Differentiation under integral sign (DUIS rule)

: Differentiation under integral sign is an operation in calculus used to evaluate certain integrals.

There are two rules of DUIS:

Rule 1) If $I(\alpha) = \int_a^b f(x,\alpha)\,dx$

Here α = parameter, x = variable of integration & a, b are the limits of integration which are constant

then $\frac{d}{d\alpha} I(\alpha) = \frac{d}{d\alpha} \int_a^b f(x,\alpha)\,dx$

$$= \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

i.e Total deriative becomes partial derivative under integral sign.

example :

1) show that $\int_0^1 \frac{x^a - 1}{\log x}\,dx = \log(a+1)$

Here ~~f(x,a)~~ a is parameter & limits 0 & 1 are constants.

$\therefore$ consider $I(a) = \int_0^1 \frac{x^a-1}{\log x}\,dx$ ——— (1)

By 1^{st} rule of DUIS

$$\frac{d}{da} I(a) = \frac{d}{da} \int_0^1 \frac{x^a-1}{\log x}\,dx$$

$$= \int_0^1 \frac{\partial}{\partial a} \frac{x^a-1}{\log x}\,dx$$

$$= \int_0^1 \frac{x^a \log x}{\log x}\,dx$$ (derivative of $\frac{x^a-1}{\log x}$ w.r.t a taking x as constant)

$\therefore \frac{d}{da} I(a) = \int_0^1 x^a \, dx$

$= \left[\frac{x^{a+1}}{a+1} \right]_0^1$

$\frac{d}{da} I(a) = \frac{1}{a+1}$

Now $d(I(a)) = \frac{1}{a+1} da$ (VSF)

integrate both side

$\therefore I(a) = \log(a+1) + c$ —— ②

Substitute suitable value of a in ① & ② to find c

Put $a=0$ in ① & ②

from ① $I(0) = \int_0^1 \frac{x^0 - 1}{\log x} dx = 0$

from ⑪ $I(0) = \log(0+1) + c = c$

$\Rightarrow c = 0$

$\therefore I(a) = \int_0^1 \frac{x^a - 1}{\log x} dx = \log(a+1)$

ex-② Using DUIS, evaluate $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ & deduce that $\int_0^\infty \frac{\sin x}{x} dx$, $a \geq 0$

$\rightarrow$ consider $I(a) = \int_0^\infty \frac{e^{-ax} \sin x}{x} dx$ —— ①

Here a is parameter, limits 0 to ∞ are constant

$\therefore$ By 1st rule of DUIS

$\frac{d}{da} I(a) = \frac{d}{da} \int_0^\infty \frac{e^{-ax} \sin x}{x} dx$

$= \int_0^\infty \frac{\partial}{\partial a} \frac{e^{-ax} \sin x}{x} dx$

$= \int_0^\infty \frac{-x e^{-ax} \sin x}{x} dx$

$= -\int_0^\infty e^{-ax} \sin x \, dx$

$$\therefore \frac{d}{da} I(a) = -\left[\frac{e^{-ax}}{a^2+1}(-a\sin x - \cos x)\right]_0^{\infty}$$

$$= -\cancel{\left[\frac{e^{-a(\infty)}}{a^2+1}\right.}(-$$

$$= -\left[\frac{0-(-1)}{a^2+1}\right] = \frac{-1}{a^2+1}$$

$$\therefore \frac{d}{da} I(a) = \frac{-1}{a^2+1}$$

$\therefore$ Integrating w.r.t a

$$\therefore I(a) = -\tan^{-1} a + c \quad \text{--- (2)}$$

put $a = \infty$ in (1) & (2).

$\therefore$ from (1) $I(a) = 0$

from (2) $I(a) = -\frac{\pi}{2} + c$

$$\Rightarrow \quad c = \pi/2 .$$

$$\therefore I(a) = -\tan^{-1} a + \pi/2$$

i.e $\displaystyle\int_0^{\infty} \frac{e^{-ax}\sin x}{x}\, dx = -\tan^{-1} a + \pi/2$

put $a=0$ $\therefore$ $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

eg. (3) show that $\displaystyle\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log \frac{b}{a}$.

