Curvilinear Motion Theory

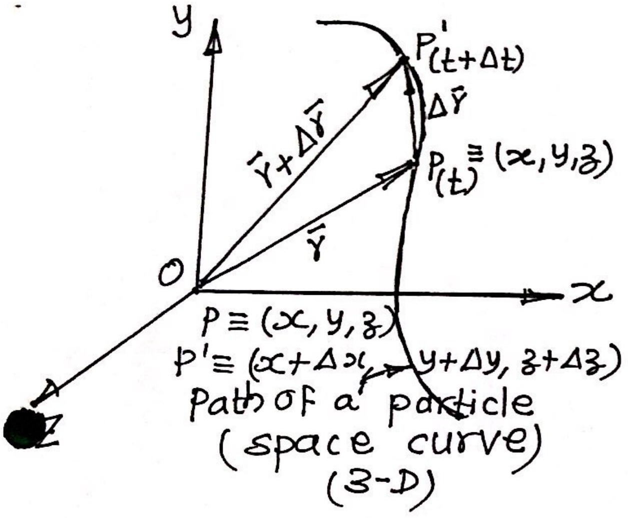
Dynamics Theory

Curvilinear Motion

1. Rectangular Co-ordinates
2. Motio of Projectiles
3. Path Variables : Tangential and Normal Components of Acceleration
4. Polar Co-ordinates : Radial and Trans-verse Components of Velocity and Acceleration

CURVILINEAR MOTION

A) Rectangular Components:-



$x, y, z \rightarrow f(t)$

Position Vector:

$$\boxed{\bar{r} = OP = xi + yj + zk \quad m}$$

Velocity

$$\boxed{\bar{v} = \frac{dr}{dt} = \dot{x}i + \dot{y}j + \dot{z}k \ \ m/s}$$

Velocity is always tangential to the path

Acceleration

$$\boxed{\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = \ddot{x}\,i + \ddot{y}\,j + \ddot{z}\,k \quad m/s^2}$$

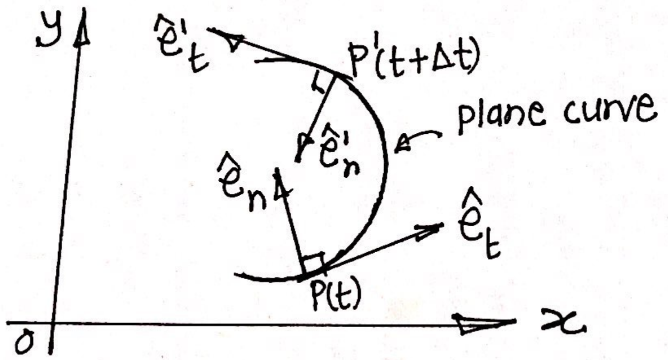
Acceleration is not tangential to the path, but it is tangential to an imaginary curve called as hodograph, **Hodograph** is an imaginary curve obtained by joining the tips to all velocity vectors. Because of this, it becomes necessary for us to resolve the acceleration along two known directions. There are two methods for resolving acceleration.

I) Tangential and Normal Components of Acceleration. (Path variables)

II) Radial and Transverse Components of Velocity and Acceleration (Polar co- ordinates)

B) Path Variables:-

Tangential and Normal Components of Acceleration

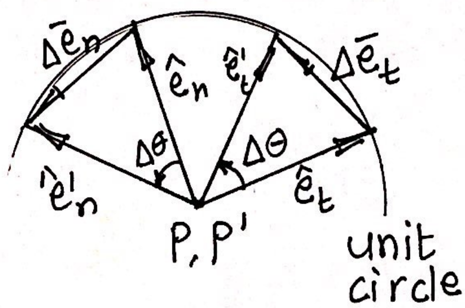


Consider a particle traveling along a plane curve. P is its position at any time t and P' at time $(t+\Delta t)$. $\hat{e}_t$ and $\hat{e}_n$ are the unit vectors along the along the tangent and normal to the path at point P. Similarly, $\hat{e}_t'$ and $\hat{e}_n'$ are the unit vectors along the tangent and the normal to the path at point P'. These unit vectors are changing with time (Hence they are not considered as constant vectors).

As $\Delta t \rightarrow 0$,

$P' \rightarrow P$

and we can draw,

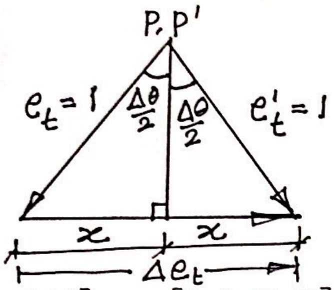


$$\hat{e}_t + \Delta\overline{e_t} = \hat{e}'_t$$

$$\hat{e}_n + \Delta\overline{e_n} = \hat{e}'_n$$

$$\Delta e_t = 2x$$

$$\sin\frac{\Delta\theta}{2} = \frac{x}{1}$$

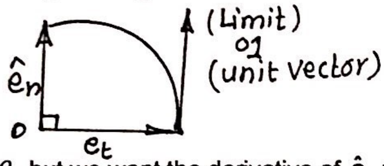


$$\Delta e_t = 2\sin\frac{\Delta\theta}{2}$$

$$\lim_{\Delta\theta\to 0}\left[\frac{\Delta \bar{e}_t}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta/2}\right]$$

$$= 1\,(\text{unit vector}) = \hat{e}_n$$

$$\therefore \boxed{\frac{d\hat{e}_t}{d\theta} = \hat{e}_n}$$



This is the derivative of $\hat{e}_t$ w.r.t. θ, but we want the derivative of $\hat{e}_t$ w.r.t. time,

Hence,

$$\dot{e}_t = \frac{de_t}{dt} = \frac{de_t}{d\theta}\cdot\frac{d\theta}{ds}\cdot\frac{ds}{dt}$$

$$= e_n \cdot \frac{1}{\rho}.v$$

As $s = \rho\cdot\theta$

$ds = \rho\cdot d\theta$

$\therefore \frac{d\theta}{ds} = \frac{1}{\rho}$

Now the velocity of the particle in curvilinear translation is tangential to the path.

$$\boxed{\bar{v} = v.\,\hat{e}_t \quad m/s} \qquad \text{.................(i)}$$

Differentiating eq^n (i) w.r.t. time we get the acceleration of a particle in curvilinear translation.

$$\bar{a} = \frac{dv}{dt}.e_t + v.\dot{e}_t$$

$$\boxed{\bar{a} = \left(\frac{dv}{dt}\right)\hat{e}_t + \left(\frac{v^2}{\rho}\right)\hat{e}_n \quad m/s^2} \qquad \text{........(ii)}$$

Thus,

$$\therefore \bar{a} = \bar{a}_t + \bar{a}_n$$

$\therefore$ Resultant accelaratбn, $\quad a = \sqrt{a_t^2 + a_n^2}$

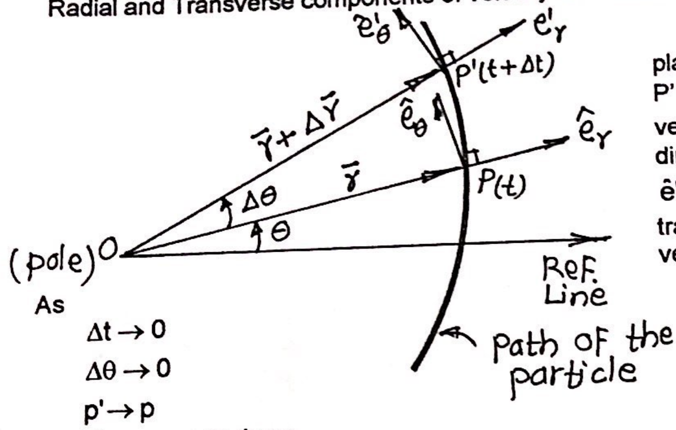
Where,

$a_t = \frac{dv}{dt}$ = tangential component of acceleration. It gives us the idea about the rate of change of the speed of the particle.

