Beta and gamma function

* In this unit, we are coming across two special type of improper integrals
  (1) Beta &
  (2) Gamma

functions which plays important role to evaluate certain complicated integrals by Handy techniques by expressing in terms of these integrals.

Also used in double integrals, Fourier transforms, Laplace transforms etc.

Gamma Function

Def^n :-

The definite improper integral $\int_0^\infty e^{-t}\, t^{n-1}\, dt$ & denoted by 'Γn', called as Gamma 'n' or Euler's integral of second kind.

i.e.

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt \quad , \quad n > 0$$

Here, we can see 'n'

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx \quad , \quad n > 0$$

This is in integration w.r.t. x but 'Γn' is Real No.

* properties of Gamma Function :-

① $\Gamma 1 = 1$

proof,

We know

$$\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt \quad , n > 0$$

then

$$\Gamma 1 = \int_0^\infty e^{-t}\, t^{1-1}\, dt$$

$$= \int_0^\infty e^{-t}\, dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^\infty$$

$$= \left[0 - (-1)\right]$$

$$\boxed{\Gamma 1 = 1}$$

2]

$\Gamma n+1 = n \Gamma n$

proof :-

We know that $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$ $(\because n>0)$

$$\Rightarrow \Gamma n+1 = \int_0^\infty e^{-t} t^{(n+1)-1} dt$$

$$= \int_0^\infty \underset{v}{e^{-t}} \underset{u}{t^n} dt$$

integrating by parts $\left[\int u \cdot v = u \int v - \int u' \int v \right]$

$$= \left[t^n \cdot \frac{e^{-t}}{-1} \right]_0^\infty - \int_0^\infty n t^{n-1} \cdot \frac{e^{-t}}{-1} dt$$

But we know that $\lim_{t \to \infty} \frac{t^n}{e^t} = 0$

$$= 0 + n \int_0^\infty t^{n-1} \cdot e^{-t} dt$$

$$\Gamma n+1 = n \int_0^\infty e^{-t} \cdot t^{n-1} dt$$

$$\Rightarrow \boxed{\Gamma n+1 = n \Gamma n} \quad \text{.... by def}^n.$$

Remark :-

if n is positive integer

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1) \quad \because \Gamma(n) = (n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2) \quad \because \Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$= \text{------}$$

$$= \text{----}$$

$$= n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot \Gamma(1)$$

$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1 \quad \because \Gamma(1) = 1$$

$\Gamma(n+1) = n!$ when n is positive integer.

Now,

$$\Gamma(0) = \infty$$

proof :- we know that $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

put $n=0 \Rightarrow \Gamma(0) = \dfrac{\Gamma(1)}{0} = \dfrac{1}{0} = \infty$

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3] Alternative definition of Gamma function:-

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

proof :- We know that, $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}\, dt \quad , \quad n > 0$ — (1)

Now put $t = x^2$

$\Rightarrow dt = 2x\, dx$

$\Rightarrow$

when $t =$	0	∞
then $x =$	0	∞

$\therefore$ From eq^n (1) $\Gamma n = \int_0^\infty e^{-x^2} (x^2)^{n-1}\, 2x\, dx$

$$= 2\int_0^\infty e^{-x^2} x^{2n-2} \cdot x\, dx$$

$$\boxed{\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx}$$

* Remark

NOTE THAT : $\boxed{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$

4] $\Gamma 1/2 = \sqrt{\pi}$

We know that, (By property 3)

$$\Gamma n = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx$$

put $n = 1/2$

$$\Gamma \tfrac{1}{2} = 2\int_0^\infty e^{-x^2} \cdot x^{\cancel{2}(1/\cancel{2})-1}\, dx$$

$$= 2\int_0^\infty e^{-x^2}\, dx$$

$$= 2 \cdot \frac{\sqrt{\pi}}{2} \quad \text{-----} \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\boxed{\Gamma \tfrac{1}{2} = \sqrt{\pi}}$$

