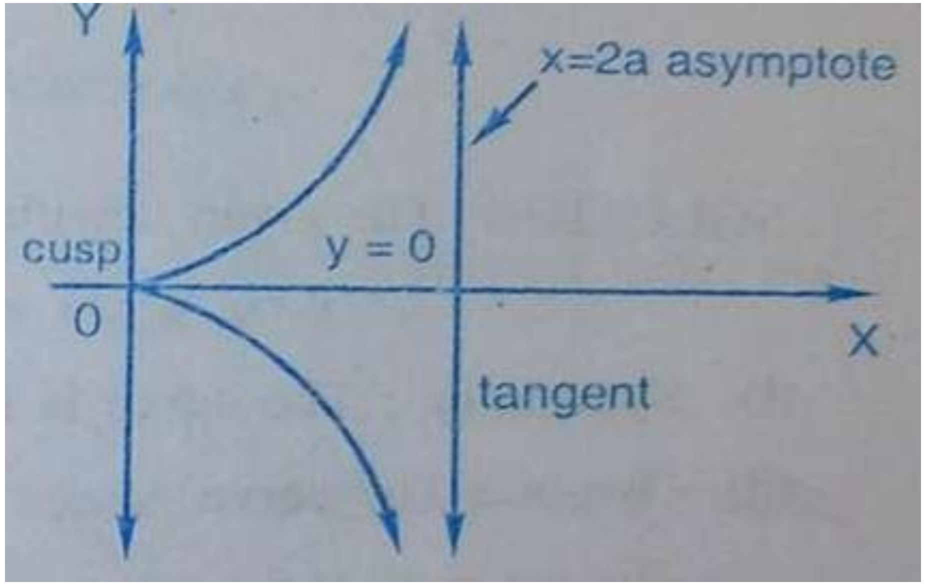
- If $\left[\frac{dy}{dx}\right]_{x_1,y_1} = 0$ Then tangent is parallel to X-axis or X-axis itself
- If $\left[\frac{dy}{dx}\right]_{x_1,y_1} = \pm\infty$ Then tangent is parallel to Y-axis or Y-axis itself
- If $\left[\frac{dy}{dx}\right]_{x_1,y_1} > 0$ Then curve strictly increasing in that interval
- If $\left[\frac{dy}{dx}\right]_{x_1,y_1} < 0$ Then curve strictly Decresingin that interval

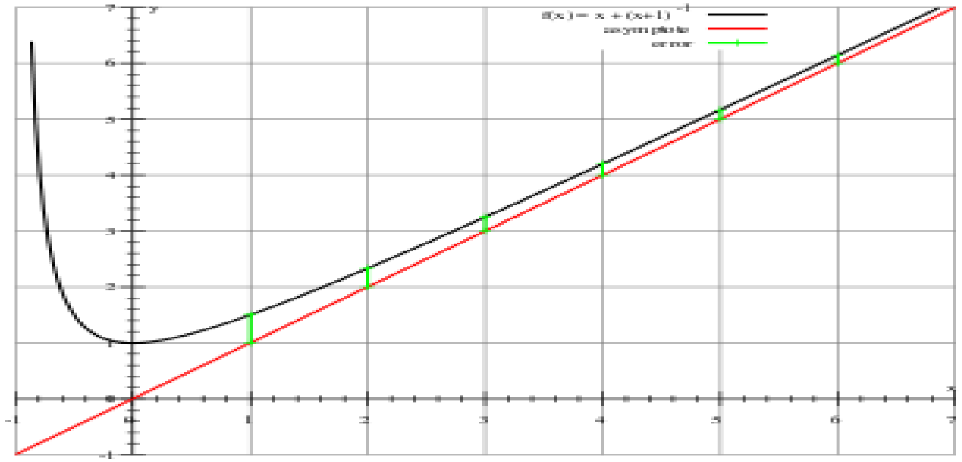
Asymptotes

Asymptotes are the tangents to the curve at infinity.

We shall consider separately the cases which arises when an asymptote is

(a) parallel to either co-ordinate axis or
(b) an oblique asymptote.





- To find the asymptotes parallel to x-axis, equate the coefficient of the highest degree terms in x to zero.
- **Example:** The curve $y^2(a^2+x^2)=a^2x^2$ has asymptote at y $= \pm a$.

- To find the asymptotes parallel to y-axis, equate the coefficient of the highest degree terms in y to zero.

Example: The curve $y^2(2a-x)=x^3$ has asymptote at $x=2a$.

- For oblique asymptotes:-Put y = mx + c in the equation f(x, y)=0 and equate to zero the coefficients of two successive highest powers of x, giving equations to determine m and c.

Example: The curve $x^3+y^3=3axy$ has a an oblique asymptote $y=-x-a$.

EXAMPLE : **Find the asymptotes of the curve $y^3 - x^2 (6 - a) = 0$.**

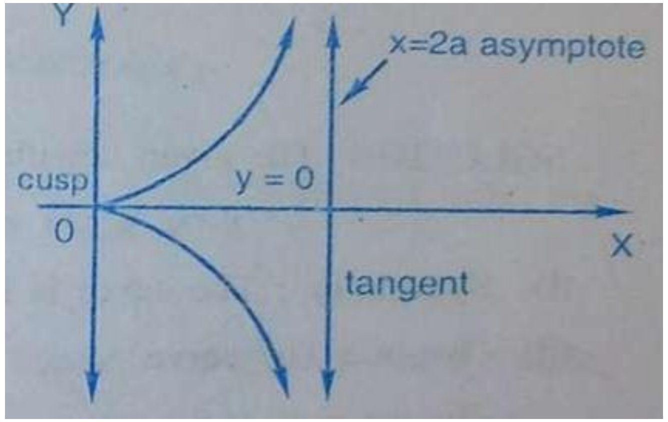
Example

Regions where no part of the Curve lies (Region of absence):

(a) If it is possible to express the equation as y = f(x) and if y becomes imaginary for some value of x > a ,then no part of the curve exists beyond x = a.

Example: In the curve $y^2(2a - x) = x^3$,

for $x < 0$ and $x > 2a,$ y is imaginary.

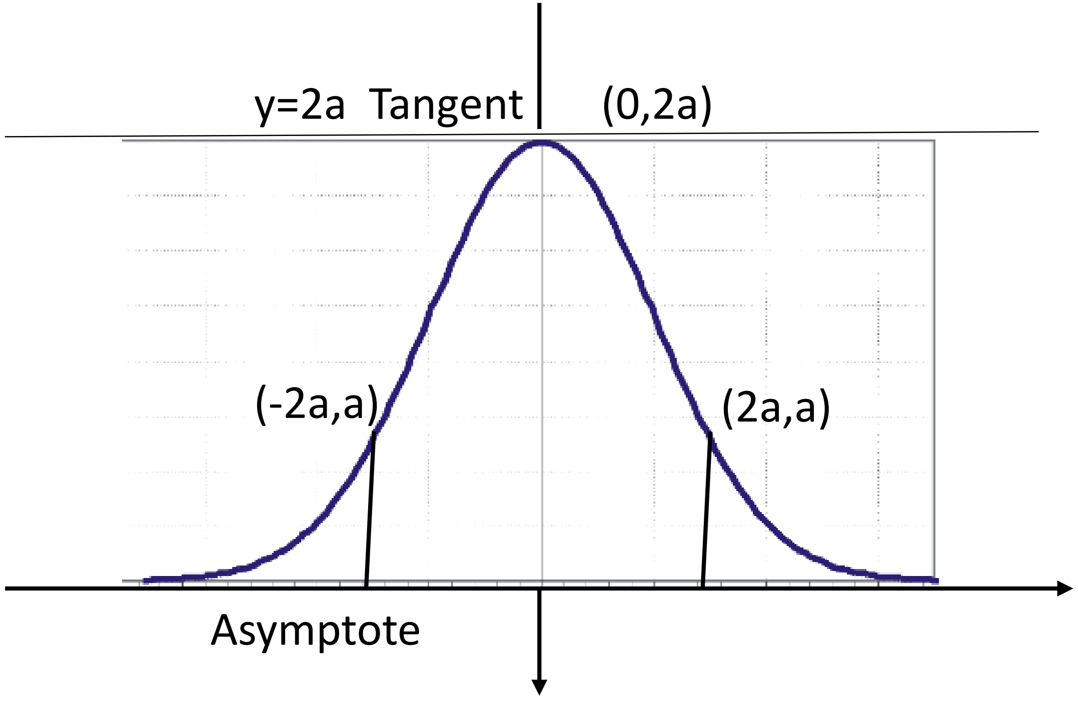


(b) Similarly , if it it possible to express the equation as x = f(y) and if x becomes imaginary for some value of y > b, then no part of the curve exists beyond y = b.

Example: In the curve $a^2x^2 = y^2(2a - y)$.

for $y > 2a,$ x is imaginary.

Example: Trace the curve $y(x^2 + 4a^2) = 8a^3$



1. **Origin:** Does not pass through origin

2. **Points of intersection:** (0, 2a).

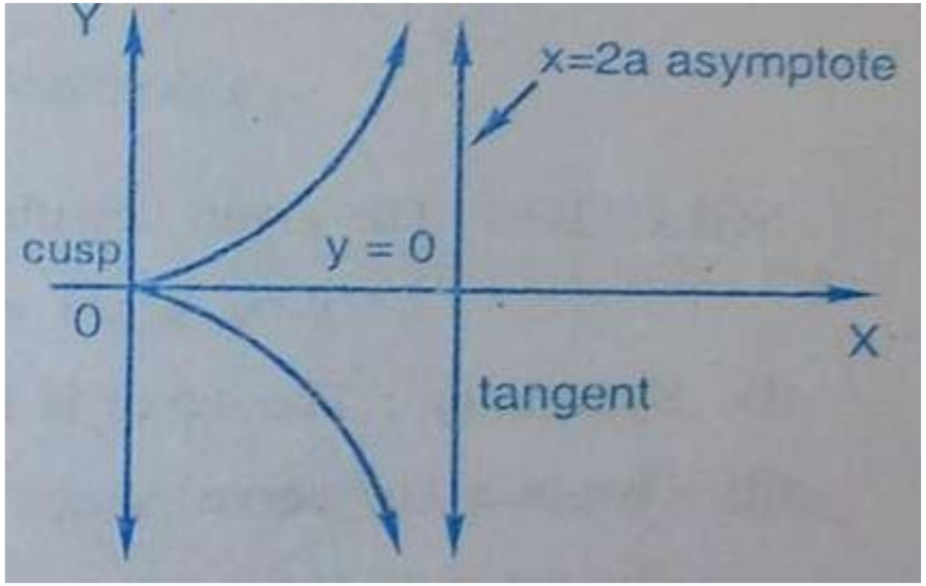
3. Y=2a is **tangent** at origin.