$a_n = \frac{v^2}{\rho}$ = Normal component of accleration. It is releated to the change in direction of the path of the particle.

C) **Polar co-ordinates:-**

Radial and Transverse components of velocity and accelerations-



Consider a particle traveling along a plane curve. 'P' is its position at any time 't' and P' at time $(t+\Delta t)$, $\hat{e}_r$ and $\hat{e}_\theta$ are the unit vectors along the radial and transverse directions at point P. Similarly $\hat{e}'_r$ and $\hat{e}'_\theta$ are the unit vectors along the radial and transverse directions at position P'. These unit vectors are changing with time.

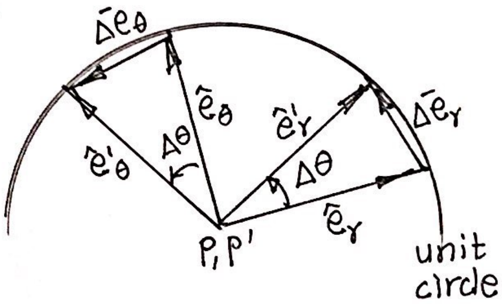
As

$\Delta t \to 0$

$\Delta\theta \to 0$

$p' \to p$

Then we can draw:



$\hat{e}_r + \overline{\Delta e_r} = e'_r$

$\hat{e}_\theta + \overline{\Delta e_\theta} = e'_\theta$

Here also, $\Delta e_r = \Delta e_\theta = 2\sin\dfrac{\Delta\theta}{2}$

$$\lim_{\Delta\theta\to 0}\left[\frac{\Delta e_t}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta/2}\right] = 1\ (\text{Unit vector}) = \hat{e}_\theta$$

$$\therefore \left[\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta\right] \qquad \text{......................(a)}$$

$$\lim_{\Delta\theta\to 0}\left[\frac{\Delta e_\theta}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta}\right] = \lim_{\Delta\theta\to 0}\left[\frac{2\sin\Delta\theta/2}{\Delta\theta/2}\right] = 1\ (\text{Unit vector}) = -\hat{e}_r$$

$$\therefore \left[\frac{d\hat{e}_r}{d\theta} = -\hat{e}_r\right] \qquad \text{......................(b)}$$

But these are the derivatives w.r.t. θ, and we are interested in the derivatives w.r.t. time, hence,

$$\dot{e}_r = \frac{d\overline{e_r}}{dt} = \frac{d\overline{e_r}}{d\theta}\frac{d\theta}{dt}$$

$$\dot{e}_r = \dot{\theta}\,.\hat{e}_\theta \qquad \text{..............................(c)}$$

$$\dot{e}_\theta = \frac{d\overline{e_\theta}}{dt} = \frac{d\overline{e_\theta}}{d\theta}\frac{d\theta}{dt}$$

$$= -\dot{\theta}\hat{e}_r \qquad \text{..............................(d)}$$

Now, the position vector of the particle at any time, 't' is given by

$$\boxed{\bar{r} = \overline{OP} = r.\hat{e}_r \quad m} \qquad \text{.........................(i)}$$

Differentiating eq^n (i) w.r.t. time, we get the velocity of the particle at any time 't'

$$\boxed{\bar{v} = \frac{d\bar{r}}{dt} = \dot{r}\,\hat{e}_r + r.\dot{\theta}\,\hat{e}_\theta \quad m/s} \qquad \text{.............(ii)}$$

Thus,

$$\therefore \bar{v} = \overline{v_r} + \overline{v_\theta}$$

$$\therefore \text{speed}, v = \sqrt{v_r^2 + v_\theta^2}$$

where, $v_r = \dot{r}$ = radial componentof velocity

$v_\theta = r\dot{\theta}$ = transversecomponentof velocity

Differentiating eq^n (ii) w.r.t time we get, acceleration of the particle

$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{r}.e_r + \dot{r}\dot{e}_r + \dot{r}\dot{\theta}e_\theta + r\ddot{\theta}e_\theta + r\dot{\theta}\dot{e}_\theta$$

$$= \ddot{r}e_r + \dot{r}\dot{\theta}e_\theta + \dot{r}\dot{\theta}e_\theta + r\ddot{\theta}e_\theta - r\dot{\theta}^2 e_r$$

$$\boxed{\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \quad m/s^2} \qquad \text{.........(iii)}$$

Thus,

$$\therefore \bar{a} = \overline{a_r} + \overline{a_\theta}$$

$$\text{Resultant acceleration } a = \sqrt{a_r^2 + a_\theta^2}$$

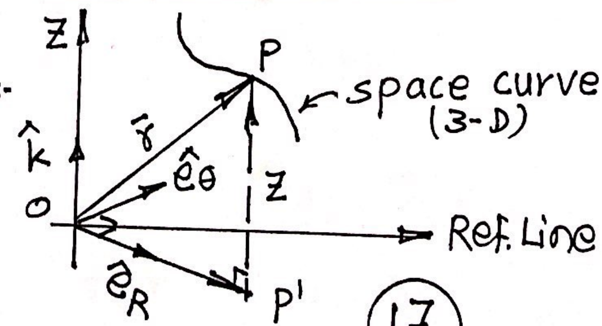
Where,

$a_r = \left(\ddot{r} - r\dot{\theta}^2\right)$ = Radial component of acceleration

$a_\theta = \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$ = Transverse component of acceleration.

The term $(2\dot{r}\dot{\theta})$ is called as <u>Coriollis component of acceleration</u>. It is the combined effect of the radial speed and angular speed.

D) <u>Cylindrical co-Ordinates</u>:-



Extension of polar frame to space is called as cylindrical frame of reference. In this case the position of the particle at any time 't' is defined by the co-ordinates $(R, \theta \text{ and } Z)$

The position vector of the particle at any time 't' is given by

$$\boxed{\bar{r} = \overline{op} = R.\hat{e}_R + Z.\hat{k} \ \ m} \quad \text{.........................(i)}$$

Differentiating (i) w.r.t. time, we get the velocity of the particle at any time 't'

$$\boxed{\bar{v} = \dot{R}.\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}k \quad m/s} \quad \text{.........................(ii)}$$

Differentiating $eq^b(2)$ w.r.t. time we get the acceleration of the particle at any time 't'

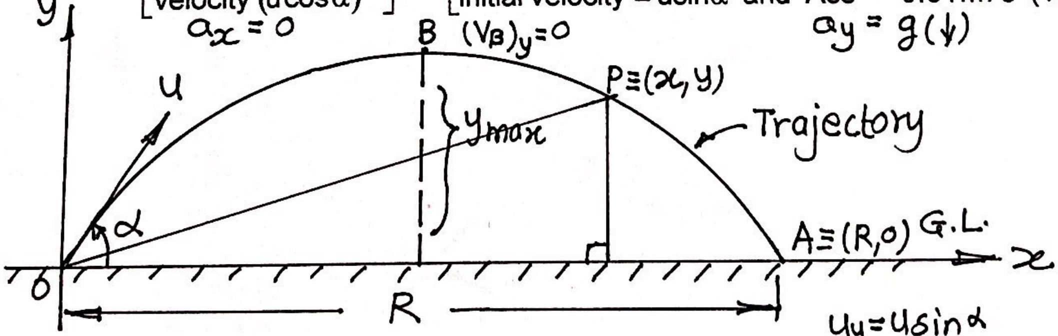
$$\boxed{\bar{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta + \ddot{z}\,k \quad m/s^2} \quad \text{.............(iii)}$$

Motion of Projectiles:-

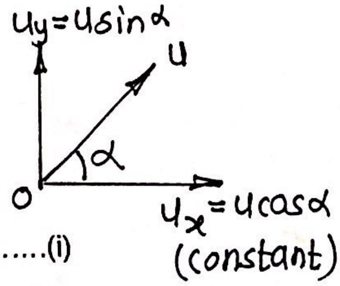
Any body projected in space is called as a projectile. When it is in space, the only force acting on it is its weight i.e. gravitational force. Because of this the motion of the projectile is having acceleration only in vertical direction i.e. gravitational acceleration. The path of the projectile is called as a trajectory and it is assumed that it lies in the same plane during the entire motion .Because of this it is considered as a plane curve. Thus, the motion of a projectile is considered as a (2 – D) motion with constant acceleration.

$$[\text{Motion of projectile}] = \begin{bmatrix} \text{Horizontal motion} \\ \text{uniform motion} \\ \text{with a constant} \\ \text{velocity } (u\cos\alpha) \end{bmatrix} + \begin{bmatrix} \text{vertical motion} \\ \text{uniformly accelareted motion} \\ \text{i.e. motion under gravity} \\ \text{initial velocity} = u\sin\alpha \text{ and Acc}^n = 9.81\,m/s^2(\downarrow) \end{bmatrix}$$

$a_x = 0$ $\qquad$ $a_y = g(\downarrow)$



Equations of motion:-



Let P = (x , y) be the position of the projectile at any time 't' where ,
x = horizontal distance traveled in time 't'
y = vertical distance traveled in time 't'

$$\boxed{x = (u\cos\alpha)t} \quad \text{...................(i)}$$

Scanned with CamScanner

$$\boxed{y = (u\sin\alpha)t - \frac{1}{2}gt^2}$$ ....(ii)

Equations (i) and (2) are called as equations of motion.

Equation of path:-

From equation (i) we get,

$$t = \left[\frac{x}{u\cos\alpha}\right]$$

substituting in equation (ii) , we get,

$$y = \left[u\sin\alpha\,\frac{x}{u\cos\alpha}\right] - \left[\frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}\right]$$

$$\boxed{y = [x\cdot\tan\alpha] - \left[\frac{1}{2}\frac{g\,x^2}{u^2\cos^2\alpha}\right]}$$ ....(iii)

Maximum Height:-

When the projectile is at point B, it is at the topmost point of the trajectory. At this point the projectile stops rising up. After this point the projectile starts falling towards the earth. $(V_B)_y = 0$
Consider only vertical motion of the projectile from O to B,
Applying,

$$v^2 = u^2 + 2as, \quad \text{we get,}$$

$$(v_B)^2_y = (u\sin\alpha)^2 - 2gy_{max}$$

$$0 = u^2\sin^2\alpha - 2gy_{max}$$

$$\boxed{y_{max} = \frac{u^2\sin^2\alpha}{2g}}$$ ....(iv)

Total time of flight:-

The total time for which the projectile is in space is called the total time of flight .It is the time required to travel form O to A

But at point A, $y = 0$

$$0 = (u\sin\alpha)t - \frac{1}{2}gt^2_A \quad \text{from this we get,}$$

$$\boxed{t_A = \frac{2.u.\sin\alpha}{g}}$$ ....(v)

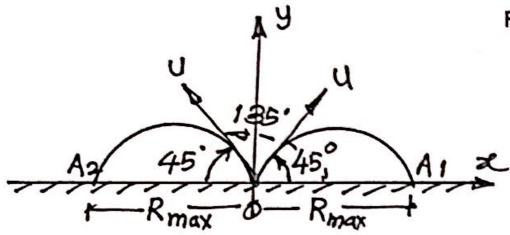
Range of the projectile:-

The maximum horizontal distance traveled by the projectile is called as its horizontal range. At point A, At point A,

$$x = R = (u\cos\alpha)t_A$$

$$= (u\cos\alpha)\,(2u\sin\alpha)$$

$$\boxed{R = \frac{u^2 \sin 2\alpha}{g}} \quad \text{..................(vi)}$$



For maximum range

$\sin 2\alpha = \pm 1$

Hence $2\alpha = 90^\circ$ or 270°

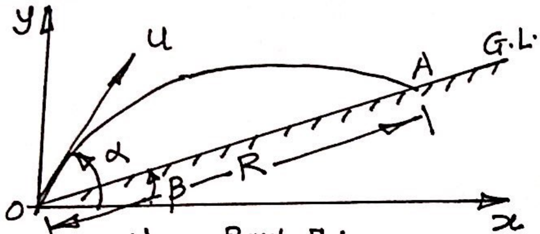
$\alpha = 45^\circ$ or 135°

$$\boxed{R_{max} = \frac{u^2}{g}}$$

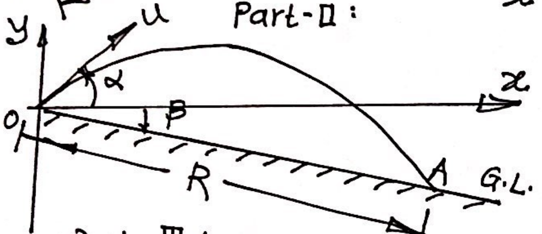
Thus, the range of the projectile on horizontal ground is maximum when the angle of projection is 45° with it.

Projectiles On Inclined Planes

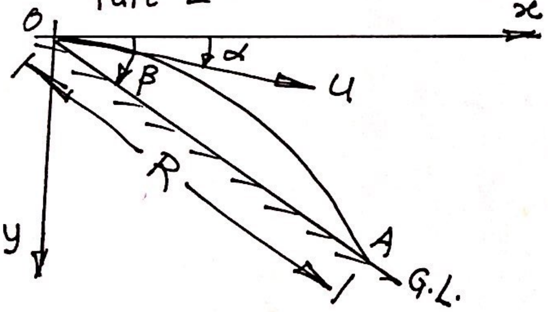
Part –(I) :



Part-II :



Part-III :



$A = (R\cos\beta, R\sin\beta)$

At po int A,

$$x = R\cos\beta = (u\cos\alpha)t_A \text{.......... .... (i)}$$

$$y = R\sin\beta = (u\sin\alpha)t_A - \frac{1}{2}g t_A^{\,2} \text{.......... .... (ii)}$$

From,(i) we get,

$$t_A = \left(\frac{R\cos\beta}{u\cos\alpha}\right)$$

Substuting it in eq^n(2) we get

$$y = u\sin\alpha . \frac{R\cos\beta}{u\cos\alpha} - \frac{1}{2}g\frac{R^2\cos^2\beta}{u^2\cos^2\alpha}$$

$$R\sin\beta = R\cos\beta\tan\alpha - \frac{1}{2}g\frac{R^2\cos^2\beta}{u^2\cos^2\alpha}$$

$$R\sin\beta = R\cos\beta\left[\tan\alpha - \frac{1}{2}g\frac{R\cos\beta}{u^2\cos^2\alpha}\right]$$

$$\tan\beta = -\frac{1}{2}g\frac{R\cos\beta}{u^2\cos^2\alpha} + \tan\alpha$$

$$\frac{1}{2}g\frac{R\cos\beta}{u^2\cos^2\alpha} = (\tan\alpha - \text{Tan}\beta)$$

$$\boxed{R = \frac{2u^2\cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta)}$$

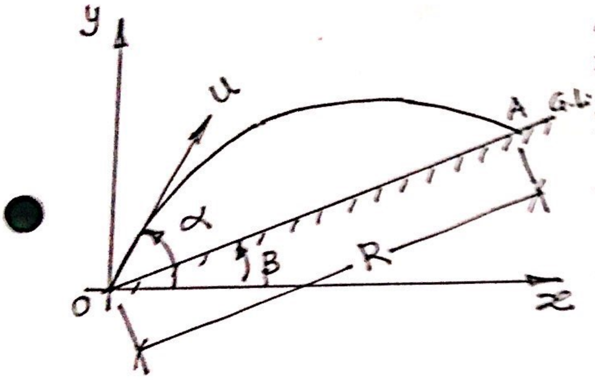
Part-II

$$R = \left(\frac{2u^2\cos^2\alpha}{g\cos\beta}\right)(\tan\alpha + \tan\beta)$$

Part-III

$$R = \left(\frac{2u^2\cos^2\alpha}{g\cos\beta}\right)(\tan\beta - \tan\alpha)$$

Condition For Maximum Range Of A Projectile On Inclined Planes:-



At point A

$$x = R\cos\beta = (u\cos\alpha)\,t_A \quad \text{.....(i)}$$

$$y = R\sin\beta = (u\sin\alpha)t_A - \frac{1}{2}g t_A^2 \quad \text{.....(ii)}$$

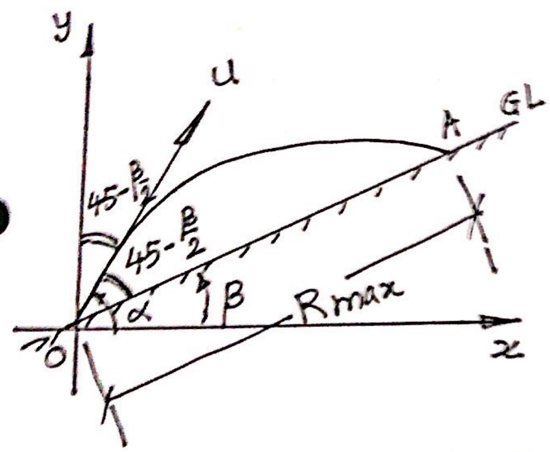
$$y = R\sin\beta = (u\sin\alpha)\left(\frac{R\cos\beta}{u\cos\alpha}\right) - \frac{1}{2}g\frac{(R\cos\beta)^2}{u^2\cos^2\alpha}$$

$$\sin\beta = \cos\beta\frac{\sin\alpha}{\cos\alpha} - \frac{1}{2}g\frac{R\cos^2\beta}{u^2\cos^2\alpha}$$

$$\frac{1}{2}g\frac{R\cos^2\beta}{u^2\cos^2\alpha} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}$$

$$\frac{1}{2}g\frac{R\cos^2\beta}{u^2\cos\alpha} = \sin(\alpha - \beta)$$

$$R = \left(\frac{u^2}{g\cos^2\beta}\right)[2\sin(\alpha-\beta)\cos\alpha]$$



using,

$$2\sin C\cos D = \sin(C + D) + \sin(C - D),$$

$$\boxed{R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha - \beta) - \sin\beta]}$$

For R to be maximum

$$\sin(2\alpha - \beta) = 1$$

$$2\alpha - \beta = 90^\circ$$

$$\boxed{\alpha = 45 + \frac{\beta}{2}}$$

On the horizontal ground $\beta = 0$ hence, we get, For maximum range $\alpha = 45^\circ$ Thus, If the velocity of projection is bisecting the angle between the sloping ground and the vertical then the range on the inclined plane will be maximum.

Dynamics

1. Introduction to Dynamics
2. kinematics and kinetics
3. Rectilinear Motion
   a. Uniform Motion
   b. Uniformly Accelerated Motion
   c. Motion under Gravity
   d. Motion with variable accelera-tion
      i) $a = f(t)$
      ii) $a = f(v)$
      iii) $a = f(x)$
   e. Simple Harmonic Motion
4. Motion Curves

DYNAMICS

Introduction:-

So far in our study in the previous chapters, we have analyzed bodies at rest, which we called as the statics part of mechanics. From this chapter onwards, we will analyze bodies under motion, which we call as the dynamics part of mechanics. For a body to move from its state of rest or change its motion, it cannot do so by itself but must be acted on by some external force. This was stated by Newton in his famous laws of motion. As it is the external force acting on a body which causes the motion of a body, we can understand that the resulting motion of a body is dependent upon the force acting on the body.

In statics, we have seen that a system of forces acting on a body can be replaced by single resultant force acting at the centre of gravity and a couple. This resultant force is equal to the summation of all individual forces and the couple is equal to the summation of all individual couples. When both the resultant force and couple are zero, then the body will be at rest or under static equilibrium. The study of bodies at rest is termed as statics. This was covered in the previous chapters, where we treated bodies at rest and analyzed their equilibrium conditions. However when the resultant force or couple or both are non-zero, then the body will be under motion. The study of bodies under motion is termed as dynamics.

The resulting motion of a body under the action of a system of forces can be either translational, rotational or a combination of both depends upon the nature of resultant of system of forces. When the resultant is single force R acting at the centre of gravity, i.e., the couple being zero, then the body is in pure translational motion in the direction of the resultant and there is no rotational motion in the direction of the resultant and there is no rotational motion.

