# Application of Partial Differentiation

- Jacobians and functional dependence.
- Errors and Approximation.
- Maxima and Minima of a function of two independent variable.
- Lagrange’s method of undetermined multipliers.

# JACOBIANS

In vector calculus, the Jacobian matrix is the matrix of all first-order partial derivatives of a vector-valued function.

**Definition**

**(German mathematician Carl Gustav Jacobi-Jacobi** (1804 − 1851)**)**
Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $f(x) = (f_1, f_2, \cdots f_m) \in \mathbb{R}^m$ for every $x \in \mathbb{R}^n$. Then the Jacobian matrix $J$ of $f$ is an $m \times n$ matrix, defined as follows:

$$J = \frac{\partial(f_1, f_2, \cdots f_m)}{\partial(x_1, x_2, \cdots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \frac{\partial f_m}{\partial x_3} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Note:If $m = n$, then determinant of Jacobian matrix $J$ is called as the Jacobian or functional determinant.

# JACOBIANS

- Definition:

If u and v be continuous and differential function of two other independent variable x and y such as $u = \phi_1(x, y), v = \phi_2(x, y)$ then we define the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

as Jocobian of u, v with respect to x,y.

And often denote this as $\frac{\partial(u, v)}{\partial(x, y)}$ or sometimes also

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

In the same way if u, v, w be the continuous and differential Functions of other variable x, y, z then we can define

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

as a Jocobian of u, v, w with respect to x, y, z.

Also define as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

# Examples on Jacobians

**Example 1:** *Compute the Jacobian of the polar coordinates*
$x = r\cos\theta$, $y = r\sin\theta$.

**Sol**. Here, x,y are the functions of r and θ

So, $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \qquad \frac{\partial y}{\partial r} = \sin(\theta),$$
$$\frac{\partial x}{\partial \theta} = -r\sin(\theta), \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta),$$

our Jacobian is

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,.$$

**Example 2 :** *Compute the Jacobian for*

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

***Sol. (Hint)***

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$$\therefore \frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial x}{\partial \phi} = -z\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2 [\cos\theta \{\cos\theta\sin\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi\}$$
$$+ \sin\theta \{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\}]$$

$$= r^2 [\sin\theta\cos^2\theta + \sin^3\theta] = r^2\sin\theta [\cos^2\theta + \sin^2\theta]$$

$$= r^2 \sin\theta$$

## Example 3 :

*If* $x = a\cosh\theta\cos\phi,\ y = a\sinh\theta\sin\phi,$

*show that* $\dfrac{\partial(x, y)}{\partial(\theta, \phi)} = \dfrac{a^2}{2}[\cosh 2\theta - \cos 2\phi].$

$$J = \frac{\partial(x, y)}{\partial(\theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} a\sinh\theta\cos\phi & -a\cosh\theta\sin\phi \\ a\cosh\theta\sin\phi & a\sinh\theta\cos\phi \end{vmatrix}$$

$$= a^2[\sinh^2\theta\cos^2\phi + \cosh^2\theta\sin^2\phi]$$

$$= a^2[\sinh^2\theta(1 - \sin^2\phi) + (1 + \sinh^2\theta)\sin^2\phi]$$

$$= a^2[\sinh^2\theta - \sinh^2\theta\sin^2\phi + \sin^2\phi + \sinh^2\theta\sin^2\phi]$$

$$= a^2[\sinh^2\theta + \sin^2\phi] = \frac{a^2}{2}[\cosh 2\theta - 1 + 1 - \cos 2\phi]$$

$$= \frac{a^2}{2}[\cosh 2\theta - \cos 2\phi]$$

# Example 4:

Example ;

if $ux = yz$, $vy = zx$, ~~wx=~~ $wz = xy$ —— ①

find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

sol$^n$ :- Here, we have to find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$\therefore$ we write eq$^n$ ① in the form of

$u, v, w \longrightarrow x, y, z$

( u,v,w are the function's of x,y,z)

$\therefore$ eq$^n$ ① $\Rightarrow$ $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $W = \frac{xy}{z}$

$$\therefore\ J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$u_x = \frac{-yz}{x^2}, \quad u_y = \frac{z}{x}, \quad u_z = \frac{y}{x}$$

$$v_x = \frac{z}{y}, \quad v_y = \frac{-zx}{y^2}, \quad v_z = \frac{x}{y}$$

$$w_x = \frac{y}{z}, \quad w_y = \frac{x}{z}, \quad w_z = -\frac{xy}{z^2}$$

$$J = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\Big[(-yz)(x^2yz - x^2yz) - (xz)(-xy^2z - xy^2z) + xy(xyz^2 + xyz^2)\Big]$$

$$= \frac{1}{x^2y^2z^2}\Big[0 + x^2y^2z^2 + x^2y^2z^2 + x^2y^2z^2 + x^2y^2z^2\Big]$$

$$= \frac{4(x^2y^2z^2)}{(x^2y^2z^2)} = 4$$

$$\boxed{J = 4}$$

# PROPERTIES OF JACOBIANS

If $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Then $$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

This implies ,

$$J \cdot J' = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

# Examples

1. **If x + y² = u , y + z² = v , z + x² = w find** $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

**Sol.**

We have $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

$= 1(1\text{-}0) – 2y(0 – 4xz) + 0$

$= 1 – 2y(\text{-}4xz)$

$= 1 + 8xyz$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]} = \frac{1}{1+8xyz}$$

**2.** **If x + y + z = u , y + z = uv , z = uvw then evaluate** $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: x + y + z = u

y + z = uv

z = uvw

y = uv – uvw = uv (1 – w)

x = u – uv = u (1 – v)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

= uv [ u –uv +uv]

= $u^2v$

**3.** **If $u = x^2 - y^2$ , $v = 2xy$ where $x = r\cos\theta$ , $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$**

Sol: Given $u = x^2 - y^2$ , $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\qquad = 2r\cos\theta\, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\qquad = r^2\sin 2\theta$

$= r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$$= 4r^2[r\cos^2 2\theta + r\sin^2 2\theta]$$

$$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$$

$$= 4r^3$$

**4.** **If $x = e^r \sec\theta$ , $y = e^r \tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$**

Sol: Given $x = e^r \sec\theta$ , $y = e^r \tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \qquad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r \sec\theta = x$ , $x_\theta = e^r \sec\theta \tan\theta$

$y_r = e^r \tan\theta = y$ , $y_\theta = e^r \sec^2\theta$

$x^2 - y^2 = e^{2r} (\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log (x^2 - y^2)$

⇨ $r = ½ \log (x^2 - y^2)$

$$r_x = \frac{1}{2}\frac{1}{x^2 - y^2}(2x) = \frac{x}{(x^2 - y^2)}$$

$$r_y = \frac{1}{2}\frac{1}{x^2 - y^2}(-2y) = \frac{-y}{(x^2 - y^2)}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$$

⇨ $\sin\theta = \frac{y}{x}$ , $\theta = \sin^{-1}(\frac{y}{x})$

$$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\,(1/x) \qquad = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta & \tan\theta \\ & e^r\sec 2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\, e^r\sec\theta\tan\theta$$

$$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x^2-y^2)} & \frac{-y}{(x^2-y^2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2-y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

# Homework Examples

1) If $x = ar\sin\theta\cos\emptyset\,, y = br\sin\theta\sin\emptyset\,, z = cr\cos\theta$ show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} =$ $abcr^2\sin\theta$
2) If $ux = yz, vy = zx, wz = xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ Answer = 4
3) If $u = x + 2y^2 - z^3, v = x^2yz,\ w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}\ at\ (1,-1,0)$

   Answer = 6
4) If $x = u - v + w,\ y = u^2 - v^2 - w^2,\ z = u^3v$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

   Answer = $6u^2(v + w) + 2u + 2w$
5) If $u = x + y + z, v = x^2 + y^2 + z^2, w = xy + yz + zx$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

   Answer = 0

# Jacobians of composite functions

- **Chain Rule for Jacobians:**

if x, y are the functions of u, v and u, v are the function of r, s

such that, $x=\phi_1(u,v), y=\phi_2(u,v)$ and $u=\varphi_1(r,s), v=\varphi_2(r,s)$

Then ,

$$\frac{\partial(x,y)}{\partial(r,s)}=\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(r,s)}$$

Similarly,

if x, y, z are the functions of u, v, w and u, v, w are the function of r, s, t then,

$$\frac{\partial(x,y,z)}{\partial(r,s,t)}=\frac{\partial(x,y,z)}{\partial(u,v,w)}\cdot\frac{\partial(u,v,w)}{\partial(r,s,t)}$$

# Examples

## Example.1

If $x = a(u+v), y = b(u-v) where\ u = r^2\cos 2\theta, v = r^2\sin 2\theta$ , $a\ and\ b$ being constant, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$

**Sol. :** Here x, y $\to$ u, v $\to$ r, θ.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a & a \\ b & -b \end{vmatrix} \cdot \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= (-2ab)\times(4r^3) = -8abr^3$$

## Example.2

For $x = e^u \cos v$, $y = e^u \sin v$ prove that $\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1$

**Sol. :** Given : $x = e^u \cos v$, $y = e^u \sin v$.

We first find

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$= e^{2u} \begin{vmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{vmatrix} = e^{2u} (\cos^2 v + \sin^2 v)$$

$$= e^{2u}$$

Next, we find $$J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

We note from the given transformations,

$$\frac{y}{x} = \tan v \quad \text{and} \quad x^2 + y^2 = e^{2u}$$

or $$v = \tan^{-1} \frac{y}{x} \quad \text{and} \quad u = \frac{1}{2} \log (x^2 + y^2)$$

$\therefore$ $$J' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} = \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2} = \frac{1}{e^{2u}}$$

From (1) and (2), we have

$$J J' = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = e^{2u} \frac{1}{e^{2u}} = 1$$

# Home work

1. If $x = \sqrt{vw}, y = \sqrt{uw}, z = \sqrt{uv}$ and $u = rsin\theta cos\emptyset, v = rsin\theta sin\emptyset, z = rcos\theta \; then \; find \; \frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$

2. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$, prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.

3. If $x + y = 2e^{\theta}\cos\phi$, $x - y = 2ie^{\theta}\sin\phi$, prove that $JJ' = 1$.

# Jacobians of Implicit functions

- **Chain Rule for Jacobians:**

if $u_1, u_2, u_3 \ldots u_n$ be the implicit functions of the variable $x_1, x_2, x_3 \ldots x_n$ connected by $f_1, f_2, f_3 \ldots f_n$ such that,

$$f_1(u_1, u_2, \ldots u_{n,} x_{1,} x_2, \ldots x_n) = 0,\ f_2(u_1, u_2, \ldots u_{n,} x_{1,} x_2, \ldots x_n) = 0, \ldots$$

$$f_3(u_1, u_2, \ldots u_{n,} x_{1,} x_2, \ldots x_n) = 0$$

Then,

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots f_n) \big/ \partial(x_1, x_2, \ldots x_n)}{\partial(f_1, f_2, \ldots f_n) \big/ \partial(u_1, u_2, \ldots u_n)}$$

# Remarks

- if $u_1, u_2, u_3$ be the implicit functions of the variable $x_1, x_2, x_3$

connected by $f_1, f_2, f_3$ such that,

$$f_1(u_1,u_2,u_{3,}x_{1,}x_2,x_3)=0,\ f_2(u_1,u_2,u_{3,}x_{1,}x_2,x_3)=0,\ f_3(u_1,u_2,u_{3,}x_{1,}x_2,x_3)=0$$

Then, $$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)\big/\partial(x_1,x_2,x_3)}{\partial(f_1,f_2,f_3)\big/\partial(u_1,u_2,u_3)}$$

- if $u_1, u_2$ be the implicit functions of the variable $x_1, x_2$

connected by $f_1, f_2$ such that, $f_1(u_1,u_2,x_{1,}x_2)=0,\ f_2(u_1,u_2,x_{1,}x_2)=0$

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = (-1)^2 \frac{\partial(f_1,f_2)\big/\partial(x_1,x_2)}{\partial(f_1,f_2)\big/\partial(u_1,u_2)}$$

# Examples

**Example.1** If $x = u(1-v), y = uv$ show that $JJ' = 1$

**Sol. :** We first find J : Here x, y $\rightarrow$ u, v.

Let
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

Next, we shall obtain $J' = \dfrac{\partial(u, v)}{\partial(x, y)}$ by the method of Implicit function.

From given relation, $f_1 = x - u + uv,\ f_2 = y - uv$

$$J' = \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Here
$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

and
$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} -1+v & u \\ -v & -u \end{vmatrix} = u$$

Putting in equation (1) the values of these Jacobians, we have $J' = \dfrac{1}{u}$.

$$\therefore \quad J J' = u \cdot \frac{1}{u} = 1$$

**Example.2** if $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$ then prove that $JJ' = 1$

**Sol. : We first find J :**

Here, $x, y, z \;\rightarrow\; u, v, w$

$$\therefore \qquad J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 0 & 2v & 3w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$$= 8uvw \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 16\, uvw$$

We next find J' :

From given relations, we have,

$$f_1 = v^2 + w^2 - x$$
$$f_2 = w^2 + u^2 - y$$
$$f_3 = u^2 + v^2 - z$$

$$J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)/\partial(x, y, z)}{\partial(f_1, f_2, f_3)/\partial(u, v, w)}$$

$$J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)/\partial(x, y, z)}{\partial(f_1, f_2, f_3)/\partial(u, v, w)}$$

$$J' = (-1)^3 \frac{\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}}$$

$$= \frac{1}{16uvw}$$

Multiplying (1) and (2), we have

$$J J' = 1$$

# Home work

1. If $x+y+z=u,\ y+z=uv,\ z=uvw$, show that $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}=u^2v$.

2. Show that $JJ' = 1$ for the following

   i) $x = uv\,, y = \frac{u}{v}$

   ii) $u = xy, v = x + y$

# FUNCTIONAL DEPENDENCE

Let $u = f_1(x, y)$ and $v = f_2(x, y)$ be any two function of $x, y$ .
Sometimes we study under what condition $u = f_1(x, y)$ and $v = f_2(x, y)$ Will be functionally dependant or independent.

If there exists a functional relation between $u = f_1(x, y)$ and $v = f_2(x, y)$ of the type $v = F(u)$ , then we say that $u = f_1(x, y)$ and $v = f_2(x, y)$ are **functionally dependent.**

**For example ,**

$$if\ , u = \frac{y}{x}, v = \frac{x}{y}$$

$$\Rightarrow v = \frac{1}{u}$$

**are Functionally Dependent**

# Remark

❑ Let $u = f_1(x, y)$ and $v = f_2(x, y)$ be any two function of $x, y$ .
are **functionally dependent** if

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f_1, f_2)}{\partial(x,y)} = 0$$

For examples ,

$$if\ , u = \frac{y}{x}, v = \frac{x}{y}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} -y/x^2 & 1/x \\ 1/y & -x/y^2 \end{vmatrix} = \left[ \frac{xy}{x^2 y^2} - \frac{1}{xy} \right] = 0$$

**=> u and v are Functionally Dependent.**

# Examples

**Example 1** Check whether the following functions are functionally dependent, if so find the relation between them, $u = \frac{x+y}{1-xy}$ , $v = \tan^{-1} x + \tan^{-1} y$

**Sol.**
$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} + 0 = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Thus $J = \dfrac{\partial(u, v)}{\partial(x, y)} = 0$, hence u and v are functionally dependent.

**Relation between u and v :** We have

$$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} u$$

$$v = \tan^{-1} u$$

**Example 2** Check whether the following functions are functionally dependent, if so find the relation between them, $u = \sin^{-1} x + \sin^{-1} y$ , $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

**Sol.**

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial v}{\partial x} = \sqrt{1-y^2} + y \cdot \frac{-2x}{2\sqrt{1-x^2}} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial v}{\partial y} = \frac{-2y}{2\sqrt{1-y^2}} + \sqrt{1-x^2} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{\sqrt{1-x^2}} & \dfrac{1}{\sqrt{1-y^2}} \\ \left(\sqrt{1-y^2} - \dfrac{xy}{\sqrt{1-x^2}}\right) & \left(-\dfrac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right) \end{vmatrix}$$

$$= \left(\frac{-xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}} + 1\right) - \left(1 - \frac{xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\right) = 0$$

**=> u and v are Functionally Dependent.**

**Relation between u and v :**

Let $\sin^{-1}x = \alpha \Rightarrow x = \sin\alpha$

$\sin^{-1}y = \beta \Rightarrow y = \sin\beta$

$$v = \sin\alpha \cdot \cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta)$$

$$= \sin(\sin^{-1}x + \sin^{-1}y)$$

$$= \sin u$$

$$v = \sin u$$

**Example 3.** If $u = x + y + z, v = x^2 + y^2 + z^2, w = xy + yz + zx$ examine whether the above functions are functionally dependent; if so find the relation between them.

**Sol. :** For functional dependence, we must have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial u}{\partial z} = 2z$$

$$\frac{\partial w}{\partial x} = y + z \qquad \frac{\partial w}{\partial y} = x + z \qquad \frac{\partial w}{\partial z} = x + y$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y + z & z + x & x + y \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

Perform $R_3 + R_2$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence u, v, w are functionally dependent.

**Relation between u, v and w :**

We have $\quad u^2 = (x+y+z)^2$

or $\quad u^2 = x^2 + y^2 + z^2 + 2(yz + zx + xy) = v + 2w$

$$u^2 = v + 2w$$

**Example 4.** Show that the function $u = x + y + z, v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx, w = x^3$ are functionally related

Sol: Given $u = x + y + z$

$$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$$

$$w = x^3 + y^3 + z^3 - 3xyz$$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$=6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \to c_1 - c_2$

$c_2 \to c_2 - c_3$

$$=6\begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$$=6[2(x-y)(y^2+xy-xz-z^2)-2(y-z)(x^2+xz-yz-y^2)]$$

$$=6[2(x-y)(y-z)(x+y+z)-2(y-z)(x-y)(x+y+z)]$$

$$=0$$

Hence there is a relation between u,v,w.

# Home work

- Determine whether the following functions are functionally dependent or not. If they are functionally dependent , find a relation between them.

  i) $u = e^x \sin y$ , $v = e^x \cos y$ ii) $u = \frac{x}{y}$ , $v = \frac{x+y}{x-y}$

- Verify if $u = 2x - y + 3z$ , $v = 2x - y - z$ , $w = 2x - y + z$ are functionally dependent and if so , find the relation between them.

- Show that the functions $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2zx$ and $w = x^3+y^3+z^3-3xyz$ are functionally related.

