Dr. Vishwanath Karad MIT WORLD PEACE UNIVERSITY | PUNE

TECHNOLOGY, RESEARCH, SOCIAL INNOVATION & PARTNERSHIPS

Basics of Electrical and Electronics Engineering ECE101B

UNIT VI - A.C. CIRCUITS

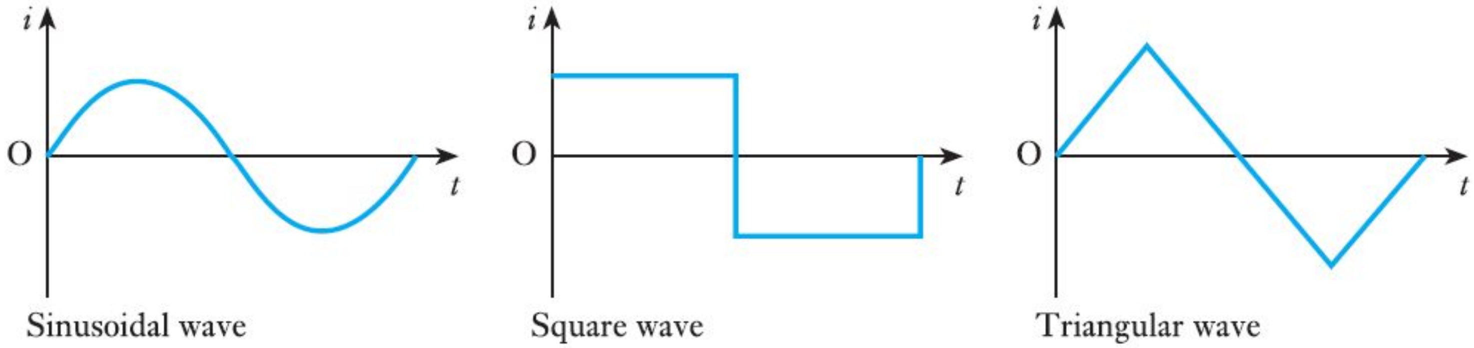
Topics

- Generation of alternating EMF
- Equation of alternating quantity, Waveforms, phasor representation
- Concept of impedance and admittance and power triangle
- Series RL, RC, RLC circuits
- Series resonance
- Parallel circuits
- Generation of three phase EMF

Alternating systems

- Alternating current can be abbreviated to a.c., hence a system with such an alternating current is known as an a.c. system. The curves relating current to time are known as waveforms.

Fig. 9.1 Alternating current waveforms



Generation of an alternating e.m.f.

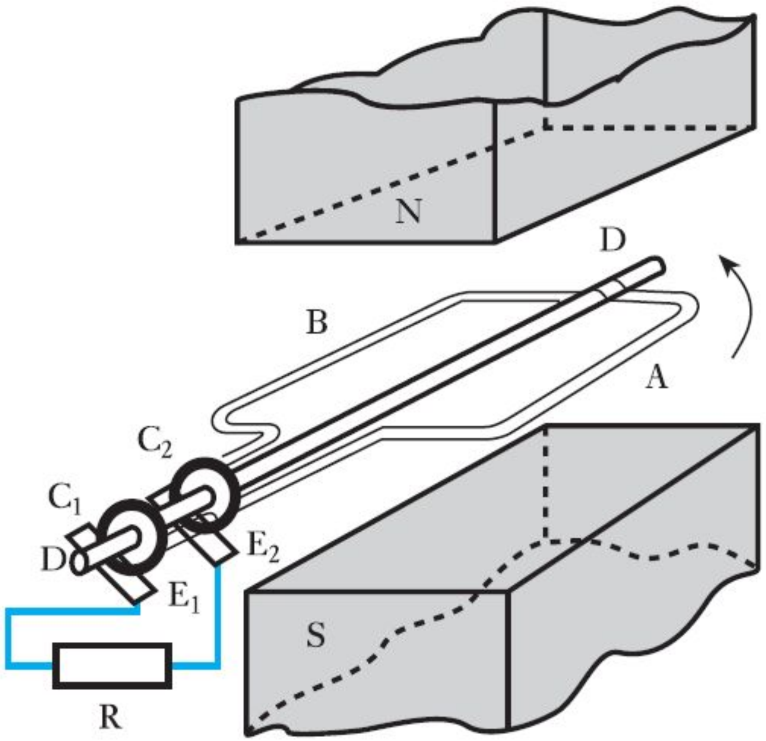


Fig. Generation of an alternating e.m.f.

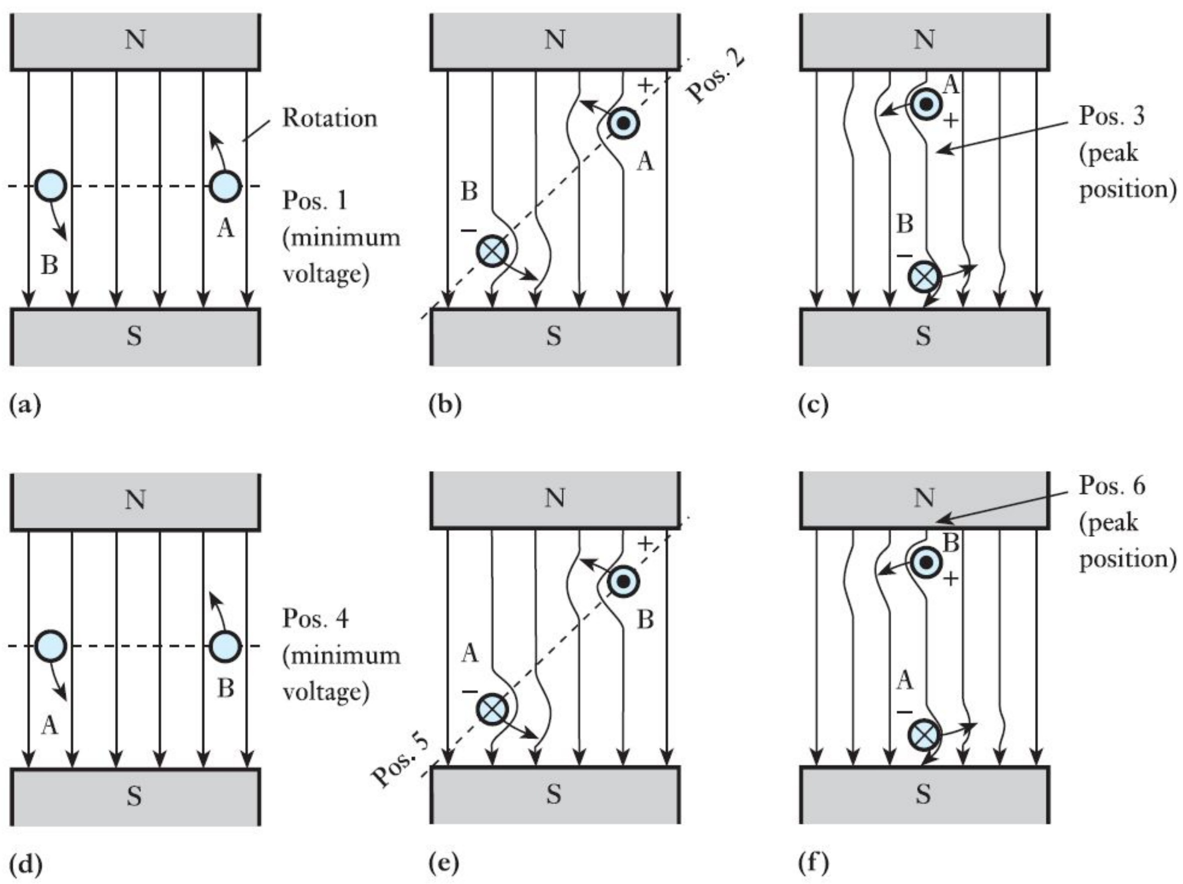


Fig. EMF in rotating coil

Generation of an alternating e.m.f.

- In Fig. (a), coil AB is shown after it has rotated through an angle θ from the horizontal position, namely the position of zero e.m.f.
- Suppose the peripheral velocity of each side of the loop to be u metres per second; then at the instant shown in Fig. 9.4, this peripheral velocity can be represented by the length of a line AL drawn at right angles to the plane of the loop.
- We can resolve AL into two components, AM and AN, perpendicular and parallel respectively to the direction of the magnetic flux, as shown in Fig. (b).
- Since The e.m.f. generated in A is due entirely to the component of the velocity perpendicular to the magnetic field.

$$\angle MLA = 90° - \angle MAL = \angle MAO = \theta$$

$$\therefore \quad AM = AL \sin\theta = u \sin\theta$$

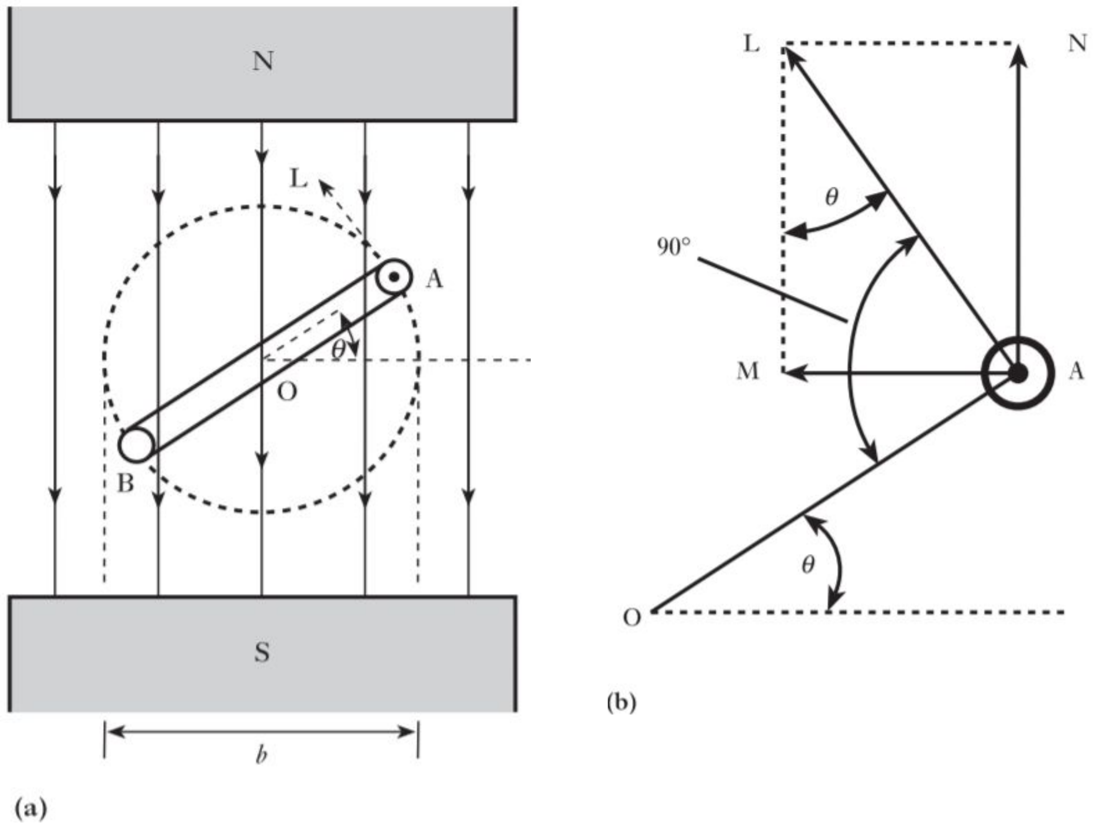


Fig. Instantaneous value of generated e.m.f.

Generation of an alternating e.m.f.

- Hence, if B is the flux density in tesla and if l is the length in metres of each of the parallel sides A and B of the loop,

Then, e.m.f. generated in one side of loop is

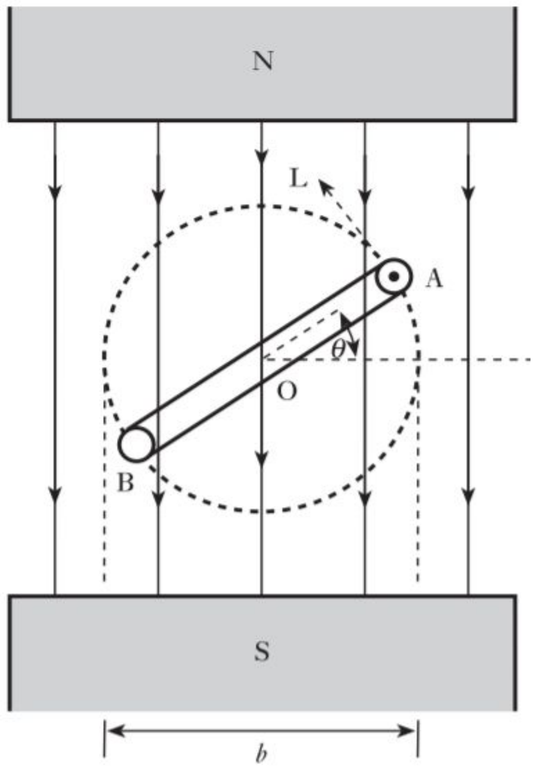
$$Blu \sin \theta \text{ volts}$$

and total e.m.f. generated in loop is

$$2Blu \sin \theta \text{ volts}$$

$$\therefore \qquad \boxed{e = 2Blu \sin \theta}$$

i.e. the generated e.m.f. is proportional to *sin θ*



(a)

Generation of an alternating e.m.f.