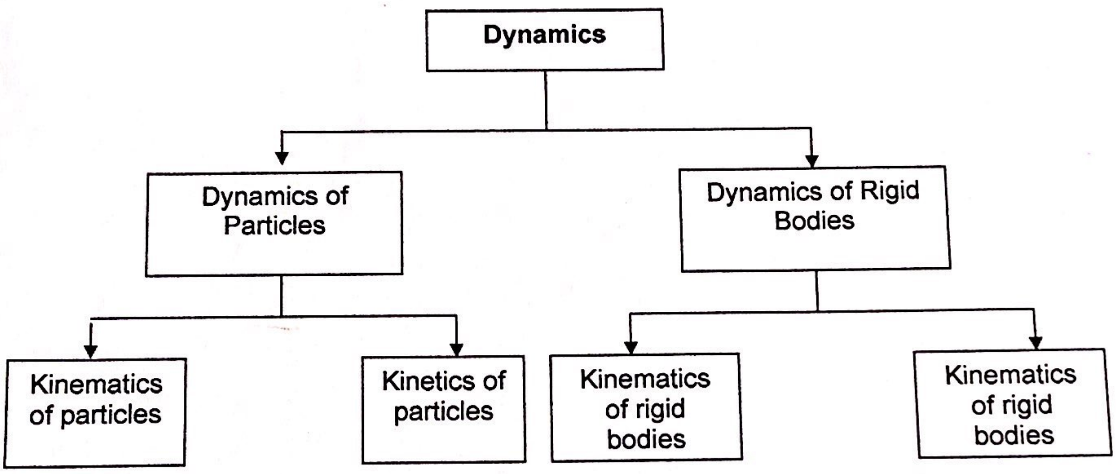
Further if the direction of the resultant force is constant, then the translational motion is along a straight line, which we term as rectilinear or one-dimensional motion. For instance, a ball thrown vertically upwards and a car traveling on a straight road are examples of rectilinear motion. However if the direction of the resultant force varies, then the motion will not be in a straight line and such a motion we term as curvilinear or two-dimensional motion. A golf ball hit from the ground and a motorist traveling on a curved road are examples of such curvilinear motion.

When the resultant of a system of forces is a couple $\vec{M}$, then the body will rotate about its centre of gravity. Further if the body is fixed at its centre of gravity, then the motion is pure rotational motion. The motion of a pulley fixed at its centre of gravity is an example for fixed axis rotation.

When the resultant of a system of forces is a centroidal force $\vec{R}$ and a couple $\vec{M}$, then the body will have general motion which is neither pure translational nor pure rotational. However, such motions can be thought of as a combination of translational and rotational motions. A cylinder rolling down on an inclined plane is an example for such a motion. If the motion of the body lies in a plane, then it is termed as planer motion, otherwise it is termed as general three dimensional or spatial motions.

When a body is in pure translation motion, all the particles in the body will move in parallel paths with the same displacement, velocity and acceleration. Hence, instead of treating the body as a whole, we can analyze the motion of the whole body by idealizing it as a particle. However if it is in pure rotational motion or a combination of both translational and rotational motions, the particles in the body move with different velocities and accelerations and hence, we can no more describe its motion by idealizing it as a particle but treating it as a rigid itself. Thus we can divide dynamics of bodies into dynamics of particles and dynamics of rigid bodies.

The dynamics of particles or rigid bodies can further be divided into two parts namely, kinematics and kinetics. If we are interested only in the motion of bodies without considering the forces causing the motion, then that branch of dynamics is termed as kinematics. It deals wit relationship between displacement, velocity and acceleration, and their variation with time. However if we want to relate the motion the motion of bodies with the forces causing the motion, then it is termed as kinetics.



Chapter 1

KINEMATICS OF PARTICLES

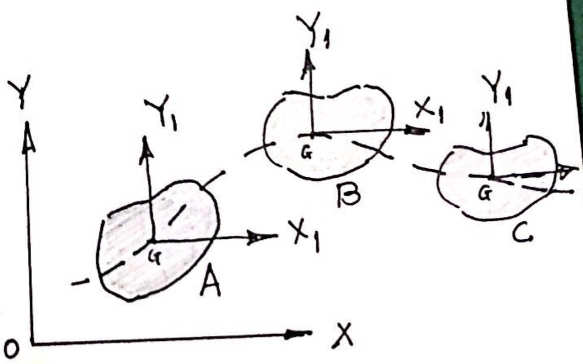
The Study of bodies in motion or the study of motion itself is called as **Dynamics**.

Motion of a particle:-

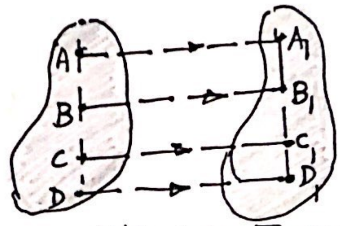
Consider the motion of a body in X – Y plane from A to B and B to C as shown in figure, where X1 – Y1 are local coordinate axes or axes fixed to the body. If throughout the motion, the X1 – Y1 axes always remain parallel to the fixed reference axes X – Y, and then we see that the body is in pure translational motion and there is no rotational motion involved. In such a motion, all the particles in the body move in parallel paths with the same displacement, velocity and acceleration. Hence, instead of analyzing the motion of the body as a whole, we can analyze the motion of a single particle in the body which is a representative of the motion of the entire body. This is termed as idealization of the body as a particle, i.e., body without extent. Whenever we say particle, we should keep inn mind that we are not dealing with minute bodies but rather gross bodies which do not have rotational motion at all or even if they have, they have been neglected. Mathematically a particle is treated as a point and normally the centre of gravity of the body is chosen as this point.

To describe the motion of a particle, we must specify its position at any instant of time, and also its velocity and acceleration at that instant. Hence, we proceed to define each of these terms in the following section. We will define these terms considering a two dimensional motion or plane motion, which can later on be extended to a more general three dimensional or spatial motion.

Displacement:-



To describe the motion of a particle, we must specify its position at any instant of time with respect to a reference frame. As we are measuring the motion of the body, the reference frame should be such that it is fixed, i.e., having no motion. For this reason, in astronomical studies, distant stars are chosen as non – inertial reference frames as they are considered to be fixed. However for normal engineering analysis, we can choose a point on the earth's surface as the fixed origin for the reference frame, even earth is not fixed but rotating about itself and about the sun.



Rectilinear Translation

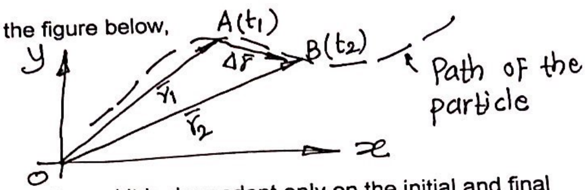
The position of a particle is then given by position vector, drawn from the origin of that reference frame to the particle. If at any instant of time t_1, say the particle is at A, whose position vector is $\vec{r_1}$ and at a later time t_2, it is at B, whose position vector is $\vec{r_2}$. Then we say that the particle has displaced from point A to point B in time ($t_2 - t_1$). Thus we define displacement vector as the change in position of the

curvillinear Translation

particle during this interval of time. In the figure below, we see that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\therefore \overrightarrow{\Delta r} = \overrightarrow{r_2} - \overrightarrow{r_1}$$



The displacement is a vector quantity and it is dependent only on the initial and final positions of the particle. It doesn't tell us any thing about the path traced. It could be a straight line or curved. On the other hand, the distance traveled being a scalar quantity, is dependent on the actual path traced by the particle. In the figure above, the arc length AB (shown by dotted line) denotes the distance traveled by the particle in this time. Hence, if a particle moves from A to B and then back to A, then the net displacement is zero as the particle is back to its initial position. However we can see that the distance traveled is not zero. As displacement is a measurement of length, its SI unit is meter 'm'.

Velocity:-

To describe the motion of a particle at any instant of time, it would not suffice to define its position alone, but also the rate at which it gets displaced. For instance, a bullock cart will take longer time to get displaced from A to B than a car. We define this change in displacement with respect to time as velocity.

Thus velocity of a particle can be defined as the rate of change of displacement with time. In the figure, as the particle moves from A to B, the average velocity during this time interval is then given as the ratio of net displacement and elapsed time, i.e.,

$$\vec{V}_{ave} = \left(\frac{\text{net displacement}}{\text{elapsed time}}\right)$$

$$= \frac{\Delta \bar{r}}{\Delta t} = \frac{\Delta \bar{r}_2 - \Delta \bar{r}_1}{t_2 - t_2}$$

We can see that as the average velocity vector is a ratio of vector displacement and scalar time, its magnitude is equal to $\left|\Delta \bar{r} / \Delta t\right|$ and direction same as $\Delta \bar{r}$.

This average velocity is determined from the initial and final positions of the particle. Hence, it doesn't say anything about the velocity of the particle at intermediate points. If the average velocity measured between any two points along the path remains the same in magnitude and direction, then the particle is said to move with constant velocity. It should be noted that as velocity is a vector quantity [being a ratio of vector displacement and time], constant velocity can be maintained only when both magnitude and direction are constant. This is possible then only when the motion is along a straight line or rectilinear motion.

On the contrary, if average velocity measured between any two points along the path does not remain constant, then the particle is said to move with variable velocity. In such a case, we must specify the velocity of the particle at a particular instant of time, called the instantaneous velocity.

To determine velocity, let us consider smaller time increments such that point B approaches A. In the limiting case as $\Delta t \to 0$, the average velocity then very closely defines the instantaneous velocity at point A. Mathematically,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

From calculus, we know that this can be expressed as:

$$\boxed{\vec{v} = \left(\frac{d\vec{r}}{dt}\right)}$$

Also in the limiting condition, we can see that the direction of $\Delta \vec{r}$ approaches that of the tangent to the path of the particle at A. Hence, the direction of instantaneous velocity is always tangential to the path of the particle and its magnitude is called as speed of the particle. The unit of velocity in SI units is m/s, but sometimes it is also expressed in km/hr. The conversion factor for which is given as:

$$\boxed{1\,\text{km/hr} = \frac{1000}{3600}\frac{\text{m}}{\text{s}} = \frac{5}{18}\,\text{m/s}}$$

Similarly, the conversion factor of m/s to km/hr is:

$$\boxed{1\,\text{m/s} = \frac{1/1000\,\text{km}}{1/3600\,\text{hr}} = \frac{18}{5}\,\text{km/hr}}$$

Acceleration:-

A particle may not always move with constant velocity throughout its motion. For instance, if it is starting from rest, it normally increases its velocity till it reaches a maximum velocity and then moves at this velocity. Thus we see that the particle changes in velocity with time. We define this change in velocity with respect to time as acceleration.

Thus acceleration of a particle can be defined as the rate of change of velocity vector with time. The velocity vector may change either in magnitude, in direction or both as the motion proceeds. Suppose at time t1, the particle is at point A with instantaneous velocity v1 (whose direction is tangential to the path at A) and at a later time t2, it is at point B with instantaneous velocity V2 (again direction is tangential to the path at B), then average acceleration is defined as the ratio of net change in velocity and time elapsed, i.e.,

$$\vec{a}_{ave} = \left(\frac{\text{net change in velocity}}{\text{time interval}}\right)$$

The net change in velocity can be obtained by drawing the velocity vector triangle as shown in figure. Hence,

$$\vec{a}_{ave} = \frac{\vec{V}_2 \quad \vec{V}_1}{t_2 \quad t_1} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a}_{ave} = \frac{d\vec{v}}{dt}$$

Its direction is same as $\Delta\vec{V}$ and its magnitude is $|\Delta\vec{v}/\Delta t|$. Its unit as seen from above equation is m/s^2.

This average acceleration is based on the initial and final positions of the particle. Hence, it doesn't say anything about the acceleration of the particle at the particle at intermediate points. If the average acceleration measured between any two points along the path remains the same in magnitude and direction, then the particle is said to move with constant acceleration. A body falling freely under gravity is an example for constant acceleration.

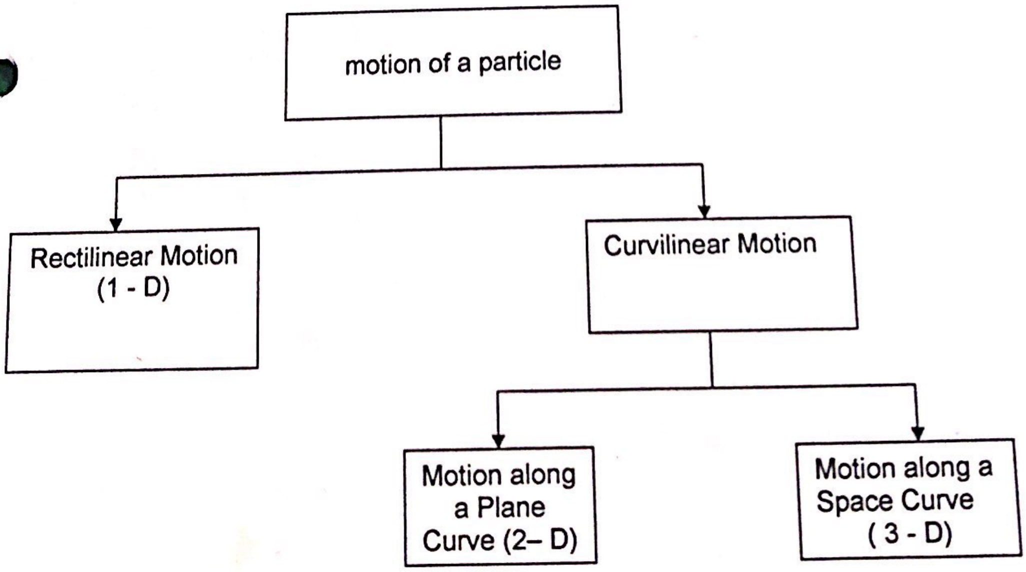
On the contrary, if average acceleration measured between any two points along the path does not remain constant, then the particle is said to move with variable acceleration. In such a case, we must specify the acceleration of a particle at a particular instant of time, called the instantaneous acceleration. To determine the instantaneous acceleration, let us consider smaller time increments such that point B approaches A. In the limiting case as $\Delta t \to 0$, the average acceleration very closely defines the instantaneous acceleration at point A. Mathematically,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

and from calculus, we know that this can be expressed as:

$$\boxed{\vec{a} = \left(\frac{d\vec{v}}{dt}\right)}$$

The magnitude of instantaneous acceleration is $|d\vec{v}/dt|$ and its direction is the limiting direction of the change in velocity vector $\Delta\vec{v}$. If the velocity of the particle decreases as it moves from point A to B, then the acceleration of the particle is negative, which we also call as deceleration.



