7 Tracing of Curves

1. Introduction

For evaluating areas, volumes of revolution etc. we need to know the general nature of the given curve. In this chapter we shall learn the methods of tracing a curve in general and the properties of some standard curves commonly met in engineering problems.

2. Procedure for tracing curves given in Cartesian equations

I. Symmetry : Find out whether the curve is symmetrical about any line with the help of the following rules :

(1) The curve is symmetrical about the x-axis if the equation of the curve remains unchanged when y is replaced by $-y$ *i.e.* if the equation contains only even powers of y.

(2) The curve is symmetrical about the y-axis if the equation of the curve remains unchanged when x is replaced by $-x$ *i.e.* if the equation contains only even powers of x.

(3) The curve is symmetrical in opposite quadrants if the equation of the curve remains unchanged when both x and y are replaced by $-x$ and $-y$.

(4) The curve is symmetrical about the line $y = x$ if the equation of the curve remains unchanged when x and y are interchanged.

II. Origin : Find out whether the origin lies on the curve. If it does, find the equations of the tangents at the origin by equating to zero the lowest degree terms.

III. Intersection with the coordinate axes : Find out the points of intersection of the curve with the coordinate axes. Find also the equations of the tangents at these points.

IV. Asymptotes : Find out the asymptotes if any.

V. Regions where no part of the curve lies : Find out the regions of the plane where no part of the curve lies.

VI . Find out dy/dx : Find out dy/dx and the points where the tangents are parallel to the coordinate axes.

3. Common Curves

You have already studied quite in detail straight line, circle, parabola, ellipse and hyperbola, including rectangular hyperbola. We shall briefly review these curves.

Rectangular Coordinates

(a) Straight Line :

General equation of straight line is of the form $ax + by + c = 0$.

To plot a straight line we put $y = 0$ and find x. Also we put $x = 0$ and find y. We plot these two points and join them to get the required line.

Some particular cases of straight lines are shown below.

(i) Lines parallel to the coordinate axes.

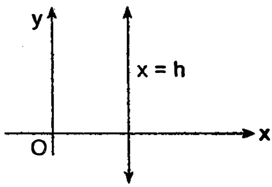


Fig. (7.1)

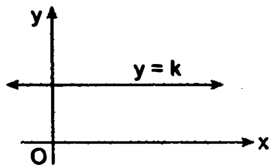


Fig. (7.2)

(ii) Lines passing through origin

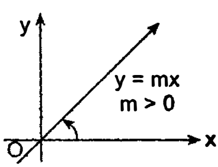


Fig. (7.3)

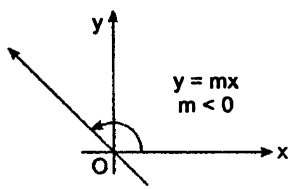


Fig. (7.4)

(iii) Lines making given intercepts on coordinate axes.

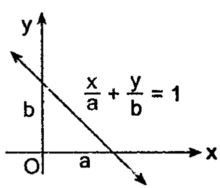


Fig. (7.5)

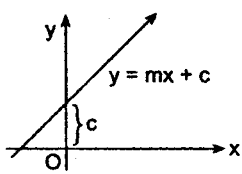


Fig. (7.6)

(b) Circle :

General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its centre is $(-g, -h)$ and radius $= \sqrt{g^2 + f^2 - c}$

(i) Circle with centre at origin and radius a and

(ii) Circle with centre at $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

are shown below.

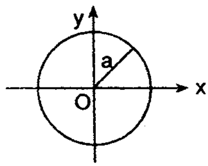


Fig. (7.7)

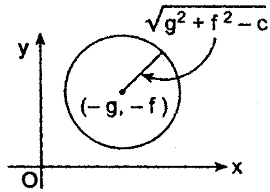


Fig. (7.8)

(iii) Circle with centre on the x- axis and passing through origin.

Its equation is of the form $x^2+y^2 \pm 2ax = 0$.