- When θ = 90°, the plane of the loop is vertical and both sides of the loop are cutting the magnetic flux at the maximum rate, so that the generated e.m.f. is then at its maximum value Em.

  when θ = 90°, *Em = 2Blu* volts.

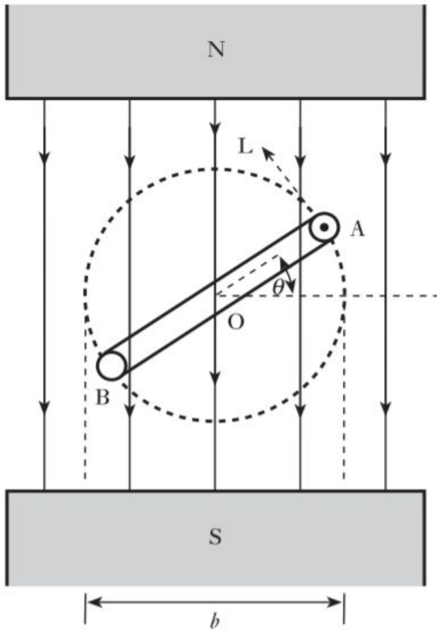
- If b is the breadth of the loop in metres, and n the speed of rotation in revolutions per second, then u is *πbn* metres per second and

$$E_m = 2Bl \times \pi bn \text{ volts}$$

$$= 2\pi BAn \text{ volts}$$

where

$$A = lb = \text{area of loop in square metres}$$



(a)

Generation of an alternating e.m.f.

- If the loop is replaced by a coil of N turns in series, each turn having an area of A square metres, maximum value of e.m.f. generated in coil is

$$\boxed{E_m = 2\pi BAnN} \quad \text{volts}$$

and instantaneous value of e.m.f. generated in coil is

$$e = E_m \sin\theta = 2\pi BAnN \sin\theta \text{ volts}$$

$$\therefore \quad \boxed{e = 2\pi BAnN \sin\theta}$$

Instantaneous value of generated e.m.f.

- Em - maximum value of the e.m.f.
- e - value after the loop has rotated through an angle θ from the position of zero e.m.f.
- e.m.f. as positive while θ is varying between 0 and 180°.
- It is negative while θ is varying between 180° and 360°, i.e. when θ varies

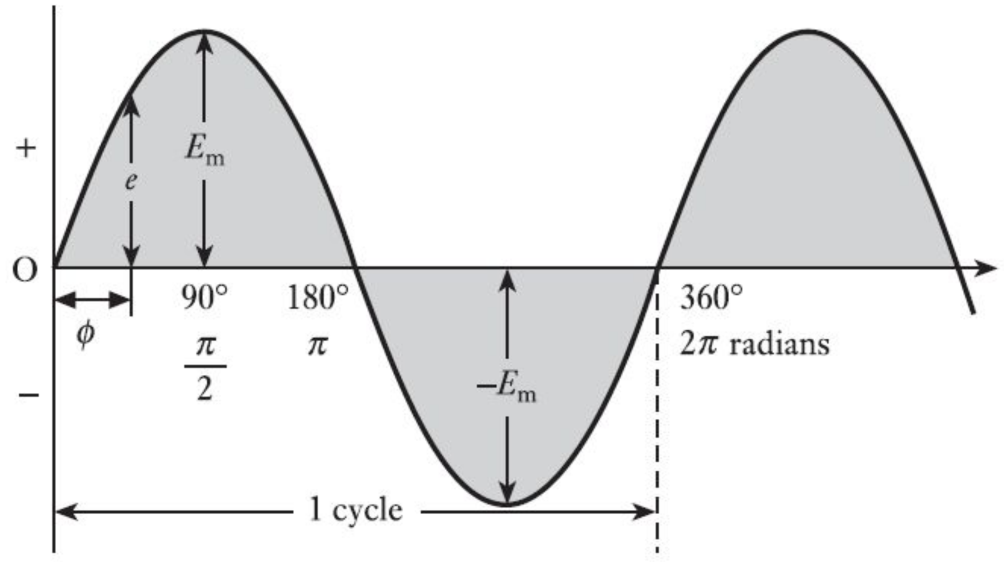


Fig. Sine wave of e.m.f.

Waveform terms and definitions

- **Waveform:** The variation of a quantity such as voltage or current shown on a graph to a base of time or rotation is a waveform.
- **Cycle:** Each repetition of a variable quantity, recurring at equal intervals, is termed a cycle.
- **Period:** The duration of one cycle is termed its period. (Cycles and periods need not commence when a waveform is zero. Figure illustrates a variety of situations in which the cycle and period have identical values.)
- **Instantaneous value:** The magnitude of a waveform at any instant in time (or position of rotation). Instantaneous values are denoted by lower-case symbols such as e, v and i.
- **Peak value:** The maximum instantaneous value measured from its zero value is known as its peak value.
- **Peak-to-peak value:** The maximum variation between the maximum positive instantaneous value and the maximum negative instantaneous value is the

peak-to-peak value. For a sinusoidal waveform, this is twice the peak value. The peak-to-peak value is Epp or Vpp or Ipp.

- **Peak amplitude:** The maximum instantaneous value measured from the mean value of a waveform is the peak amplitude. Later we will find how to determine the mean value, but for most sinusoidal alternating voltages and currents the mean value is zero.

  The peak amplitude is Em or Vm or Im. The peak amplitude is generally described as the maximum value, hence the maximum voltage has the symbol Vm.

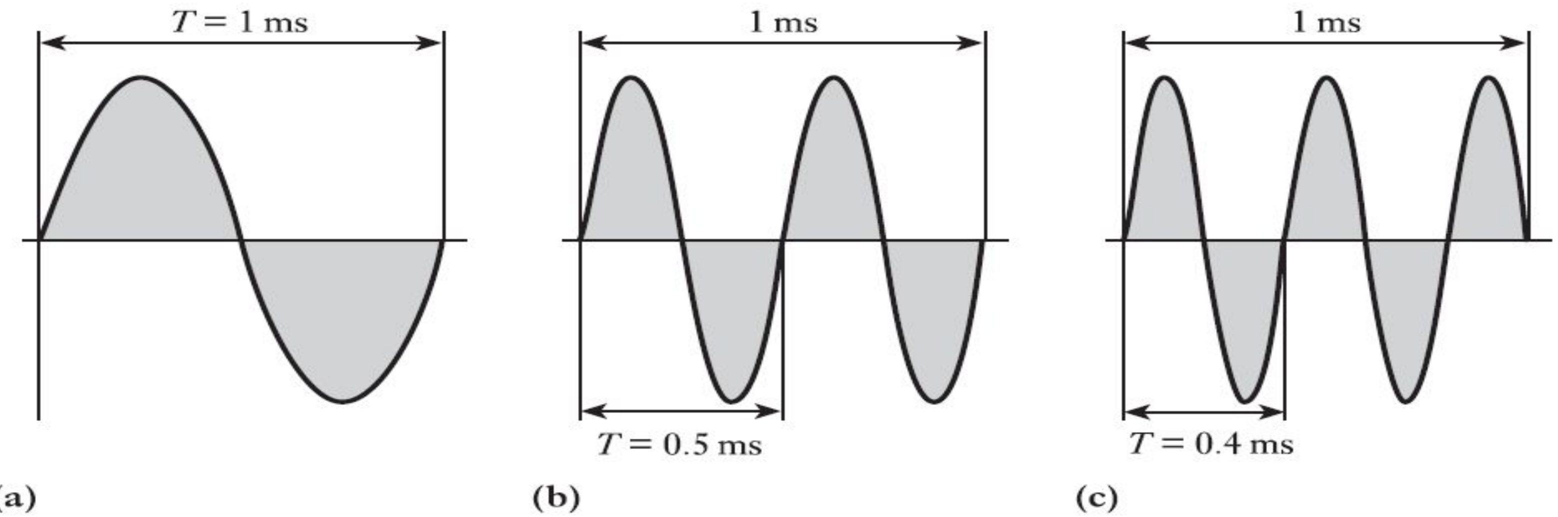
- **Frequency**: The number of cycles that occur in 1 second is termed the frequency of that quantity. Frequency is measured in hertz (Hz)

  Frequency f is related to the period T by the relation

  $$f = 1/ T$$

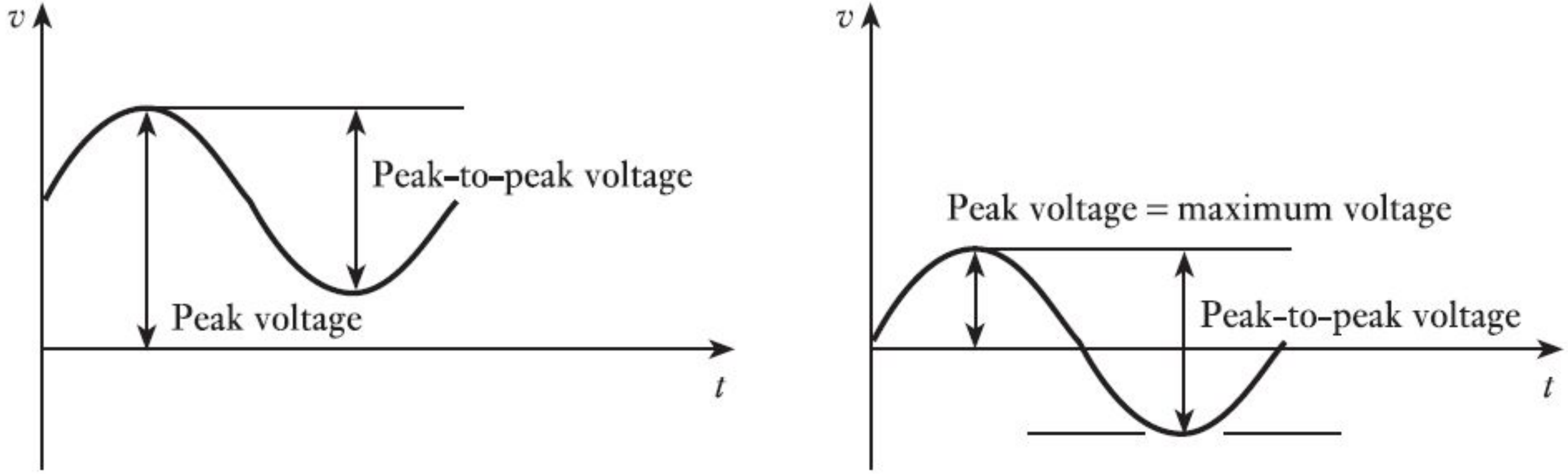
  where f is the frequency in hertz(Hz) and T is the period in seconds.

Fig. Cycles and periods,
Effect on waveforms by varying frequency



The diagrams assume frequencies of 1000 Hz (1 kHz), 2000 Hz (2 kHz) and 2500 Hz (2.5 kHz).

Fig. Peak values



Example 1

A coil of 100 turns is rotated at 1500 r/min in a magnetic field having a uniform density of 0.05 T, the axis of rotation being at right angles to the direction of the flux. The mean area per turn is 40 cm^2. Calculate

(a) the frequency;
(b) the period;
(c) the maximum value of the generated e.m.f.;
(d) the value of the generated e.m.f. when the coil has rotated through 30° from the position of zero e.m.f.

(a) Since the e.m.f. generated in the coil undergoes one cycle of variation when the coil rotates through one revolution,

$$\therefore \quad \text{Frequency} = \text{no. of cycles per second}$$

$$= \text{no. of revolutions per second}$$

$$= \frac{1500}{60} = \mathbf{25\ Hz}$$

(b) $$\text{Period} = \text{time of 1 cycle}$$

$$= \frac{1}{25} = \mathbf{0.04\ s}$$

(c) From expression [9.2]

$$E_m = 2\pi \times 0.05 \times 0.004 \times 100 \times 1500/60 = \mathbf{3.14\ V}$$

(d) For $\theta = 30°$, $\sin 30° = 0.5$,

$$\therefore \quad e = 3.14 \times 0.5 = \mathbf{1.57\ V}$$

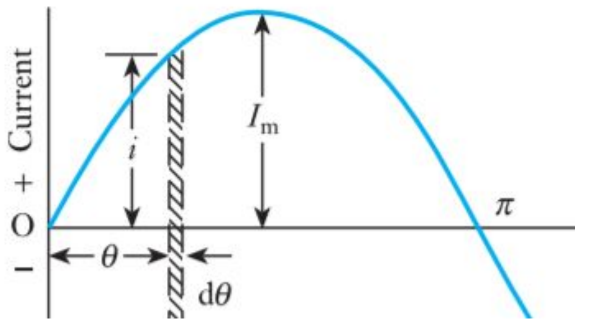
Average and r.m.s. values of sinusoidal currents and voltages

- If Im is the maximum value of a current which varies sinusoidally, the instantaneous value i is represented by,

  $i = I_m \sin\theta$

  where θ is the angle in radians from instant of zero current.

