Reduction Formulae :

Integration by reduction formula in integral calculus is technique of integration, in the form of a recurrence relation.

For example : If $I_n = \int \sin^n x \, dx$, $n \in Z^+$

We will prove that

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

So we have recurrence relation bet^n I_n & I_{n-2}

This will help us to calculate integration of higher powers of $\sin x$.

In short, reduction formulae help us to find integration of powers of function.

* Reduction formula for $\int \sin^n x \, dx$ $n \in Z^+$

Let $I_n = \int \sin^n x \, dx = \int \underset{u}{\sin^{n-1} x} \, \underset{v}{(\sin x)} \, dx$

Applying by part

$$I_n = \sin^{n-1} x (-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} x \cos x \, dx$$

$$I_n = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x \, dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \underline{(1 - \sin^2 x)} \, dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - (n-1) \underbrace{\int \sin^n x \, dx}_{I_n}.$$

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$$\therefore \; I_n = -\sin^{n-1}x\cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\sin^{n-1}x\cos x + (n-1) I_{n-2}$$

$$n I_n = -\sin^{n-1}x\cos x + (n-1) I_{n-2}$$

$$\boxed{I_n = -\frac{\sin^{n-1}x\cdot\cos x}{n} + \frac{(n-1)}{n} I_{n-2}} \quad \text{— } (1)$$

H.W. obtain reduction formula for $I_n = \int \cos^n x \, dx$

Ans: $I_n = \dfrac{\cos^{n-1}x\sin x}{n} + \dfrac{(n-1)}{n} I_{n-2}$

* Find reduction formula for

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

→ We know that, if ~~I_n~~

$$I_n = \int \sin^n x \, dx$$

then

$$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{— } (*)$$

$\therefore$ if $I_n = \int_0^{\pi/2} \sin^n x \, dx$

$$\circledast \Rightarrow \quad I_n = \int_0^{\pi/2} \sin^n x \, dx = \underbrace{\left[-\frac{\sin^{n-1} x \cos x}{n} \right]_0^{\pi/2}}_{0} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$$

$$\therefore \boxed{I_n = \frac{n-1}{n} I_{n-2}} \quad \text{———} \quad (2)$$

Replace n by $n-2$ in (2)

$$\boxed{I_{n-2} = \frac{n-3}{n-2} I_{n-4}}$$

Replace n by $n-4$ in (2)

$$\boxed{I_{n-4} = \frac{n-5}{n-4} I_{n-6}}$$

Resubstituting I_{n-2} in (2)

We get.

$$\boxed{I_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} I_{n-6}} \quad — \quad (3)$$

If we keep on reducing above formula we get.

$$I_n = \frac{(n-1)}{n} \frac{(n-3)}{n-2} \frac{(n-5)}{n-4} \frac{(n-7)}{n-6} \cdots I_0$$

(n is even)

and

$$I_n = \frac{(n-1)}{n} \frac{(n-3)}{n-2} \frac{(n-5)}{n-4} \frac{(n-7)}{n-6} \cdots I_1$$

(n is odd)

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where $I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \pi/2$

& $I_1 = \int_0^{\pi/2} \sin^1 x \, dx = [-\cos x]_0^{\pi/2} = 1$

Finally,

$$\boxed{I_n = \frac{n-1}{n} \, \frac{n-3}{n-2} \, \frac{n-5}{n-4} \cdots \frac{\pi}{2}} \quad (n \text{ is even})$$

$$\boxed{I_n = \frac{n-1}{n} \, \frac{n-3}{n-2} \, \frac{n-5}{n-4} \cdots 1} \quad (n \text{ is odd}).$$

H.W. Derive reduction formula for $I_n = \int_0^{\pi/2} \cos^n x \, dx$

We know that

$$\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx.$$

$\therefore$ We have $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right) dx$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx.$$

This mean that betn the limits 0 to $\pi/2$ reduction formula for $\sin^n x$ & $\cos^n x$ are same.

$$\therefore I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{\pi}{2}$$

(n is even)

$$\& \quad I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots 1$$

(n is odd)

* some examples:

① Evaluate ⓐ $\int_0^{\pi/2} \sin^5 x\,dx$ ⓒ $\int_0^{\pi/2} \sin^4 x\,dx.$

ⓑ $\int_0^{\pi/2} \cos^8 x\,dx.$ ⓓ $\int_0^{\pi/2} \cos^7 x\,dx.$

→ ⓐ $\int_0^{\pi/2} \sin^5 x\,dx$

$\underline{n=5} \rightarrow$ odd.

$$I_5 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{15}$$

ⓑ $\int_0^{\pi/2} \cos^8 x\,dx.$

$n=8 \rightarrow$ even.

$$I_8 = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{105\pi}{768}$$

c) $\int_0^{\pi/2} \sin^4 x \, dx$

$n = 4 \rightarrow$ even

$I_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$

d) $\int_0^{\pi/2} \cos^7 x \, dx$

$n = 7 \rightarrow$ odd.