* Find $\Gamma \frac{5}{2}$, As we know $\Gamma n+1 = n\Gamma n$

$\Rightarrow \Gamma n = (n-1)\,\Gamma n-1$

$= \Gamma \frac{5}{2} = \left(\frac{5}{2}-1\right)\Gamma \frac{5}{2}-1$

$= \frac{3}{2}\Gamma 3/2 \Rightarrow \frac{3}{2}\cdot\frac{1}{2}\Gamma 1/2 \Rightarrow \frac{3}{4}\sqrt{\pi}$

5] $$\int_0^\infty e^{-kx} \cdot x^{n-1}\, dx = \frac{\Gamma n}{k^n}$$

Consider $L.H.S. = \int_0^\infty e^{-kx} \cdot x^{n-1}\, dx$ —— (1)

put $kx = t \Rightarrow x = \frac{t}{k}$

$\Rightarrow dx = \frac{dt}{k}$

when $x =$	0	∞
then $t =$	0	∞

$\therefore$ eqⁿ (1) becomes,

$$= \int_0^\infty e^{-t} \cdot \left(\frac{t}{k}\right)^{n-1} \frac{dt}{k}$$

$$= \int_0^\infty e^{-t} \cdot \frac{t^{n-1}}{k^{n-1}} \cdot \frac{1}{k}\, dt$$

$$= \frac{1}{k^n} \int_0^\infty e^{-t} \cdot t^{n-1}\, dt$$

$$= \frac{\Gamma n}{k^n}$$

$\boxed{LHS = RHS.}$

* Remark

Note that :-

$$\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi} \quad , \quad 0 < p < 1$$

e.g.

① $\Gamma 1/4 \cdot \Gamma 3/4 = \Gamma\frac{1}{4} \cdot \Gamma 1-\frac{1}{4} \quad , \quad 0 < \frac{1}{4} < 1$

$$= \frac{\pi}{\sin\left(\frac{1}{4}\cdot\pi\right)}$$

$$= \frac{\pi}{\sin(\pi/4)} = \frac{\pi}{1/\sqrt{2}}$$

$$\boxed{\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4} = \pi\cdot\sqrt{2}}$$

② $\Gamma\frac{1}{3} \cdot \Gamma\frac{2}{3} = \Gamma\frac{1}{3} \, \Gamma 1-\frac{1}{3} \quad , \quad 0 < \frac{1}{3} < 1$

$$= \frac{\pi}{\sin(\pi/3)} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$$

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Examples :- Type - 1

① Evaluate $\int_0^\infty e^{-x^4}\, dx$ —— ①

put $x^4 = t$

$\Rightarrow \quad x = t^{1/4}$

$$dx = \frac{1}{4} t^{1/4 - 1}\, dt$$

$$\boxed{dx = \frac{1}{4} t^{-3/4}\, dt}$$

When x =	0	∞
then t =	0	∞

$\therefore$ eqn ① becomes,

$$\int_0^\infty e^{-x^4}\, dx = \int_0^\infty e^{-t} \left(\frac{1}{4} t^{-3/4}\right) dt$$

We can write $-\frac{3}{4} = \frac{1}{4} - 1$

$$= \frac{1}{4} \int_0^\infty e^{-t}\, t^{1/4 - 1}\, dt$$

$$\boxed{\int_0^\infty e^{-x^4}\, dx = \frac{1}{4} \Gamma\left(\frac{1}{4}\right)} \quad \ldots \because \int_0^\infty e^{-t}\, t^{n-1}\, dt = \Gamma n$$

② Evaluate $\int_0^{\infty} \sqrt[4]{x} \cdot e^{-\sqrt{x}} \, dx$ ——— ①

Sol^n put $\sqrt{x} = t$

$\Rightarrow \quad x = t^2$

$\Rightarrow \quad dx = 2t \, dt$

when x =	0	∞
then t =	0	∞

$\therefore$ eq^n ① becomes

$$\int_0^{\infty} \sqrt[4]{x} \, e^{-\sqrt{x}} dx = \int_0^{\infty} (t^2)^{1/4} \cdot e^{-t} \cdot 2t \, dt$$

$$= 2 \int_0^{\infty} e^{-t} \, t^{1/2} \cdot t \, dt$$

$$= 2 \int_0^{\infty} e^{-t} \, t^{3/2} \, dt$$

$$= 2 \int_0^{\infty} e^{-t} \, t^{5/2 - 1} \, dt$$

$$= 2 \left[\Gamma \tfrac{5}{2} \right]$$

$$= \cancel{2} \cdot \left[\frac{3}{\cancel{2}} \cdot \frac{1}{2} \cdot \Gamma \frac{1}{2} \right] = \frac{3}{2} \sqrt{\pi}$$