- Therefore, total area enclosed by the current wave over half-cycle



$$\int_0^{\pi} i \cdot d\theta = I_m \int_0^{\pi} \sin\theta \cdot d\theta = -I_m \left[\cos\theta\right]_0^{\pi}$$

$$= -I_m[-1-1] = 2I_m \text{ ampere radians}$$

- Average value of current over a half-cycle is,

$$\frac{2I_m \text{ [ampere radians]}}{\pi \text{ [radians]}}$$

i.e. $\boxed{I_{av} = 0.637 I_m}$ amperes

- RMS value of current over a half-cycle is,

$$\therefore \quad \boxed{I = \frac{I_m}{\sqrt{2}} = 0.707 I_m}$$

Example 2

An alternating voltage has the equation $v = 141.4 \sin 377t$; what are the values of:

(a) r.m.s. voltage;
(b) frequency;
(c) the instantaneous voltage when $t = 3$ ms?

The relation is of the form $v = V_m \sin \omega t$ and, by comparison,

(a) $V_m = 141.4 \text{ V} = \sqrt{2}V$

hence $V = \dfrac{141.4}{\sqrt{2}} = 100 \text{ V}$

(b) Also by comparison

$\omega = 377 \text{ rad/s} = 2\pi f$

hence $f = \dfrac{377}{2\pi} = 60 \text{ Hz}$

Example 2

(c) Finally

$$v = 141.4 \sin 377t$$

When $t = 3 \times 10^{-3}$ s

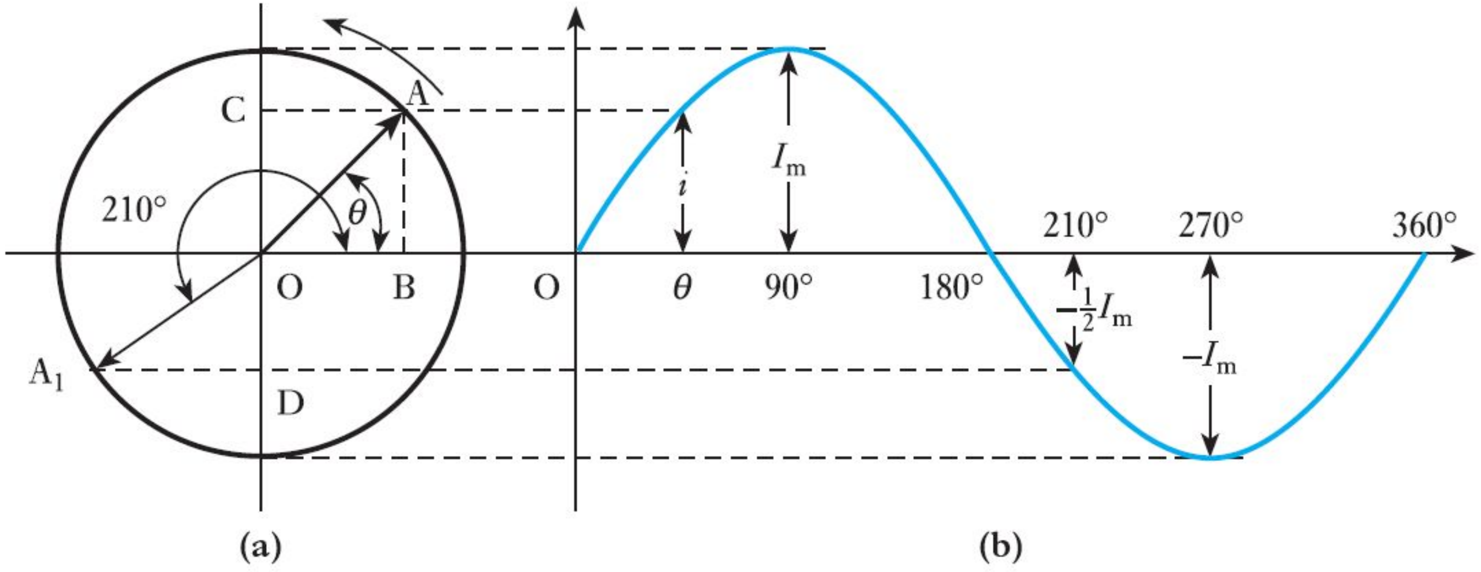
$$v = 141.4 \sin(377 \times 3 \times 10^{-3}) = 141.4 \sin 1.131$$

$$= 141.4 \times 0.904 = \mathbf{127.8\ V}$$

Note that, in this example, it was necessary to determine the sine of 1.131 rad, which could be obtained either from suitable tables, or from a calculator. Alternatively, 1.131 rad may be converted into degree measurement, i.e.

$$1.131 \text{ rad} \equiv 1.131 \times \frac{180}{\pi} = 64.8°$$

Representation of an alternating quantity by a phasor



AB and AC are drawn perpendicular to the horizontal and vertical axes respectively:

$OC = AB = OA \sin\theta$

$= I_m \sin\theta$

$= i$

i is the value of the current at that instant

Representation of an alternating quantity by a phasor

If *f* is the frequency in hertz, then OA rotates through *f* revolutions of *2π f* radians in 1 s. Hence the angular velocity of OA is *2π f* radians per second and is denoted by the symbol *ω (omega),* i.e.

$$\boxed{\omega = 2\pi f} \quad \text{radians per second}$$

If the time taken by OA to rotate through an angle θ radians is t seconds, then

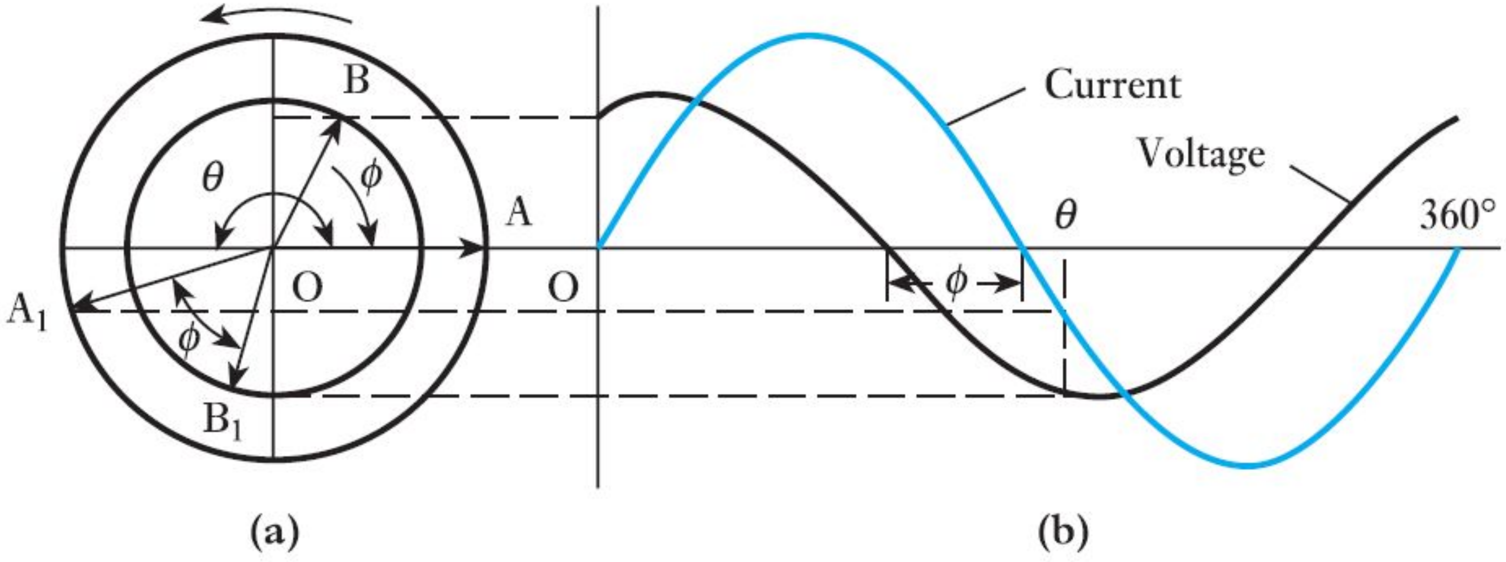
$$\theta = \text{angular velocity} \times \text{time}$$
$$= \omega t = 2\pi f t \text{ radians}$$

We can therefore express the instantaneous value of the current thus:

$$i = I_m \sin \theta = I_m \sin \omega t$$

$$\therefore \qquad \boxed{i = I_m \sin 2\pi f t} \qquad [9.20]$$

Phasor representation of quantities differing in phase



If the instantaneous value of the current is represented by

$$i = Im \sin\theta$$

then the instantaneous value of the voltage is represented by

$$v = Vm \sin(\theta + \phi)$$

where Im = OA and Vm = OB

Alternating current in a resistive circuit

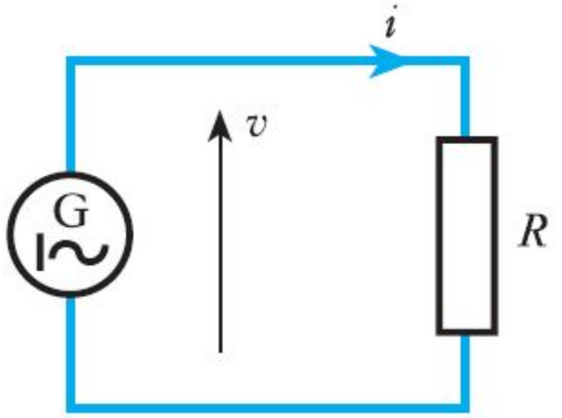
- Consider a circuit having a resistance R ohms connected across the terminals of an a.c. generator G, as in Fig. and suppose the alternating voltage to be represented by the sine wave

- If the value of the voltage at any instant is v volts, the value of the current at that instant is given by

$$i = \frac{v}{R} \text{ amperes}$$

- If Vm and Im are the maximum values of the voltage and current respectively, it follows that

$$\boxed{I_m = \frac{V_m}{R}} \quad \text{---------------[1]}$$



Alternating current in a resistive circuit

But the r.m.s. value of a sine wave is 0.707 times the maximum value, so that

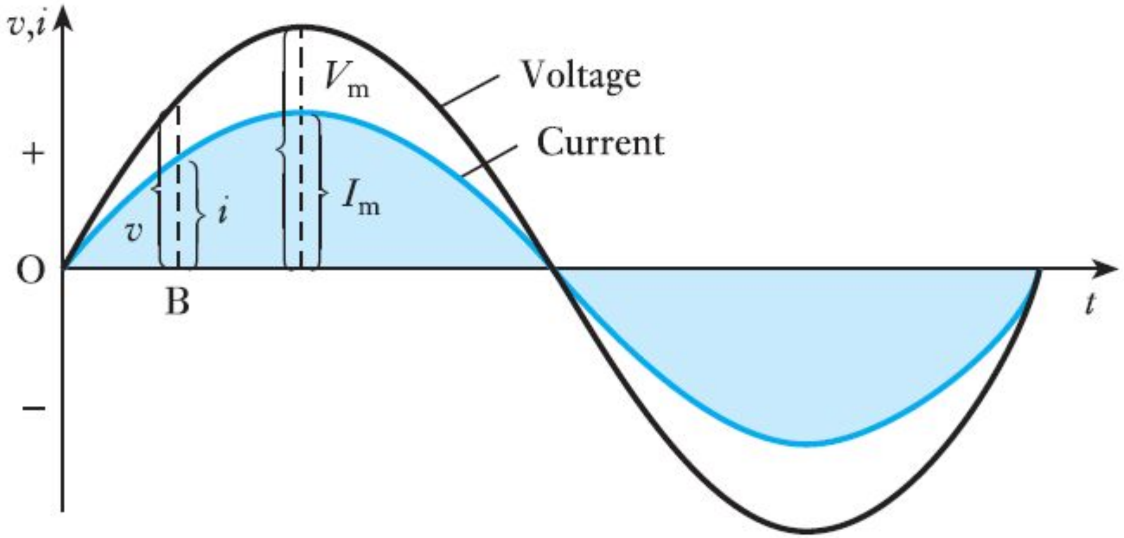
RMS value of voltage = *V = 0.707Vm*

and RMS value of current = *I = 0.707Im*

Substituting for Im and Vm in equation [1] we have

$$\frac{I}{0.707} = \frac{V}{0.707R}$$

$$\boxed{I = \frac{V}{R}}$$



Alternating current in a resistive circuit

Hence Ohm's law can be applied without any modification to an a.c. circuit possessing resistance only.
If the instantaneous value of the applied voltage is represented by

$$v = V_m \sin \omega t$$

then instantaneous value of current in a resistive circuit is

$$i = \frac{V_m}{R} \sin \omega t$$

Fig. 10.3 Phasor diagram for a resistive circuit

Alternating current in an inductive circuit

- Let us consider the effect of a sinusoidal current flowing through a coil having an inductance of L henrys and a negligible resistance, as in Fig.he e.m.f., in volts, induced in a coil is

$$e = L \times \text{rate of change of current in amperes per second}$$

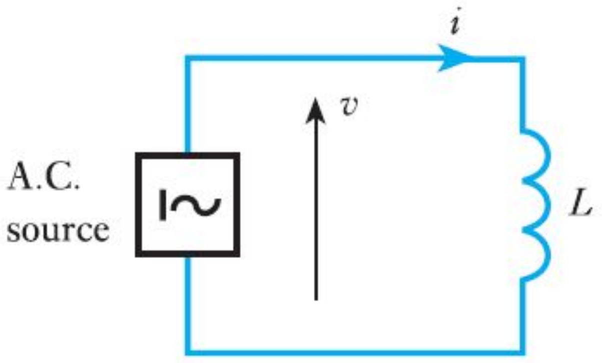
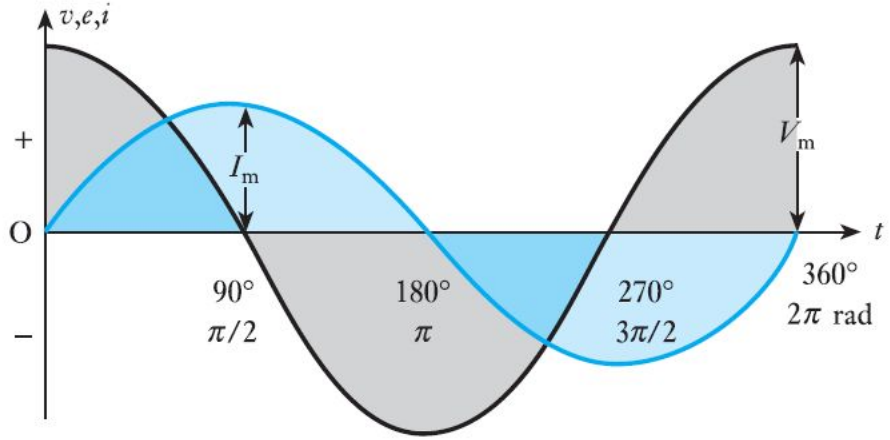