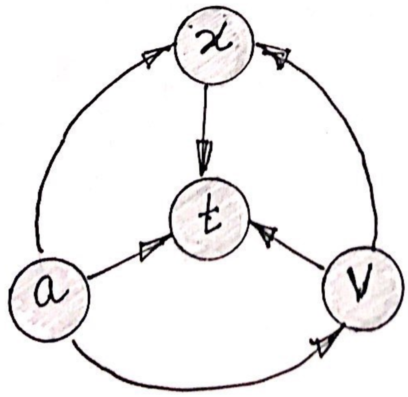
RECTILINEAR MOTION

When the motion of a particle is restricted along a straight line, the motion is said to be one dimensional or rectilinear motion. We will consider motions either along a horizontal axis i.e., X-axis or along a vertical axis i.e., Y – axis. We will discuss about rectilinear motion along X – axis such as a car moving on a straight road or a sprinter running a race on a straight track etc., we will discuss about rectilinear motion along Y – axis such as a ball thrown vertically upwards or dropped from a building etc.

Rectilinear Motion

Uniform Motion	Uniformly accelerated motion	Motion with variable acceleration
V = constant	a = constant	1) a = f (t)
$a = 0$	$v = u + at$………..(i)	2) a = f (v)
$s = vt$	$s = ut + \frac{1}{2}at^2$ …..(ii)	3) a = f (x)
	$v^2 = u^2 + 2as$…..(iii)	e.g. S.H.M.
	e.g. Motion Under gravity	

Motion curves:-



Radial relations are primary motion equations.

X = f (t), v = f (t), a = f (t)

They are commonly known as equations of motion. 'Peripheral' relations are secondary equations of motion.

A= f (v), v = f (x), a = f (x)

The graphical display of 'equations of motion' and 'secondary equations of motion' are respectively known as 'curves of motion'. Secondary motion curves are more practically useful than motion curves in motion analysis. We must understand the relationship between these curves, so that if one of the curves is known, then the other two can be obtained.

We know that,

(I) Rate of Change of displacement w.r.t. time = velocity, $\boxed{v = \frac{dx}{dt}}$

(II) Rate of change of velocity w.r.t. time = acceleration $\boxed{a = \frac{dv}{dt}}$

(III) Rate of change of acceleration w.r.t. time = jerk $\boxed{j = \frac{da}{dt}}$

A) As $v = \frac{dx}{dt}$

$$\boxed{\text{Ordinate of (v – t) diagram = slope of (x – t) diagram}}$$

B) As $dx = v\,dt$

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v.dt \quad \therefore (x_2 - x_1) = v\,(t_2 - t_1) = \int_{t_1}^{t_2} v\,.dt$$

$$\boxed{\text{change in displacement} = \int_{t1}^{t2} v\,.dt = \text{Area under (v-t) diagram}}$$

C) As, $a = \frac{dv}{dt}$

$$\boxed{\text{ordinate of (a-t) diagram = slope of (v-t) diagram}}$$

D) As, $dv = a.\,dt$

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a.dt \quad \therefore$$

$$\boxed{(v_2 - v_1) = \text{change in velocity} = \text{Area under (a-t) diagram}}$$

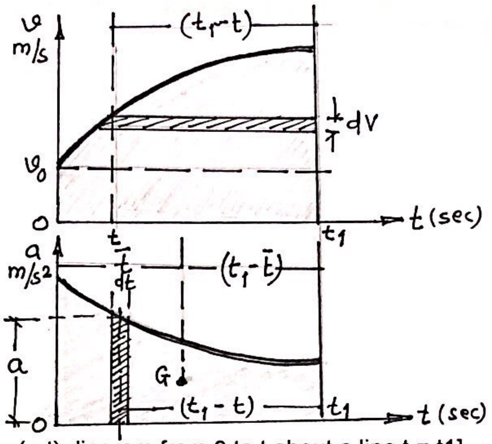
An area above the 't' axis corresponds to an increase in x or v, while the area located below the t axis measures a decrease in x or v.

In general, if the acceleration is a polynomial of degree 'n' in t, the velocity will be a polynomial of degree (n + 1) and the position co – ordinate a polynomial of degree (n + 2). These polynomial are represented by motion curves of a corresponding degree.

E) <u>Moment – Area Method</u>:-

This is a graphical method used to determine the position of a particle at a given instant directly from the (a - t) curve.

Let, x_0 = displacement at t = 0,
V_0 = velocity at t = 0,
X_1 = displacement at t = t_1,
V_1 = velocity at t = t_1.



Then, $(x_1 - x_0)$ = area under (v - t) curve

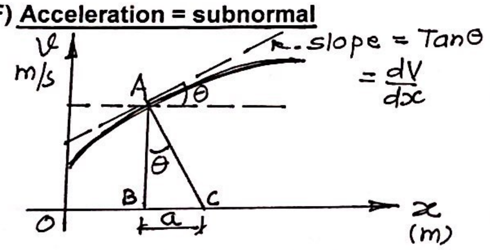
$$= v_0.t_1 + \int_{V_2}^{V_1} (t_1 - t)dv$$

But, dv = a.dt

$$\therefore (x_1 - x_0) = v_0 \cdot t_1 + \int_0^{t_1} (t_1 - t) \cdot a \cdot dt$$

$\therefore x_1 = x_0 + v_0.t_1$ + [moment of the area under (a-t) diagram from 0 to t about a line t = t1]

F) Acceleration = subnormal



By using (v-x) curve, the acceleration 'a' can be obtained at any time by drawing the normal AC to the curve and measuring the subnormal BC.
If θ = angle between the tangent at A and the horizontal.

$$BC = AB.\tan\theta = v \cdot \frac{dv}{dx} = a \text{ i.e. acceleration at A.}$$

Motion with variable acceleration:-

The general procedure for solving the problems is explained as under,

A) Acceleration as a function of time:

$$\boxed{a = f(t)} \qquad \text{.............................(i)}$$

$$a = f(t) = \frac{dv}{dt}$$

$$\int dv = \int f(t)dt$$

$$\boxed{v = \phi(t)} \qquad \text{.................................(ii)}$$

$$v = \frac{dx}{dt} = \phi(t)$$

$$\int dx = \int \phi(t)dt$$

$$\boxed{x = \phi(t)} \qquad \text{..............................(iii)}$$

Eq^n (i), (ii)& (iii) are called as equations of motion.

B) Acceleration as a Function of Velocity:

$$\boxed{a = f(v)} \quad \text{.................(i)}$$

we have, $a = v\dfrac{dv}{dx}$

$a.dx = v.dv$

$f(v)dx = v.dv$

$\int dx = \int \dfrac{vdv}{f(v)}$

$$\boxed{x = \phi(v)} \quad \text{.................(ii)}$$

Now,

$v = \phi(x) = \dfrac{dx}{dt}$

$\int dt = \int \dfrac{dx}{\phi(x)}$

$$\boxed{t = \phi(x)} \quad \text{.................(iii)}$$

Eq^n (i), (ii) & (iii) are called as equations of motion.