4. **Asymptote** is parallel to x – axis $y = 0$ is asymptote.

5. **Region of Absence:** For y <0 and y> 2a curve does not exist.

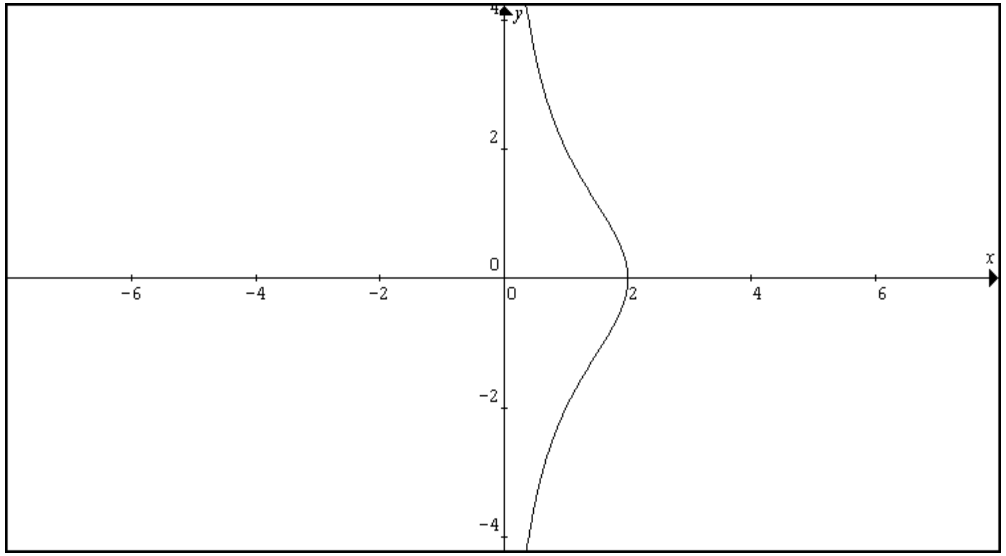
1. Trace the curve : $y^2(2a - x) = x^3$

Symmetry:	
To find Points of Intersection: Put y = 0 and x = 0 for x-axis and y-axis respectively.	
Tangents at the origin: The equations of the tangents to the curve at the origin is obtained by equating the lowest degree terms in x and y in the given equation to zero, provided the curve passes through the origin.	
Tangent at any other point: Find $\frac{dy}{dx}$ at that point.	
Asymptotes: To find asymptote parallel to x-axis (or y-axis) Equate the coefficient of the highest degree terms in x (or y) to zero).	
Region of absence: Find value of x where y becomes imaginary.	


Y
x=2a asymptote
cusp
y = 0
0
X
tangent

$$xy^2 = a^2(a-x)$$

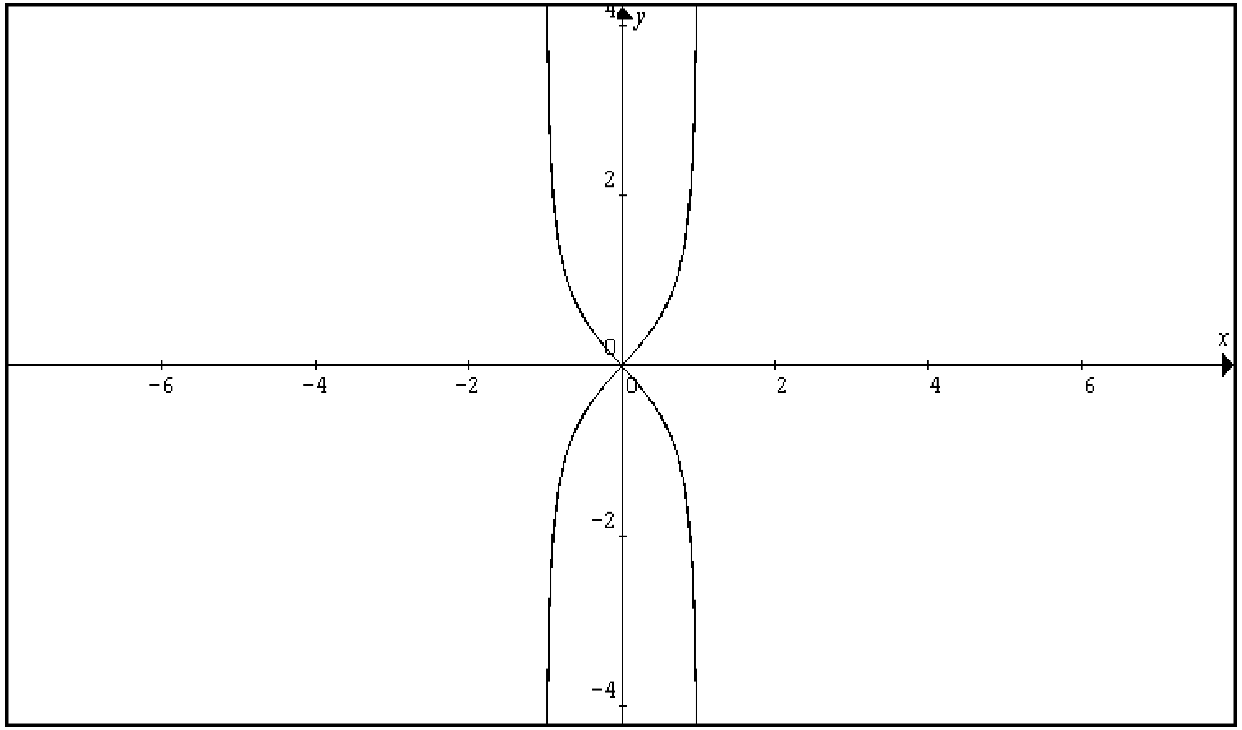
Symmetry:	
To find Points of Intersection: Put y = 0 and x = 0 for x-axis and y-axis respectively.	
Tangents at the origin: The equations of the tangents to the curve at the origin is obtained by equating the lowest degree terms in x and y in the given equation to zero, provided the curve passes through the origin.	
Tangent at any other point: Find $\frac{dy}{dx}$ at that point.	
Asymptotes: To find asymptote parallel to x-axis (or y-axis) Equate the coefficient of the highest degree terms in x (or y) to zero).	
Region of absence: Find value of x where y becomes imaginary.	



y
x
4
2
0
-2
-4
-6
-4
-2
0
2
4
6

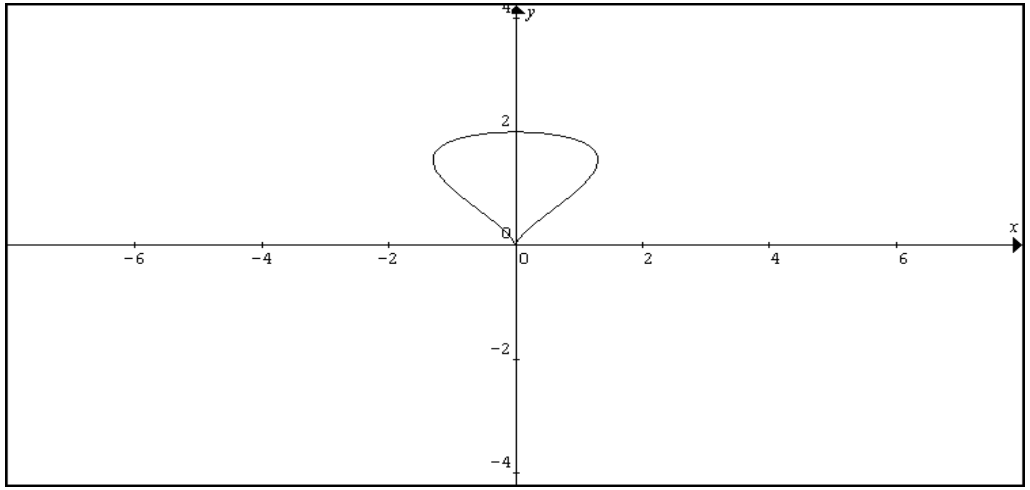
$$x^2y^2 = a^2(y^2 - x^2)$$

Symmetry:	
To find Points of Intersection: Put y = 0 and x = 0 for x-axis and y-axis respectively.	
Tangents at the origin: The equations of the tangents to the curve at the <u>origin</u> is obtained by equating the <u>lowest degree terms</u> in x and y in the given equation to zero, provided the curve passes through the origin.	
Tangent at any other point: Find $\frac{dy}{dx}$ at that point.	
Asymptotes: To find asymptote parallel to x-axis (or y-axis) Equate the coefficient of the highest degree terms in x (or y) to zero).	
Region of absence: Find value of x where y becomes imaginary.	