3] Evaluate $\int_0^{\infty} e^{-2x^2} \cdot x^9 \, dx$ —— (1)

put $2x^2 = t \Rightarrow x = \frac{\sqrt{t}}{\sqrt{2}}$

$\Rightarrow 4x\,dx = dt$

$\Rightarrow dx = \frac{dt}{4\left(\sqrt{t/2}\right)} = \frac{dt \cdot \sqrt{2}}{4\sqrt{t}}$

When $x =$	0	∞
then $t =$	0	∞

$\therefore$ eq^n (1) becomes,

$$\int_0^{\infty} e^{-2x^2} \cdot x^9 \, dx = \int_0^{\infty} e^{-t} \cdot \left(\sqrt{\frac{t}{2}}\right)^9 \cdot \frac{\sqrt{2}}{4\sqrt{t}} \, dt$$

$$= \int_0^{\infty} e^{-t} \cdot \frac{(t)^{9/2}}{(2)^{9/2}} \cdot \frac{(2)^{1/2}}{4(t)^{1/2}} \, dt$$

$$= \int_0^{\infty} \frac{e^{-t}}{4} \frac{(t)^{9/2-1/2}}{(2)^{9/2-1/2}} \, dt$$

$$= \frac{1}{4} \cdot \frac{1}{2^4} \int_0^{\infty} e^{-t} \; t^4 \, dt$$

$$= \frac{1}{64} \cdot \int_0^{\infty} e^{-t} \; t^{5-1} \, dt$$

$$= \frac{1}{64} \left(\Gamma 5\right) = \frac{1}{64} \cdot 4! = \frac{24}{64} = \frac{3}{8}$$

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* Home work Etamples

Evaluate following

① $\int_0^{\infty} x^7 e^{-2x^2} dx$ Answer $= \frac{3}{16}$

② $\int_0^{\infty} \sqrt{x} \cdot e^{-\sqrt[3]{x}} dx$ Answer $= \frac{315\sqrt{\pi}}{16}$

③ $\int_0^{\infty} \sqrt{y}\, e^{-y^3} dy$ Answer $= \frac{\sqrt{\pi}}{3}$

④ $\int_0^{\infty} x^{2/3} e^{-\sqrt[3]{x}} dx$ Answer $= 72$

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Type : 2

Evaluate $\int_0^{\infty} \frac{x^a}{a^x}\, dx \quad , (a>0)$ —— (1)

Sol^n :- put $a^x = e^t$

$\Rightarrow \quad x \cdot \log(a) = t$

$x = \frac{t}{\log a}$

$dx = \frac{dt}{\log a}$,

when $x =$	0	∞
then $t =$	0	∞

$\therefore$ eq^n (1) becomes,

$$\int_0^{\infty} \frac{x^a}{a^x}\, dx = \int_0^{\infty} \frac{x^a}{e^t} \frac{dt}{\log a}$$

$$= \cancel{\frac{1}{\log a} \int_0^{\infty} e^{-t}\, x^a\, dt}$$

$$= \frac{1}{\log a} \int_0^{\infty} e^{-t} \left(\frac{t}{\log a} \right)^a dt$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} e^{-t}\, t^{(a+1)-1}\, dt = \frac{\Gamma a+1}{(\log a)^{a+1}}$$

(2) Evaluate, $\int_0^\infty \frac{x^2}{(3)^{x^2}}\,dx$ —— (1)