Fig. 10.4 Circuit with inductance only

Fig. 10.6 Voltage and current waveforms for a purely inductive circuit



Current and voltage in an inductive circuit

- Suppose the instantaneous value of the current through a coil having inductance L henrys and negligible resistance to be represented by

$$i = I_m \sin \omega t = I_m \sin 2\pi f t$$

where t is the time, in seconds, after the current has passed through zero from negative to positive values

- Suppose the current to increase by di amperes in dt seconds, then instantaneous value of induced e.m.f. is

$$e = L \cdot \frac{di}{dt}$$

$$= L I_m \frac{d}{dt}(\sin 2\pi f t)$$

$$= 2\pi f L I_m \cos 2\pi f t$$

$$e = 2\pi f L I_m \sin\left(2\pi f t + \frac{\pi}{2}\right)$$

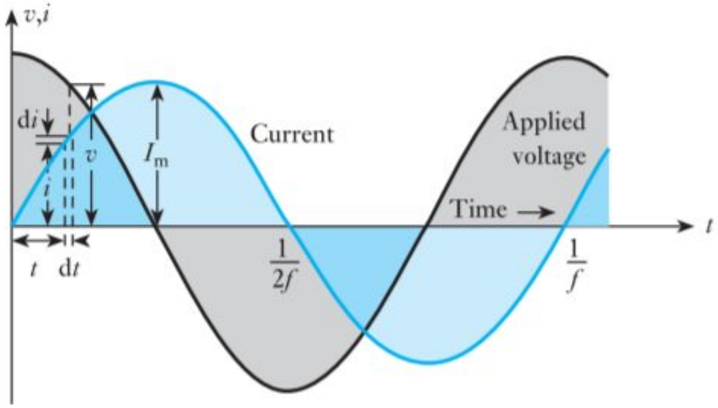
Current and voltage in an inductive circuit

$$t = 0, \quad \cos 2\pi ft = 1$$

$$\text{Induced e.m.f.} = 2\pi fLI_m$$

$$t = 1/(2f), \quad \cos 2\pi ft = \cos \pi = -1$$

$$\text{Induced e.m.f.} = -2\pi fLI_m$$



- Since the resistance of the circuit is assumed negligible, the whole of the applied voltage is equal to the induced e.m.f.,

$$v = e$$

$$= 2\pi fLI_m \cos 2\pi ft$$

$$v = 2\pi fLI_m \sin(2\pi ft + \pi/2)$$

Current and voltage in an inductive circuit

The maximum value Vm of the applied voltage is $2\pi fLI_m$, i.e.

$$V_m = 2\pi fLI_m \quad \text{so that} \quad \frac{V_m}{I_m} = 2\pi fL$$

If I and V are the r.m.s. values, then

$$\frac{V}{I} = \frac{0.707V_m}{0.707I_m} = 2\pi fL$$

$$= \textit{inductive reactance}$$

Inductive reactance Symbol: X_L Unit: ohm (Ω)

The inductive reactance is expressed in ohms and is represented by the symbol X_L. Hence

$$\boxed{I = \frac{V}{2\pi fL} = \frac{V}{X_L}} \qquad [10.7]$$

where $X_L = 2\pi fL$

Current and voltage in an inductive circuit

- The inductive reactance is proportional to the frequency and the current produced by a given voltage is inversely proportional to the frequency as shown in Fig. 10.8
- The phasor diagram for a purely inductive circuit is given in Fig.10.9 where E represents the r.m.s. value of the e.m.f. induced in the circuit, and V, equal to E, represents the r.m.s. value of the applied voltage.

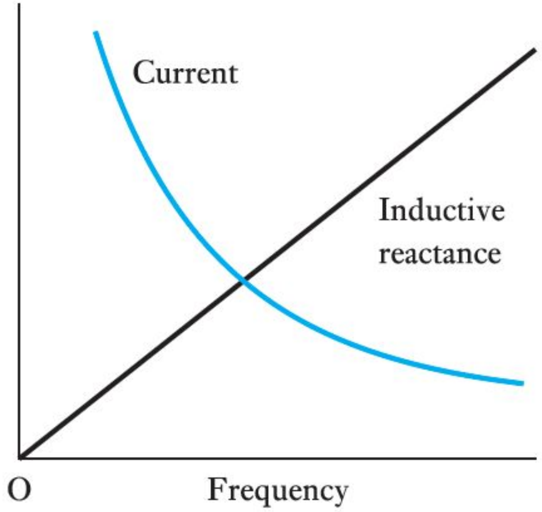


Fig. 10.8 Variation of reactance and current with frequency for a purely inductive circuit

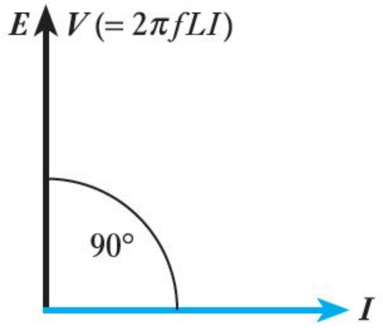
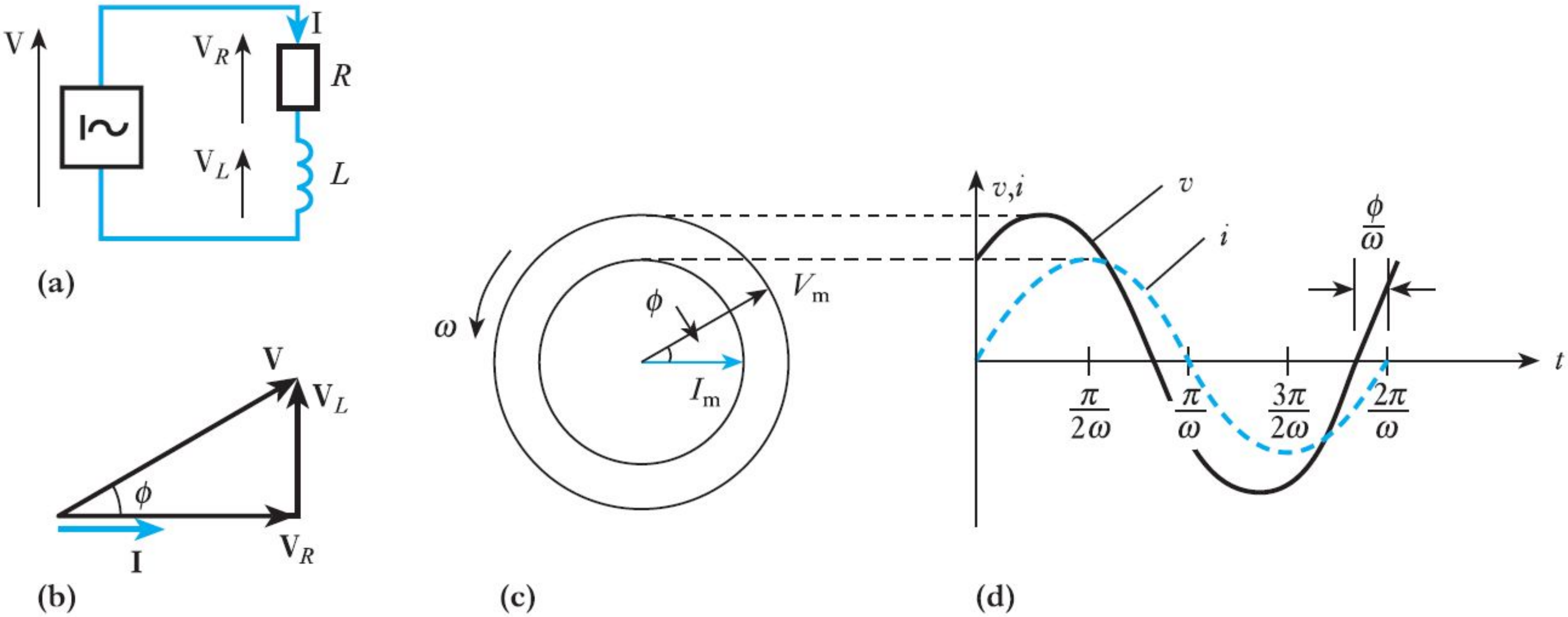


Fig. 10.9 Phasor diagram for a purely inductive circuit

Resistance and inductance in series

- Effect of Resistance and inductance connected in series.



(a) Circuit diagram; (b) phasor diagram; (c) instantaneous phasor diagram; (d) wave diagram

Resistance and inductance in series

The current is taken as reference since it is common to all the elements of a series circuit. The circuit voltage may then be derived from the following relations:

$$\mathbf{V}_R = \mathbf{I}R, \text{ where } \mathbf{V}_R \text{ is in phase with } \mathbf{I}$$

$$\mathbf{V}_L = \mathbf{I}X_L, \text{ where } \mathbf{V}_L \text{ leads } \mathbf{I} \text{ by } 90^\circ$$

$$\boxed{\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L} \quad \text{(phasor sum)}$$

Also,

$$\begin{aligned} V &= (V_R^2 + V_L^2)^{\frac{1}{2}} \\ &= (I^2R^2 + I^2X_L^2)^{\frac{1}{2}} \\ &= I(R^2 + X_L^2)^{\frac{1}{2}} \end{aligned}$$

Hence $\boxed{V = IZ}$ volts

where $Z = (R^2 + X_L^2)^{\frac{1}{2}}$

or $\boxed{Z = (R^2 + \omega^2L^2)^{\frac{1}{2}}}$ ohms

- Here Z is termed the impedance of the circuit.
- However, for any given frequency, the impedance is constant and hence Ohm's law also applies to a.c. circuit analysis.

- Impedance Symbol: Z
- Unit: ohm (Ω)

Resistance and inductance in series

Phase angle and is represented by ϕ.

$$\phi = \tan^{-1}\frac{V_L}{V_R} = \tan^{-1}\frac{IX_L}{IR}$$

$$\phi = \tan^{-1}\frac{X_L}{R}$$

Example 3

A resistance of 7.0 Ω is connected in series with a pure inductance of 31.8 mH and the circuit is connected to a 100 V, 50 Hz, sinusoidal supply (Fig. 10.13). Calculate:

(a) the circuit current;
(b) the phase angle.

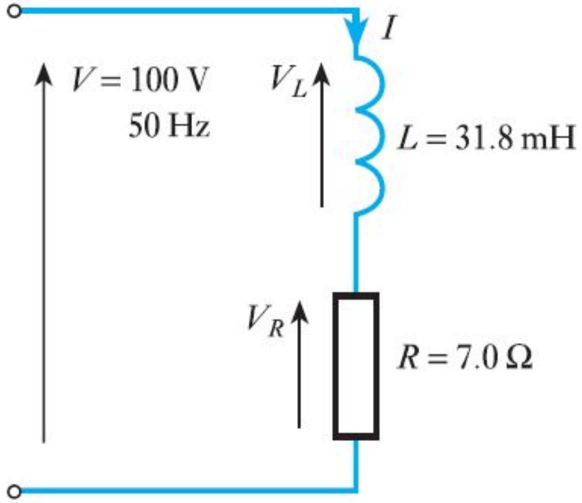


Fig. 10.13 Circuit diagram for Example 10.1

$$X_L = 2\pi fL = 2\pi 50 \times 31.8 \times 10^{-3} = 10.0\ \Omega$$

$$Z = (R^2 + X_L^2)^{\frac{1}{2}} = (7.0^2 + 10.0^2)^{\frac{1}{2}} = 12.2\ \Omega$$

$$I = \frac{V}{Z} = \frac{100}{12.2} = 8.2\ \mathrm{A}$$

$$\phi = \tan^{-1}\frac{X_L}{R} = \tan^{-1}\frac{10.0}{7.0} = 55.1° \text{ lag or } -55.1°$$

