Curve Tracing: Cartesian Curves

BASIC DEFINITIONS

1. **Convex Upwards:** If the portion of the curve on both sides of point 'A' lies below the tangent at A, then the curve is convex upwards.
2. **Convex Downwards:** If the portion of the curve on both sides of point 'A' lies above the tangent at A, then the curve is convex downwards.
3. **Singular points:-** An unusual point on a curve is called a singular point such as a point of inflexion, a double point, a multiple point, cusp, node or a conjugate point.
4. **Point of Inflexion:-** The point that separate the convex part of a continuous curve from the concave part is called the point of inflexion of the curve. i.e. A point where the curve unusually crosses its tangent is called a point of inflexion.
5. **Multiple Point:** A point through which more than one branches of a curve pass is called a multiple point of the curve.
6. **Double point:** A point on a curve is called a double point, if two branches of the curve pass through it. If r branches pass through a point, the point is called a multiple point of r^{th} order.
7. **Node:** A double point is called node if the branches of curve passing through it are real and the tangents at the common point of intersection are distinct.
8. **Cusp:** A double point is called Cusp if the tangents at that point to the two branches of the curve are coincident.
9. **A Conjugate Point:** A Point P is called a conjugate point on the curve if there are no real points on the curve in the vicinity of the point P. It is also called as an isolated Point.

CURVE TRACING

Tracing of Cartesian curves

The following rules will help in tracing a Cartesian curve.

Rule 1: Symmetry

(a) ***Symmetry about X-axis*:** If the equation of curve containing all even power terms in ‘y’ then the curve is symmetric about X-axis.

(b) ***Symmetry about Y-axis* :** If the equation of curve containing all even power terms in ‘x’ then the curve is symmetric about Y-axis

(c) ***Symmetry about both X and Y axes*:** If the equation of curve containing all even power terms in ‘x’ and ‘y’ then the curve is symmetric about both axes.

(d) ***Symmetry in opposite quadrants*:** If the equation of curve remains unchanged when x and y are replaced by –x and –y respectively then the curve is symmetric in opposite quadrants.

(e) ***Symmetry about the line*** $\boldsymbol{y = x}$ **:** If the equation of curve remains unchanged when x is replaced by y and y is replaced by x then the curve is symmetric about the line y=x.

(f) ***Symmetry about the line*** $\boldsymbol{y = -x}$ **:** If the equation remains unchanged when x is replaced by -y and y is replaced by - x then the curve is symmetric about the line y=-x.

Rule 2: *Origin*:

(g) If the equation of curve does not contain any absolute constant then the curve passes through the origin.

(h) If the curve passes through origin then the equations of the tangent at origin can be obtained by equating the lowest degree terms taken together to zero.

Rule 3: Coordinate axes

***(a) Intersection with the co-ordinate axes*:**
Intersection with X-axis: put y = 0 in the given equation and find the value of x.
Intersection with Y-axis: put x = 0 in the given equation and find the value of y.

***(b) Points on the line of symmetry*:** If y = x is the line of symmetry then put y = x to find the points on line of symmetry.

(c) ***Other points*:** To find nature of tangent at any point find $\frac{dy}{dx}$ at that point.

Case 1: If $\frac{dy}{dx} = 0$ then tangent is parallel to X-axis.

Case 2: If $\frac{dy}{dx} = \infty$ then tangent is parallel to Y-axis.

Case 3: If $\frac{dy}{dx} = +ve$ then tangent makes acute angle with X-axis.

Case 4: If $\frac{dy}{dx} = -ve$ then tangent makes obtuse angle with X-axis.

Rule 4: Asymptotes:

(d) ***Parallel to X-axis*****:** Asymptotes parallel to X-axis are obtained by equating the coefficient of highest degree term in x to zero.

(e) ***Parallel to Y-axis*****:** Asymptotes parallel to Y-axis are obtained by equating the coefficient of highest degree term in y to zero.

(f) ***Oblique asymptote*****:** Asymptotes which are not parallel to co-ordinate axes are called as *Oblique asymptotes*.

Method 1: Let y = mx+c be the asymptote. The point of intersection with the curve f(x, y) = 0 are given by f(x, mx+c) = 0. Equate to zero the coefficients of two successive highest power of 'x', giving equations to determine m & c.

Method 2:

(g) Let y = mx+c be the equation of asymptote.

(h) Find $\phi_n(m)$ by putting x = 1 and y = m in the highest degree (n) terms of the equation.

(i) Similarly find $\phi_{n-1}(m)$.

(j) Solve $\phi_n(m)$= 0 to determine m.

(k) Find 'c' by the formula $c = -\dfrac{\phi_{n-1}(m)}{\phi_n(m)}$.

(l) If the roots of m are equal, then find 'c' by $\dfrac{c^2}{2}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$.

Rule 5: Special points on the curve: Find out such points on the curve whose presence can be easily detected.

Rule 6: Region of absence of the curve: Find the values of x (or y) where y(or x) becomes imaginary, then the curve does not exists in that region.

Q1. Trace the following curves

(i) $y^2(2a - x) = x^3$

Solution:

$$y^2(2a - x) = x^3 \qquad \text{------ ---------} \qquad (1)$$

We check the following points for tracing of the above curve.

1. Symmetry:- Since the power of y is even.

$\therefore$ The curve is symmetry about x-axis.

No any other symmetry.

2. Origin:-

(i) Since there is no constant term in the given equation.
$\therefore$ The curve passes through origin.

(ii) Tangent at origin is given by
Lowest degree term $= 0$

$\Rightarrow 2ay^2 = 0$

$\Rightarrow y^2 = 0$

$\Rightarrow y = 0,\ y = 0$

$\Rightarrow y = 0$

(iii) Nature of origin:-
Since there is only one common tangent at origin.
$\therefore$ Origin is cusp.

(iv) Intersection with coordinate axes:-
Put $y = 0$ in (1), we have $x = 0$.

Put $x = 0$ in (1), we have $y = 0$.

Hence curve meets coordinate axes only at origin $(0,\ 0)$.

3. Asymptotes:-

(i) Asymptotes parallel to x-axis:-
Equating the coefficient of highest power of x, we obtain the asymptotes parallel to x-axis. Here highest power of x is x^3 and whose coefficient is 1 cannot equate to zero. Hence no asymptotes parallel to x-axis.

(ii) Asymptotes parallel to y-axis:-
Equating the coefficient of highest power of y, we obtain the asymptotes parallel to y-axis. Here highest power of y is y^2 and whose coefficient is $(2a - x)$. Hence the asymptotes parallel to y-axis is

$2a - x = 0$

$\Rightarrow x = 2a$

(iii) Oblique asymptotes:-
Asymptotes not parallel to x and y axes are called oblique asymptotes.
Here since parallel asymptotes are present, so no oblique asymptotes.

4. Region:-

From (1), we have

$$y^2 = \frac{x^3}{2a - x}$$

$$\therefore \qquad y = \frac{x\sqrt{x}}{\sqrt{2a - x}}$$

From the above expression, it is clear that

(i) If x is negative, then y will be imaginary. So, there is no part of the curve for which x is negative i.e. left hand side of y-axis.

(ii) If $2a - x < 0$ *i.e.* $x > 2a$, then y will be imaginary. So, there is no part of the curve right hand side of the line $x = 2a$.

Hence the approximate shape of the curve is as follows: