Differential Equations

Def^n :- An equation involving the dependant variable, an independant variable and the differential coefficients of various orders is called D.E.

\# Order of D.E : The order of D.E is the order of the highest derivative.

\# Degree of D.E : The degree of D.E is the degree of highest order derivative, when D.E is free from radicals & fractions.

Ex. 1) $\frac{dy}{dx} = \frac{x+2y+3}{x-y+1}$ O : 1 , D : 1

2) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = ny$ O : 2 , D : 1

3) $\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2} = \frac{d^2y}{dx^2}$

Squaring on both sides

$\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$ $\therefore$ O : 2 , D : 2

4) $\left(\frac{dy}{dx}\right)^4 = \left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}$ $\therefore$ O : 2 , D : 1

Solution of D.E : It is a relation betⁿ dependant variable & independant variable which is free from derivatives and satisfies D.E

Types of solutions :-

A) General Solution : It is a solution which contains arbitrary constants equal to the order of the D.E.

B) Particular solution :- It is a solution obtained by putting particular values in G.S.

Formation of D.E :-

Working rule

1) If G.S contains 'n' arbitrary consts. then diff. G.S n-times.

2) Remove all arbitrary constants

★ In general, a G.S involving 'n' arbit. constants will give rise to a n^{th} order D.E.

Ex Form a D.E whose G.S is given by

1) $y = ax^2 + bx + c$

$\Rightarrow$ no. of arbit. const. $= 3$ (a, b, c)

$\therefore$ we have to differentiate 3-times

$$\frac{dy}{dx} = 2ax + b, \quad \frac{d^2y}{dx^2} = 2a, \quad \boxed{\frac{d^3y}{dx^3} = 0}$$

this eq^n doesn't contain any arbitrary constant.

$$\therefore \frac{d^3y}{dx^3} = 0 \qquad O:3 \quad D:1.$$

2) $y = a\sin t + b\cos t$

$\Rightarrow$ no. of arbit. const $= 2$ (a, b)

$\therefore$ Diff. twice wrt 't'

$$\frac{dy}{dt} = a\cos t - b\sin t$$

$$\frac{d^2y}{dt^2} = -a\sin t - b\cos t = -(a\sin t + b\cos t)$$

$$\therefore \frac{d^2y}{dt^2} = -y \Rightarrow \frac{d^2y}{dt^2} + y = 0 \text{ is req. D.E.}$$

3) Form the D.E of all circles which touch the y-axis at the origin & centre lies X-axis.

$\Rightarrow$ Eq^n of circle is given as

$$(x-a)^2 + (y-0)^2 = a^2$$

$$x^2 - 2ax + a^2 + y^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 = 2ax$$

Diff. w.r.t x

$$2x + 2y\frac{dy}{dx} = 2a$$

$$x + y\frac{dy}{dx} = a = \frac{x^2+y^2}{2x}$$

$$\therefore \frac{dy}{dx} = \cancel{\frac{x^2+y^2}{2xy}} \ \frac{y^2-x^2}{2xy}.$$

O.D.E of first order and first degree

It is of the form

$M + N\frac{dy}{dx} = 0$ or $M\,dx + N\,dy = 0$ where

M, N are functions of x, y or constants.

The G.S will contain only one arbi. const.

Methods of solution :-

A) Variable separable form (V.S form) :-

In this method, D.E can be written in $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ form by algebraic manipulations.

G.S is given by.

$$\int g(y)\,dy = \int f(x)\,dx + c$$

Ex 1) $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{1+y^2}{1+x^2}$$

$$\therefore \frac{dy}{1+y^2} = -\frac{dx}{1+x^2}$$

On integrating both sides

$$\int \frac{dy}{1+y^2} = -\int \frac{dx}{1+x^2} + c$$

$$\tan^{-1}y = -\tan^{-1}x + c$$

$$\Rightarrow \tan^{-1}y + \tan^{-1}x = c$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}c_1$$

$$\boxed{\frac{x+y}{1-xy} = c_1}$$

2) $\frac{dy}{dx} = e^{x-y} + 3x^2 e^{-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{3x^2}{e^y}$$

$$e^y \frac{dy}{dx} = e^x + 3x^2$$

$$\therefore e^y dy = (e^x + 3x^2)\,dx$$

$$\int e^y dy = \int (e^x + 3x^2)\,dx + c \Rightarrow \boxed{e^y = e^x + x^3 + c}$$

3) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

$\Rightarrow \quad y - x\dfrac{dy}{dx} = ay^2 + a\dfrac{dy}{dx}$

$y - ay^2 = (a+x)\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{dx}{a+x} = \dfrac{dy}{y(1-ay)} = dy\left(\dfrac{1}{y} - \dfrac{a}{1-ay}\right)$

$\displaystyle\int \frac{dx}{a+x} = \int\left(\frac{1}{y} \mp \frac{a}{1-ay}\right)dy + \log c$

$\log(a+x) = \log y - \log(1-ay) + \log c$

$a+x = \dfrac{y}{1-ay}\cdot c$

$\Rightarrow \boxed{(a+x)(1-ay) = yc}$ is G.S.