Here consider a as parameter or b as parameter.

$$I(a) = \int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx \quad \text{--- (1)}$$

$$\therefore \frac{dI}{da} = \int_0^{\infty} \frac{\partial}{\partial a}\left(\frac{e^{-ax} - e^{-bx}}{x}\right) dx$$

$$= \int_0^{\infty} \frac{-\cancel{x} e^{-ax}}{\cancel{x}}\, dx = -\int_0^{\infty} e^{-ax}\, dx$$

$$= \left[\frac{-e^{-ax}}{-a}\right]_0^{\infty} = \frac{-1}{a}$$

$\therefore I(a) = -\log a + c$ — ②

Put $a=b$ in ① & ②

$\therefore$ from ① $I(a)=0$

from ⑪ $I(a) = -\log b + c$

$\therefore 0 = -\log b + c$

$\Rightarrow c = +\log b$

$\therefore I(a) = -\log a + \log b$

$$\therefore \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\left(\frac{b}{a}\right)$$

4) Prove that $\int_0^1 \frac{x^a - x^b}{\log x}\,dx = \cancel{\log\left[\frac{a+1}{a+1}\right]}$, $a>0, b>0$

$= \log\left(\frac{a+1}{b+1}\right)$

$$I(b) = \int_0^1 \frac{x^a - x^b}{\log x}\,dx \quad \text{— ①}$$

$$\frac{dI}{db} = \int_0^1 \frac{\partial}{\partial b}\left(\frac{x^a - x^b}{\log x}\right)dx$$

$$= \int_0^1 \frac{0 - x^b \log x}{\log x}\,dx$$

$$= \int_0^1 -x^b\,dx = \left[\frac{-x^{b+1}}{b+1}\right]_0^1 = \frac{1}{b+1}$$

$$\therefore dI = \frac{-1}{b+1}\,db$$

integrate both side

$\therefore I(b) = -\log(b+1) + c$ — ②

put $a=b$ in ① & ②

$\therefore$ from ① $I(b) = 0$

② $I(b) = -\log(a+1) + c$

$\therefore 0 = -\log(a+1) + c$

$\Rightarrow c = \log(a+1)$

$\therefore I(b) = -\log(b+1) + \log(a+1)$

$$\therefore \int_0^1 \frac{x^a - x^b}{\log x}\,dx = \log\left(\frac{a+1}{b+1}\right)$$

HW: ① Evaluate $\displaystyle\int_0^\infty \left(\frac{1-e^{-ax}}{x}\right) e^{-x}\,dx$

② Prove that $\displaystyle\int_0^\infty \left(\frac{1-\cos mx}{x}\right) e^{-x}\,dx = \log\sqrt{1+m^2}$

Rule (III) :- If $\displaystyle I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$

Here a & b are the limits which are functions of parameter α

then $\displaystyle\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx$

$$= \left[\int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx\right] + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

It is also called as Leibnitz rule.

eg. ① Verify Leibnitz rule of DUIS for the integral $\displaystyle\int_a^{a^2} \frac{dx}{x+a}$

$$I(a) = \int_a^{a^2} \frac{dx}{x+a}$$

$$= \left[\log(x+a)\right]_a^{a^2}$$

$$= \log(a+a^2) - \log(2a)$$

$$= \log\frac{a+a^2}{2a}$$

$$= \log\frac{a(1+a)}{2a}$$

$$\therefore I(a) = \log\left(\frac{a+1}{2}\right)$$

$$\therefore \boxed{\frac{dI}{da} = \frac{1}{a+1}} \quad \text{—— (1)}$$

da a a

$$\therefore \int_0^1 \frac{x^a - x^b}{\log x}\,dx = \log\left(\frac{a+1}{b+1}\right)$$

HW: ① Evaluate $\int_0^\infty \left(\frac{1-e^{-ax}}{x}\right) e^{-x}\,dx$

② Prove that $\int_0^\infty \left(\frac{1-\cos mx}{x}\right) e^{-x}\,dx = \log\sqrt{1+m^2}$

Rule (III) :- If $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$

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It is also called as Leibnitz rule.