Example 4

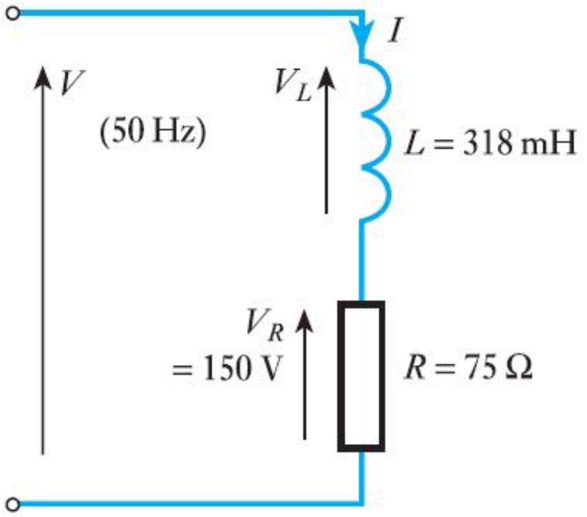


Fig. 10.14 Circuit diagram for Example 10.2

A pure inductance of 318 mH is connected in series with a pure resistance of 75 Ω. The circuit is supplied from a 50 Hz sinusoidal source and the voltage across the 75 Ω resistor is found to be 150 V (Fig. 10.14). Calculate the supply voltage.

$$V_R = 150 \text{ V}$$

$$I = \frac{V}{R} = \frac{150}{75} = 2 \text{ A}$$

$$X_L = 2\pi f L = 2\pi 50 \times 318 \times 10^{-3} = \mathbf{100\ \Omega}$$

$$V_L = IX_L = 2 \times 100 = 200 \text{ V}$$

$$V = (V_R^2 + V_L^2)^{\frac{1}{2}} = (150^2 + 200^2)^{\frac{1}{2}} = \mathbf{250\ V}$$

Alternatively

$$Z = (R^2 + X_L^2)^{\frac{1}{2}} = (75^2 + 100^2)^{\frac{1}{2}} = 125\ \Omega$$

$$V = IZ = 2 \times 125 = \mathbf{250\ V}$$

Alternating current in a capacitive circuit

- Figure shows a capacitor C connected in series with an ammeter A across the terminals of an a.c. source;
- The alternating voltage applied to C is positive when it makes plate D positive relative to plate E.
- If the capacitance is C farads, the charging current i is given by

$$i = C \cdot \text{rate of change of p.d.}$$

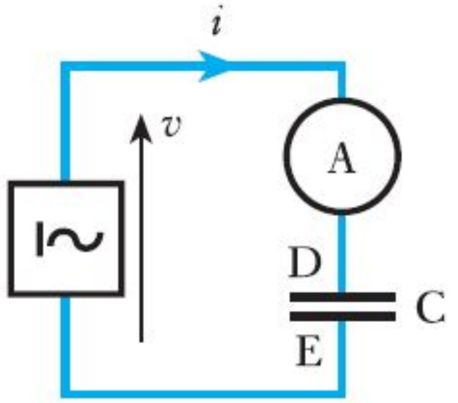


Fig. 10.17 Circuit with capacitance only

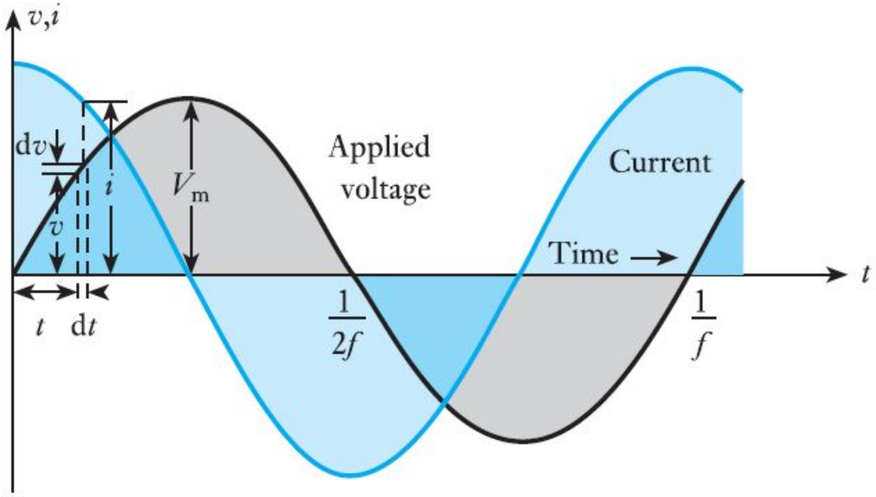


Fig. 10.18 Voltage and current waveforms for a purely capacitive circuit

Current and voltage in a capacitive circuit

Suppose that the instantaneous value of the voltage applied to a capacitor having capacitance C farads is represented by

$$v = V_m \sin \omega t = V_m \sin 2\pi f t$$

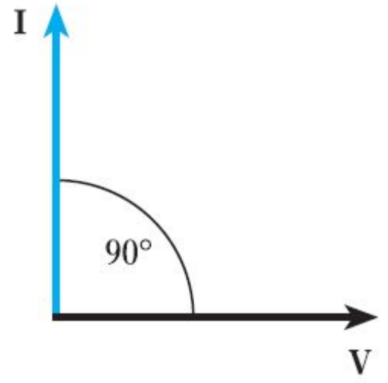


Fig. 10.19 Phasor diagram for a purely capacitive circuit

If the applied voltage increases by dv volts in dt seconds then, instantaneous value of current is

$$i = C\frac{dv}{dt}$$

$$= C\frac{d}{dt}(V_m \sin 2\pi f t)$$

$$= 2\pi f C V_m \cos 2\pi f t$$

$$i = 2\pi f C V_m \sin\left(2\pi f t + \frac{\pi}{2}\right)$$

Current and voltage in a capacitive circuit

$$\frac{V_m}{I_m} = \frac{1}{2\pi f C}$$

Hence, if I and V are the r.m.s. values

$$\frac{V}{I} = \frac{1}{2\pi f C} = \textit{capacitive reactance} \qquad [10.15]$$

The capacitive reactance is expressed in ohms and is represented by the symbol X_C. Hence

$$I = 2\pi f C V = \frac{V}{X_C}$$

$$\therefore \qquad X_C = \frac{1}{2\pi f C} \qquad [10.16]$$

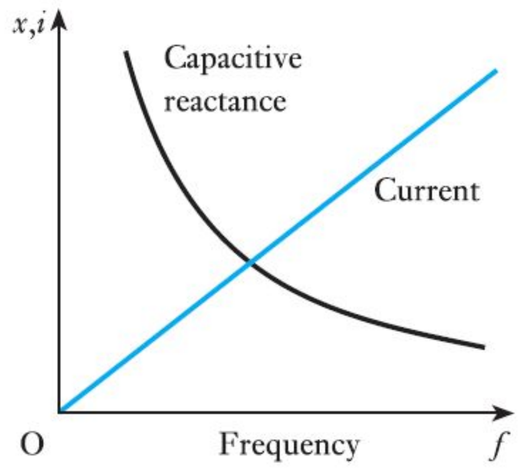


Fig. 10.20 Variation of reactance and current with frequency for a purely capacitive circuit

The capacitive reactance is inversely proportional to the frequency, and the current produced by a given voltage is proportional to the frequency, as shown in Fig. 10.20.

Capacitive reactance Symbol: X_C Unit: ohm (Ω)

Example 5

A 30 μF capacitor is connected across a 400 V, 50 Hz supply. Calculate:

(a) the reactance of the capacitor;
(b) the current.

(a) From expression [10.16]:

$$\text{reactance } X_C = \frac{1}{2 \times 3.14 \times 50 \times 30 \times 10^{-6}} = \mathbf{106.2\ \Omega}$$

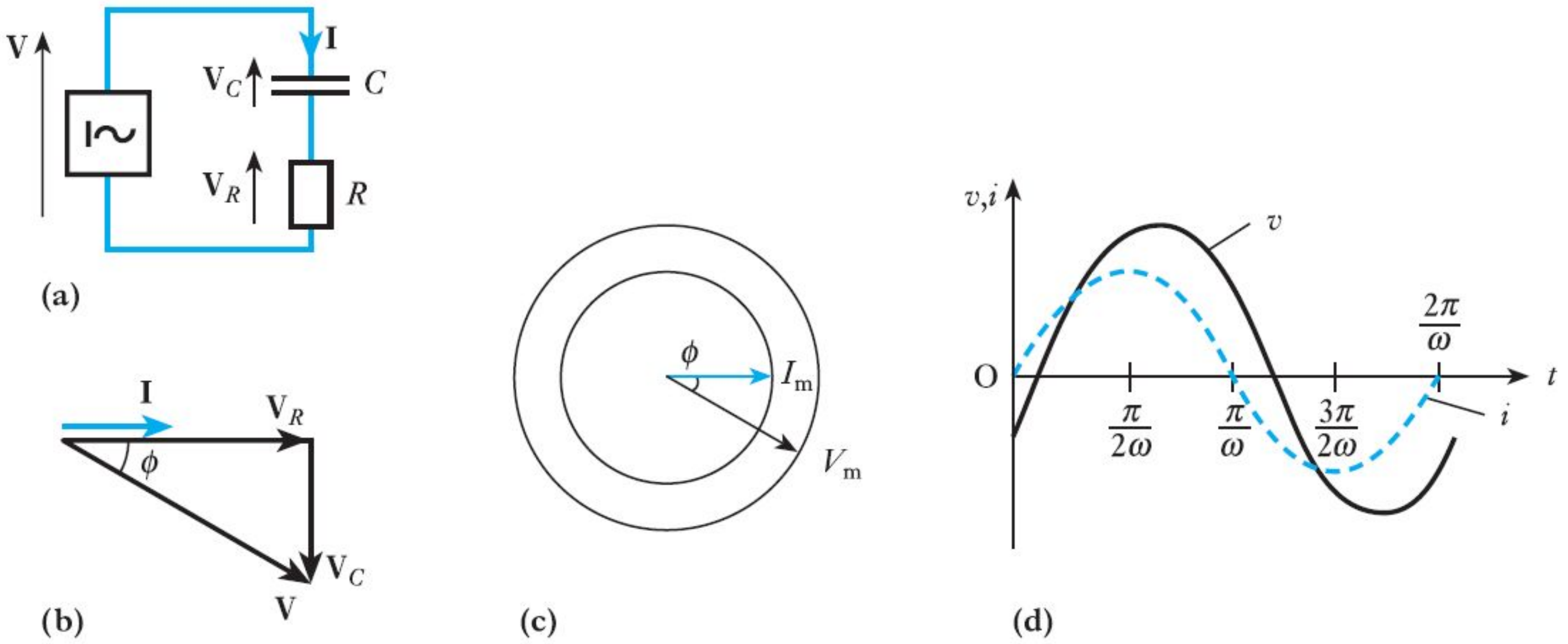
(b) From expression [10.15]:

$$\text{Current} = \frac{400}{106.2} = \mathbf{3.77\ A}$$

Resistance and capacitance in series

- The effect of connecting resistance and capacitance in series is illustrated in Fig.
- The current is again taken as reference.

Fig. 10.22 Resistance and capacitance in series. (a) Circuit diagram; (b) phasor diagram; (c) instantaneous phasor diagram; (d) wave diagram



Resistance and capacitance in series

- The circuit voltage is derived from the following relations:

$$\mathbf{V}_R = \mathbf{I}R, \text{ where } \mathbf{V}_R \text{ is in phase with } \mathbf{I}$$

$$\mathbf{V}_C = \mathbf{I}X_C, \text{ where } \mathbf{V}_C \text{ lags } \mathbf{I} \text{ by } 90^\circ$$

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_C \quad \text{(phasor sum)}$$

Also

$$\begin{aligned} V &= (V_R^2 + V_C^2)^{\frac{1}{2}} \\ &= (I^2R^2 + I^2X_C^2)^{\frac{1}{2}} \\ &= I(R^2 + X_C^2)^{\frac{1}{2}} \end{aligned}$$

Hence

$$V = IZ$$

where

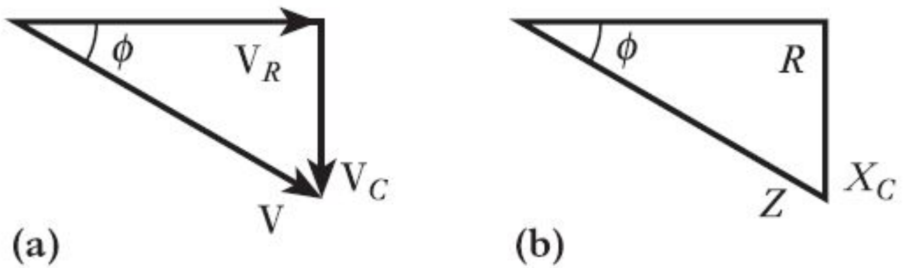
$$\boxed{Z = (R^2 + X_C^2)^{\frac{1}{2}}}$$

and

$$Z = \left(R^2 + \frac{1}{\omega^2 C^2}\right)^{\frac{1}{2}}$$

Resistance and capacitance in series

Fig. 10.23 Voltage and impedance diagrams.
(a) Voltage diagram;
(b) impedance diagram



By the geometry of the diagram:

$$\phi = \tan^{-1}\frac{V_C}{V_R} = \tan^{-1}\frac{IX_C}{IR}$$

$$\boxed{\phi = \tan^{-1}\frac{X_C}{R}}$$

Example 6

- A capacitor of 8.0 μF takes a current of 1.0 A when the alternating voltage applied across it is 230 V. Calculate:

(a) the frequency of the applied voltage;

(b) the resistance to be connected in series with the capacitor to reduce the current in the circuit to 0.5 A at the same frequency;