H.W 1) $y\dfrac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$

2) $\dfrac{dy}{dx} = \dfrac{x\sin x}{2e^y\sinh y}$ $\left(\text{Hint } \sinh y = \dfrac{e^y - e^{-y}}{2}\right)$

3) $(xy^2 - x)\,dx = (y + x^2y)\,dy$

4) $3e^x\tan y\,dx + (1+e^x)\sec^2 y\,dy = 0$

B) Homogenous D.E :-

Consider $M\,dx + N\,dy = 0$. It is said to be homogenous if M & N are homogenous functions of x, y of same degree.

Method : Put $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

then it is reduced in V.S form.

Ex: $(x^2+y^2)\,dx + 8xy\,dy = 0$

$\Rightarrow$ It is homo. D.E with degree 2.

~~Put~~ $\therefore \frac{dy}{dx} = -\frac{x^2+y^2}{8xy}$ ——— (1)

Put $y = ux$ $\therefore \frac{dy}{dx} = u + x\frac{du}{dx}$

$\therefore$ eq^n (1) becomes

$$u + x\frac{du}{dx} = \frac{x^2+u^2x^2}{8ux^2} = \frac{1+u^2}{8u}$$

$$x\frac{du}{dx} = \frac{1+u^2}{8u} - u = \frac{1-7u^2}{8u}$$

$$\frac{8u}{1-7u^2}\,du = \frac{dx}{x}$$

$$\frac{8}{-14}\int \frac{-14u}{1-7u^2}\,du = \int \frac{dx}{x} + \log c.$$

$$\frac{8}{-14}\log(1-7u^2) = \log x + \log c.$$

$$(1-7u^2)^{-\frac{4}{7}} = xc.$$

$$\boxed{\left(1-7\frac{y^2}{x^2}\right)^{-\frac{4}{7}} = xc}$$

2) Solve $x\frac{dy}{dx} + \frac{y^2}{x} = y$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$ —————— (1)

It is homo. in x & y.

Put $y = ux$ $\therefore \frac{dy}{dx} = u + x\frac{du}{dx}$

$\therefore$ eq^n (1) becomes.

$$u + x\frac{du}{dx} + u^2 = u \Rightarrow x\frac{du}{dx} + u^2 = 0$$

$$\Rightarrow x\frac{du}{dx} = -u^2 \Rightarrow -\frac{du}{u^2} = \frac{dx}{x}$$

On integrating both sides

$$-\int\frac{du}{u^2} = \int\frac{dx}{x} + \log c \Rightarrow \frac{1}{u} = xc \Rightarrow \frac{x}{y} = xc$$

No.

c) Exact D.E :-

Consider $M\,dx + N\,dy = 0$.

It is said to be an exact D.E iff it satisfies following condition.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

When the condition of exactness is satisfied the general solution can be obtained by

$$\int\limits_{y\text{-constant}} M\,dx + \int \begin{bmatrix}\text{Terms of } N \text{ free} \\ \text{from } x\end{bmatrix} dy = c$$

Ex Solve $(x+y-2)\,dx + (x-y+4)\,dy = 0$

$\Rightarrow$ Compare with $M\,dx + N\,dy = 0$

$\therefore M = x+y-2$, $N = x-y+4$.

$$\frac{\partial M}{\partial y} = 1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$\therefore$ given D.E is EXACT.

$\therefore$ G.S is given by

$$\int\limits_{y\text{-const}} M\,dx + \int \left[\text{Terms of } N \text{ free } x\right] dy = c$$

$$\int (x+y-2)\,dx + \int (-y+4)\,dy = c$$

$$\boxed{\frac{x^2}{2} + xy - 2x - \frac{y^2}{2} + 4y = c}$$

2) $\dfrac{dy}{dx} = \dfrac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$.

$\Rightarrow (\tan y - 2xy - y)\,dx + (x\tan^2 y - x^2 - \sec^2 y)\,dy = 0$

$M = \tan y - 2xy - y \quad , \quad N = x\tan^2 y - x^2 - \sec^2 y$

$\dfrac{\partial M}{\partial y} = \sec^2 y - 2x - 1 \quad , \quad \dfrac{\partial N}{\partial x} = \tan^2 y - 2x$

$= \tan^2 y - 2x$.

$\therefore \boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

$\therefore$ given D.E is EXACT

G.S is given by

$\int\limits_{y\text{-const}} M\,dx + \int [\text{Terms of } N \text{ free from } x]\,dy = C$

$\int (\tan y - 2xy - y)\,dx + \int -\sec^2 y\,dy = C$

$\boxed{x\tan y - x^2 y - xy - \tan y = C}$.