$I_7 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{48}{105}$

* some properties of definite integral

① $\int_0^a f(x) dx = \int_0^a f(a-x) dx.$

② $\int_{-a}^a f(x) dx = \int_0^a f(x) dx + \int_0^a f(-x) dx.$

③ $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx.$

Using these properties we have following important result.

(Observe limits)

a) $$\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\pi/2} \sin^n x \, dx \quad \text{for all } n$$

b) $$\int_0^{\pi} \cos^n x \, dx = \begin{cases} 2\int_0^{\pi/2} \cos^n x \, dx & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ odd integer.} \end{cases}$$

c) $$\int_0^{2\pi} \cos^n x \, dx = 4\int_0^{\pi/2} \cos^n x \, dx \quad \text{if } n \text{ is even}$$

$$= 0 \quad \text{if } n \text{ is odd}$$

d) $$\int_0^{2\pi} \sin^n x \, dx = 4\int_0^{\pi/2} \sin^n x \, dx \quad \text{if } n \text{ is even}$$

$$= 0 \quad \text{if } n \text{ is odd.}$$

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* some examples :

a) $$\int_0^{\pi} \sin^8 x \, dx = 2\int_0^{\pi/2} \sin^8 x \, dx = 2\left[\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

b) $$\int_0^{\pi} \cos^5 x \, dx = 0 \qquad n=5 \text{ odd.}$$

c) $$\int_0^{2\pi} \cos^7 x \, dx = 0 \qquad n=7 \text{ odd.}$$

d) $$\int_0^{2\pi} \sin^6 x \, dx = 4\int_0^{\pi/2} \sin^6 x \, dx = 4\left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

* Reduction Formula For $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$.

Let $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$.

(proof skipped)

$$I_{m,n} = \frac{[(m-1)(m-3)(m-5)\cdots 2 \text{ or } 1][(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} K$$

where $K = \frac{\pi}{2}$ — if m, n both are even

$K = 1$ otherwise.

* Some examples :

1) Find $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

$m=4, \quad n=6$ both even. $(K=\pi/2)$

$$I = \frac{(3 \cdot 1)(5 \cdot 3 \cdot 1)}{(10)(8)(6)(4)(2)} \cdot \frac{\pi}{2}$$

2) $\int_0^{\pi/2} \sin^5 x \cos^8 x \, dx$

$m=5, \quad n=6 \qquad K=1$

$$I = \frac{(4 \cdot 2)(5 \cdot 3 \cdot 1)}{(11)(9)(7)(5)(3) \cdot 1} \cdot 1$$

* Using properties of definite integral following can be proved.

① $\int_0^{\pi} \sin^m x \cos^n x \, dx = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ n-even

& No condition on m.

$= 0$ if n is odd

No condition on m.

② $\int_0^{2\pi} \sin^m x \cos^n x \, dx = 4 \int_0^{\pi/2} \sin^m x \cos^n x \, dx$; $\boxed{m, n \text{ both even}}$

$= 0$ otherwise.

* Some examples

① Evaluate $\int_0^{\pi} \sin^3 x \cos^4 x \, dx$.

$m = 3$, $n = 4$ (even)

$I = \int_0^{\pi} \sin^3 x \cos^4 x \, dx = 2 \int_0^{\pi/2} \sin^3 x \cos^4 x \, dx$

$= 2 \left[\frac{2 \cdot 3}{7 \cdot 5 \cdot 3} \cdot 1 \right] = \frac{12}{105}$

② Evaluate $\int_0^{\pi} \sin^5 x \cos^7 x \, dx$.

$m = 5$, $n = 7$ (odd)

$\therefore I = 0$

3) Evaluate $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$.

$m = 4$, $n = 6$ both even.