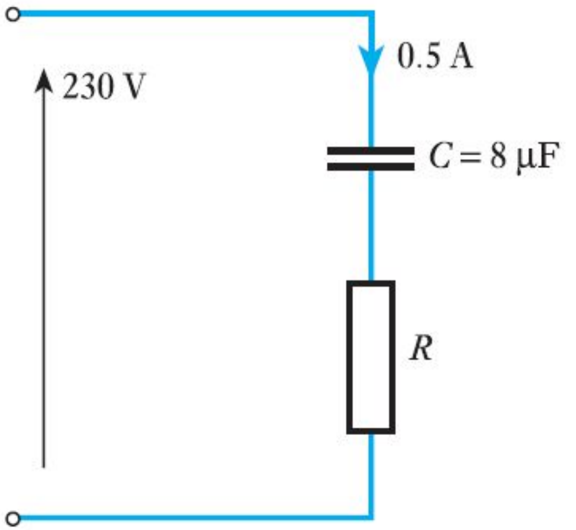


Fig. 10.24 Circuit diagram for Example 10.5

(a) $$X_C = \frac{V}{I} = \frac{230}{1.0} = 230\ \Omega$$

$$= \frac{1}{2\pi f C}$$

$$\therefore \quad f = \frac{1}{2\pi C X_C} = \frac{1}{2\pi \times 8 \times 10^{-6} \times 230} = 86.5\ \text{Hz}$$

(b) When a resistance is connected in series with the capacitor, the circuit is now as given in Fig. 10.24.

$$Z = \frac{V}{I} = \frac{230}{0.5} = 460\ \Omega$$

$$= (R^2 + X_C^2)^{\frac{1}{2}}$$

but $\quad X_C = 230\ \Omega$

hence $\quad R = 398\ \Omega$

Alternating current in an RLC circuit

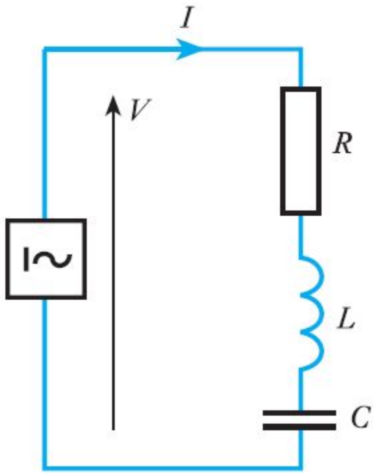


Fig. 10.25 Circuit with R, L and C in series

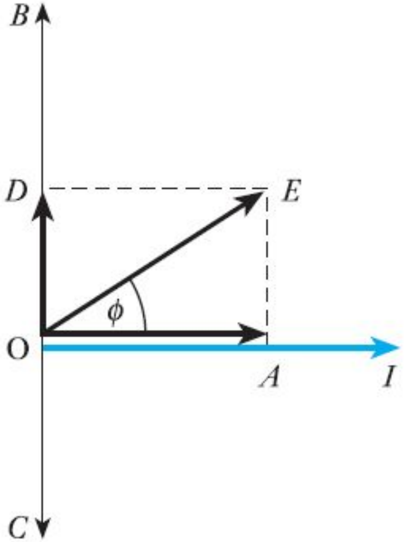


Fig. 10.26 Phasor diagram for Fig. 10.25

$$\tan\phi = \frac{AE}{OA} = \frac{OD}{OA} = \frac{OB - OC}{OA} = \frac{2\pi fLI - I/(2\pi fC)}{RI}$$

$$= \frac{\text{inductive reactance} - \text{capacitive reactance}}{\text{resistance}}$$

$$\therefore \quad \boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

$$OE^2 = OA^2 + OD^2 = OA^2 + (OB - OC)^2$$

$$\therefore \quad V^2 = (RI)^2 + \left(2\pi fLI - \frac{I}{2\pi fC}\right)^2$$

so that

$$\boxed{I = \frac{V}{\sqrt{\left\{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2\right\}}} = \frac{V}{Z}}$$

where Z = *impedance* of circuit in ohms

$$\boxed{Z = \frac{V}{I} = \sqrt{\left\{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2\right\}}}$$

From this expression it is seen that

$$\text{Resultant reactance} = 2\pi fL - \frac{1}{2\pi fC}$$

= inductive reactance – capacitive reactance

Example 7

- A coil having a resistance of 12 Ω and an inductance of 0.1 H is connected across a 100 V, 50 Hz supply. Calculate:

(a) the reactance and the impedance of the coil;

(b) the current;

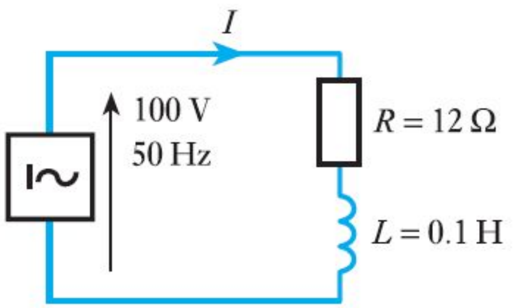


Fig. 10.27 Circuit diagram for Example 10.6

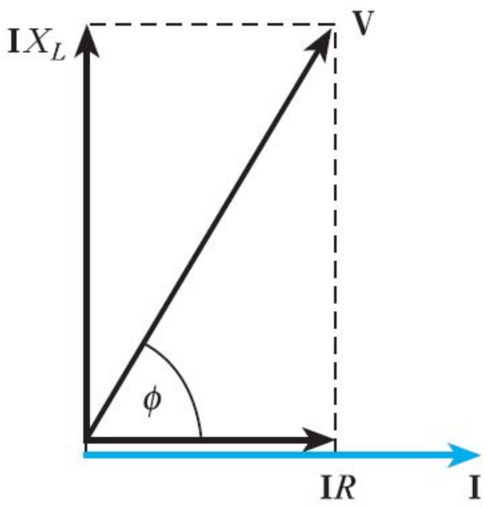


Fig. 10.28 Phasor diagram for Example 10.6

(a) $\text{Reactance} = X_L = 2\pi fL$

$= 2\pi \times 50 \times 0.1 = \mathbf{31.4\ \Omega}$

$\text{Impedance} = Z = \sqrt{(R^2 + X_L^2)}$

$= \sqrt{(12^2 + 31.4^2)} = \mathbf{33.6\ \Omega}$

(b) $\text{Current} = I = \dfrac{V}{Z} = \dfrac{100}{33.6} = \mathbf{2.97\ A}$

(c) $\tan\phi = \dfrac{X}{R} = \dfrac{31.4}{12} = 2.617$

$\therefore \quad \phi = \mathbf{69°}$

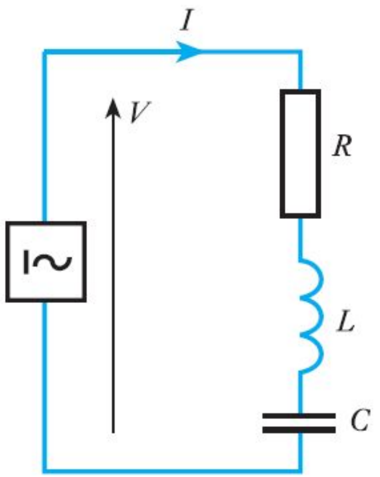
Example 8

- A circuit having a resistance of 12 Ω, an inductance of 0.15 H and a capacitance of 100 μF in series, is connected across a 100 V, 50 Hz supply. Calculate:

  (a) the impedance;

  (b) the current;

  (c) the voltages across R, L and C;



(a) From equation [10.21],

$$Z = \sqrt{\left\{(12)^2 + \left(2 \times 3.14 \times 50 \times 0.15 - \frac{10^6}{2 \times 3.14 \times 50 \times 100}\right)^2\right\}}$$

$$= \sqrt{\{144 + (47.1 - 31.85)^2\}} = \mathbf{19.4\ \Omega}$$

(b) Current $= \dfrac{V}{Z} = \dfrac{100}{19.4} = \mathbf{5.15\ A}$

Example 8

(c) Voltage across $R = V_R = 12 \times 5.15 = 61.8$ V

Voltage across $L = V_L = 47.1 \times 5.15 = 242.5$ V

and Voltage across $C = V_C = 31.85 \times 5.15 = 164.0$ V

Simple Parallel Circuits

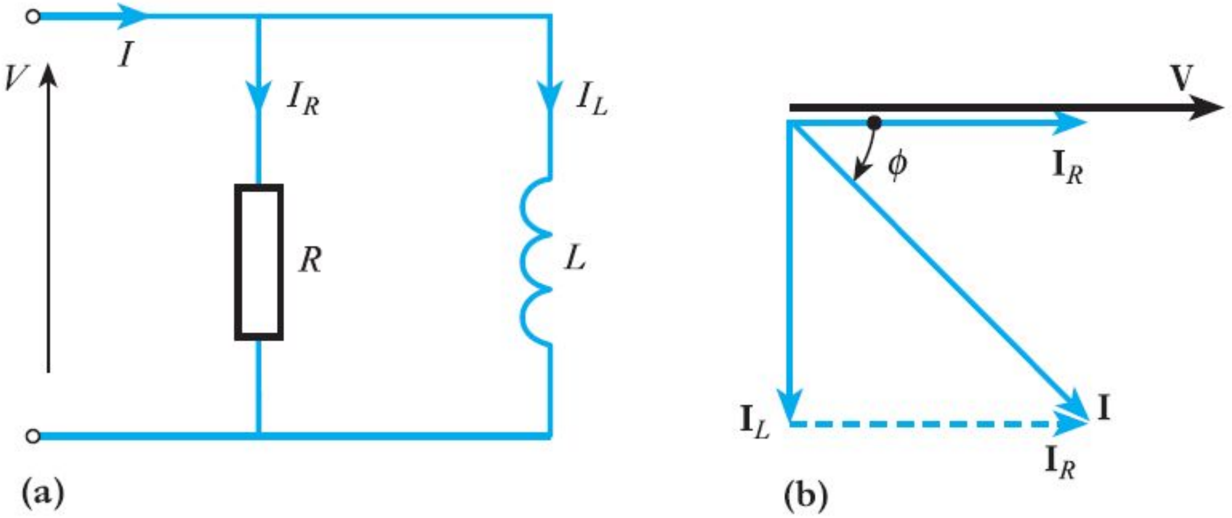


Fig. (a) Circuit diagram; (b) phasor diagram

In the inductive branch, the current is given by

$$I_L = \frac{V}{X_L}, \text{ where } \mathbf{I}_L \text{ lags } \mathbf{V} \text{ by } 90°$$

In the resistive branch, the current is given by

$$I_R = \frac{V}{R}, \text{ where } I_R \text{ and } V \text{ are in phase}$$

The total supply current *I* is obtained by adding the branch currents

$$I = I_R + I_L \text{ (phasor sum)}$$

Simple Parallel Circuits

From the complexor diagram:

$$I = (I_R^2 + I_L^2)^{\frac{1}{2}}$$

$$= \left\{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_L}\right)^2\right\}^{\frac{1}{2}}$$

$$= V\left(\frac{1}{R^2} + \frac{1}{X_L^2}\right)^{\frac{1}{2}}$$

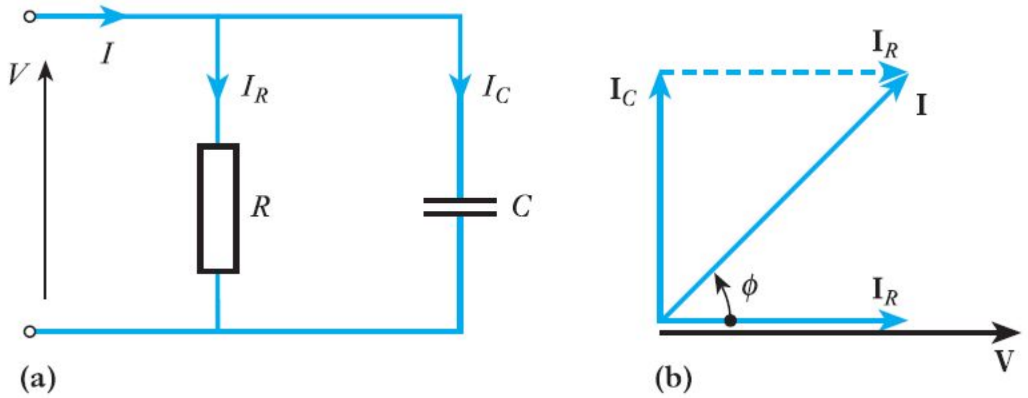
$$\frac{V}{I} = Z = \frac{1}{\left(\frac{1}{R^2} + \frac{1}{X_L^2}\right)^{\frac{1}{2}}}$$

It can be seen from the phasor diagram that the phase angle ϕ is a lagging angle.

$$\phi = \tan^{-1}\frac{I_L}{I_R} = \tan^{-1}\frac{R}{X_L} = \tan^{-1}\frac{R}{\omega L}$$

Simple Parallel Circuits

- Resistance and capacitance in parallel. (a) Circuit diagram; (b) phasor diagram



In the resistive branch, the current is given by

$$I_R = \frac{V}{R}, \text{ where } \mathbf{I}_R \text{ and } \mathbf{V} \text{ are in phase}$$

In the capacitive branch, the current is given by

$$I_C = \frac{V}{X_C}, \text{ where } \mathbf{I}_C \text{ leads } \mathbf{V} \text{ by } 90^\circ$$