Reducible to Exact form :-

Consider $Mdx + Ndy = 0$ ---- (1)

which is NOT EXACT. In this case we find integrating factor [I.F] by one of the four rule & multiply it to eq^n (1). Then eq^n (1) becomes EXACT & we solve by method of Exactness

Rules :-

R1) If given D.E is HOMOGENOUS & $xM + yN \neq 0$ then

$$I.F = \frac{1}{xM + yN}$$

R2) If given D.E is of type $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$ and $xM - yN \neq 0$ then

$$I.F = \frac{1}{x.M - y.N}$$

e.g.) ✓ $f(xy) = x^2y^2 + xy + 1$ (As $1 = x^0y^0$)

✗ $f(xy) \neq x^2y^3 + xy$ (Power of x & y should be same in each term)

R3) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ i.e function of only 'x'

then $I.F = e^{\int f(x)\,dx}$

R4) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ i.e fun^n of only 'y'

then $I.F = e^{\int g(y)\,dy}$

(If given D.E is NOT EXACT, only then find I.F.)

standard form : $M\,dx + N\,dy = 0$

P.No. ②

Solve the following D.E

1) Solve $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$ ...... ①

$\Rightarrow$ Here $M = xy - 2y^2$, $N = -(x^2 - 3xy)$

$\therefore \frac{\partial M}{\partial y} = x - 4y$, $\frac{\partial N}{\partial x} = -2x + 3y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ given D.E is NOT EXACT.

Now,

Given D.E is homogenous &

$x \cdot M + y \cdot N = x^2y - 2xy^2 + 3xy^2 - x^2y = xy^2 \neq 0$

$\therefore$ by R1)

$I.F = \frac{1}{xM + y \cdot N} = \frac{1}{xy^2}$

Now, multiply eqn ① by I.F

$\therefore \frac{1}{xy^2}(xy - 2y^2)\,dx - \frac{1}{xy^2}(x^2 - 3xy)\,dy = 0$

i.e $\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

i.e $\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0$ ...... ②

* No need to show ~~it is~~ exact

Now equation ② ie EXACT.

$\therefore$ general solution is given as

$\int M\,dx + \int \left[\begin{matrix}\text{Terms of } N \\ \text{free from } x\end{matrix}\right] dy = c$

$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = 0$

$$\boxed{\therefore \frac{x}{y} - 2\ln x + 3\ln y = c}$$

is required sol^n

Ex2 Solve $(x\sec^2 y - x^2\cos y)dy = (\tan y - 3x^4)dx$

$\Rightarrow$ standard form

$$(3x^4 - \tan y)dx + (x\sec^2 y - x^2\cos y)dy = 0 \quad \text{---} \ (1)$$

$\therefore M = 3x^4 - \tan y$, $N = x\sec^2 y - x^2\cos y$

$\therefore \frac{\partial M}{\partial y} = -\sec^2 y$, $\frac{\partial N}{\partial x} = \sec^2 y - 2x\cos y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ eq^n (1) isn't EXACT.

Now,

R1) is not applicable since eq^n (1) is not homo.

R2) is not applicable since eq^n (1) is not in $yf_1(xy)dx + xf_2(xy)dy = 0$ form

$\therefore$ we go for R3)

Consider

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\sec^2 y + 2x\cos y - \sec^2 y}{x\sec^2 y - x^2\cos y}$$

$$= \frac{-[2\sec^2 y - 2x\cos y]}{x(\sec^2 y - x\cos y)}$$

$$= \frac{-2}{x} = f(x)$$

$$\therefore I.F = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$\therefore$ I.F $= \frac{1}{x^2}$

$\therefore$ eq^n (1) becomes

$$\frac{1}{x^2}\left(3x^4 - \tan y\right)dx + \frac{1}{x^2}\left(x\sec^2 y - x^2\cos y\right)dy = 0$$

$$\left(3x^2 - \frac{\tan y}{x^2}\right)dx + \left(\frac{\sec^2 y}{x} - \cos y\right)dy = 0$$

which is exact.

$\therefore$ G.S is given by

$$\int M\,dx + \int \begin{bmatrix}\text{Terms of } N \\ \text{free from } x\end{bmatrix} dy = C$$

$$\int\left(3x^2 - \frac{\tan y}{x^2}\right)dx + \int(-\cos y)\,dy = C$$

$$\frac{3x^3}{3} + \frac{\tan y}{x} - \sin y = C$$

$$\boxed{x^3 + \frac{\tan y}{x} - \sin y = c}$$

3) Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ ——— (1)

$M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ NOT EXACT

Now,

~~Consider~~.