- Prove that $u = \frac{x^2 - y^2}{x^2 + y^2}$ , $v = \frac{2xy}{x^2+y^2}$ are functionally dependent and find the relation between them.

- Under which condition $u = a_1x + b_1y + c_1$ , $and\ v = a_2x + b_2y + c_2$ are functionally dependent.

# ERRORS AND APPROXIMATION

Let $z = f(x, y)$ ......(i)

If $\delta x, \delta y$ are small increments in $x$ and $y$ respectively and $\delta z$ is small increments in $z$.

then, $z + \delta z = f(x + \delta x, y + \delta y)$ ........$(ii)$

Subtracting (i) from (ii), we get,

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x, y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + ...... - f(x, y)$$

$$= \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} \text{ (Approximately)}$$

As neglecting Higher power of $\delta x, \delta y$

so, $\delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$ (Approximately)

$\Rightarrow$ If $\delta x, \delta y$ are small changes (Error) in $x$ and $y$ respectively

then an approximate change (or Error) in $z$ is $\delta z$

Now, Replacing $\delta x, \delta y, \delta z$ by $dx, dy, dz$ respectively.

We have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Here, $dz, dx, dy$ are the Actual Error in $z, x, y$ respectively

*Note that* :

- $\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}$ are known as Relative Error in $x, y, z$ respectively.
- $\frac{100dx}{x}, \frac{100dy}{y}, \frac{100dz}{z}$ are known as Percentage Error in $x, y, z$ respectively.

**Example 1** The time T of a complete oscillation of a simple pendulum of length L is governed by the equation $T = 2\pi\sqrt{\frac{L}{g}}$,

Find the maximum error in T due to possible errors upto 1% in L and 2% in g.

**Solution :** We have $T = 2\pi\sqrt{\frac{L}{g}}$

$$\log T = \log 2\pi + \frac{1}{2}\log_e L - \frac{1}{2}\log_e g$$

Differentiating

$$\frac{dT}{T} = 0 + \frac{1}{2}\frac{dL}{L} - \frac{1}{2}\frac{dg}{g}$$

$$\Rightarrow \left(\frac{dT}{T}\right)\times 100 = \frac{1}{2}\left[\left(\frac{dL}{L}\right)\times 100 - \left(\frac{dg}{g}\right)\times 100\right]$$

$$\Rightarrow \begin{cases} \text{Given that,} \\ \text{\% Error in } L = 1 \\ \text{\% Error in } G = 2 \end{cases}$$

But $\frac{dL}{L}\times 100 = 1, \frac{dg}{g}\times 100 = 2$

so

$$\frac{dT}{T}\times 100 = \frac{1}{2}[1 \pm 2] = \frac{3}{2}$$

Maximum error in T = 1.5% Answer.

**Example 2:** The power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Find by using calculus the approximate percentage change in P when E is increased by 3% and R is decreased by 2%.

**Solution:** Here given $P = \frac{E^2}{R}$

Taking logarithm we have

$\log P = 2 \log E - \log R$

on differentiating, we get

$$\frac{\delta P}{P} = \frac{2}{E}\delta E - \frac{\delta R}{R}$$

or $100\frac{\delta P}{P} = 2 \times \frac{100\delta E}{E} - \frac{100\delta R}{R}$

$$\Rightarrow \begin{cases} \text{Given that,} \\ \text{\% Error in } E = 3 \\ \text{\% Error in } R = -2 \end{cases}$$

or $100\frac{\delta P}{P} = 2(3) - (-2)$

$= 8$

Percentage change in P = 8% Answer.

**Example 3:** In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m), count of bricks is taken as 100 bricks per $m^3$. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is Rs. 2,000 per thousand bricks.

**Solution:** We have volume V = xyz

Taking log of both sides, we have

log V= log x + log y + log z

Differentiating, we get

$$\frac{\delta V}{V}=\frac{\delta x}{x}+\frac{\delta y}{y}+\frac{\delta z}{z}$$

$$100\frac{\delta V}{V}=100\frac{\delta x}{x}+100\frac{\delta y}{y}+100\frac{\delta z}{z}$$

= 2 + 2 + 2 (As given)

= 6

$\Rightarrow \begin{cases}\text{Given that,}\\ \text{\% Error in } x, y, z = 2\end{cases}$

$$\Rightarrow \delta V = 6\frac{V}{100} = 6\frac{(5\times10\times5)}{100}$$

= 15 cubic metre

Number of bricks in δV = 15×100

= 1500

Error in cost $\frac{1500\times2000}{1000}$

= 3000

This error in cost, a loss to the seller of bricks = Rs. 3000.

**Example 4:** What error in the common logarithm of a number will be produced by an error of 1% in the number.

**Solution:** Consider x as any number and, let

$y = \log_{10} x$

Then $\delta y = \frac{1}{x} \log_{10} e \; \delta x$

$= \frac{\delta x}{x} \log_{10} e$

$= \left( \frac{\delta x}{x} \times 100 \right) \left( \frac{1}{100} \log_{10} e \right)$

$= \frac{1}{100} \log_{10} e \qquad \therefore$ as given $\frac{\delta x}{x} \times 100 = 1$

$= \frac{0.43429}{100}$

$= 0.0043429$

which is the required error. Answer

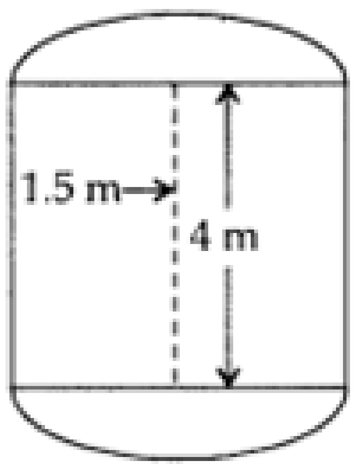
**Example 5:** A balloon is in the form of right circular cylinder of radius 1.5m and length 4m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05m, find the percentage change in the volume of balloon.

**Solution:** Here given,

radius of the cylinder (r) = 1.5m

length of the cylinder (h) = 4m

$\delta r = 0.01m$, $\delta h = 0.05m$




Let V be the volume of the balloon then.

V= volume of the circular cylinder + 2 (volume of the hemisphere)

$$= \pi r^2 h + 2\left(\frac{2}{3}\pi r^3\right)$$

$$= \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\text{Now } \delta V = \pi 2r\delta r h, h + \pi r^2 \delta h + \frac{4}{3}\pi .3r^2 \delta r$$

$$= \pi r[2h\delta r + r\delta h + 4r\delta r]$$

$$\Rightarrow \frac{\delta V}{V} = \frac{\pi r\left[2(h+2r)\delta r + r\delta h\right]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\delta r + r\delta h}{rh + \frac{4}{3}r^2}$$

$$= \frac{2(4+3)(0.01) + (1.5)(0.05)}{(1.5\times 4) + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

$$\therefore \frac{\delta V}{V}\times 100 = \frac{0.215}{9}\times 100 = \frac{21.5}{9} = 2.389\% \text{ Answer.}$$

**Example 6:** Find approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$

**Solution:** Let $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$ (i)

Taking $x = 1$, $y = 2$ and $z = 2$ so that $dx = -0.02$, $dy = 0.01$ and $dx = -0.06$ from (i)

$$\frac{\partial f}{\partial x} = x\,(x^2 + y^2 + z^2)^{-1/2},\ \frac{\partial f}{\partial y} = y\,(x^2 + y^2 + z^2)^{-1/2},\ \frac{\partial f}{\partial z} = z\,(x^2 + y^2 + z^2)^{-1/2}$$

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ (by total differentiation)

$= (x^2 + y^2 + z^2)^{-1/2}\,(xdx + ydy + zdz)$

$= \frac{1}{3}(-0.02 + 0.02 - 0.12)$

$= -0.04$

$\therefore [(0.98)^2 + (2.01)^2 + (1.94)2]^{1/2} = f(1,2,2) + df$

$= 3 + (-0.04)$

$= 2.96$ Answer.

# Home work

- **Example** : If the base radius and height of a cone are measured as 4 and 8 inches with a possible error of 0.04 and 0.08 inches respectively, calculate the percentage (%) error in calculating volume of the cone.

- **Example** : If the kinetic energy T is given by $T = \frac{1}{2}mv^2$ find approximate the change in T as the mass m change from 49 to 49.5 and the velocity v changes from 1600 to 1590.

- **Example** : If the sides and angles of a triangle ABC vary in such a way that its circum radius remains constant, Prove that $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$

- **Example** : Compute an approximate value of $(1.04)^{3.01}$

  Ans. 1.12

- **Example** : Evaluate $[(4.85)^2 + 2(2.5)^3]^{1/5}$

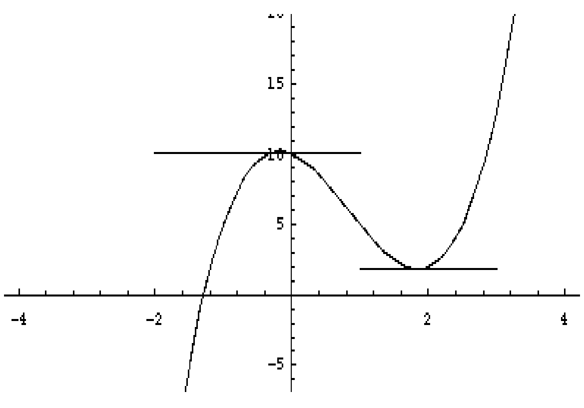
  Ans. 2.15289

- **Example** : The focal length of a mirror is given by the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. If equal errors $\delta$ are made in the determination of u and v, show that the relative error $\frac{\delta f}{f}$ in the focal length is given by $\delta\left(\frac{1}{u} + \frac{1}{v}\right)$.