$I = \int_0^{2\pi} \sin^4 x \cos^6 x \, dx = 4 \int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

$= 4 \left[\frac{(3)(5)(3)(1)}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2} \right]$

* Illustrative examples :-

(1) Find reduction Formula for :

$$I_n = \int_{\pi/4}^{\pi/2} \cot^n\theta \, d\theta$$ & Hence find $\int_{\pi/4}^{\pi/2} \cot^4\theta \, d\theta$

$$\rightarrow \quad I_n = \int_{\pi/4}^{\pi/2} \cot^n\theta \, d\theta = \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \, \cot^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cot^{(n-2)}\theta \, (\operatorname{cosec}^2\theta - 1) \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cot^{(n-2)}\theta \, \operatorname{cosec}^2\theta \, d\theta - \int_{\pi/4}^{\pi/2} \cot^{n-2}\theta \, d\theta$$

$$I_n = \int_{\pi/4}^{\pi/2} \cot^{(n-2)}\theta \, \operatorname{cosec}^2\theta \, d\theta - I_{n-2}$$

Let $\cot\theta = t$

$-\operatorname{cosec}^2\theta \, d\theta = dt$

At $\theta = \frac{\pi}{4}$ $\quad t = 1$

At $\theta = \frac{\pi}{2}$ $\quad t = 0$

$$I_n = \int_1^0 t^{n-2} (-dt) - I_{n-2}$$

$$I_n = \int_0^1 t^{n-2}\,dt - I_{n-2}$$

$$I_n = \cancel{x}\left[\frac{t^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1} - I_{n-2}}$$

To calculate $\int_{\pi/4}^{\pi/2} \cot^4\theta\, d\theta$

put $n=4$ in above Formula.

$$I_4 = \frac{1}{3} - I_2$$

$$I_2 = 1 - I_0$$

$$I_0 = \int_{\pi/4}^{\pi/2} \cot^0\theta\, d\theta = \int_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore\ I_2 = 1 - \frac{\pi}{4}$$

$$\therefore\ I_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{-2}{3} + \frac{\pi}{4}.$$

② If $I_n = \int_0^{\pi/2} x^n \cos x \, dx$ then prove that

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

& Hence evaluate $\int_0^{\pi/2} x^2 \cos x \, dx$.

→ $$I_n = \int_0^{\pi/2} \underset{u}{x^n} \; \underset{v}{\cos x} \, dx$$

$$I_n = \left[x^n (\sin x) \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \; n x^{n-1} \, dx$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n \int_0^{\pi/2} \underset{v}{\sin x} \; \underset{u}{x^{n-1}} \, dx$$

observe here

By part one more time.

$$I_n = \left(\frac{\pi}{2}\right)^n - n \left[\left[x^{n-1} (-\cos x) \right]_0^{\pi/2} + \int_0^{\pi/2} (n-1) x^{n-2} \cos x \, dx \right]$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n \left[0 + (n-1) \underbrace{\int_0^{\pi/2} x^{n-2} \cos x \, dx}_{I_{n-2}} \right]$$

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$$

To calcuate $\int_0^{\pi/2} x^2 \cos x \, dx$

put $n = 2$ in above formula,

$$I_2 = \left(\frac{\pi}{2}\right)^2 - 2 I_0$$

$$I_0 = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$$

$$\therefore \quad I_2 = \left(\frac{\pi}{2}\right)^2 - 2$$

H.W.3) If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ then show that

$$I_n = \int_0^{\pi/2} x \sin^n x \, dx = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$$

& hence find I_4.

Hint $\longrightarrow$ $I_n = \int_0^{\pi/2} \underbrace{(x \sin x)}_{v} \underbrace{\sin^{n-1} x}_{u} \, dx$ (Important ~~test~~ step)

$$\cancel{I_n} = \cancel{\left[(\sin^{n-1} x)\left[\int_0^{\pi/2} x \sin x\right]\right]_0^{\pi/2}} + \cancel{\int_0^{\pi/2}}$$

4) If $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$,

show that $n(I_{n+1} + I_{n-1}) = 1$

$\rightarrow$ $I_n = \int_0^{\pi/4} \tan^n\theta \, d\theta$

$I_{n+1} = \int_0^{\pi/4} \tan^{n+1}\theta \, d\theta$

$I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta \, \tan^2\theta \, d\theta$

$= \int_0^{\pi/4} \tan^{n-1}\theta \, (\sec^2\theta - 1) \, d\theta$

$= \int_0^{\pi/4} \tan^{n-1}\theta \, \sec^2\theta \, d\theta - \underbrace{\int_0^{\pi/4} \tan^{n-1}\theta \, d\theta}_{I_{n-1}}$

put $\tan\theta = t$

$\sec^2\theta \, d\theta = dt$

at $\theta = 0$ $t = 0$

$\theta = \pi/4$ $t = 1$

$I_{n+1} = \int_0^1 t^{n-1} \, dt - I_{n-1}$

$I_{n+1} = \left[\frac{t^n}{n}\right]_0^1 - I_{n-1} = \frac{1}{n} - I_{n-1}$

$\therefore n(I_{n+1} + I_{n-1}) = 1$

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5) If $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$ then show that