R1) isn't applicable ∵ not homo.
R2) isn't applicable ∵ not is form
R3 ∴ consider;

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{4y^3+2-y^3-4}{xy^3+2y^4-4x} = \frac{3y^3-2}{xy^3-4x+2y^4} \neq f(x)$$

(Terms of 'y' are also present)

∴ R3) isn't applicable

Now

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y}$$

$$= \frac{-3y^3-6}{y(y^3+2)} = \frac{-3(y^3+2)}{y(y^3+2)}$$

$$= \frac{-3}{y} = g(y).$$

$$\therefore \quad I.F = e^{\int g(y)\,dy} = e^{\int -\frac{3}{y}\,dy} = e^{-3\ln y}$$

$$= e^{\ln y^{-3}} = y^{-3} = \frac{1}{y^3}$$

$$\therefore \quad I.F = \frac{1}{y^3}$$

∴ equation (1) becomes

$$\frac{1}{y^3}(y^4+2y)\,dx + \frac{1}{y^3}(xy^3+2y^4-4x)\,dy = 0$$

$$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

$\therefore$ G.S is given by

$$\int M\,dx + \int \begin{bmatrix} \text{Terms of } N \\ \text{free from } 'x' \end{bmatrix} dy = c$$

$$\int \left(y + \frac{2}{y^2}\right) dx + \int 2y\,dy = c$$

$$\boxed{xy + \frac{2x}{y^2} + y^2 = c}$$

Type 5) Linear Differential Equations :-

D·E in which dependant variable and differential coeff. has degree '1' and both are not multiplied together then it is called L·D·E

* Types :-

1) 'y' dependant L·D·E :- In this type 'y' is dependant variable.

stand. form $\boxed{\frac{dy}{dx} + P(x)\cdot y = Q(x)}$

where $P(x), Q(x)$ are functions of 'x' or constants

Solution :- $I.F = e^{\int P(x).dx}$.

and G.S is given by.

$$y \times I.F = \int I.F \times Q(x)\, dx + c$$

i.e $y.e^{\int P(x)dx} = \int e^{\int P(x)dx} . Q(x)\, dx + C$

2) 'x' dependant L·D·E :- In this type 'x' is dependant variable

Stand. Form $\boxed{\frac{dx}{dy} + P(y).x = Q(y)}$

where $P(y), Q(y)$ are functions of 'y' or constant

Solution :- $I.F = e^{\int P(y)dy}$

and G·S is given by.

$$x \times I.F = \int I.F \times Q(y)dy + C$$

i.e $x\,e^{\int P(y)dy} = \int e^{\int P(y)dy} . Q(y)dy + C$

3) Reducible to L.D.E :-

$$f'(y)\frac{dy}{dx} + f(y).P = Q \quad ; P, Q \to fun^n \text{ of } 'x' \qquad \text{--- (1)}$$

Put $f(y) = u$
& diff. w.r.t 'x' (i.e independant vari.)

$$f'(y).\frac{dy}{dx} = \frac{du}{dx}$$

$\therefore$ eq^n (1) becomes.

$$\frac{du}{dx} + P.u = Q$$

Here "u" is dependant.
$\therefore$ ~~G.S~~ is given as

$$u.e^{\int P dx} = \int e^{\int P dx}.Q\,dx + c.$$

Finally put $u = f(y)$.

4) Bernoulli's L.D.E :-

$$\frac{dy}{dx} + P.y = Q.y^n \quad ; P, Q \to fun^n \text{ of } 'x'$$

Divide by y^n

$$y^{-n}\frac{dy}{dx} + y^{1-n}.P = Q \quad \text{--- (1)}$$

Put $y^{1-n} = u$
diff. w.r.t x

$$\therefore (1-n)\,y^{-n}\frac{dy}{dx} = \frac{du}{dx} \Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{du}{dx}$$

$\therefore$ equ^n (1) becomes

$$\frac{1}{(1-n)}\frac{du}{dx} + P.u = Q$$

i.e $\boxed{\frac{dy}{dx} + (1-n).Pu = Q.}$

Ex. 1) Solve $x\,dy = (x\sin x - y)\,dx$

$\Rightarrow$ $x\,dy = (x\sin x - y)\,dx$

$$\therefore \frac{dy}{dx} = \frac{x\sin x - y}{x}$$

$$\frac{dy}{dx} = \sin x - \frac{y}{x}$$

$$\therefore \frac{dy}{dx} + \frac{y}{x} = \sin x$$ [compare with $\frac{dy}{dx} + Py = Q$]

(y - dependant).