# MAXIMA AND MINIMA

## ❑ Maxima and Minima of Functions of y=f(x) :-

- The problem of determining the maximum or minimum of function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century.
- Let us recall the procedure for the case of a function of one variable y=f(x). First, we determine points "a" from f'(x)=0(solving this equation). These points are called **critical points / stationary points**. At critical points the tangent line is horizontal. This is shown in the figure



- *The second derivative test* is employed to determine if a **critical point / stationary points** is a relative **maximum** or a relative **minimum**.
- If f''(a)>0, then “a” is a relative minimum.
- If f''(a)<0, then “a” is a maximum.
- If f''(a)=0, then the test gives no information.
- **For example:**

**Question 1 : Find the local maxima and minima for the function $y = x^3 - 3x + 2$.**

**Answer :** We'll need to find the stationary points for this function, for which we need to calculate $\frac{df}{dx}$. We'll proceed as follows:

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

At stationary points, $\frac{dy}{dx} = 0$. Thus, we have;

$$3x^2 - 3 = 0$$

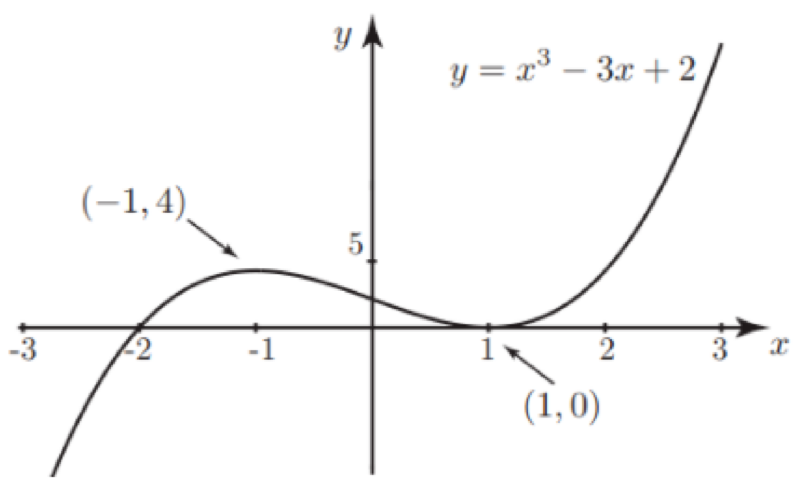
$$3x^2 - 3 = 0 \Longrightarrow 3(x^2 - 1) = 0$$
$$x^2 - 1 = 0 \Longrightarrow (x-1)(x+1) = 0$$
$$x = 1 \;\; / \;\; x = -1$$

Now we have to determine whether any of these stationary points are extremum points. We'll use the second derivative test for this :

$$\frac{\mathrm{dy}}{\mathrm{dx}} = 3x^2 - 3 \Longrightarrow \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = 6x$$

- For x = 1; $\frac{d^2y}{d^2x} = 6/times1 = 6$, which is positive. Thus the point (1, y(x = 1)) is a point of Local Minima.
- For x = -1; $\frac{d^2y}{d^2x} = 6/times{-1} = -6$, which is positive. Thus the point (-1, y(x = -1)) is a point of Local Maxima.

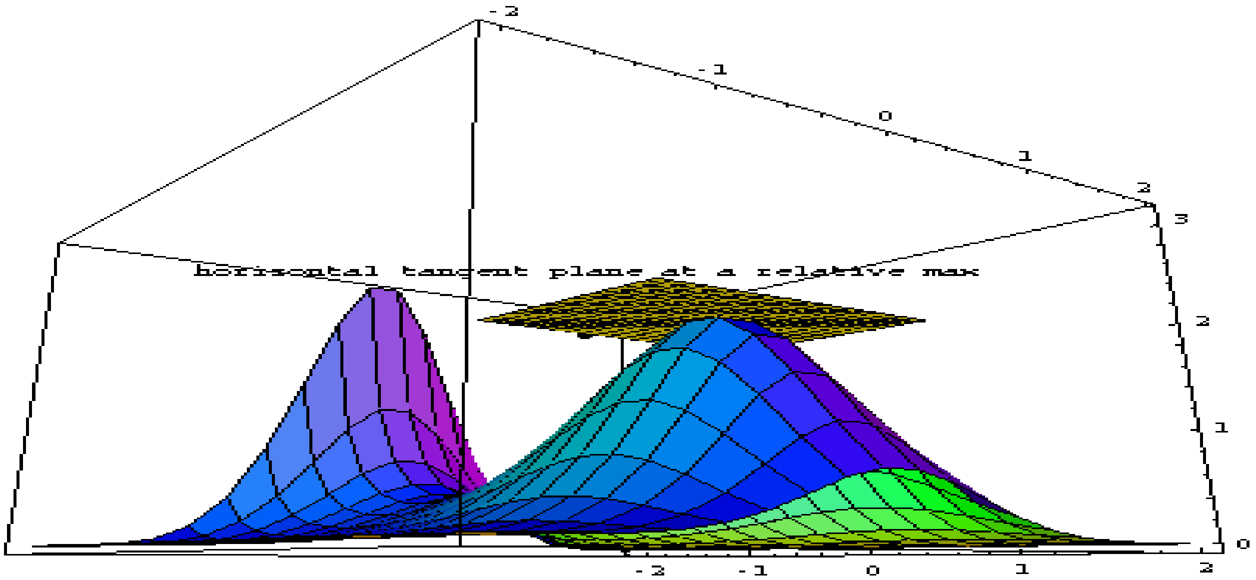
We can see from the graph below to verify our calculations:



## Maxima and Minima of Functions of z=f(x,y):-

## Critical / Stationary Points :-

- The notions of critical points and the second derivative test carry over to functions of two variables. Let z=f(x,y). Critical points are points in the xy-plane where the tangent plane is horizontal.




- Hence, *critical points are solutions of the equations*:

$$f_x(x,y) = 0 \text{ and } f_y(x,y) = 0$$

- *Lets find the critical points for*

$$z = f(x,y) = e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

*the partial derivatives are*

$$f_x(x,y) = (-x^2 + 1)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

$$f_y(x,y) = (-2y)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

$$now,\ f_x(x,y) = f_y(x,y) = 0$$

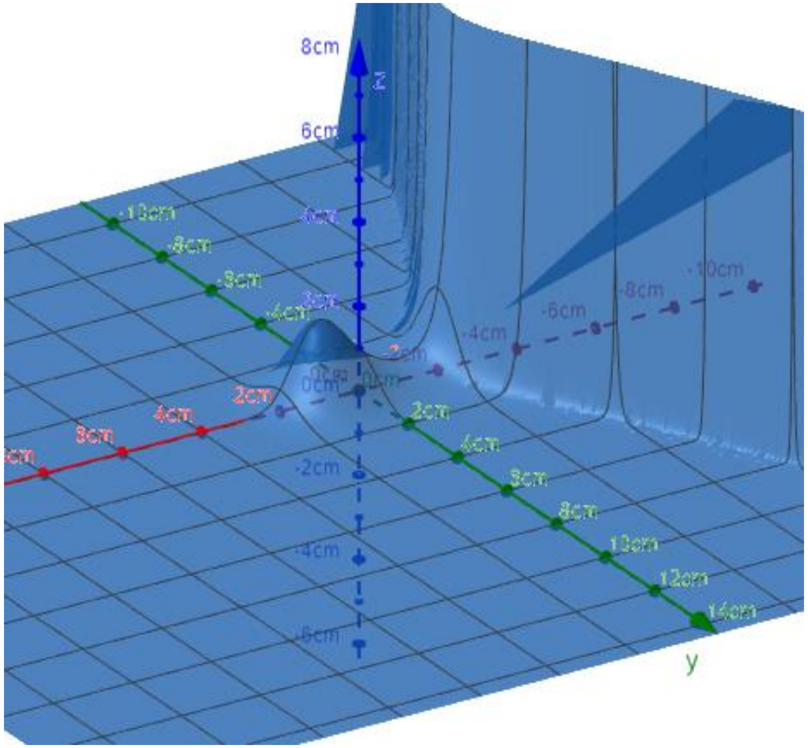
*gives us*

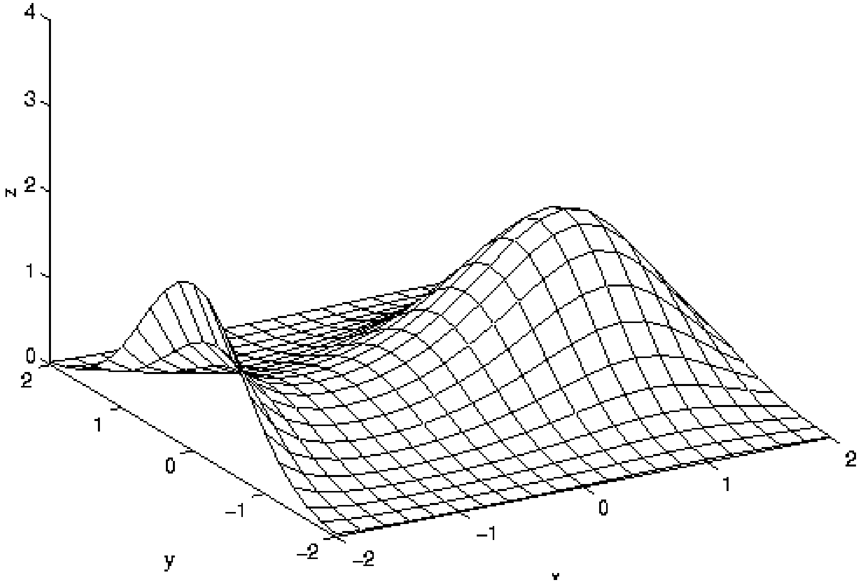
$$(-x^2 + 1) = 0 \text{ and } (-2y) = 0$$