Sol^n : put $(3)^{x^2} = e^t$

$\Rightarrow x^2 \log(3) = t \Rightarrow 2x\,dx = \frac{dt}{\log 3}$

$\Rightarrow x = \sqrt{\frac{t}{\log 3}}$

$2x\,dx = \frac{dt}{\log 3}$

$\Rightarrow x^2\,dx = \frac{\sqrt{t}\,dt}{2\sqrt{\log 3}\cdot \log 3}$

&

when $x =$	0	∞
then $t =$	0	∞

put in eq^n (1) then it becomes.

$$\int_0^\infty \frac{x^2}{(3)^{x^2}}\,dx = \int_0^\infty \frac{1}{e^t}\cdot\frac{\sqrt{t}}{2(\log 3)^{3/2}}\,dt$$

$$= \frac{1}{2(\log 3)^{3/2}}\int_0^\infty e^{-t}\, t^{1/2}\,dt$$

$$= \frac{1}{2(\log 3)^{3/2}}\int_0^\infty e^{-t}\, t^{3/2-1}\,dt$$

$$= \frac{1}{2(\log 3)^{3/2}}\cdot\left(\Gamma\frac{3}{2}\right) \Rightarrow \frac{1}{2(\log 3)^{3/2}}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{4(\log 3)^{3/2}}$$

③ Evaluate $\int_0^\infty 5^{-4x^2}\, dx$ ——— ①

Sol^n :- put $5^{-4x^2} = e^{-t}$

$\Rightarrow -4x^2 \log 5 = -t$

$\Rightarrow x^2 = \dfrac{t}{4\cdot\log 5} \Rightarrow x = \sqrt{\dfrac{t}{4\log 5}}$

$\Rightarrow 2x\, dx = \dfrac{dt}{4\cdot\log 5}$

$\Rightarrow dx = \dfrac{dt}{\cancel{2}\left(\dfrac{\sqrt{t}}{\cancel{2}\sqrt{\log 5}}\right) 4\cdot\log 5}$

$\Rightarrow dx = \dfrac{\sqrt{\log 5}\; dt}{4\sqrt{t}\,(\log 5)} \Rightarrow dx = \dfrac{dt}{4\sqrt{t}\cdot\sqrt{\log 5}}$

when $x=$	0	∞
then $t=$	0	∞

∴ From eq^n ①,

$\int_0^\infty 5^{-4x^2}\, dx = \int_0^\infty e^{-t}\cdot \dfrac{dt}{4\cdot\sqrt{t}\cdot\sqrt{\log 5}}$

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$$= \int_0^\infty e^{-t} \frac{t^{-1/2}}{4\sqrt{\log 5}} dt$$

$$= \frac{1}{4\sqrt{\log 5}} \int_0^\infty e^{-t} \cdot t^{1/2 - 1} \, dt$$

$$= \frac{1}{4\sqrt{\log 5}} \cdot \sqrt{\tfrac{1}{2}}$$

$$\int_0^\infty 5^{-4x^2} dx = \frac{\sqrt{\pi}}{4\sqrt{\log 5}} \quad \because \sqrt{\tfrac{1}{2}} = \sqrt{\pi}$$

* Example's For Home work.

Evaluate Following

① $\int_0^\infty \frac{dx}{3^{4x^2}}$ Answer $= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

② $\int_0^\infty \frac{x^2}{2^x} dx$ Answer $= \frac{2}{(\log 2)^3}$

③ $\int_0^\infty \frac{x^4}{4^x} dx$ Answer $= \frac{24}{(\log 4)^5}$

④ $\int_0^\infty 7^{-4x^2} dx$ Answer $= \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$

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Type-3,

(1) Evaluate $\int_0^1 \frac{dx}{\sqrt{x \log(\frac{1}{x})}}$ ——— (1)

Sol^n put $\log\left(\frac{1}{x}\right) = t$

$\Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = e^{-t}$ $\Rightarrow$

when $x=$	0	1
then $t=$	∞	0

$\Rightarrow dx = -e^{-t}\,dt$

So, eq^n (1) becomes.