$\therefore$ $P = \frac{1}{x}$, $Q = \sin x$

$\therefore$ $I.F = e^{\int P dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

$\therefore$ G.S is given as

$$y \times I.F = \int I.F \times Q\,dx + C$$

$$y.x = \int x.\sin x\,dx + C$$

$$xy = -x\cos x + \sin x + C$$

i.e $\boxed{xy = \sin x - x\cos x + C}$

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Ex 2 Solve $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$\Rightarrow$ $(1+y^2)=(e^{\tan^{-1}y}-x)\frac{dy}{dx}$

$$\frac{dx}{dy}=\frac{e^{\tan^{-1}y}-x}{1+y^2}$$

$$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$ [Comparre with $\frac{dx}{dy}+P_1x=Q$]

'x' dependant

$\therefore$ $P=\frac{1}{1+y^2}$, $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$

$\therefore$ $I.F=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

$\therefore$ G.S is given as

$$x\times I.F=\int I.F\times Q\,dy+C$$

$$x\cdot e^{\tan^{-1}y}=\int e^{\tan^{-1}y}\cdot\frac{e^{\tan^{-1}y}}{1+y^2}dy+C$$

$$x\cdot e^{\tan^{-1}y}=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+C$$

put $\tan^{-1}y=t$

$\therefore$ $\frac{1}{1+y^2}dy=dt$

$\therefore$ $xe^{\tan^{-1}y}=\int e^{2t}dt+C$

$$xe^{\tan^{-1}y}=\frac{e^{2t}}{2}+C$$

$\therefore$ $x\cdot e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C$

Ex 3

Solve $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^3 x$.

$\Rightarrow$ $\dfrac{\tan y}{\cos y} \dfrac{dy}{dx} + \dfrac{\tan x}{\cos y} = \cos^3 x$

$\sec y \cdot \tan y \dfrac{dy}{dx} + \sec y \cdot \tan x = \cos^3 x \quad \text{--- (1)}$

Put $\sec y = u$

$\therefore \sec y \cdot \tan y \dfrac{dy}{dx} = \dfrac{du}{dx}$

$\therefore$ eq^n (1) becomes

$\dfrac{du}{dx} + \tan x \cdot u = \cos^3 x$

'u' dependant

$\therefore P = \tan x$, $Q = \cos^3 x$.

$\therefore$ G.S is given by

$I.F = e^{\int \tan x \, dx}$
$I.F = e^{-\ln \cos x}$
$I.F = (\cos x)^{-1} = \sec x$

$u \times I.F = \int I.F \times Q \, dx + c$

$u \times \sec x = \int \sec x \cdot \cos^3 x \, dx + c$

$\sec y \cdot \sec x = \int \cos^2 x \, dx + c$

$\downarrow$ H.W

$\sec y \cdot \sec x = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$

Ex 4 Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

$\Rightarrow \frac{dy}{dx} - xy = y^3 e^{-x^2}$ ---- (1)

which is Bernoulli's D.E

Divide by y^3

$\frac{1}{y^3}\frac{dy}{dx} - \frac{x}{y^2} = e^{-x^2}$ ---- (2)

Put $\frac{-1}{y^2} = u$

$\therefore \frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$\therefore$ eq^n (2) becomes

$\frac{1}{2}\frac{du}{dx} + ux = e^{-x^2} \Rightarrow \frac{du}{dx} + 2x \cdot u = 2e^{-x^2}$

'u' dependant

$\therefore P = 2x, \ Q = 2e^{-x^2}$

$I.F = e^{\int 2x\,dx} = e^{\frac{2x^2}{2}} = e^{x^2}$

$\therefore$ G-S is given by

$u \times I.F = \int I.F \times Q \, dx + C$

$\frac{-1}{y^2} \times e^{x^2} = \int e^{x^2} \cdot e^{-x^2} dx + C$

H.W

$\boxed{2xy^2 - e^{x^2} - cy^2 = 0}$

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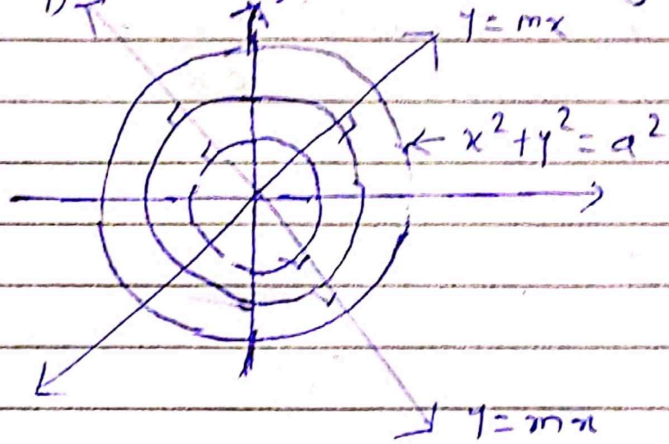
Application of D.E :-

A) Orthogonal Trajectories :-

A curve which intersects every member of given family of curve at right angle is called orthogonal trajectory to the given family of curves.

e.g $y = mx$ [straight lines passing through origin] are orthogonal trajectories to $x^2+y^2=a^2$



Method to find O.T :-

A) For cartesian curves :-

1) Given eq^n of curve $f(x,y,a)=0$ form where 'a' is arbitrary constant.