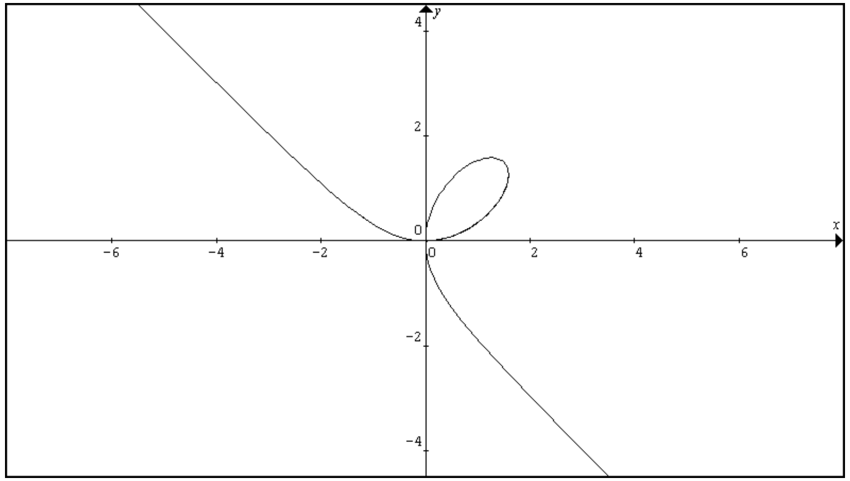


y
x
4
2
0
-2
-4
-6
-4
-2
0
2
4
6

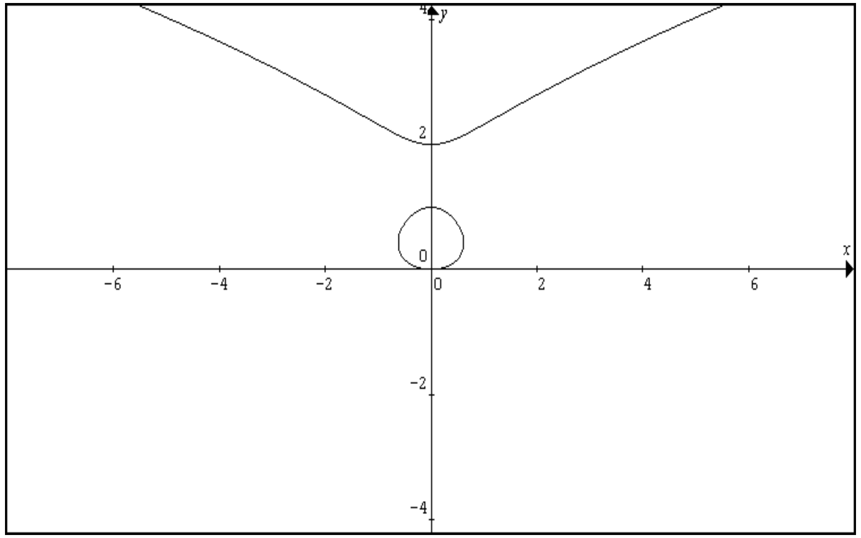
$$a^2x^2 = y^3(2a - y)$$



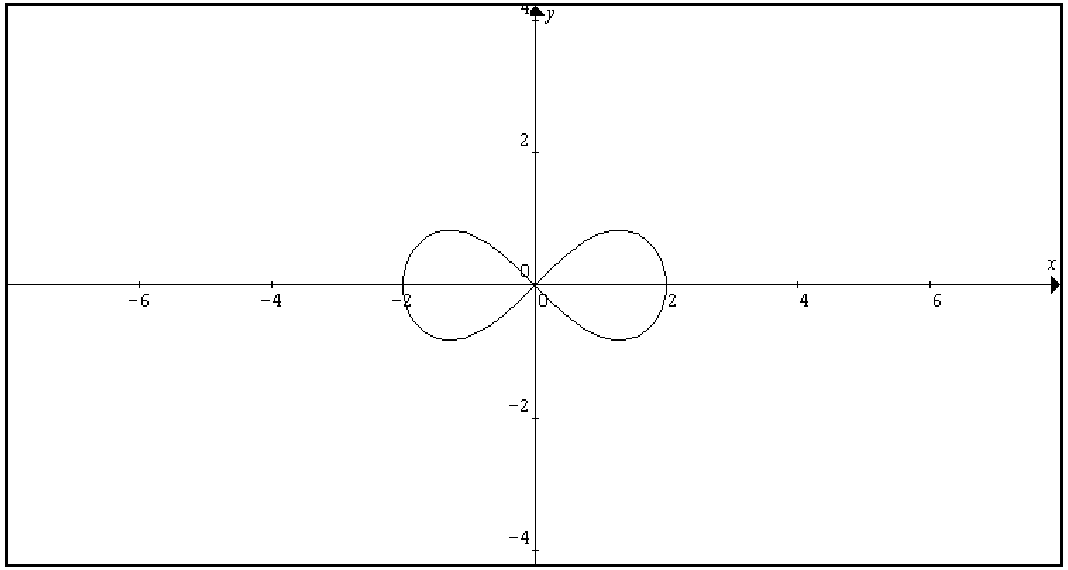
$$x^3 + y^3 = 3axy$$



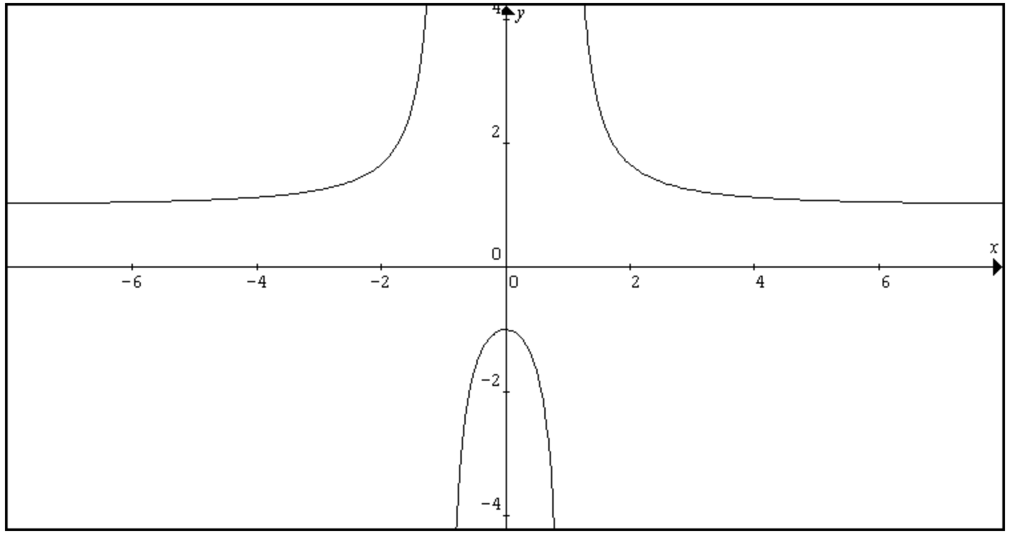
$$x^2 = y(y-1)(y-2)$$



$$y^2(a^2+x^2)=x^2(a^2-x^2)$$



$$y = \frac{(x^2+1)}{(x^2-1)}$$



Tracing of Polar Curves

- To trace a polar curve $r = f(\theta) or\ g(r, \theta) = c$, c a constant, we use the following different features of the curve.

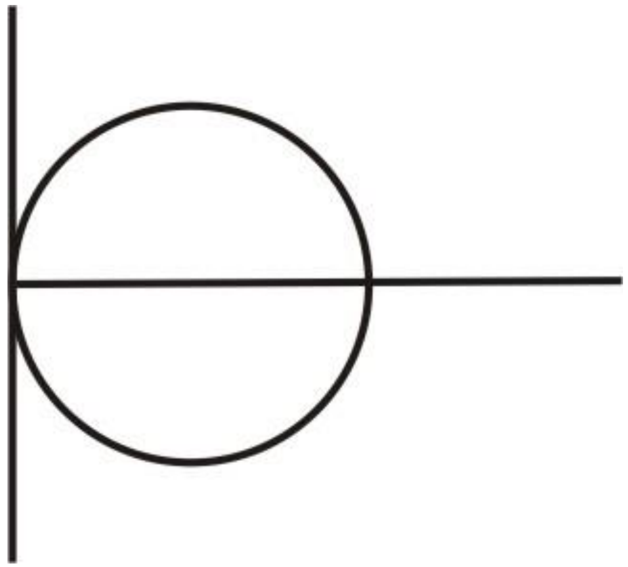
  - Symmetry.
  - Passing through the pole ?
  - Angle between r and tangent.
  - Table showing values of r and θ.

Symmetry

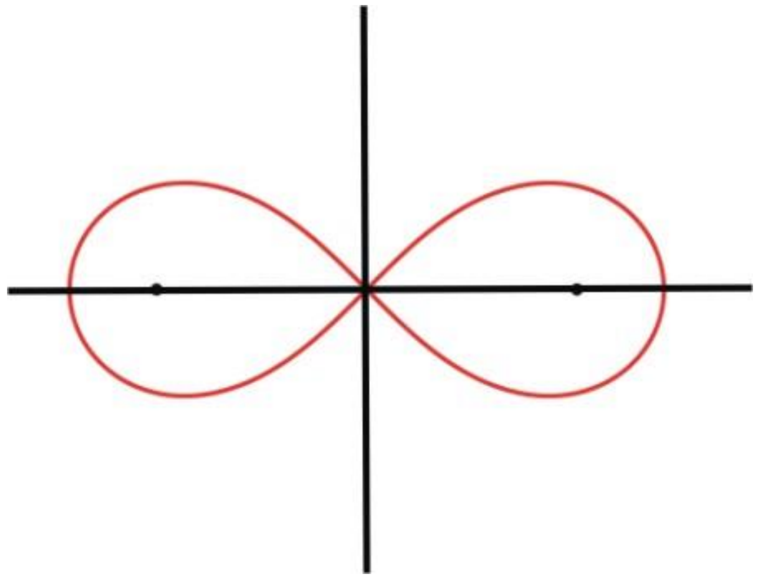
i) If the equation of the curve is an even function of θ, then the curve is symmetrical about the initial line.

ii) If the equation is an even function of r, the curve is symmetric about the pole.

$r = a\cos\theta$ Circle touching Y axis at origin



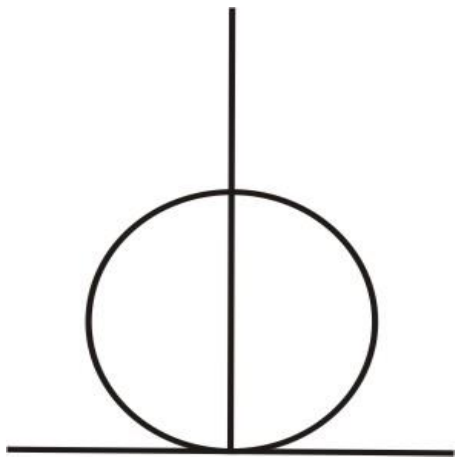
$r^2 = a^2\cos 2\theta$



iii) If the equation remains unaltered when θ is replaced by $-\theta$ and r is replace by -r then curve is symmetric about the line through the pole and perpendicular to the initial line.

iv) If the equation remains unaltered when θ is replaced by $\pi - \theta$ then curve is symmetric about the line through the pole and perpendicular to the initial line.

$r = a\sin\theta$ Circle touching X axis at origin



Passing through the Pole ?

i) The pole will lie on the curve if for some value of θ, r becomes zero.

ii) To find tangents at pole put r = 0 , the value of θ gives the tangent at the pole

iii) Care should be taken of the points where the curve cuts the initial line and the line $\theta = \pi/2$.