The phasor diagram is constructed in the usual manner based on the relation

$$I = I_R + I_C \qquad \text{(phasor sum)}$$

Simple Parallel Circuits

From the phasor diagram:

$$I = (I_R^2 + I_C^2)^{\frac{1}{2}}$$

$$= \left\{\left(\frac{V}{R}\right)^2 + \left(\frac{V}{X_C}\right)^2\right\}^{\frac{1}{2}}$$

$$= V\left(\frac{1}{R^2} + \frac{1}{X_C^2}\right)^{\frac{1}{2}}$$

$$\frac{V}{I} = Z = \frac{1}{\left(\frac{1}{R^2} + \frac{1}{X_C^2}\right)^{\frac{1}{2}}}$$

- The phase angle ϕ is a leading angle. It follows that parallel circuits behave in a similar fashion to series circuits in that the combination of resistance with inductance produces a lagging circuit while the combination of resistance with capacitance gives rise to a leading circuit.

$$\phi = \tan^{-1}\frac{I_C}{I_R} = \tan^{-1}\frac{R}{X_C} = \tan^{-1} R\omega C$$

Also $\quad \phi = \cos^{-1}\frac{I_R}{I}$

$\therefore \quad \phi = \cos^{-1}\frac{Z}{R}$

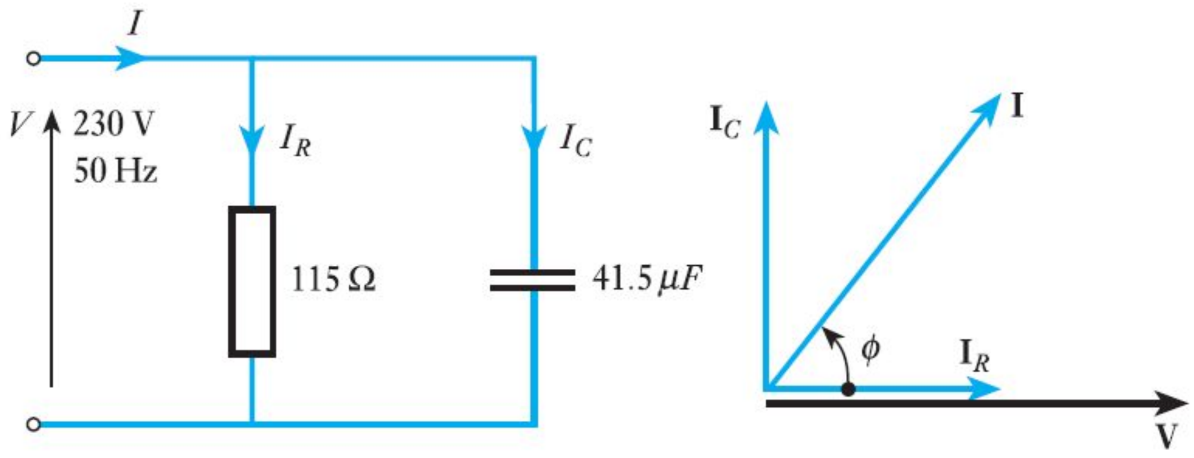
Example 9

- A circuit consists of a 115 Ω resistor in parallel with a 41.5 μF capacitor and is connected to a 230 V, 50 Hz supply (Fig. 11.3). Calculate:

  (a) the branch currents and the supply current;

  (b) the circuit phase angle;

  (c) the circuit impedance.



$$I_R = \frac{V}{R} = \frac{230}{115} = \mathbf{2.0\ A}$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi 50 \times 41.5 \times 10^{-6}} = \mathbf{76.7\ \Omega}$$

$$I_C = \frac{V}{X_C} = \frac{230}{76.7} = \mathbf{3.0\ A}$$

$$I = (I_R^2 + I_C^2)^{\frac{1}{2}} = (2.0^2 + 3.0^2)^{\frac{1}{2}} = \mathbf{3.6\ A}$$

$$\phi = \cos^{-1}\frac{I_R}{I} = \cos^{-1}\frac{2.0}{3.6} = \mathbf{56.3^\circ\ lead}$$

$$Z = \frac{V}{I} = \frac{230}{3.6} = \mathbf{63.9\ \Omega}$$

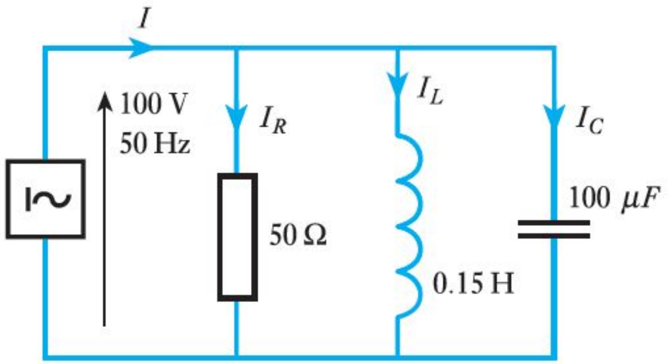
Example 10

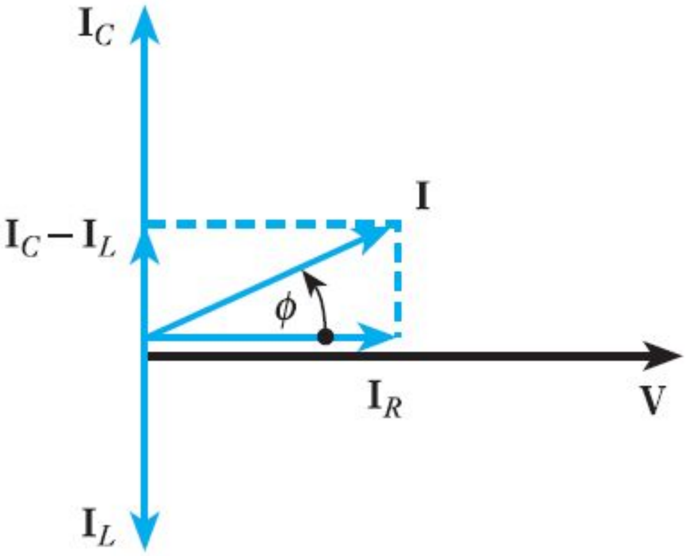
- Three branches, possessing a resistance of 50 Ω, an inductance of 0.15 H and a capacitance of 100 μF respectively, are connected in parallel across a 100 V, 50 Hz supply. Calculate:

(a) the current in each branch;

(b) the supply current;

(c) the phase angle between the supply current and the supply voltage.





(a) The circuit diagram is given in Fig. 11.4, where I_R, I_L and I_C represent the currents through the resistance, inductance and capacitance respectively.

$$I_R = \frac{100}{50} = \mathbf{2.0\ A}$$

$$I_L = \frac{100}{2 \times 3.14 \times 50 \times 0.15} = \mathbf{2.12\ A}$$

and $$I_C = 2 \times 3.14 \times 50 \times 100 \times 10^{-6} \times 100 = \mathbf{3.14\ A}$$

Example 10

(b) The capacitor and inductor branch currents are in antiphase, hence the resultant of I_C and I_L is

$$I_C - I_L = 3.14 - 2.12$$
$$= 1.02\ \text{A, leading by } 90°$$

The current I taken from the supply is the resultant of I_R and $(I_C - I_L)$, and from Fig. 11.5:

$$I^2 = I_R^2 + (I_C - I_L)^2 = 2^2 + (1.015)^2 = 5.03$$

$$\therefore \qquad I = \mathbf{2.24\ A}$$

(c) From Fig. 11.5:

$$\cos\phi = \frac{I_R}{I} = \frac{2}{2.24} = 0.893$$

$$\phi = 26°45'$$

Since I_C is greater than I_L, the supply current leads the supply voltage by **26°45′**.

Admittance and conductance

- When resistors having resistances R1, R2, etc. are in parallel, the equivalent resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots$$

- In d.c. work the reciprocal of the resistance is known as conductance. It is represented by symbol G and the unit of conductance is the *siemens*. Hence, if circuits having conductances G1, G2, etc. are in parallel, the total conductance G is given by

G = G1 + G2 + . . .

- In a.c. work the conductance is the reciprocal of the resistance only when the circuit possesses no reactance.

- If circuits having impedances Z1, Z2, etc. are connected in parallel across a supply voltage V, then

$$I_1 = \frac{V}{Z_1} \quad I_2 = \frac{V}{Z_2}, \quad \text{etc.}$$

Admittance, conductance

- If Z is the equivalent impedance of Z1, Z2, etc. in parallel and if I is the resultant current, then, using complex notation, we have

  I = I1 + I2 + . . .

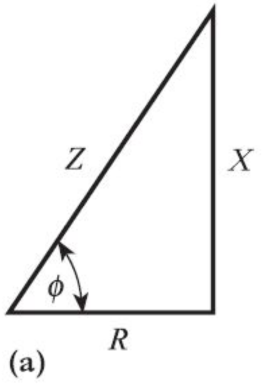
$$\therefore \qquad \frac{\mathbf{V}}{\mathbf{Z}} = \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} + \ldots$$

so that $$\boxed{\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \ldots}$$

- The reciprocal of impedance is termed admittance and is represented by the symbol Y, the unit being again the siemens (abbreviation, S). Hence,

  Y = Y1 + Y2 + . . .

RL series circuit admittance



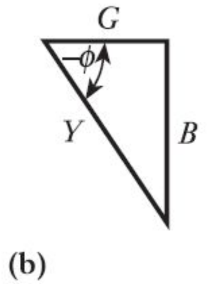


Fig. 13.10 (a) Impedance and (b) admittance triangles

$$\mathbf{V} = \mathbf{I}R + \mathrm{j}\mathbf{I}X_L$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + \mathrm{j}X_L$$

The method of transferring the j term from the denominator to the numerator is known as 'rationalizing'; thus

$$\frac{1}{a + \mathrm{j}b} = \frac{a - \mathrm{j}b}{(a + \mathrm{j}b)(a - \mathrm{j}b)} = \frac{a - \mathrm{j}b}{a^2 + b^2} \qquad [13.6]$$

If Y is the admittance of the circuit, then

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R + \mathrm{j}X_L} = \frac{R - \mathrm{j}X_L}{R^2 + X_L^2}$$

$$= \frac{R}{R^2 + X_L^2} - \frac{\mathrm{j}X_L}{R^2 + X_L^2} = G - \mathrm{j}B_L$$

$$\therefore \qquad \mathbf{Y} = G - \mathrm{j}B_L \qquad [13.7]$$

RC series circuit admittance

$$\mathbf{V} = \mathbf{I}R - \mathrm{j}\mathbf{I}X_C$$

$$\therefore \qquad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R - \mathrm{j}X_C$$

$$\text{and} \qquad \mathbf{Y} = \frac{1}{R - \mathrm{j}X_C} = \frac{R + \mathrm{j}X_C}{R^2 + X_C^2} = \frac{R}{R^2 + X_C^2} + \frac{\mathrm{j}X_C}{R^2 + X_C^2} = G + \mathrm{j}B_C$$

Parallel admittance

(a) Inductive reactance

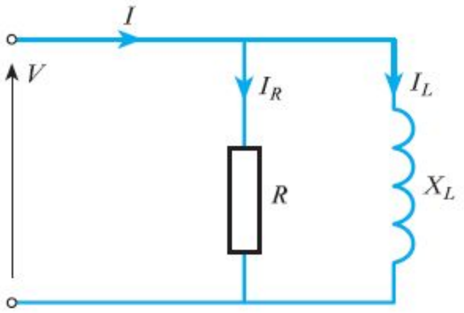


Fig. 13.11 R and L in parallel

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L = \frac{\mathbf{V}}{R} - \frac{\mathrm{j}\mathbf{V}}{X_L}$$

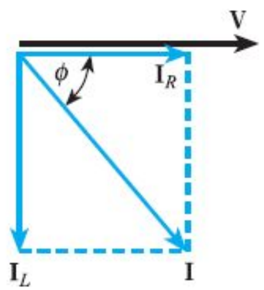


Fig. 13.12 Phasor diagram for Fig. 13.11

$$\therefore \qquad \boxed{\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} - \frac{\mathrm{j}}{X_L} = G - \mathrm{j}B_L} \qquad [13.12]$$

(b) Capacitive reactance

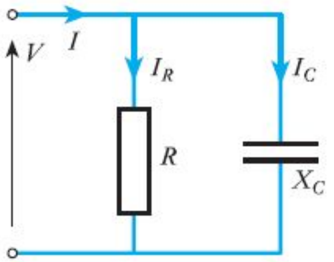


Fig. 13.13 R and C in parallel

From Figs 13.13 and 13.14 it follows that

$$\mathbf{I} = \mathbf{I}_R + \mathbf{I}_C = \frac{\mathbf{V}}{R} + \frac{\mathrm{j}\mathbf{V}}{X_C}$$

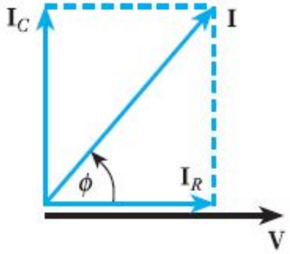


Fig. 13.14 Phasor diagram for Fig. 13.13

$$\therefore \qquad \boxed{\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + \frac{\mathrm{j}}{X_C} = G + \mathrm{j}B_C} \qquad [13.13]$$

Power and voltamperes

- The voltage drop across the resistor is in phase with the current and equal to VR = IR.
- The voltage drop across the inductor is equal to the current multiplied by the reactance of the inductor. The current lags this voltage drop by 90°.
- The reactance of the inductor is given by $X_L = 2\pi fL$
- hence the magnitude of the voltage across the inductor is $V_L = IX_L$.
- The impedance of the inductor is $Z_L = jX_L$; the phasor representing the voltage across the inductor is therefore the current phasor multiplied by the impedance, i.e. $V_L = jX_L I$. The j term produces a rotation of 90° in the complex plane; the voltage across the inductor therefore leads the current by 90°.