2) Diff. w.r.t 'x' and eliminate 'a'. Then eq^n will be in $f\left(x,y,\frac{dy}{dx}\right)=0$ form

3) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve D.E.

~~by the method~~, obtained eq^n be of O.T of given curve.

Ex Find O.T to family of curves given by
$x^2+y^2=a^2$

$\Rightarrow$ $x^2+y^2=a^2$ - - - (1)

diff. w.r.t x

$$2x+2y\frac{dy}{dx}=0$$

$$\therefore x+y\frac{dy}{dx}=0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\therefore x-y\frac{dx}{dy}=0$$

$$\therefore x=y\frac{dx}{dy}$$

$$\therefore \frac{dy}{y}=\frac{dx}{x}$$

on integrating both sides

$$\int\frac{dy}{y}=\int\frac{dx}{x}+\ln C$$

$$\ln y=\ln x+\ln C$$

$$\therefore \boxed{y=C.x}$$

2) Find O.T to family of curve given by

$$x^2 = cy^2$$

$$\Rightarrow \quad x^2 = cy^2 \quad \Rightarrow \quad c = \frac{x^2}{y^2}$$

Diff. w.r.t x

$$\cancel{2}x = \cancel{2}cy\frac{dy}{dx} \quad \text{—— (1)}$$

put $c = \frac{x^2}{y^2}$ in (1)

$$x = \frac{x^2}{y^2} \cdot y\frac{dy}{dx}$$

$$\frac{x}{x^2} = \frac{y}{y^2}\frac{dy}{dx}$$

$$\frac{1}{x} = \frac{1}{y}\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\frac{1}{x} = -\frac{1}{y}\frac{dx}{dy}$$

$$\therefore y\,dy = -x\,dx$$

on integrating both sides

$$\int y\,dy = -\int x\,dx + c$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\therefore \frac{y^2}{2} + \frac{x^2}{2} = c \Rightarrow \boxed{y^2 + x^2 = 2c = c_1^2}$$

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B) For Polar ~~coordinates~~ curves :-

1) Given eq^n of curve $f(r, \theta, a) = 0$ form where 'a' is arbitrary const.

2) Diff. w.r.t 'θ' and eliminate 'a' then equation will be in form $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ form.

3) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ and solve D.E. Obtained eq^n will be of required O.T.

Ex Find O.T of family of curves given by

$$r = a\cos^2\theta$$

$\Rightarrow$

$$r = a\cos^2\theta \Rightarrow \boxed{\therefore a = \frac{r}{\cos^2\theta}}$$

diff. w.r.t θ

$$\frac{dr}{d\theta} = -2a\cos\theta \cdot \sin\theta$$

put $a = \dfrac{r}{\cos^2\theta}$

$$\frac{dr}{d\theta} = -2 \cdot \frac{r}{\cos^2\theta} \cdot \cos\theta \cdot \sin\theta = -2\tan\theta \cdot r$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$-r^2\frac{d\theta}{dr} = -2 \cdot r \cdot \tan\theta$$

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$$r\frac{d\theta}{dr} = 2\tan\theta$$

$$\cot\theta \, d\theta = 2\frac{dr}{r}$$

on integrating both sides

$$\int \cot\theta \, d\theta = 2\int \frac{dr}{r} + \ln c$$

$$\ln|\sin\theta| = 2\ln r + \ln c$$

$$\sin\theta = c r^2$$

$$\Rightarrow \boxed{r^2 = c_1 \sin\theta}$$

H.W Find O.T for family of curves

1) $2x^2+y^2 = cx$ 2) $ay^2 = x^3$

3) $r = a(1-\sin\theta)$ 4) $r(1-\cos\theta) = 2a$.

B) Heat flow :-

* Fourier's law of heat conduction :-
The rate of heat flow across an area is proportional to the area and rate of change of temperature with respect to distance normal to the area.

Let q (cal/sec) heat flows from a slab of area A (cm^2) and the thickness δx then

$$q = -k \cdot A \cdot \frac{dT}{dx}$$

where K is called thermal conductivity constant & -ive sign indicates that 'T' decreases as 'x' is increased.

Ex 1) A pipe 20 cm in diameter contains steam at 150°c and is protected with covering 5 cm thick whose thermal conductivity is K = 0.0025. If temp of outer surface of covering is 40°c, find the ~~to~~ heat lost per hour.