- **Angle between the radius vector and tangent [ϕ] :**

Use the formula $\tan \phi = r\frac{d\Theta}{dr}$ and find ϕ also the points where $\phi = 0$ or ∞ i.e. find the points where the tangent coincides with the radius vector or is perpendicular to it

- **The table showing values of r for different values of Θ:** is very useful in plotting a polar curve .
  Also find the values of Θ at which $r = 0$ or $r = \infty$

Region of absence of the curve

i) Solve the equation for r and consider how r varies as θ increases from 0 to +∞ and also when θ decreases from 0 to- ∞. If necessary form a table of values of r and θ.

ii) If for some values of θ, say α and β, r is imaginary, this means that no branch of the curve exists between lines θ = α and θ = β.

iii) If the maximum value of r is 'a', the entire curve will lie within a circle of radius a.

iv) If the least value of r is 'b',the entire curve will lie outside the circle of radius b.

v) In most of polar equations, only periodic functions $\sin\theta$ and $cos\,\theta$ occur and hence values of θ from 0 to 2πshould only be considered.

Example: Trace the curve (Bernoulli's Lemniscate) $r^2 = a^2 \cos 2\theta$

Symmetry: If r = f(θ) remains unchanged by replacing **θ by –θ, r by –r, both and θ by $\pi - \theta$**	
If θ = 0 → $r = 0$ then the pole lies on the curve.	
Put r = 0, then values of θ gives tangent at the pole.	
Angle between the radius vector and tangent [ϕ] : Use the formula tan ϕ = r$\frac{d\Theta}{dr}$ and find ϕ also the points where ϕ = 0 or ∞ i. e. find the points where the tangent coincides with the radius vector or is perpendicular to it	
The table showing values of r for different values of θ: Find value of r for different values of θ.	

Example: Trace the curve (Pascal's Limacon) $r = a + b\cos\theta$.

Symmetry: If r = f(θ) remains unchanged by replacing **θ by –θ, r by –r, both and θ by $\pi - \theta$**	
If $\theta = 0 \rightarrow r = 0$ then the pole lies on the curve.	
Put r = 0, then values of θ gives tangent at the pole.	
Angle between the radius vector and tangent [ϕ] : Use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ also the points where $\phi = 0$ or ∞ i. e. find the points where the tangent coincides with the radius vector or is perpendicular to it	
The table showing values of r for different values of θ: Find value of r for different values of θ.	

Example: Trace the curve (Rose curve) r $= a\, cos 2\theta$

Symmetry: If r = f(θ) remains unchanged by replacing **θ by –θ, r by –r, both and θ by $\boldsymbol{\pi - \theta}$**	
If θ = 0 → $r = 0$ then the pole lies on the curve.	
Put r = 0, then values of θ gives tangent at the pole.	
Angle between the radius vector and tangent [ϕ] : Use the formula tan ϕ = r$\frac{d\Theta}{dr}$ and find ϕ also the points where ϕ = 0 or ∞ i. e. find the points where the tangent coincides with the radius vector or is perpendicular to it	
The table showing values of r for different values of θ: Find value of r for different values of θ. No of loops = n if n is odd and 2n if n is even.	

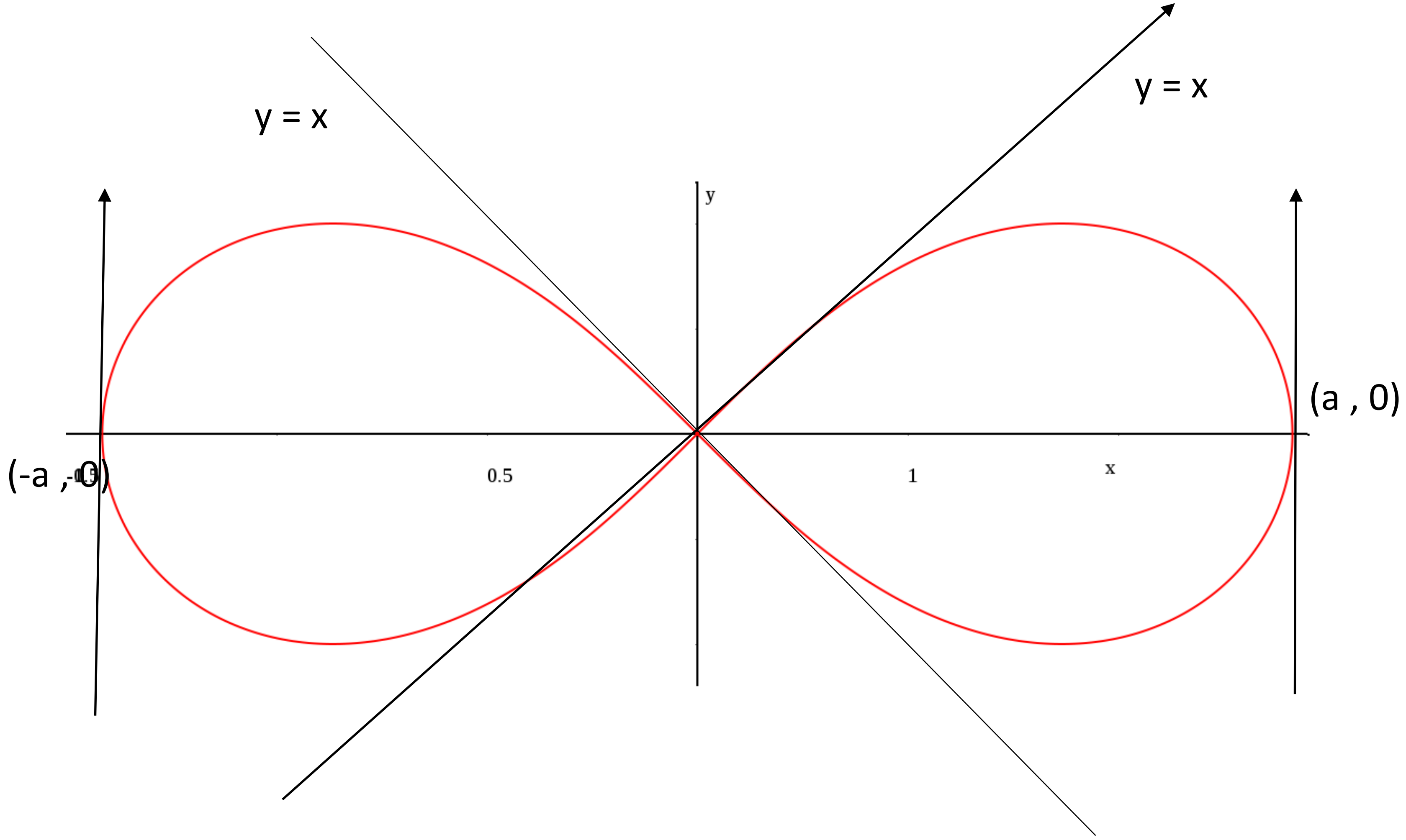
Example: Trace the curve (Rose curve) $r = a \sin 3\theta$

Symmetry: If r = f(θ) remains unchanged by replacing **θ by –θ, r by –r, both and θ by $\boldsymbol{\pi - \theta}$**	
If θ = 0 → $r = 0$ then the pole lies on the curve.	
Put r = 0, then values of θ gives tangent at the pole.	
Angle between the radius vector and tangent [ϕ] : Use the formula tan ϕ = r$\frac{d\Theta}{dr}$ and find ϕ also the points where ϕ = 0 or ∞ i. e. find the points where the tangent coincides with the radius vector or is perpendicular to it	
The table showing values of r for different values of θ: Find value of r for different values of θ. No of loops = n if n is odd and 2n if n is even.	

Example: Trace the curve (Cardioide) $r = \frac{a}{2}(1 + \cos\theta)$

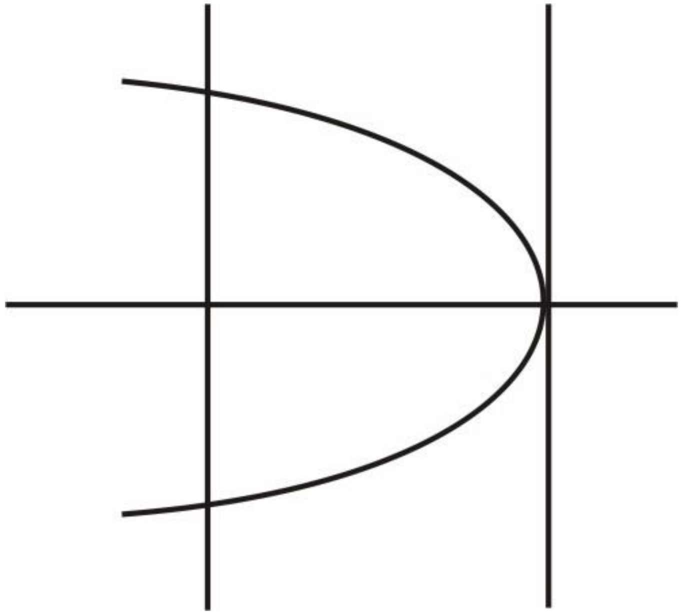
Symmetry: If r = f(θ) remains unchanged by replacing **θ by –θ, r by –r, both and θ by $\pi - \theta$**	
If θ = 0 → $r = 0$ then the pole lies on the curve.	
Put r = 0, then values of θ gives tangent at the pole.	
Angle between the radius vector and tangent [ϕ] : Use the formula tan ϕ = $r\frac{d\Theta}{dr}$ and find ϕ also the points where ϕ = 0 or ∞ i. e. find the points where the tangent coincides with the radius vector or is perpendicular to it	
The table showing values of r for different values of θ: Find value of r for different values of θ.	