$$\int_0^1 \frac{dx}{\sqrt{x \cdot \log(1/x)}} = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{e^{-t}\cdot t}}$$

$$= \int_0^\infty e^{-t}\cdot(e^{-t})^{-1/2}\cdot(t)^{-1/2}\,dt \quad ; \int_a^b f(x) = -\int_b^a f(x)$$

$$= \int_0^\infty e^{-t}\cdot e^{t/2}\cdot t^{-1/2}\,dt$$

$$= \int_0^\infty e^{-t/2}\cdot t^{-1/2}\,dt$$

$$\int_0^1 \frac{dx}{\sqrt{x \log(1/x)}} = \int_0^\infty e^{-t/2} \cdot t^{-1/2}\, dt$$

$$= \int_0^\infty e^{-\left(\frac{1}{2}\right)t} \cdot t^{\frac{1}{2}-1}\, dt$$

As we know that

$$\int_0^\infty e^{-kt} \cdot t^{n-1}\, dt = \frac{\Gamma n}{k^n}$$ Here $k = \frac{1}{2}$, $n = \frac{1}{2}$

$$= \frac{\Gamma 1/2}{\left(\frac{1}{2}\right)^{1/2}}$$

$$= \sqrt{\pi} \cdot \sqrt{2} \qquad \because \Gamma \tfrac{1}{2} = \sqrt{\pi}$$

$$\int_0^1 \frac{dx}{\sqrt{x \log(1/x)}} = \sqrt{2\pi}$$

2] Evaluate $\int_0^1 (x \log x)^4 \, dx$ ——— (1)

Sol^n :- put $\log x = -t$

$\Rightarrow \quad x = e^{-t} \quad \Rightarrow$

when $x =$	0	1
then $t =$	∞	0

$\Rightarrow \quad dx = -e^{-t} \, dt$

put in (1) we get

$$\int_0^1 (x \cdot \log x)^4 \, dx = \int_\infty^0 \left(e^{-t} \cdot (-t)\right)^4 \left(-e^{-t} \, dt\right)$$

$$= -\int_\infty^0 e^{-4t} \cdot t^4 \cdot e^{-t} \, dt$$

$$= \int_0^\infty e^{-5t} \, t^{5-1} \, dt \quad ; \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

We know that

$$\int_0^\infty e^{-kt} \cdot t^{n-1} \, dt = \frac{\Gamma n}{k^n} \quad , \quad \text{Here } k=5, \; n=5$$

$$= \frac{\Gamma 5}{(5)^5} = \frac{4!}{(5)^5} \quad \ldots \quad \because \Gamma n = (n-1)! \text{ when } n \in \mathbb{Z}^+$$

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3] Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log(x)}}$ OR $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}}$ ——— (1)

Soln put

$$-\log(x) = t$$
$$\Rightarrow \log(x) = -t$$
$$\Rightarrow x = e^{-t}$$
$$\Rightarrow dx = -e^{-t}\,dt$$

$\Rightarrow$

when $x=$	0	1
then $t=$	∞	0

so, eqn (1) becomes,

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}}$$

$$= -\int_\infty^0 e^{-t}\,t^{-1/2}\,dt$$

$$= \int_0^\infty e^{-t}\cdot t^{1/2-1}\,dt$$

$$= \Gamma\tfrac{1}{2}$$

$$= \sqrt{\pi}$$

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Examples on Home work

① prove that $\int_0^1 \left(\log\left(\frac{1}{x}\right)\right)^{n-1} dx = \Gamma n$

Evalvate the following

① $\int_0^1 \frac{x\,dx}{\sqrt{\log\left(\frac{1}{x}\right)}}$ Answer $= \sqrt{\frac{\pi}{2}}$

② $\int_0^1 x^3 \cdot (\log x)^3\, dx$ Answer $= \frac{3!}{-(4)^4}$

③ $\int_0^1 x^3 \left(\log\left(\frac{1}{x}\right)\right)^4 dx$ Answer $= \frac{3}{128}$

④ $\int_0^1 (\log x)^n\, dx$ Answer $= (-1)^n \Gamma n$