⟹ Given,

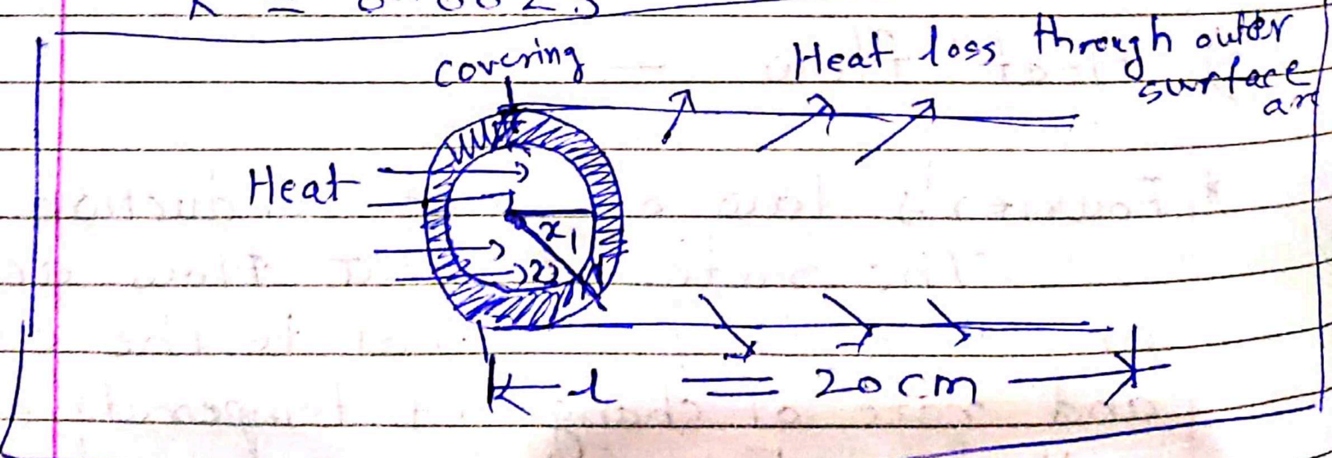
$d = 20$ cm $\therefore x_1 = 10$ cm [radius / thickness]

$x_2 = 10$ ~~c~~m + 5 cm (covering) = 15 cm

At $x_1 = 10$ cm, $T_1 = 150°$ ~~c~~ [Inner temp]

$x_2 = 15$ cm, $T_2 = 40°C$ [Outer temp]

$K = 0.0025$



By Fourier's law

$$q = -k \cdot A \cdot \frac{dT}{dx}$$

Now, heat losses through lateral surface of pipe

$$\therefore A = 2\pi x \cdot l \quad [x \leftarrow \text{radius},\ l - \text{length}]$$
$$= 2\pi x \cdot 20$$

$$\therefore q = -k \cdot 2\pi x 20 \frac{dT}{dx}$$

$$\therefore \frac{q}{40\pi \cdot k} \cdot \frac{dx}{x} = -dT$$

To find q (heat lost per sec) integrate between limits.

At $x_1 = 10$ cm, $T = 150°C$
$x_2 = 15$ cm, $T = 40°C$.

$$\therefore \frac{q}{40 \cdot \pi \cdot 0.0025} \int_{10}^{15} \frac{dx}{x} = -\int_{150}^{40} dT$$

$$\frac{q}{0.3143} [\ln x]_{10}^{15} = -[T]_{150}^{40}$$

$$\frac{q}{0.3143} [\ln 15 - \ln 10] = -[40 - 150]$$

$$\therefore q = \frac{110 \times 0.3143}{\ln 1.5} \quad \left[\because \ln 15 - \ln 10 = \ln \frac{15}{10} = \ln 1.5\right]$$

$$q = \frac{110 \times 0.3143}{0.4055} = 85.2602 \text{ cal/sec}$$

$\therefore Q$ = Heat lost through 20 cm pipe per hr

$= q \times 3600 = 306,936.6214$ cal/hr.

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c) Simple Electrical Circuits :-

1) Current $i = \frac{dq}{dt}$: rate of flow of charge

2) Voltage drop across resistor 'R' = iR

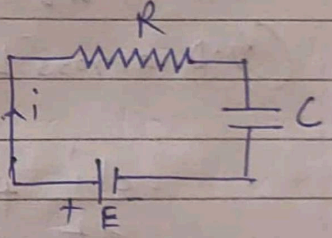
3) Voltage drop across capacitor 'C' = $\frac{q}{C}$

4) Voltage drop across inductor 'L' is = $L \cdot \frac{di}{dt}$

★ Kirchoff's Law :-

The algebraic sum of voltage drop across any closed circuit is equal to the resultant EMF in the circuit.

1) RC Circuit :- It is a circuit containing resistance 'R', capacitance 'C' along with voltage (emf) 'E' all in series.