Example: Trace the curve $r^2 = a^2 \cos 2\theta$

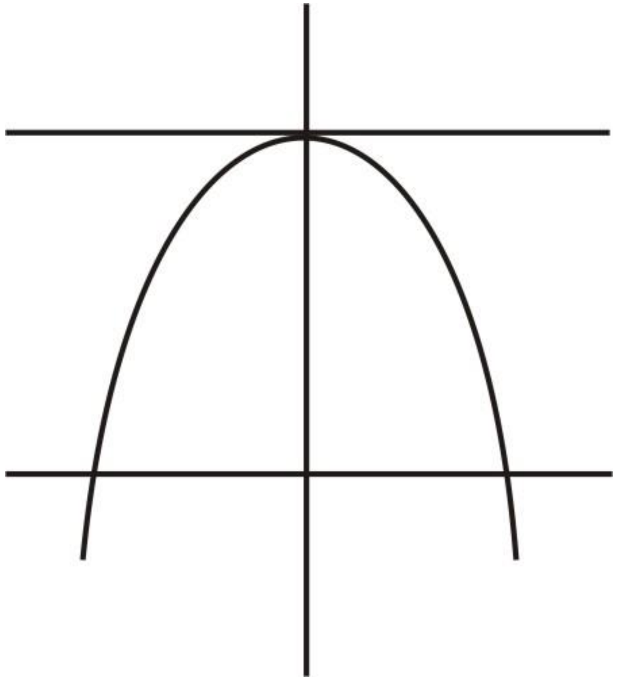


n = – ½ Parabola

$$r = \frac{2a}{1+\cos\theta}$$

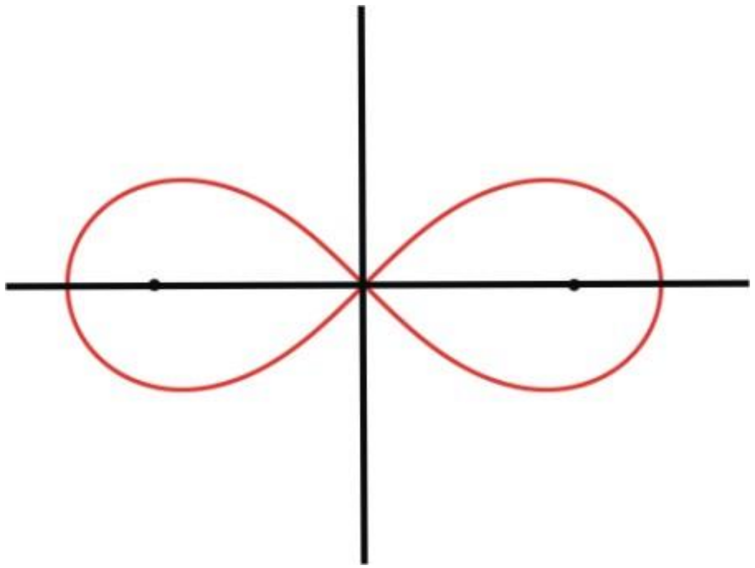


$$r = \frac{2a}{1+\sin\theta}$$

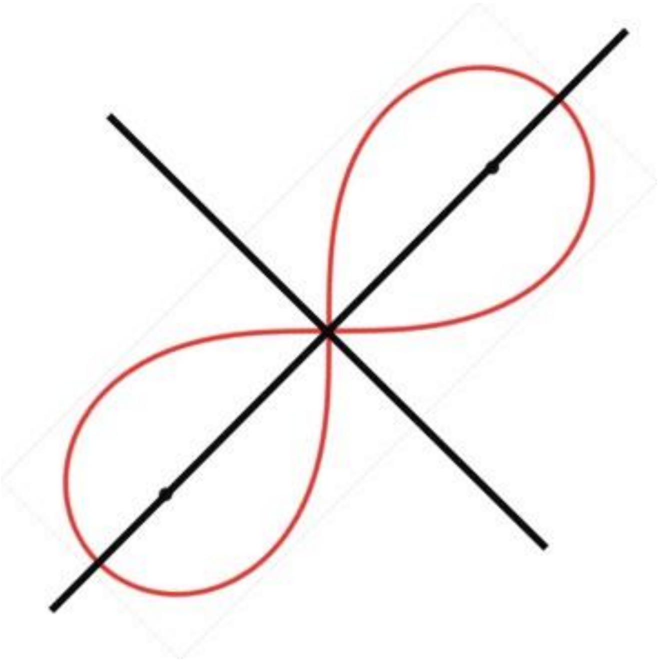


n = 2 Bernoulli's Lemniscates

$$r^2 = a^2 \cos 2\theta$$

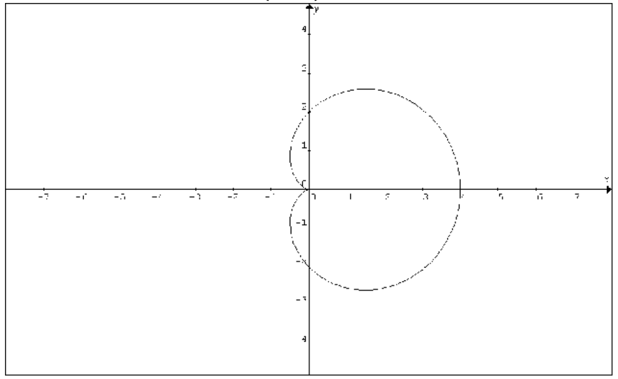


$$r^2 = a^2 \sin 2\theta$$

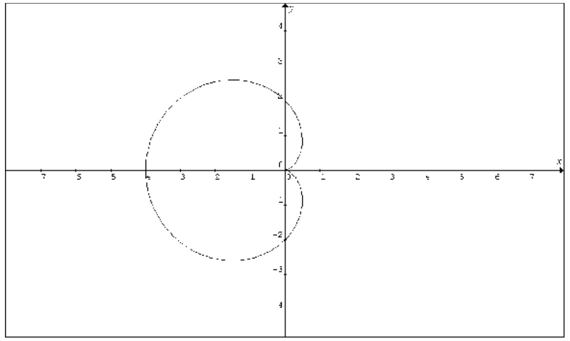


n = 1/2 Cardiode

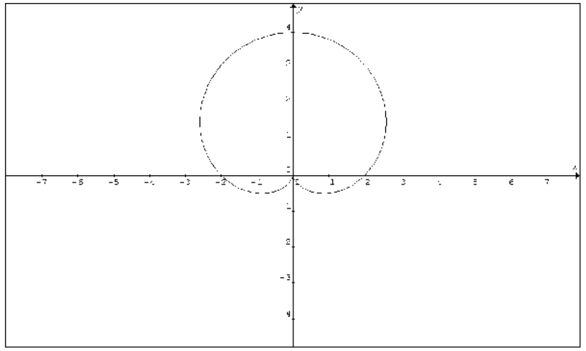
$$r^{1/2} = a^{1/2}\cos\left(\frac{\theta}{2}\right), \text{ i.e. } r = \frac{a}{2}(1+\cos\theta)$$