C) Acceleration as a function of displacement

$$\boxed{a = f(x)} \quad \text{.......... (i)}$$

$adx = vdv$

$\int f(x)dx = \int vdv$

$$\boxed{v = \phi(x)} \quad \text{...................(ii)}$$

$v = \dfrac{dx}{dt} = \phi(x)$

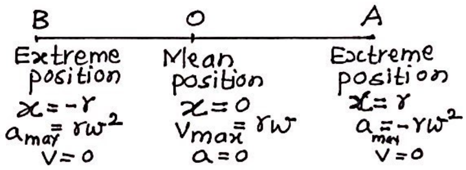
$\int dt = \int \dfrac{dx}{\phi(x)}$

$$\boxed{t = \phi(x)} \quad \text{...................(iii)}$$

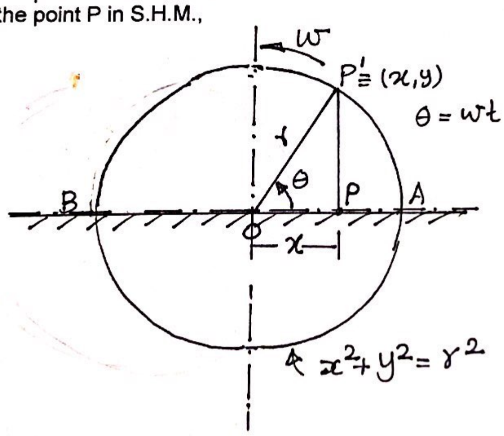
Eq^n (i), (ii) & (iii) are called as equations of motion

Simple Harmonic Motion (S.H.M.)

The rectilinear motion in which acceleration is a function of displacement in the form of $a = -\omega^2 x$ is called as **S.H.M.**. In this motion the particle moves to and fro along the same path, again & again. The midpoint of the path is called as **mean position**. The maximum distance traveled by the particle from the mean position is called as **amplitude of oscillation**. No. of oscillations per unit time is called as **Frequency**. The time required for one oscillation is called as **periodic time**.



S.H.M. can be considered as the projection of Uniform Circular Motion on any diameter. In the above figure, point P' is subjected to U.C.M. & point P is subjected to S.H.M. $\theta = \omega.t$ = Angular displacement in time 't' for the point P' in U.C.M..
Hence for the point P in S.H.M.,



Displacement $\boxed{x = r\cos\omega t = r\cos\theta}$

Velocity
$$V = \frac{dx}{dt} = -r\omega\sin t\omega t = -\omega y$$
$$\boxed{V = \left|\omega\sqrt{r^2 - x^2}\right|}$$

Acceleration
$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$
$$= r\omega^2\cos\omega t$$
$$\boxed{a = -\omega^2 x}$$

(Negative sign indicates that, at any point, the direction of acceleration is opposite to the direction of the displacement.)

$$\omega = 2\pi f \qquad \text{or} \qquad \omega = \frac{2\pi N}{60}$$

$f \rightarrow$ revolutions per second. i.e. Hz

$N \rightarrow$ revolutions per minute

$$\boxed{f = \frac{\omega}{2\pi}} \qquad \boxed{T = \frac{1}{f} = \frac{2\pi}{\omega}}$$

Relative Velocity

A & B are two moving particles in (x – y) plane. At any time 't' vectors $\overline{OA}$ and $\overline{OB}$ are representing the position vectors of A and B w.r.t a stationary frame of reference (x – o - y). Any vector expressed w.r.t. a stationary frame of reference is called as absolute vector. Hence $\bar{r}_A$ and $\bar{r}_B$ are representing the absolute position vectors of A & B at any time 't'. (X – A - Y) is a moving frame of reference attached to point A. Then $\overline{AB}$ is the position vector of 'B' w.r.t 'A'. As it is expressed w.r.t. a moving frame of reference, $\bar{r}_{B/A} = \overline{AB}$ is a relative position vector of 'B' w.r.t 'A'

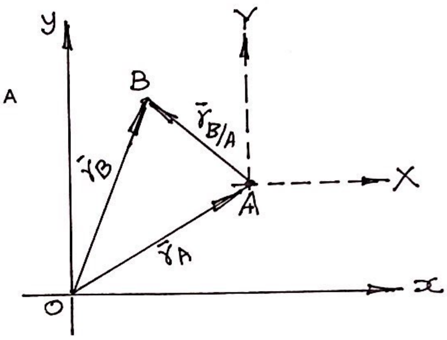
$\bar{r}_A = \overline{r_{A/o}}$

$\bar{r}_B = \overline{r_{B/o}}$

$\bar{r}_{B/A} = \overline{AB} =$ Relative Position Vector of B w.r.t A

$(x - o - y) \rightarrow$ Stationary frame of reference

$(X - A - Y) \rightarrow$ Moving frame of reference



In ΔOAB,

$\overline{OA} + \overline{AB} = \overline{OB}$

$\overline{AB} = \overline{OB} - \overline{OA}$

$$\boxed{\bar{r}_{A/B} = (\overline{r_B} - \overline{r_A})} \quad \text{.................(i)}$$

Differentiating (i) w.r.t. time, we get, the relative velocity of B w.r.t. A

$$\overline{V_{B/A}} = \frac{d\overline{r_{B/A}}}{dt} = \frac{d\bar{r}_B}{dt} - \frac{d\bar{r}_A}{dt}$$

$$\boxed{\bar{V}_{B/A} = (\overline{V_B} - \overline{V}_A)} \quad \text{............(ii)}$$

Differentiating to (i) w.r.t. time we get the relative acceleration of B w.r.t. A

$$\overline{a_{B/A}} = \frac{d\overline{v_{B/A}}}{dt} = \frac{d\bar{v}_B}{dt} - \frac{d\overline{v_A}}{dt}$$

$$\boxed{\bar{a}_{B/A} = (\overline{a_B} - \overline{a}_A)} \quad \text{.............(iii)}$$

Shortest Distance between two moving particles:-

When two particles are moving in (x-y) plane and the shortest distance between the two is to be calculated, and then the procedure is as under;

1) Assume one of the particle to be stationary
2) Then the second particle is supposed to travel with its relative velocity w.r.t the first particle along the relative path.
3) Then draw a perpendicular from the stationary particle on to the relative path of other.

4) The length of the perpendicular gives us the shortest distance between the two particles.

5) The time to attain the shortest distance = $\frac{\text{Rrelative distance travelled}}{\text{Relative velocity}}$

Dependant Motion:-

Some times the position of the particle depends upon the position of another particle or several other particles. Then the motion is called as dependant motion.

When the relation existing between the position co – ordinates of several particles is linear, a similar relation holds between the velocities and between the accelerations of the particles.

In dependant motion, choose a fixed point on the line of motion. The velocity and acceleration equations, obtained on differentiating the geometry equations are strictly not vector equations (i.e. are scalar equations).

The 'degree of freedom' of the given system of bodies / particles can be decided on the number of co – ordinates which can be chosen arbitrarily.

For example, if $n_1 \cdot x_A + n_2 \cdot x_B + n_3 \cdot x_C = \text{constant}$
then differentiating w.r.t. time we get,

$$n_1 \cdot v_A + n_2 \cdot v_B + n_3 \cdot v_C = 0$$

And

$$n_1 \cdot a_A + n_2 \cdot a_B + n_3 \cdot a_C = 0$$

Scanned with CamScanner