$(n^2+1) I_n = n(n-1) I_{n-2}$ & evaluate I_4

→ $I_n = \int_0^\infty \underset{v}{e^{-x}} \underset{u}{\sin^n x} \, dx$

$I_n = \left[\sin^n x \, (-e^{-x})\right]_0^\infty - \int_0^\infty n \sin^{n-1} x \cos x \, (-e^{-x}) \, dx$

$I_n = 0 + n \int_0^\infty \underset{v}{e^{-x}} \underset{u}{(\sin^{n-1} x \cos x)} \, dx$

By part again

$I_n = n \left[\sin^{n-1} x \cos x \, (-e^{-x})\right]_0^\infty - n \int_0^\infty \left[\sin^{n-1} x (-\sin x) + (n-1) \sin^{n-2} \cos^2 x\right] (-e^{-x}) dx$

$I_n = 0 + n \int_0^\infty (-\sin^n x) e^{-x} + (n-1)(1-\sin^2 x) \overset{\sin^{n-2}}{} e^{-x} \, dx$

$I_n = -n \int_0^\infty \underbrace{e^{-x} \sin^n x \, dx}_{I_n} + n(n-1) \left[\int_0^\infty e^{-x} [\sin^{n-2} x - \sin^n x] dx\right]$

$I_n = -n I_n + n(n-1) \left[\int_0^\infty \underbrace{e^{-x} \sin^{n-2} x \, dx}_{I_{n-2}} - \int_0^\infty \underbrace{e^{-x} \sin^n x \, dx}_{I_n}\right]$

$I_n = -n I_n + n(n-1) [I_{n-2} - I_n]$

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$$I_n + nI_n + n(n-1)I_n = n(n-1)I_{n-2}$$

$$(1+\cancel{n}+n^2-\cancel{n})I_n = n(n-1)I_{n-2}$$

$$\boxed{I_n = \frac{n(n-1)}{n^2+1} I_{n-2}}$$

To find I_4 put $n=4$

$$I_4 = \frac{4(3)}{17} I_2$$

$$I_2 = \frac{2(1)}{5} I_0$$

$$I_0 = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 0+1 = 1$$

$$\therefore I_2 = \frac{2}{5}$$

$$I_4 = \frac{12}{17} \times \frac{2}{5} = \frac{24}{85}.$$

* Important note :-

① $$\int_0^{\pi/2} \sin^m x \cos x\,dx = \frac{1}{m+1}$$

② $$\int_0^{\pi/2} \sin x \cos^n x\,dx = \frac{1}{n+1}$$

* Homework :

① IF $I_n = \int_0^{\pi/4} \sin^{2n} x \, dx$

P.T. $I_n = \left(1 - \frac{1}{2n}\right) I_{n-1} - \frac{1}{n\, 2^{n+1}}$

Hint $I_n = \int_0^{\pi/4} \underset{\uparrow\, u}{\sin^{2n-1} x} \; \underset{\uparrow\, v}{\sin x} \, dx$

2) Find reduction formula for $I_n = \int_0^{\pi/3} \cos^n x \, dx$ & hence evaluate $\int_0^{\pi/3} \cos^6 x \, dx$.

Hint $I_n = \int_0^{\pi/3} \underset{\uparrow\, u}{\cos^{n-1} x} \; \underset{\uparrow\, v}{\cos x} \, dx$

3) Find reduction formula for $I_n = \int_0^{\pi/4} \sin^n x \, dx$

& hence show that $\int_0^{\pi/4} \sin^6 x \, dx = \frac{5\pi}{64} - \frac{11}{48}$

4) Find reduction formula for $I_n = \int_0^{\pi/4} \cos^{2n} x \, dx$

& hence show that $\int_0^{\pi/4} \cos^6 x \, dx = \frac{11}{48} + \frac{5\pi}{64}$

5) P.T.

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

Hint : $\int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx.$

$$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx.$$

6) Find reduction formula for

$I_n = \int \sec^n x \, dx$ & hence find

$$\int_0^{\pi/4} \sec^6 x \, dx$$

Hint : $\int \sec^n x \, dx = \int \underset{u}{\sec^{n-2} x} \; \underset{v}{\sec^2 x} \, dx.$

Apply by part