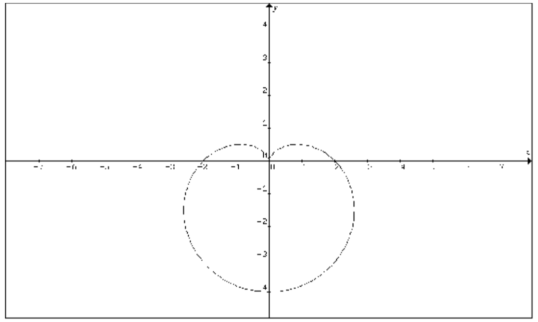
$$r = \frac{a}{2}(1-\cos\theta)$$



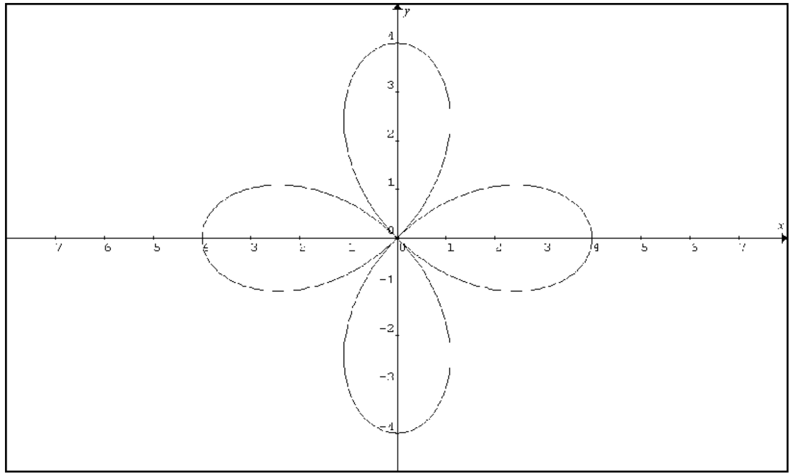
$$r = \frac{a}{2}(1+\sin\theta)$$



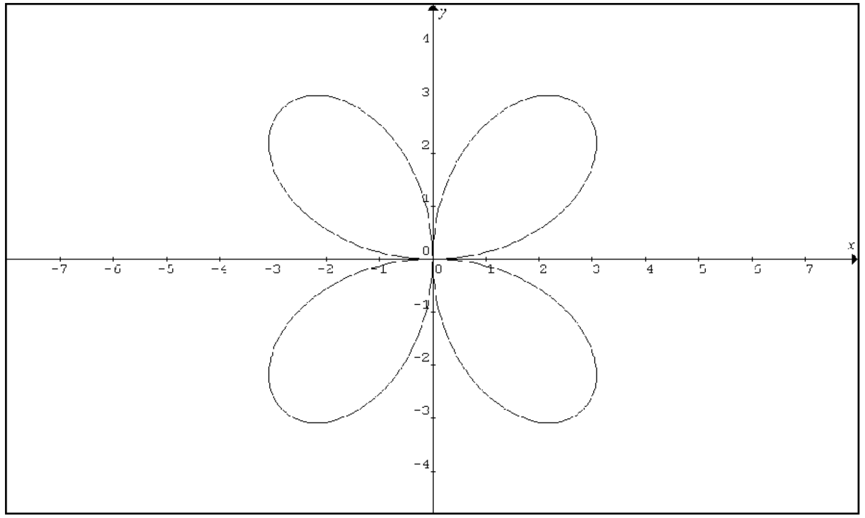
$$r = \frac{a}{2}(1-\sin\theta)$$



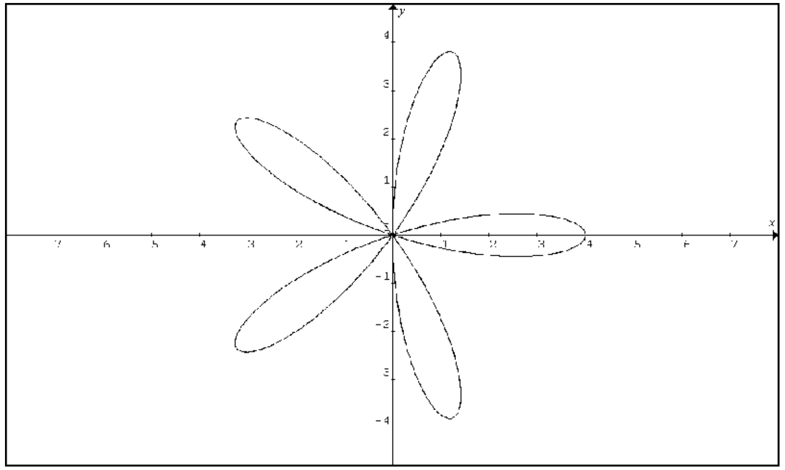
$$r = a\cos 2\theta$$



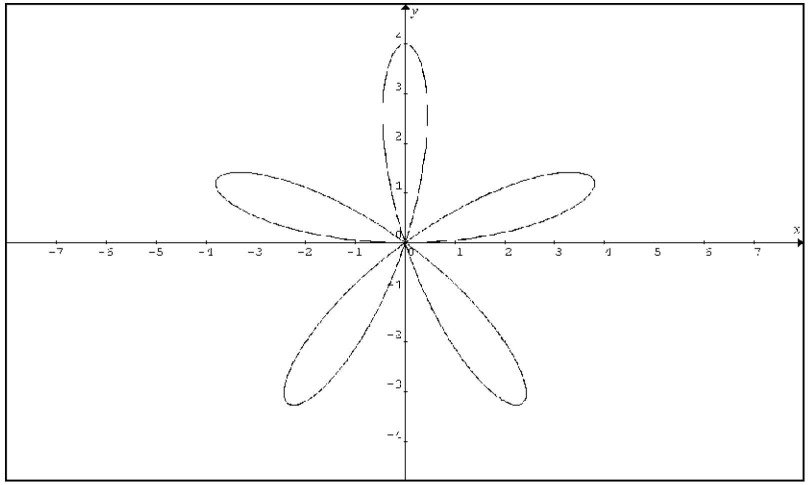
$$r = a\sin 2\theta$$



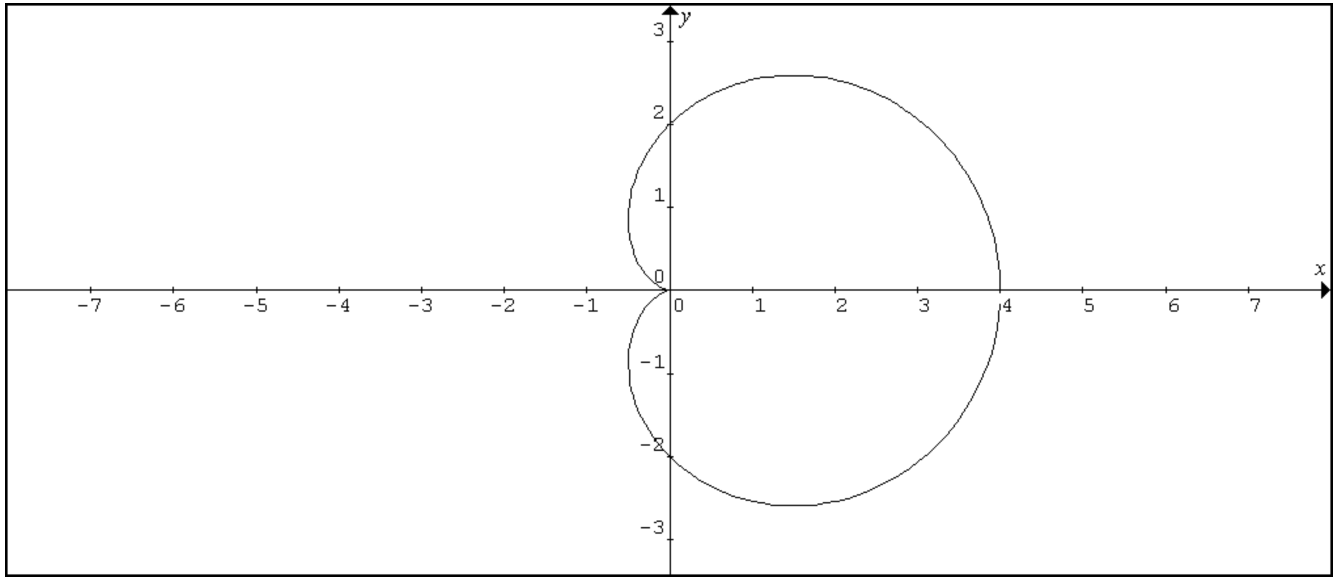
$$r = a\cos 5\theta$$



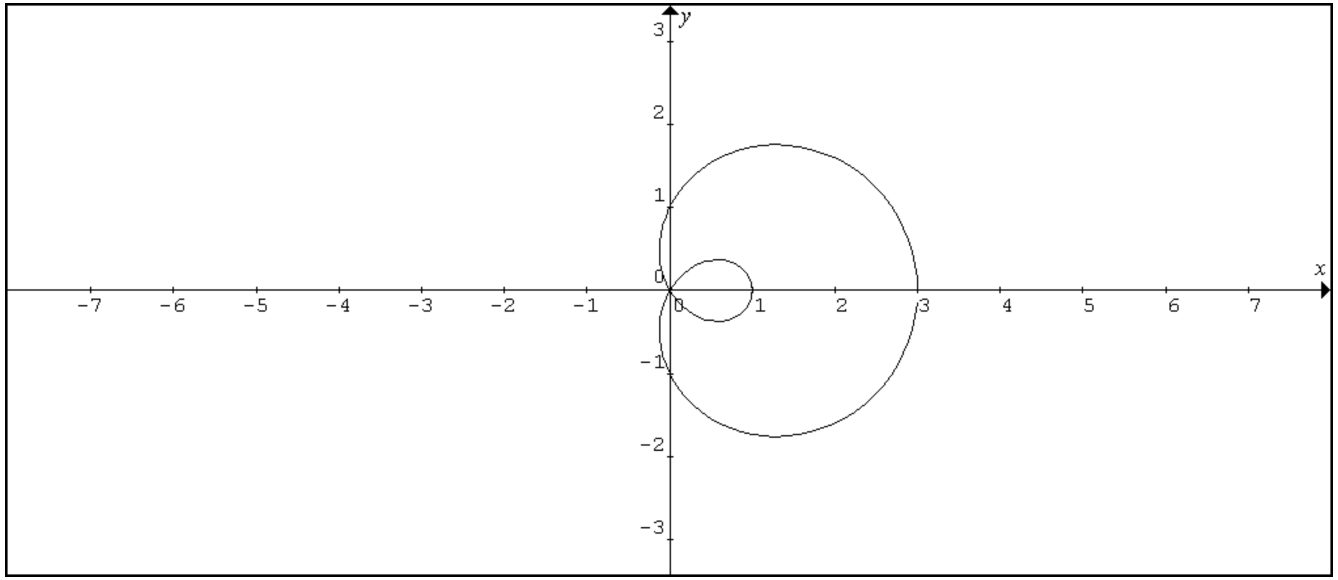
$$r = a\sin 5\theta$$



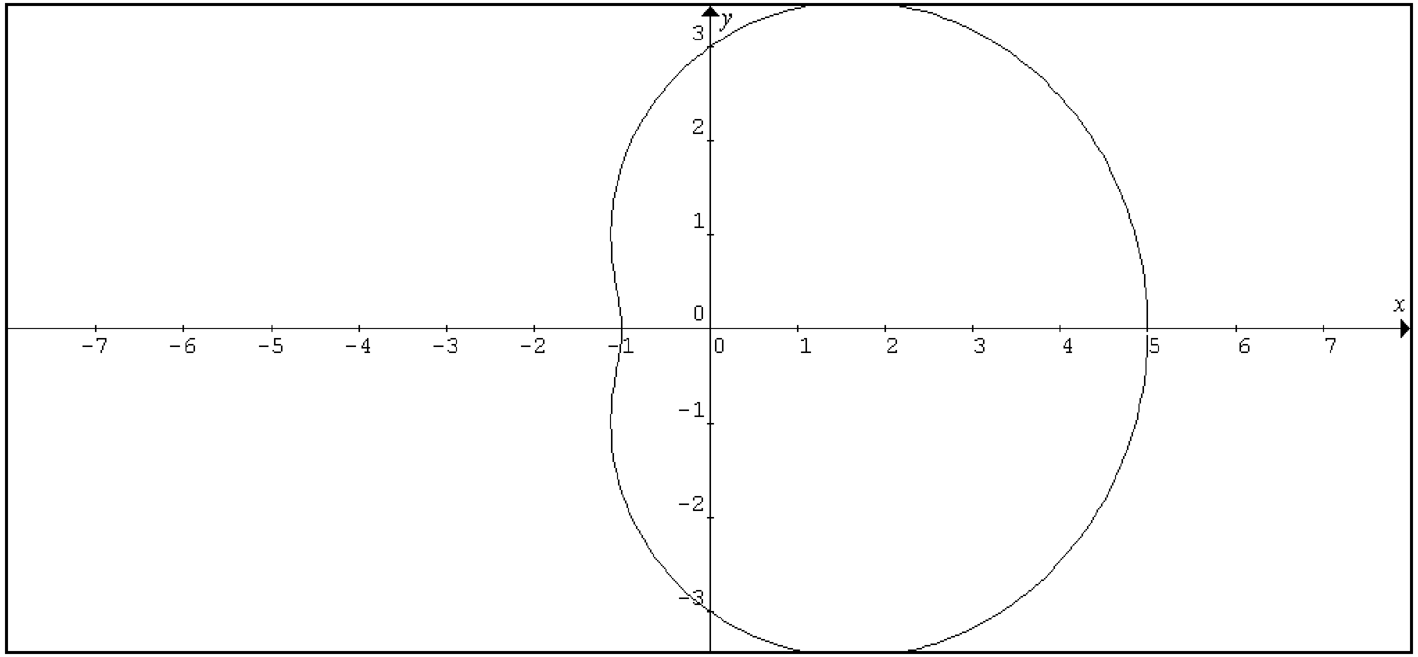
$$r = a(1+\cos\theta)$$



$$r = a + b\cos\theta,\ a < b$$



$$r = a + b\cos\theta,\ a > b$$



Tracing of Parametric Curves

PARAMETRIC EQUATIONS:-

- Some curves—such as the cycloid—are best handled when both x and y are given in terms of a third variable t called a parameter [$x = f(t)$, $y = g(t)$].

- Suppose x and y are both given as functions of a third variable t (called a parameter) by the equations $x = f(t)$ and $y = g(t)$

- Each value of t determines a point (x, y),which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C.

Tracing of Parametric Curves:

- SYMMETRY
- ORIGIN
- SPECIAL POINTS AND TANGENTS
- REGION OF ABSENCE

SYMMETRY

1. If $f(t)$ be an even function of t and $g(t)$ an odd then curve is symmetrical to X-axis

2. If $f(t)$ be an odd function of t and $g(t)$ an Even then curve is symmetrical to Y-axis

3. If both $f(t)$ and $g(t)$ are even or odd then curve is symmetrical in opposite quadrants.

PASSING THROUGH THE ORIGIN?