$$\Rightarrow x = \pm 1, y = 0$$

*critical / stationary points* are

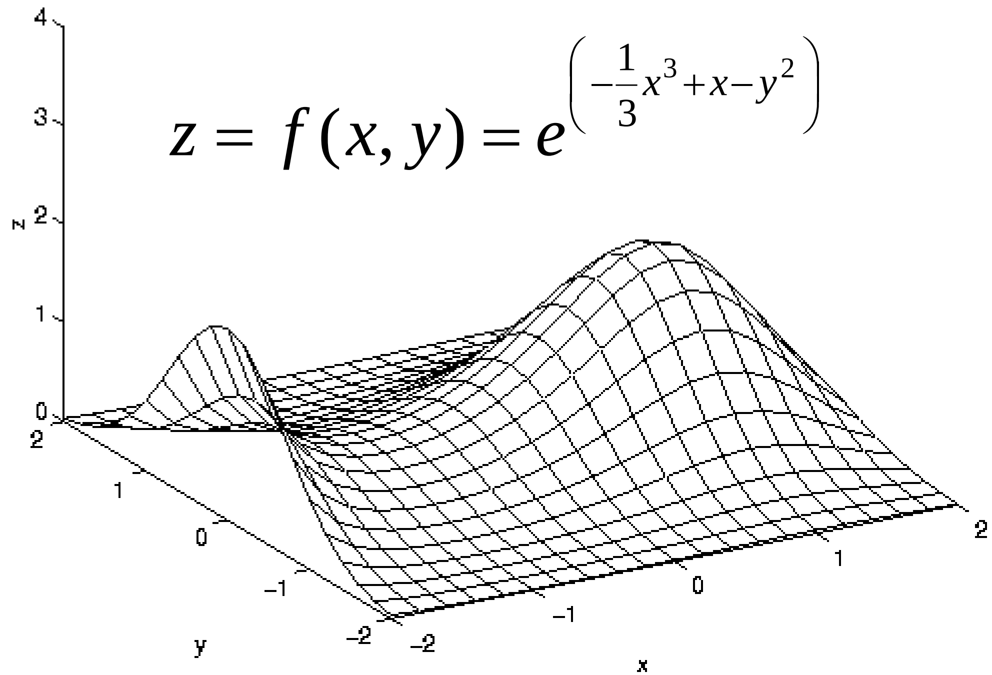
(1,0) and (-1,0)



- Notice the relative maximum at (x=1,y=0). (x=-1,y=0) is a relative maximum if one travels in the y direction and a relative minimum if one travels in the x-direction. Near (-1,0) the surface looks like a saddle(neither maxima nor minima).

$$z = f(x, y) = e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$



# ❑ The Second Derivative Test for Functions of Two Variables:-

- How can we determine if the critical points found above are relative **Maxima or Minima**?
- We apply a second derivative test for functions of two variables.

Let (a,b) be a critical point and define $rt - s^2$

$$where,$$

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}, s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$$

or

$$r = f_{xx}(a,b),\ t = f_{yy}(a,b),\ s = f_{xy}(a,b)$$

*now*

- if $rt - s^2 > 0$, and $\mathrm{r} = f_{xx}(a,b) > 0 \Rightarrow f(x,y)$ is Minimum at $(a,b)$ and $f(a,b)$ its Minimum value.
- if $rt - s^2 > 0$, and $\mathrm{r} = f_{xx}(a,b) < 0 \Rightarrow f(x,y)$ is Maximum at $(a,b)$ and $f(a,b)$ its Maximum value.
- if $rt - s^2 < 0 \Rightarrow f(x,y)$ is neigther Maxima nor Minimum at $(a,b)$

(Such a point is called as saddle point)

- if $rt - s^2 = 0 \Rightarrow$ the case is undecided

$$Lets\ continued\ \ previous\ example\ \ z = f(x,y) = e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

$$f_x = \left(-x^2 + 1\right)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)} \Rightarrow f_{xx}(x,y) = \left(-2x + \left(1 - x^2\right)^2\right)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

$$f_y = \left(-2y\right)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)} \Rightarrow f_{yy}(x,y) = \left(-2 + 4y^2\right)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

$$f_{xy}(x,y) = -2y\left(1 - x^2\right)\cdot e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$

- $Now,\ \ for\ \ point\ (1,0)$

$$r = f_{xx}(1,0) = -2\cdot e^{\left(\frac{2}{3}\right)}, s = f_{xy}(1,0) = 0, t = f_{yy}(1,0) = -2\cdot e^{\left(\frac{2}{3}\right)}$$

$$\Rightarrow rt - s^2 = 4e^{\left(\frac{4}{3}\right)} > 0\ and\ r = f_{xx}(1,0) = -2\cdot e^{\left(\frac{2}{3}\right)} < 0$$

$$\Rightarrow (1,0)\ is\ Maximum\ Point.$$

$$Lets\ continued\ \ previous\ example\ \ z = f(x,y) = e^{\left(-\frac{1}{3}x^3+x-y^2\right)}$$

$$f_x = \left(-x^2+1\right)\cdot e^{\left(-\frac{1}{3}x^3+x-y^2\right)} \Rightarrow f_{xx}(x,y) = \left(-2x+\left(1-x^2\right)^2\right)\cdot e^{\left(-\frac{1}{3}x^3+x-y^2\right)}$$

$$f_y = \left(-2y\right)\cdot e^{\left(-\frac{1}{3}x^3+x-y^2\right)} \Rightarrow f_{yy}(x,y) = \left(-2+4y^2\right)\cdot e^{\left(-\frac{1}{3}x^3+x-y^2\right)}$$

$$f_{xy}(x,y) = -2y\left(1-x^2\right)\cdot e^{\left(-\frac{1}{3}x^3+x-y^2\right)}$$

- $Now,\ \ for\ \ po\mathrm{int}\ (-1,0)$

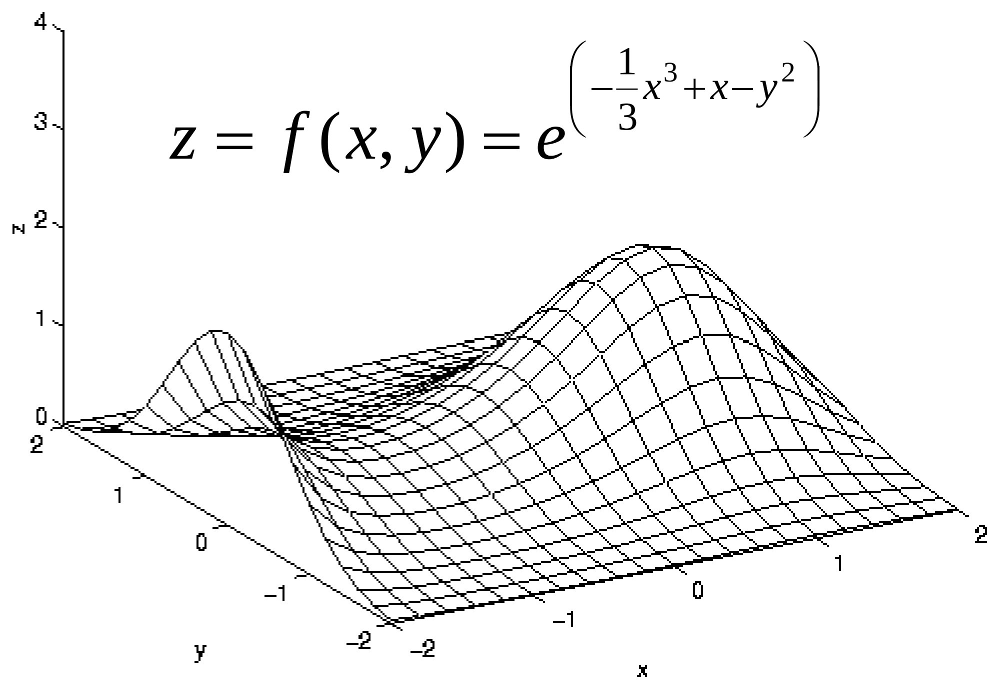
$$r = f_{xx}(-1,0) = 2\cdot e^{\left(\frac{-2}{3}\right)}, s = f_{xy}(-1,0) = 0, t = f_{yy}(-1,0) = -2\cdot e^{\left(\frac{-2}{3}\right)}$$

$$\Rightarrow rt - s^2 = -4e^{\left(-\frac{4}{3}\right)} < 0$$

$$\Rightarrow (-1,0)\ is\ \ Saddle\ Po\mathrm{int}.$$

- Notice the relative maximum at (x=1,y=0).  (x=-1,y=0) is a relative maximum if one travels in the y direction and a relative minimum if one travels in the x-direction. Near (-1,0) the surface looks like a saddle(neither maxima nor minima).

$$z = f(x,y) = e^{\left(-\frac{1}{3}x^3 + x - y^2\right)}$$



Revision : -

Working rule to find Extremum Values for a function $z = f(x, y)$ :

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ i.e. p and q equate then to zero. Solve these simultaneous equations for x and y Let $a_1, b_1; a_2, b_2;$ ………be the pairs of roots.

Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$, (i.e. r, s and t respectively) and substitute in them by turns $a_1, b_1; a_2, b_2;$………for x, y. Calculate the value of $rt - s^2$ for each pair of roots.

If $rt - s^2 > 0$ and r is negative for a pair of roots, f(x,y) is a maximum for this pair.

If $rt - s^2 > 0$ and r is positive, f(x, y) is a minimum. If $rt - s^2 < 0$, the function has a saddle point there.

If $rt - s^2 = 0$, the case is undecided, and further investigation is necessary to decide it.

# Examples

Example 1: Find the Maximum and Minimum of the function

$$f(x, y) = x^3 + y^3 - 3axy$$

**Solution:** Given that

$f(x,y) = x^3 + y^3 - 3\,axy$ (i)

Therefore

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay,\quad q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x,\ s = \frac{\partial^2 f}{\partial x \partial y} = -3a,\ \ t = \frac{\partial^2 f}{\partial y^2} = 6y$$

for maxima and minima, we have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0 \Rightarrow x^2 = ay \qquad \text{(ii)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow y^2 = ax \qquad \text{(iii)}$$

putting the value of y in equation (iii), we get

$x^4 = a^3 x$

$\Rightarrow x\,(x^3 - a^3) = 0$

$\Rightarrow x\,(x - a)\,(x^2 + ax + a^2) = 0$

$\Rightarrow x = 0,\ x = a$

Putting x =0 in (ii), we get y =0, and putting x =a in (ii) we get y =a. Therefore (0,0) and (a, a) are the stationary points (i.e. critical points) testing at (0, 0).

$r = 0,\ t = 0,\ s = -3a \Rightarrow rt - s^2 =$ Negative

Hence there is no extemum value at (0,0).

Testing at (a, a)

$r = 6a,\ t = 6a,\ s = -3a$

$\Rightarrow rt - s^2 = 6a \times 6a - (-3a)^2$

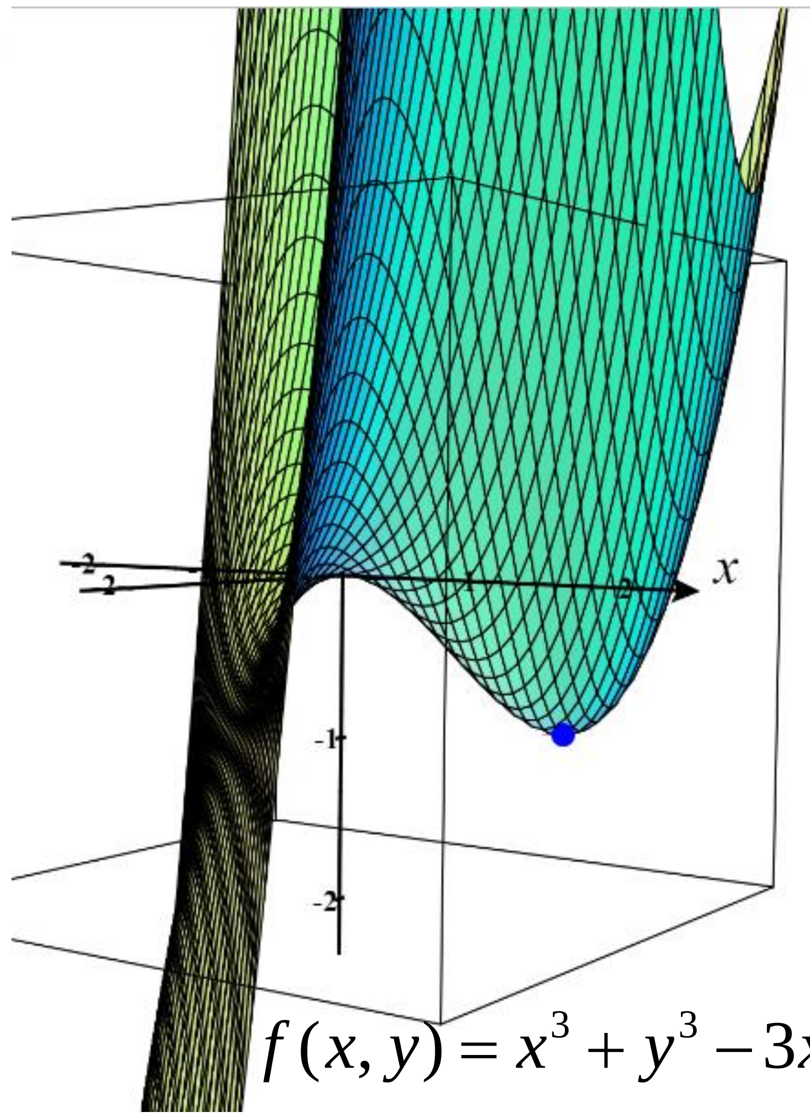
$= 36a^2 - 9a^2$

$= 27a^2 > 0$ and also $r = 6a > 0$

Therefore (a, a) is a minimum point.

The minimum value of $f(a, a) = a^3 + a^3 - 3a^3$

$= - a^3$ Answer.




$f(x,y) = x^3 + y^3 - 3xy$

**Example 2:** Find the extreme value of $f(x,y) = y^2 - x^2$

**Solution:** we have

$f(x, y) = y^2 - x^2$

Differentiating partially with respect to x and y, we get

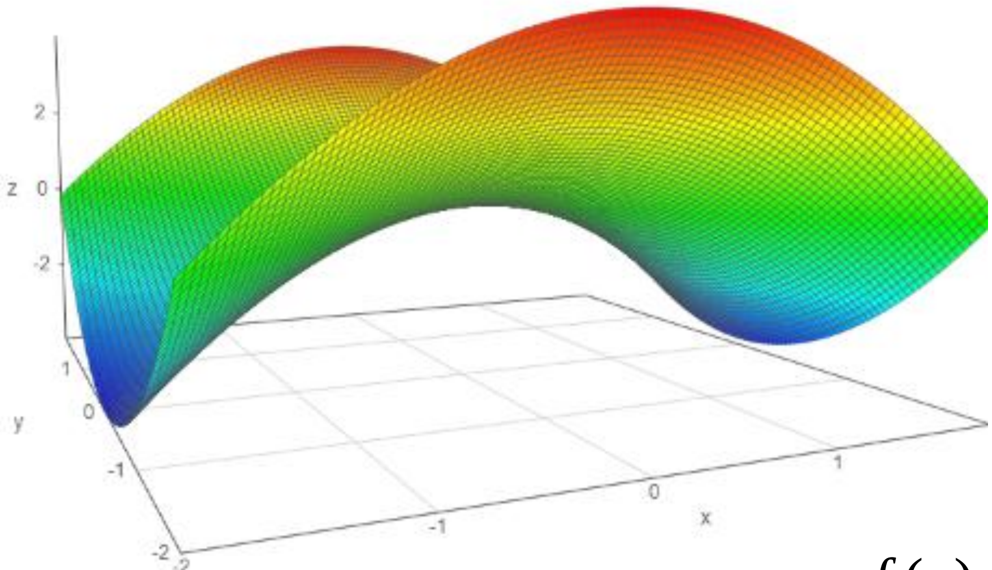
$f_x = -2x,\ f_y = 2y$

which follows $fx(0,0) = 0$ and $fy\ (0,0) = 0$

Therefore, (0,0) is the only critical point.

However the function f has neither local maximum nor a local minima at (0,0).

Thus (0,0) is called saddle point of f.




$$f(z) = y^2 - x^2$$

Example 3: Test the function $f(x, y) = x^3 y^2 (6 - x - y)$ for Maxima and minima for points not at the origin

**Solution:**

Here $f(x,y) = x^3 y^2 (6 - x - y)$

$$= 6x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$\therefore p = \frac{\partial f}{\partial x} = 18x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$q = \frac{\partial f}{\partial y} = 12x^3 y - 2x^4 y - 3x^3 y^2$$

$$r = \frac{\partial^2 f}{\partial x^2} = 36xy^2 - 12x^2 y^2 - 6xy^3$$

$$= 6xy^2 (6 - 2x - y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 36x^2 y - 8x^3 y - 9x^2 y^2$$

$$= x^2 y (36 - 8x - 9y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12x^3 - 2x^4 - 6x^3 y$$

$$= x^3 (12 - 2x - 6y)$$

Now, for maxima or minima, we have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x^2 y^2 (18 - 4x - 3y) = 0 \qquad \text{(i)}$$

and $\dfrac{\partial f}{\partial y} = 0$

$$\Rightarrow x^3 y(12 - 2x - 3y) = 0 \qquad \text{(ii)}$$

From (i) & (ii)

$4x + 3y = 18$

$2x + 3y = 12$ and $x = 0 = y$

solving, we get $x = 3$, $y = 2$ and $x = 0 = y$. Leaving $x = 0 = y$, we get $x = 3$, $y = 2$. Hence (3, 2) is the only stationary point under consideration.