If 'i' is current flowing in the circuits then by Kirchoff's law,

Voltage drop across 'R' + Voltage drop across 'C' = EMF

i.e $Ri + \frac{q}{c} = E$

$$R \cdot \frac{dq}{dt} + \frac{q}{c} = E$$

$$\therefore \frac{dq}{dt} + \frac{1}{RC} q = \frac{E}{R}$$

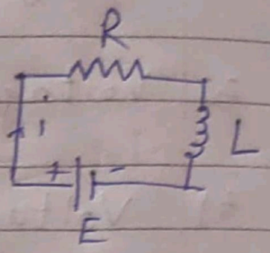
(which is L.D.E where q - dependant, t - independant

$\therefore$ $I.F = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$

and solution is given by.

$$\boxed{q \times I.F = \int I.F \times Q \, dt + k}, \quad k\text{-const.}$$

2) RL circuit :- It is a circuit containing resistance 'R', inductance 'L' with emf 'E' all in series.



If 'i' is current flowing in the circuit then by Kirchoff's law,

$$L \frac{di}{dt} + Ri = E$$

(V.D across 'L') + (V.D across 'R') = emf.

$$\therefore \frac{di}{dt} + \frac{R}{L} \cdot i = \frac{E}{L}$$

which is linear in 'i'.

$$\therefore I.F = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

general sol^n is given as

$$i \times (I.F) = \int (I.F) \times Q \, dt + k, \quad k\text{-const}$$

Note :- 1) i_{max} i.e maximum current is obtained as $t \to \infty$

2) $i_{max} = \frac{E}{R}$.

Ex In an electric circuit containing an inductance $L = 640 H$, a resistance $R = 250 \Omega$ & voltage $E = 500 V$. Find current 'i' at any time 't'. Also find time that ellapses before the current reaches 90% of it's max. value.

$\Rightarrow$ Given $R = 250 \Omega$, $L = 640 H$, $E = 500 V$.

$\therefore$ By Kirchoff's Law.

(V.D across R) + (V.D across L) = E

$$\cancel{Ri}\ Ri + L\frac{di}{dt} = E$$

$$\therefore \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

$$\therefore \frac{di}{dt} + \frac{250}{640} i = \frac{500}{640}$$

$$\therefore \frac{di}{dt} + \frac{25}{64} i = \frac{50}{64}$$

which is LDE in 'i'

$$\therefore P = \frac{25}{64}, \quad Q = \frac{50}{64}$$

$$\therefore I.F = e^{\int P dt} = e^{\int \frac{25}{64} dt} = e^{\frac{25}{64}t}.$$

$\therefore$ general sol^n is given as

$$i \times I.F = \int (I.F) \times Q\, dt + k$$

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$$i \cdot e^{\frac{25}{64}t} = \int e^{\frac{25}{64}t} \cdot \frac{50}{64}\, dt + k.$$

$$i \cdot e^{\frac{25}{64}t} = \frac{25}{32} \int e^{\frac{25}{64}t}\, dt + k$$

$$i \cdot e^{\frac{25}{64}t} = \frac{25}{32} \cdot e^{\frac{25}{64}t} \times \frac{64}{25} + k$$

$$\therefore i \cdot e^{\frac{25}{64}t} = 2 \cdot e^{\frac{25}{64}t} + k$$

$$\therefore i = 2 + k \cdot e^{-\frac{25}{64}t} \quad \text{--- (1)}$$

Now, we find particular value of k (since 'i' depends on only t).

$\therefore$ At $t = 0$, $i = 0$ put in (1)

$$\therefore 0 = 2 + k \cdot e^{-\frac{25}{64} \cdot 0}$$

$$\Rightarrow \boxed{k = -2}$$

$$\therefore i = +2 - 2e^{-\frac{25}{64}t}$$

$$\boxed{\therefore i = 2\left(1 - e^{-\frac{25}{64}t}\right)} \quad \text{--- (2)}$$

is eq^n of current 'i' at anytime 't'.

Now,

i_{max} is obtained at $t \to \infty$.

Apply $t \to \infty$ in eq^n (2)

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$\therefore i_{max} = 2(1 - e^{-\infty})$

$\boxed{i_{max} = 2.}$

and.

90% of $i_{max} = \frac{90}{100} \times 2 = 1.8$

at $i = 1.8$, $t = ?$

$\therefore 1.8 = 2(1 - e^{-\frac{25}{64}t})$

$\Rightarrow 0.9 = 1 - e^{-\frac{25}{64}t}$

$\Rightarrow e^{-\frac{25}{64}t} = 0.1$

$\Rightarrow -\frac{25}{64}t = \ln(0.1)$

$\therefore t = -\ln(0.1) \times \frac{64}{25}$

$\therefore \boxed{t = 5.89 \text{ sec}}$

$\therefore$ at $t = 5.89$ sec current reaches 90% of it's maximum value.