If on putting $x = 0$ we obtain $y = 0$ for some value of t, then curve passes through origin. Also find the points of intersection and the axes.

SPECIAL POINTS /NATURE OF TANGENTS

Try to find few points on the curve by observation and also those points where $dy/dx=0$ or ∞. Here $x=f(t)$, $y=g(t)$, hence use the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

REGION OF ABSENCE OF CURVE

1. Make a table of values of t, x, y, dx/dt, dy/dt.

2. Find those regions where curve does not exist .

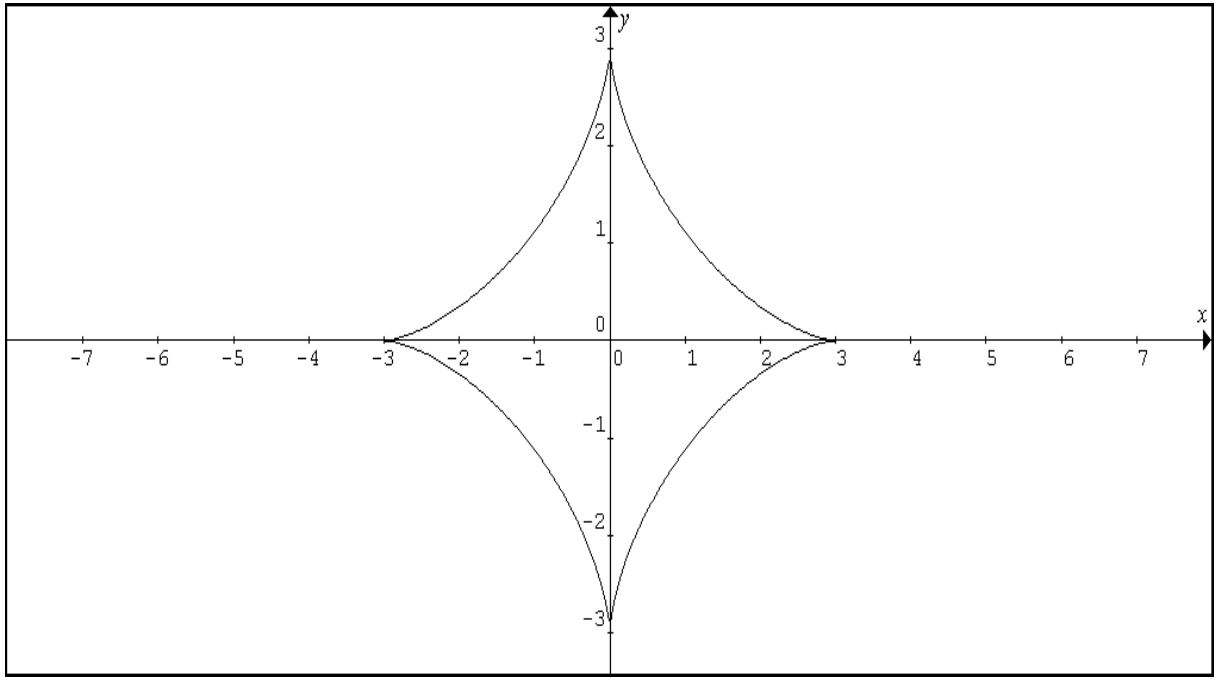
3. If both x and y are periodic function of t, with a common period we need to study the position of the curve for one period only.

$$Trace\ the\ curve\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

$$OR\ x = acos^3 t, y = asin^3 t$$

Symmetry: Put **t = - t . Only x(-t) =-x(t) or** $\boldsymbol{x(\pi - \theta) = -x(\theta)}$ $\rightarrow$ **about y-axis, only y(-t) =-y(t)** $\rightarrow$ **about x-axis, both or** $\rightarrow$ **about origin**	
Find t at which x = y = 0 then the origin lies on the curve.	
Try to find few points on the curve by observation and also those points where dy/dx=0 or ∞.	
Make a table of values of t, x, y, dx/dt, dy/dt.	

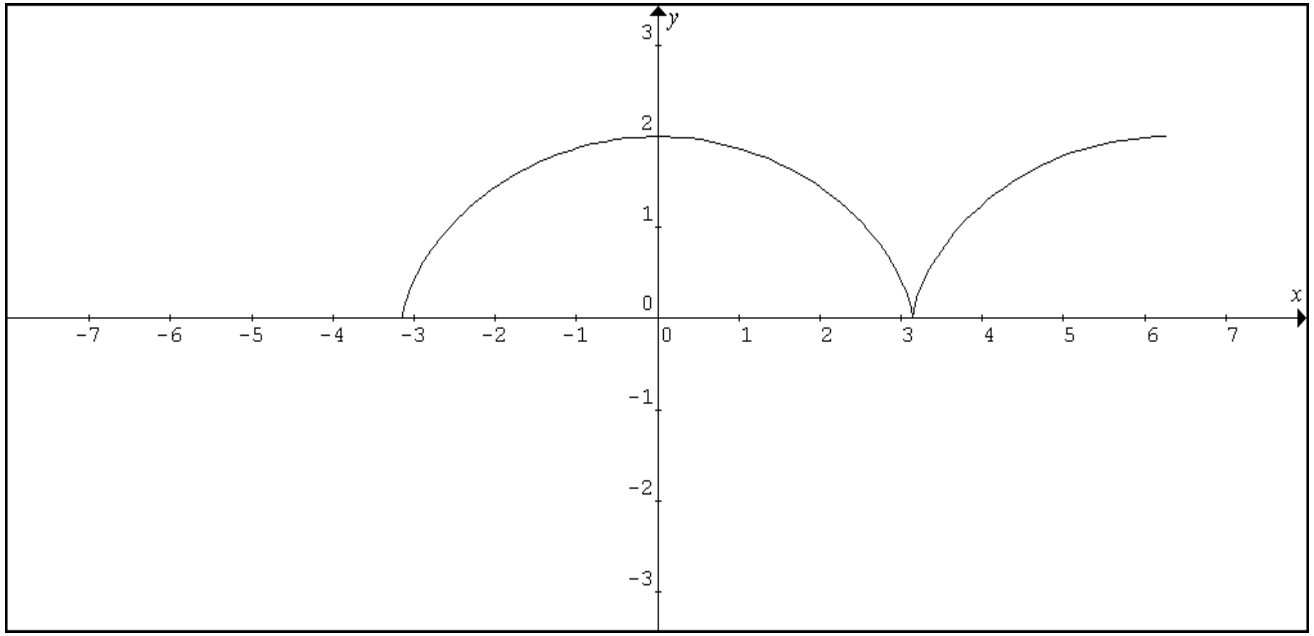
$$x = a\cos^3\theta,\ y = a\sin^3\theta$$



Example: Trace the curve x $= a(\theta + sin\theta)$ and $y = a(1 + cos\theta)$.

Symmetry: Put **t = - t . Only x(-t) =-x(t)or $x(\pi - \theta) = -x(\theta)$ → about y-axis, only y(-t) =-y(t) → about x-axis, both or → about origin**	
Find t at which x = y = 0 then the origin lies on the curve.	
Try to find few points on the curve by observation and also those points where dy/dx=0 or ∞.	
Make a table of values of t, x, y, dx/dt, dy/dt.	