eg. ① Verify Leibnitz rule of DUIS for the integral

$$\int_a^{a^2} \frac{dx}{x+a}$$

$$I(a) = \int_a^{a^2} \frac{dx}{x+a}$$

$$= \left[\log(x+a)\right]_a^{a^2}$$

$$= \log(a+a^2) - \log(2a)$$

$$= \log\frac{a+a^2}{2a}$$

$$= \log\frac{\cancel{a}(1+a)}{2\cancel{a}}$$

$$\therefore I(a) = \log\left(\frac{a+1}{2}\right)$$

$$\therefore \boxed{\frac{dI}{da} = \frac{1}{a+1}} \quad \text{—— ①}$$

Now By DUIS

$$\frac{dI}{da} = \frac{d}{da}\int_a^{a^2} \frac{dx}{x+a}$$

$$= \left[\int_a^{a^2} \frac{\partial}{\partial a} \frac{dx}{x+a}\right] + \frac{1}{a^2+a}\frac{d}{da}(a^2) - \frac{1}{a+a}\frac{d}{da}(a)$$

$$= \cancel{\int_a^{a^2} \log(x+a)}$$

$$= \int_a^{a^2} \frac{-1}{(x+a)^2}\,dx + \frac{2a}{a^2+a} - \frac{1}{2a}$$

$$= \left(\frac{1}{x+a}\right)_a^{a^2} + \frac{2}{1+a} - \frac{1}{2a}$$

$$= \frac{1}{a^2+a} - \frac{1}{2a} + \frac{2}{a+1} - \frac{1}{2a}$$

$$\boxed{\frac{dI}{da} = \frac{1}{a+1}} \quad \text{--- (2)}$$

$\therefore$ from (1) & (2) DUIS rule is verified.

(2) find $\frac{dI}{da}$ if $I(a) = \int_a^{a^2} \frac{\sin ax}{x}\,dx$.

$$\frac{dI}{da} = \left[\int_a^{a^2} \frac{\partial}{\partial a}\frac{\sin ax}{x}\,dx\right] + \frac{\sin a^3}{a^2}\frac{d}{da}(a^2) - \frac{\sin a^2}{a}\frac{d}{da}(a)$$

$$= \int_a^{a^2} \frac{\cancel{x}\cos ax}{\cancel{x}}\,dx + \frac{\sin a^3}{a^2}(2a) - \frac{\sin a^2}{a}(1)$$

$$= \left(\frac{\sin ax}{a}\right)_a^{a^2} + \frac{2\sin a^3}{a} - \frac{\sin a^2}{a}$$

$$= \frac{\sin a^3}{a} - \frac{\sin a^2}{a} + \frac{2\sin a^3}{a} - \frac{\sin a^2}{a}$$

$$\frac{dI}{da} = \frac{3\sin a^3}{a} - \frac{2\sin a^2}{a}$$

1) If $y = \int_0^x f(t)\sin[a(x-t)]\,dt$ then show that
$\frac{d^2y}{dx^2} + a^2 y = a f(x)$.

$\rightarrow$ $y = \int_0^x f(t)\sin[a(x-t)]\,dt$, Here x is parameter

By 2^{nd} rule of DUIS

$$\frac{dy}{dx} = \left[\int_0^x \frac{\partial}{\partial x} f(t)\sin a(x-t)\,dt\right] + f(x)\sin a(\cancel{x}-\cancel{x})\frac{d}{dx}(x) - f(0)\sin a(x-0)\frac{d}{dx}(0)$$

$$= \int_0^x f(t)\cos a(x-t)\cdot a\,dt + 0 - 0$$

$$= a\int_0^x f(t)\cos a(x-t)\,dt$$

Again by DUIS

$$\frac{d^2y}{dx^2} = a\left[\int_0^x \frac{\partial}{\partial x} f(t)\cos a(x-t)\,dt + f(x)\cos a(\cancel{x}-\cancel{x})\frac{d}{dx}(x) - f(0)\cos a(x-0)\frac{d}{dx}(0)\right]$$

$$\therefore \frac{d^2y}{dx^2} = -a^2\int_0^x f(t)\sin a(x-t)\,dt + af(x) - 0$$

$$= -a^2 y + af(x)$$

$$\therefore \boxed{\frac{d^2y}{dx^2} + a^2 y = a f(x)}$$

HW :- ① show that $\int_{\frac{\pi a}{6}}^{\frac{\pi a}{2}} \frac{\sin ax}{x}\,dx$ is independent of a

(i.e To show $I'(a) = 0$)

② If $f(x) = \int_0^x (x-t)^2 G(t)$ then prove that

$\frac{d^3f}{dx^3} = 2G(x)$.

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