Now,

$rt - s^2 = 6x^4 y^4 (6 - 2x - y)(12 - 2x - 6y) - x^4 y^2 (36 - 8x - 9y)^2$

At (3, 2)

$rt - s^2 = +$ ive $(> 0)$

Also, $r = 6(3)(4)(6 - 6 - 4) = -$ ive $(< 0)$

$\therefore f(x,y)$ has a maximum value at (3, 2).

Example 4. Examine for minimum and maximum value for

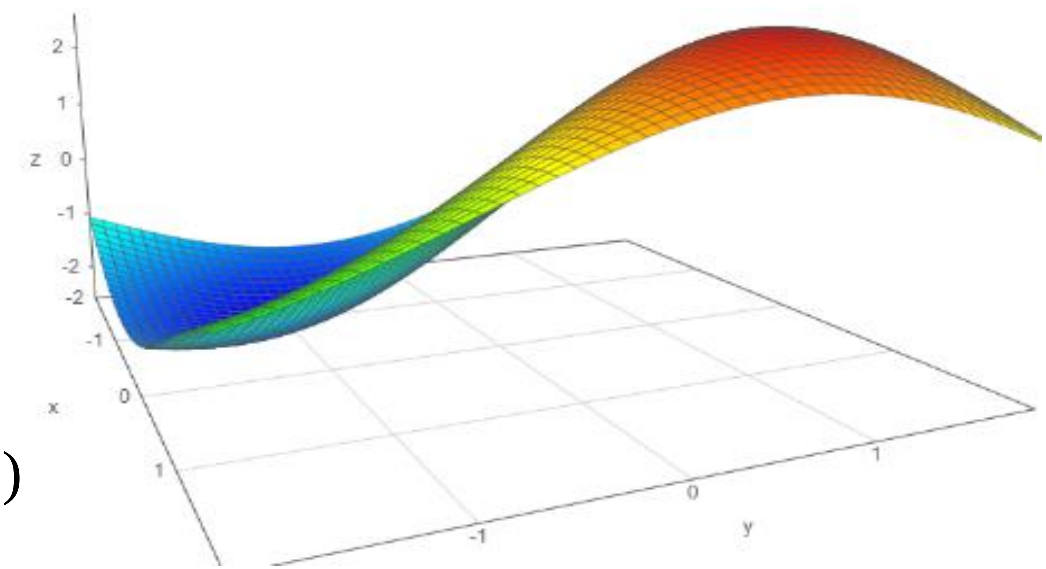
$$f(x, y) = \sin x + \sin y + \sin(x + y)$$

**Solution:** Here, f(x,y) = sin x + sin y + sin ( x + y)

$$\therefore p = \frac{\partial f}{\partial x} = \cos x + \cos(x + y)$$

$$q = \frac{\partial f}{\partial y} = \cos y + \cos(x + y)$$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x + y)$$




$$f(x, y) = \sin x + \sin y + \sin(x + y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x + y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x + y)$$

Now $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow \cos x + \cos(x + y) = 0$……….(i) and $\cos y + \cos(x + y) = 0$ ……..(ii)
subtracting (ii) from (i) we have
$\cos x - \cos y = 0$ or $\cos x = \cos y$
$\Rightarrow x = y$
From (i), $\cos x + \cos 2x = 0$
or $\cos 2x = -\cos x = \cos(\pi - x)$

or $2x = \pi - x \therefore x = \frac{\pi}{3}$

$\therefore x = y = \frac{\pi}{3}$ is a stationary point

$\therefore x = y = \dfrac{\pi}{3}$ is a stationary point

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $r = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3}$, $s = \dfrac{\sqrt{3}}{2}$

$t = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore rt - s^2 = 3 - \dfrac{3}{4} = \dfrac{9}{4} > 0$

Aslo $r < 0$

$\therefore f(x, y)$ has a maximum value at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$

Maximum value $= f\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$

$= \sin\dfrac{\pi}{3} + \sin\dfrac{\pi}{3} + \sin\dfrac{2\pi}{3}$

$= \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2}$ Answer.

**Ex. 5 :** *Show that the minimum value of* $xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ *is* $3a^2$.

**Sol. :** Here $\quad f(x, y) = xy + a^3/x + a^3/y$

**Step I :** Extreme values of f (x, y) are given by

$$\frac{\partial f}{\partial x} = y - a^3/x^2 = 0$$

$$\frac{\partial f}{\partial y} = x - a^3/y^2 = 0$$

We now solve (1) and (2) simultaneously.

Equations (1) and (2) are equivalent to

$$x^2y = a^3 \text{ and } xy^2 = a^3$$

Substituting the value of x from second equation into the first,

$$\frac{a^6}{y^4}\, y = a^3 \text{ or } y = a$$

From second equation, $x = a$.

Hence the stationary point is (a, a).

**Step II** : We have

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \qquad t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

At $(a, a)$, $r = 2$, $s = 1$, $t = 2$

**Step III :**

At $(a, a)$, $rt - s^2 = (2)(2) - (1)^2 = 3 > 0$ and $r = 2 > 0$

Hence at the point $(a, a)$, function $f(x, y)$ has minimum value and

$f_{min} = (a)(a) + a^3(1/a + 1/a) = 3a^2$, which is the required result.

*Ex. 6. : Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.*

**Sol. :** Consider x, y, z be the numbers whose sum is 120

i.e. $$x + y + z = 120 \quad \dots (1)$$

Next, consider the sum of their products as

$$u = xy + yz + zx \quad \text{(function of three variables)} \quad \dots (2)$$

Converting the function of two variables using condition (1), we have

$$\begin{aligned} u &= xy + y(120 - x - y) + x(120 - x - y) \\ &= 120x + 120y - xy - x^2 - y^2 = f(x, y) \text{ (function of two variables)} \end{aligned} \quad \dots (3)$$

**Step I :** Now for maximum value of u, we solve

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad 120 - y - 2x = 0 \quad \dots (4)$$

$$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad 120 - x - 2y = 0 \quad \dots (5)$$

Solving (4) and (5) simultaneously, we have

$$x = 40 \quad \text{and} \quad y = 40$$

$$x = 40 \quad \text{and} \quad y = 40$$

Hence the point (40, 40) is the stationary point of the function.

**Step II :** We have

$$r = \frac{\partial^2 f}{\partial x^2} = -2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

**Step III :** At (40, 40), $rt - s^2 = (-2)(-2) - (-1)^2 = 3 > 0$ and $r = -2 < 0$

Hence the function f (x, y) and therefore u is maximum at $x = 40$, $y = 40$ and $z = 40$

## Homework Examples

Find Maximum and minimum value of following functions

1. $(x-y)(x^2+y^2)(x+y-1)$ Ans: No maxima , No Minima
2. $2(x^2-y^2)-x^4+y^4$ Ans: Max at $(\pm 1,0)$ minima at $(0,\pm 1)$
3. $(x^2+y^2)^2-2(x^2-y^2)$ , Ans: Min value -1 at (1,0) and (-1,0)

2) test the function $f(x,y)=(x^2+y^2)e^{-(x^2+y^2)}$ for Maxima or Minima for points not on the circle $x^2+y^2=1$

3. In plane triangle ABC find the maximum value of $\cos A \cdot \cos B \cdot \cos C$

# LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

We know how to find the critical points of a function of two variables:

look for where ∇f = 0. That is, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

Sometimes, however, we have a constraint which restricts us from choosing variables freely

- ❑.Maximize volume subject to limited material costs
- ❑. Minimize surface area subject to fixed volume
- ❑.Maximize utility subject to limited income

Consider the Example :

*Maximize the function* $f(x, y) = \sqrt{xy}$ *subject to the constraint* $20x + 10y = 200$.

Solution

*Solve the constraint for* $y$ *and make* $f$ *a single-variable function:* $2x + y = 20$, *so* $y = 20 - 2x$. *Thus*

$$f(x) = \sqrt{x(20 - 2x)} = \sqrt{20x - 2x^2}$$
$$f'(x) = \frac{1}{2\sqrt{20x - 2x^2}}(20 - 4x) = \frac{10 - 2x}{\sqrt{20x - 2x^2}}.$$

*Then* $f'(x) = 0$ *when* $10 - 2x = 0$, *or* $x = 5$. *Since* $y = 20 - 2x$, $y = 10$. $f(5, 10) = \sqrt{50}$.

# LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

- In many problems, a function of two or more variables is to be optimized, **subjected to a restriction or constraint on the variables**, here we will consider a function of three variables to study Lagrange's method of undetermined multipliers

- Suppose it is required to find the stationary values for the function $f(x , y , z)$ subject to condition $\Phi( x , y , z) = 0$ -------- (1)
- .Step 1 : Form a lagrangean function

$F(x , y , z) = f(x , y , z) + \lambda \Phi ( x , y , z)$ where "$\lambda$ " is called Lagrange's constant, which is determined by the following conditions.

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

- . Step 2 : Solving the equations (1) (2) (3) & (4) we get the stationary point (x, y, z).
- . Step 3 : Substitute the value of x , y , z so obtained in equation (1) we get the stationary Values.