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$$

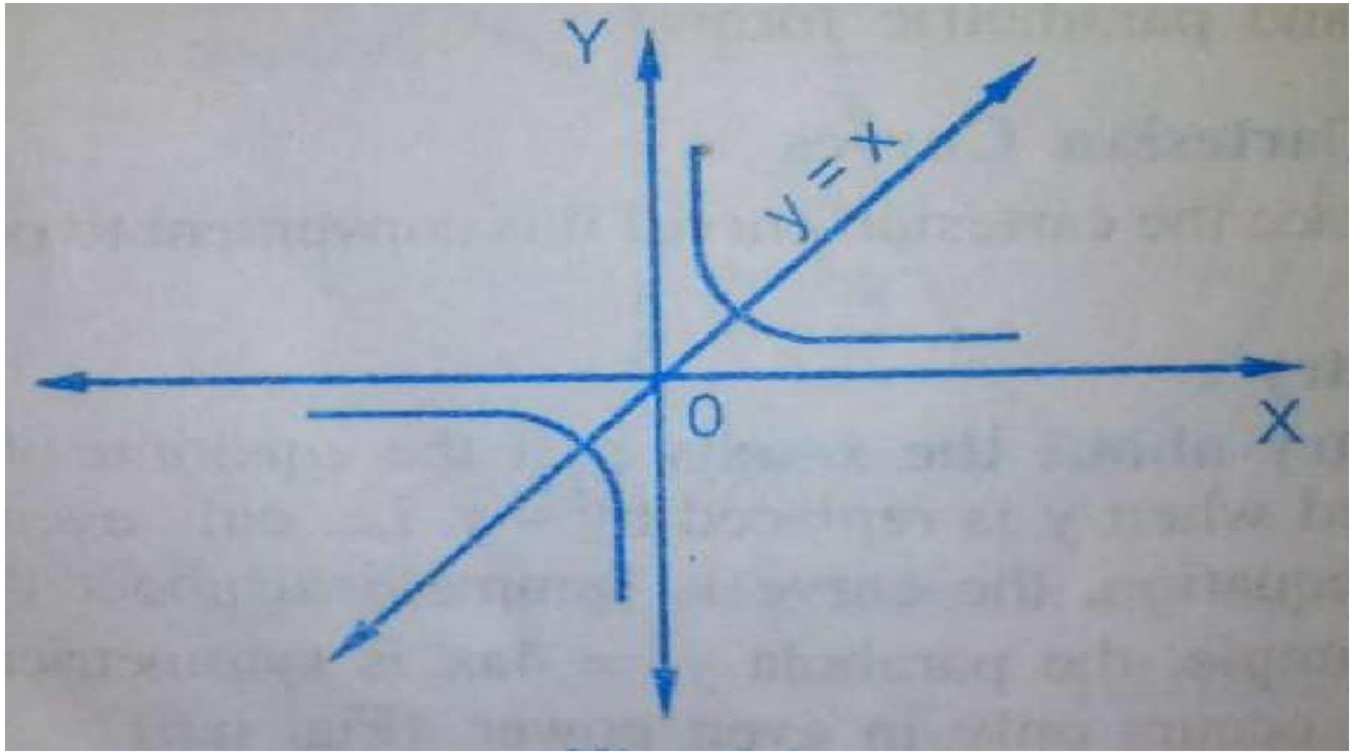


Example: Trace the curve x $= at\ and\ y = \frac{a}{t^2}$

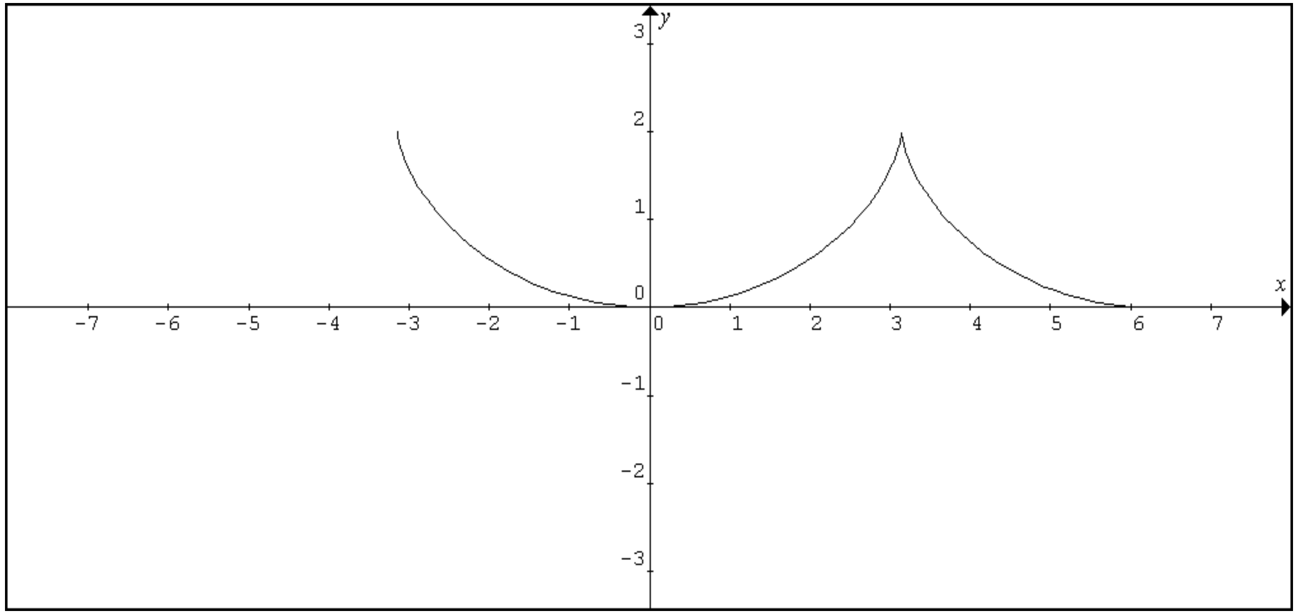
Symmetry: Put **t = - t . Only x(-t) =-x(t)or $\boldsymbol{x(\pi - \theta) = -x(\theta)} \rightarrow$ about y-axis, only y(-t) =-y(t) $\rightarrow$ about x-axis, both or $\rightarrow$ about origin**	
Find t at which x = y = 0 then the origin lies on the curve.	
Try to find few points on the curve by observation and also those points where dy/dx=0 or ∞.	
Make a table of values of t, x, y, dx/dt, dy/dt.	

Examples:-

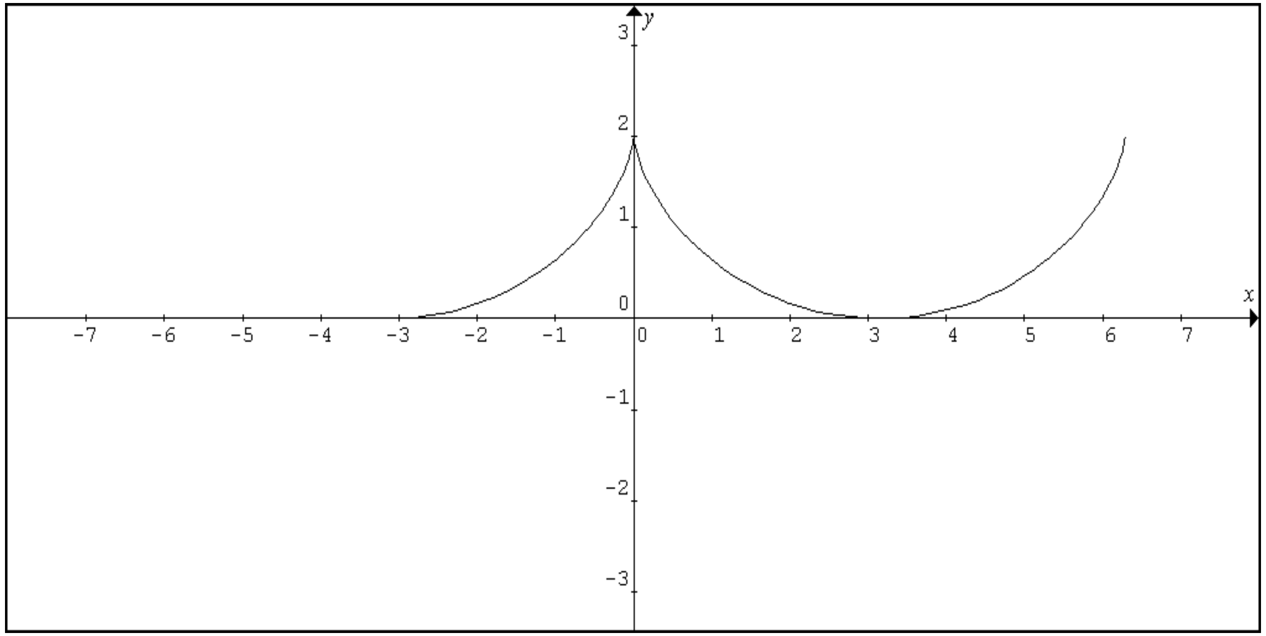
1) **Trace the Curve :** $\mathrm{x} = at \; and \; y = \frac{a}{t^2}$



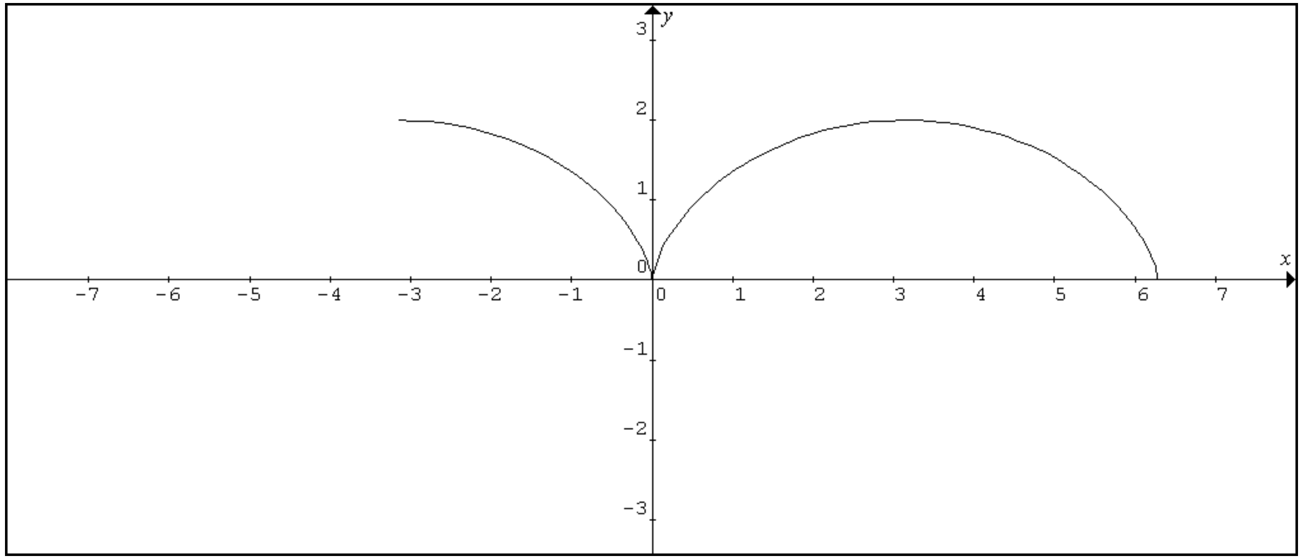
$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$



$$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$$



$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$



Examples for Home Work:-

Trace the following Curves

$$1)\, y^2(2a - x) = x^3$$

$$2)\, r(1 + \sin\theta) = 2a$$

$$3)\, x = a(t + \sin t),\ y = a(1 + \cos t)$$

Rectification of Curves

- We shall consider the applications of integration to measure the length of ARC of plane curves, this process is called Rectification
  - **Rectification of Plane Curve for Cartesian Equation**

Sr. No.	Equation of Curve	Formula in Differential Calculus	Formula in Integral Calculus
1)	$y = f(x)$	$ds = \sqrt{1+(\frac{dy}{dx})^2}.dx$	$s = \int_{x_1}^{x_2} \sqrt{1+(\frac{dy}{dx})^2}.dx$
2)	$x = g(y)$	$ds = \sqrt{1+(\frac{dx}{dy})^2}.dy$	$s = \int_{y_1}^{y_2} \sqrt{1+(\frac{dx}{dy})^2}.dy$
3)	$x = f_1(t), y = f_2(t)$	$ds = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}.dt$	$s = \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}.dt$

• Rectification of Plane Curve for Polar Equation

Sr. No.	Equation of Curve	Formula in Differential Calculus	Formula in Integral Calculus
1)	$r = f(\theta)$	$ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2}.d\theta$	$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (\frac{dr}{d\theta})^2}.d\theta$
2)	$\theta = f(r)$	$ds = \sqrt{1 + r^2(\frac{d\theta}{dr})^2}.dr$	$s = \int_{r_1}^{r_2} \sqrt{1 + r^2(\frac{d\theta}{dr})^2}.dr$
3)	$r = f_1(\alpha), \theta = f_2(\alpha)$	$ds = \sqrt{(\frac{dr}{d\alpha})^2 + r^2(\frac{d\theta}{d\alpha})^2}.d\alpha$	$s = \int_{\alpha_1}^{\alpha_2} \sqrt{(\frac{dr}{d\alpha})^2 + r^2(\frac{d\theta}{d\alpha})^2}.d\alpha$

- **Examples:-**

1) *Find* the length of upper arc of one loop of curve $r^2 = a^2 \cos 2\theta$

2) *Find* the length of cardioid $r = a(1+\cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$

3) *Find* the length of loop of curve $x = t^2, y = t(1-\frac{t^2}{3})$

Thank you