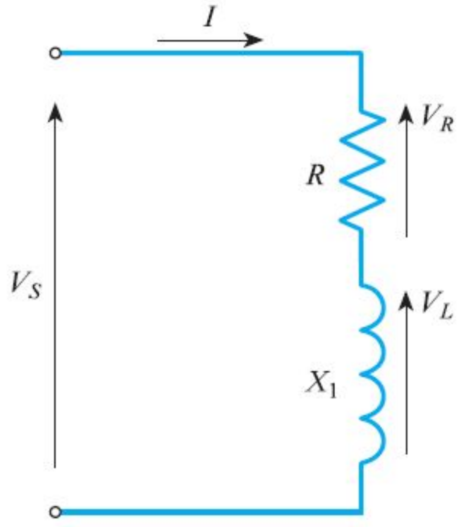


Fig. 13.20 Series *RL* circuit

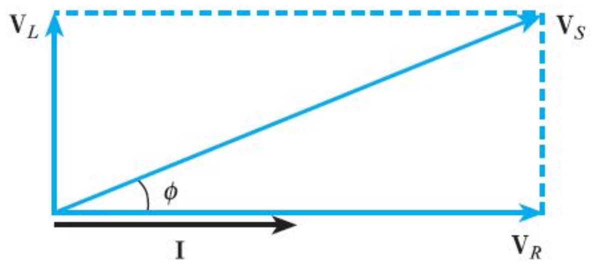
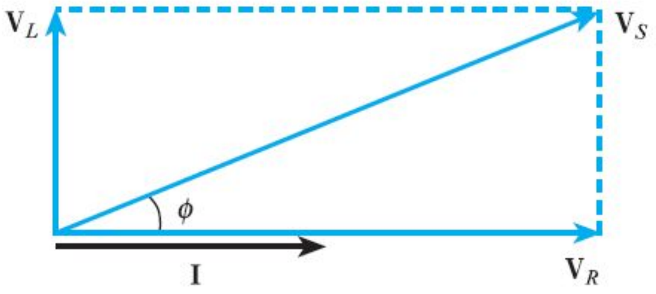


Fig. 13.21 Series *RL* circuit – phasor diagram

Power and voltamperes

Fig. 13.21 Series RL circuit – phasor diagram

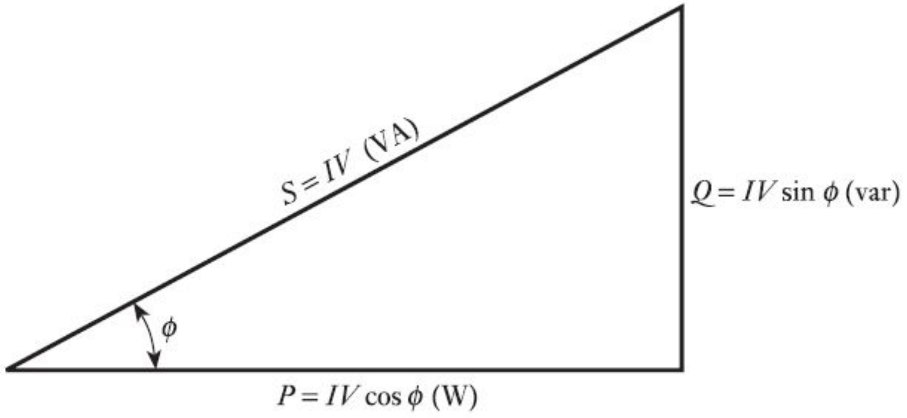


If each of the current phasors (I, $I \cos \phi$ and $I \sin \phi$) is multiplied by the applied voltage V, then the *power triangle* is obtained, shown in Fig. 13.22.

The component $VI \cos \phi$ is the *real* or *active power* and has the units of watts; symbol P.

The component $VI \sin \phi$ is called the *reactive power* and is referred to as 'voltamperes reactive' or 'var'; symbol Q.

Fig. 13.22 Power triangle



Frequency variation in a series RLC circuit

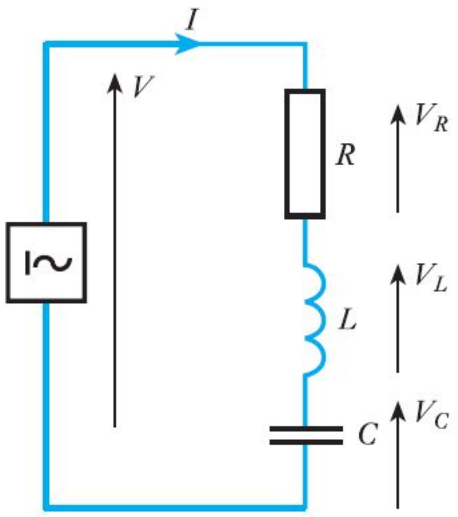


Fig. 14.1 Circuit with R, L and C in series

The impedance Z of this circuit is given by

$$Z = \sqrt{\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}}$$

The value of the reactance X of the circuit

$\omega L - 1/(\omega C)$ (i.e. inductive reactance − capacitive reactance) will depend on frequency.

For the inductive reactance:

$|X_L| = \omega L = 2\pi fL$

which will increase with frequency.

For the capacitive reactance:

$$|X_C| = \frac{1}{\omega C} = \frac{1}{2\pi fC}$$

which is largest at low frequencies.

Frequency variation in a series RLC circuit

- at frequency fr, |XL| = |XC | so the impedance Z, is purely resistive;
- below fr, |XL| < |XC | so the circuit is capacitive;
- above fr, |XL| > |XC | so the circuit is inductive.

Fig. 14.2 Inductive reactance increases linearly with frequency

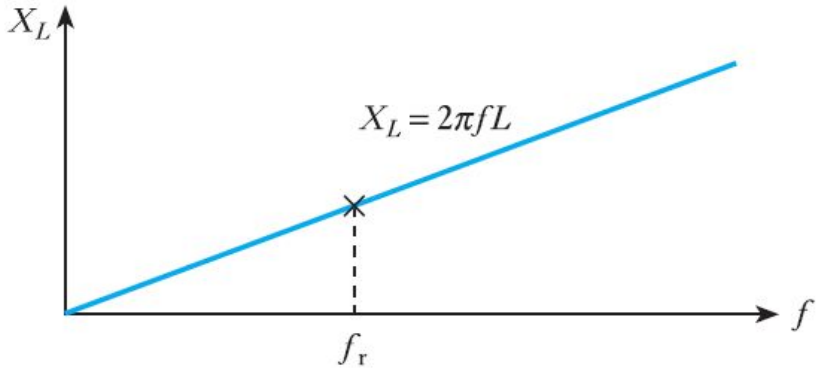
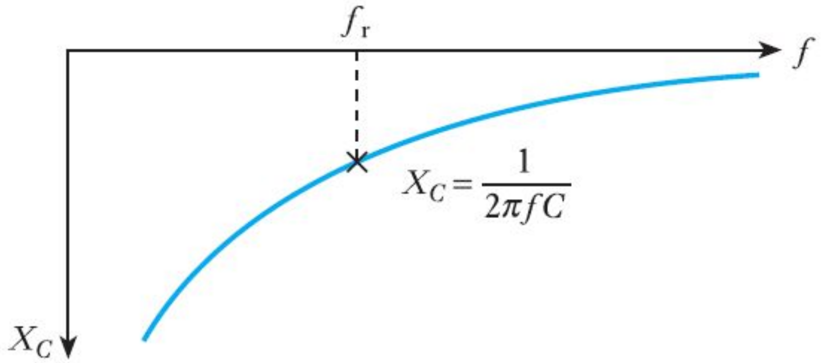


Fig. 14.3 Capacitive reactance decreases with frequency



Frequency variation in a series RLC circuit

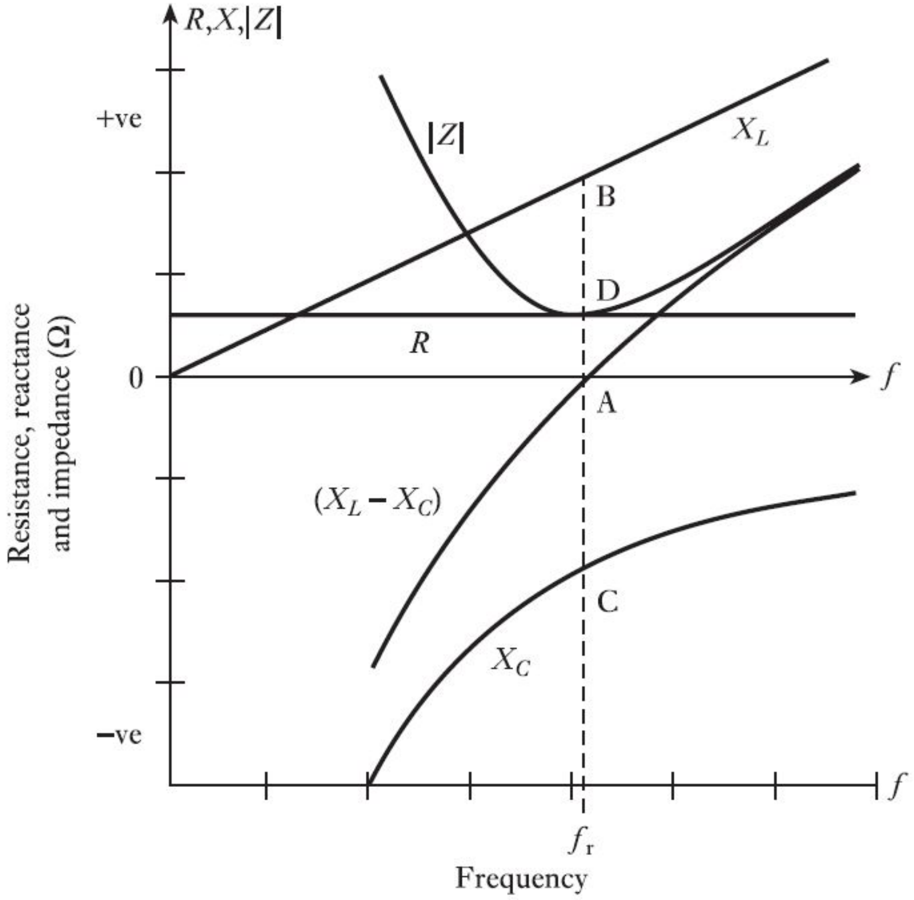


Fig. 14.4 Variation of reactance and impedance with frequency

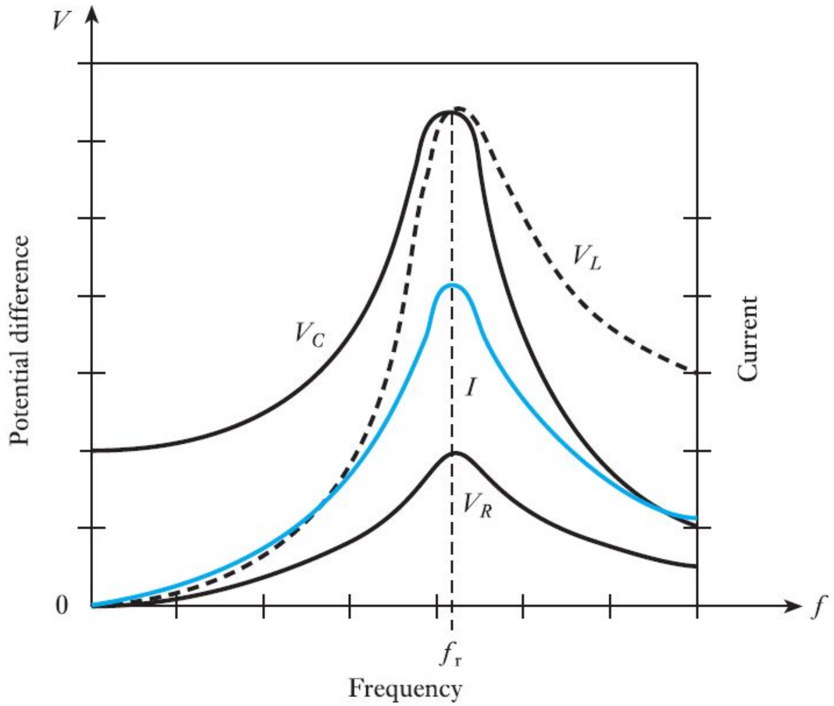


Fig. 14.5 Effect of frequency variation on voltages across *R*, *L* and *C*

The resonant frequency of a series RLC circuit

14.3 **The resonant frequency of a series *RLC* circuit**

At the frequency f_r, $|X_L| = |X_C|$:

$$2\pi f_r L = \frac{1}{2\pi f_r C}$$

so

$$f_r = \frac{1}{2\pi\sqrt{(LC)}} \qquad [14.3]$$

At this frequency f_r, known as the resonant frequency, $Z = R$ and $I = V/R$. The angular frequency ω_r, at resonance, is

$$\omega_r = \frac{1}{\sqrt{(LC)}}$$