# Examples

Example 1: As dimension of a triangle ABC are varied, Show that the maximum value of $\cos A \cdot \cos B \cdot \cos C$ is obtained when the triangle is equilateral

**Sol. : Step I :** Let $u = f(A, B, C) = \cos A \cos B \cos C$ ... (1)

Under the condition $\phi = A + B + C - \pi = 0$ ... (2)

Construct the function $F = ut + \lambda\phi$

$$F = \cos A \cos B \cos C + \lambda (A + B + C - \pi)$$

Form the equations :

$$\frac{\partial F}{\partial A} = 0 \quad \therefore \quad -\sin A \cos B \cos C + \lambda = 0 \qquad \text{... (3)}$$

$$\frac{\partial F}{\partial B} = 0 \quad \therefore \quad -\cos A \sin B \cos C + \lambda = 0 \qquad \text{... (4)}$$

$$\frac{\partial F}{\partial C} = 0 \quad \therefore \quad -\cos A \cos B \sin C + \lambda = 0 \qquad \text{... (5)}$$

**Step II :** We eliminate A, B, C and $\lambda$ using equations (1) to (5).

From equations (3), (4) and (5)

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$$

Dividing by $\cos A \cos B \cos C$, we have

$$\tan A = \tan B = \tan C$$

$$\Rightarrow \qquad A = B = C$$

$\Rightarrow$ $\Delta$ ABC is equilateral.

**2. Find the minimum value of $x^2 + y^2 + z^2$ , given that $xyz = a^3$**

Sol: Let $u = x^2 + y^2 + z^2$ ...... (1)

And $\emptyset = xyz - a^3 = 0$ ...... (2)

Consider the lagrangean function $F(x,y,z) = u(x,y,z) + \lambda\,\phi(x,y,z)$

i.e , $F(x,y,z) = x^2 + y^2 + z^2 + \lambda\,(xyz - a^3)$ ......(3)

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 2x + \lambda yz = 0$ ......(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 2y + \lambda xz = 0$ ......(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 2z + \lambda yx = 0$ ......(6)

From (4) , (5) and (6) ,we have $\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = -\frac{\lambda}{2}$ ......(7)

From the first two members , we have $\frac{x}{yz} = \frac{y}{xz} \Rightarrow x^2 = y^2$ ...(8)

From the last members , we have $\frac{y}{xz} = \frac{z}{xy} \Rightarrow y^2 = z^2$ ......(9)

From (8) and (9) , we have $x^2 = y^2 = z^2 \Rightarrow x = y = z$ ......(10)

on solving (2) and (10) , we get , $x = y = z = a$

$\therefore$ Minimum value of $u = a^2 + a^2 = 3a^2$

**3. Find the maximum value of $u = x^2y^3z^4$ if $2x + 3y + 4z = a$**

Sol: Given $u = x^2y^3z^4$ ...... (1)

Let $\phi(x, y, z) = 2x + 3y + 4z - a = 0$ ......(2)

Consider the lagrangean function $F(x,y,z) = u(x,y,z) + \lambda\,\phi(x, y, z)$

i.e, , $F(x,y,z) = x^2y^3z^4 + \lambda\,(2x + 3y + 4z - a)$ ......(3)

for maxima or minima $\frac{\partial F}{\partial x} = 0$ , $\frac{\partial F}{\partial y} = 0$ , $\frac{\partial F}{\partial z} = 0$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 2xy^3z^4 + 2\lambda = 0 \Rightarrow xy^3z^4 = -\lambda$ ......(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + 3\lambda = 0 \Rightarrow x^2y^2z^4 = -\lambda$ .......(5)

and $\frac{\partial F}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + 4\lambda = 0 \Rightarrow x^2y^3z^3 = -\lambda$ .......(6)

From (4) and (5) , we have $x = y$ .......(7)

From (5) and (6) , we have $y = z$ .......(8)

Hence from (7) and (8) , we get $x = y = z$ .......(9)

On solving (2) and (9) ,we get $x = y = z = \frac{a}{9}$

$$\therefore \text{Maximum value of } u = \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^9$$

**4. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.**

Sol: Let 2x ,2y, 2z are the length , breadth and height of rectangular solid

Then its volume V = 8 xyz ........(1)

Let the sphere have a radius of 'r' so that $x^2 + y^2 + z^2 = r^2$ ......(2)

Consider the lagrangean function F(x,y,z) = u( x,y,z ) + λ ϕ(x, y, z)

i.e, $F(x,y,z) = V + \lambda (x^2 + y^2 + z^2 - r^2)$

$$= 8xyz + \lambda (x^2 + y^2 + z^2 - r^2) \quad .......(3)$$

For maxima or minima $\frac{\partial F}{\partial x} = 0$ , $\frac{\partial F}{\partial y} = 0$ , $\frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + 2\lambda x = 0$ ......(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8zx + 2\lambda y = 0$ ......(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xz + 2\lambda z = 0$ ......(6)

From (4) ,(5) and (6) we have $2x^2\lambda = -8xyz = -2y^2\lambda = -2z^2\lambda$

$$\Rightarrow x = y = z$$

Thus for a maximum value x = y = z which shows that the rectangular solid is a cube.

Example 5 : Use Lagranges method to find the minimum distance from origin to the plane $3x+2y+z=12$

**Sol. : Step I :** Let P (x, y, z) be any point on the plane $3x + 2y + z = 12$

$$\therefore \quad d(O, P) = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad [d(O, P)]^2 = x^2 + y^2 + z^2$$

Let $\quad u = f(x, y, z) = x^2 + y^2 + z^2 \quad \ldots (1)$

Under the condition $\quad \phi = 3x + 2y + z - 12 = 0 \quad \ldots (2)$

Construct the function $\quad F = u + \lambda\phi$

$$= x^2 + y^2 + z^2 + \lambda[3x + 2y + z - 12]$$

Form the equations :

$$\frac{\partial F}{\partial x} = 0 \quad \therefore \quad 2x + 3\lambda = 0 \quad \ldots (3)$$

$$\frac{\partial F}{\partial y} = 0 \quad \therefore \quad 2y + 2\lambda = 0$$

$$y + \lambda = 0 \quad \ldots (4)$$

$$\frac{\partial F}{\partial z} = 0 \quad \therefore \quad 2z + \lambda = 0 \quad \ldots (5)$$

**Step II :** We eliminate x, y, z and $\lambda$ using equations (1) to (5).

From equations (3), (4) and (5),

$$\frac{2x}{3} = y = 2z$$

Let $$\frac{2x}{3} = y = 2z = k$$

$\therefore$ $$x = \frac{3k}{2}, \quad y = k, \quad z = \frac{k}{2} \qquad \ldots (6)$$

Substituting in (2)

$$\frac{9k}{2} + 2k + \frac{k}{2} = 12$$

$\therefore$ $$k = \frac{12}{7}$$

From (6), $$x = \frac{18}{7}$$

$$y = \frac{12}{7}$$

$$z = \frac{6}{7} \qquad \therefore \qquad P \equiv \left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$$

$\therefore$ $$d(OP) = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{\sqrt{504}}{7}$$

Example 6 : Divide 24 into three parts such that continued product of the first, square of second, and cube of third may be maximum

**Sol. : Step I :** Divide 24 into three parts as

$$\text{first part} = x$$
$$\text{second part} = y$$
$$\text{third part} = z$$

Let $u = f(x, y, z) = xy^2z^3$ ... (1)

Under the condition $\phi = x + y + z - 24 = 0$ ... (2)

Construct the function $F = u + \lambda\phi$

$\therefore$ $F = xy^2z^3 + \lambda(x + y + z - 24)$

Form the equations :

$$\frac{\partial F}{\partial x} = 0 \quad \therefore \quad y^2z^3 + \lambda = 0 \quad \text{... (3)}$$

$$\frac{\partial F}{\partial y} = 0 \quad \therefore \quad 2xyz^3 + \lambda = 0 \quad \text{... (4)}$$

$$\frac{\partial F}{\partial z} = 0 \quad \therefore \quad 3xy^2z^2 + \lambda = 0 \quad \text{... (5)}$$

**Step II :** We eliminate x, y, z and $\lambda$ using equations (1) and (5).

From equations (3), (4) and (5)

$$y^2 z^3 = 2xyz^3 = 3xy^2z^2$$

Dividing by $xy^2 z^3$

$$\Rightarrow \qquad \frac{1}{x} = \frac{2}{y} = \frac{3}{z} = k$$

$$\left.\begin{array}{lll} \dfrac{1}{x} = k & \therefore & x = \dfrac{1}{k} \\ \dfrac{2}{y} = k & \therefore & y = \dfrac{2}{k} \\ \dfrac{3}{z} = k & \therefore & z = \dfrac{3}{k} \end{array}\right\} \qquad \cdots (6)$$

Substituting in equation (2),

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} = 24$$

$$\therefore \qquad 6 = 24\,k$$

$$\therefore \qquad k = \frac{1}{4}$$

From equation (6) $\quad x = 4$

$$y = 8$$

$$z = 12$$

$\therefore$ We divide 24 into three parts 4, 8, 12.

# Home Work Examples

Example 1. FInd the Maximum Value of $u = x^2 y^3 z^4$

such that $2x + 3y + 4z = a$ Answer: $\left(\frac{a}{9}\right)^9$

Example 2. Find the Maximum & Minimum value of

$x^2 + y^2$ when $3x^2 + 4xy + 6y^2 = 140$

Answer: Max value: 70, Min Value : 20

Example 3. Find Stationary values of $u = x + y + z$

if $xy + yz + zx = 12$ Ans: 6