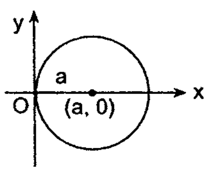


Fig. (7.9)

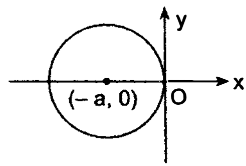


Fig. (7.10)

(iv) Circle with centre on the y-axis and passing through origin.

Its equation is of the form $x^2+y^2 \pm 2by = 0$.

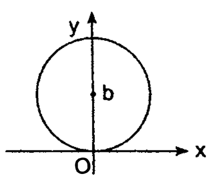


Fig. (7.11)

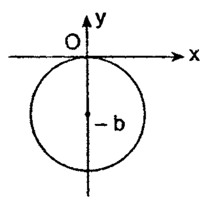


Fig. (7.12)

(v) Circle with centre on the axes but not passing through origin.

Its equation is of the form $x^2+y^2 \pm 2gx + c = 0$, $x^2+y^2 \pm 2fy + c =$

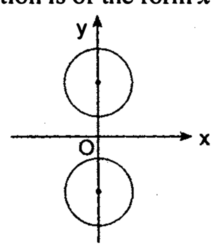


Fig. (7.13)

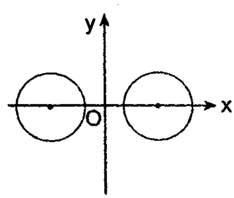


Fig. (7.14)

(vi) Circle touching both axes.

Its equation is of the form $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$.

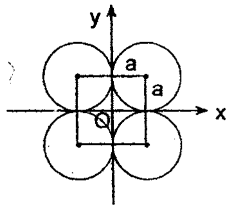


Fig. (7.15)

Ex. 1 : Draw the circle $x^2 + y^2 - 4x + 4y + 4 = 0$

Sol. : We write the equation as $(x-2)^2 + (y+2)^2 = 2^2$

Its centre is (2, –2) and radius = 2.

It is shown on the right.

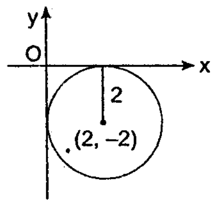


Fig. (7.16)

Polar Coordinates

(a) Straight line

If we put $x = r\cos\theta$, $y = r\sin\theta$ in the equation of the straight line

$$ax + by + c = 0$$

we get, $ar\cos\theta + br\sin\theta + c = 0$ *i.e.* $a\cos\theta + b\sin\theta + \dfrac{c}{r} = 0$

(i) A line parallel to the initial line.

If a line is parallal to the initial line (or the x – ax is) at a ditance p from it then, from the figure we see that its equation is $r\sin\theta = \pm$ p (It is clear that $r\sin\theta = p$, if the line is above the initial line and $r\sin\theta = -p$, if the line is below the initial line.)

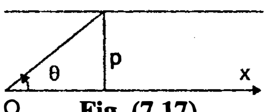


Fig. (7.17)

(ii) A line perpendicular to the initial line.

If a line is perpendienlar to the initial line (or the x – axis) at a ditance p from it then from the figure we see that $r\cos\theta = \pm$ p

(It is clear that $r\cos\theta = p$, if the line is to the right of the pole and $r\cos\theta = -p$, if the line is to the left of the pole.)

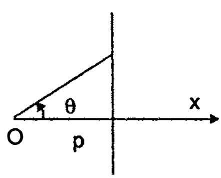


Fig. (7.18)

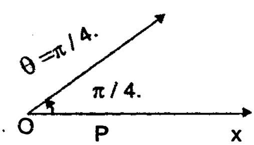


Fig. (7.19)

(iii) A line through the pole (or the origin)

If a line passes through the pole and makes an angle α with the inital line then every point on the line has coordnates (r, α), where r is positive or negative. Hence the equation of a line is $\theta = \alpha$

In particular the equation of a line making an angle of 45° is $\theta = \pi/4$.

(b) Circle

If we put $x = r\cos\theta$, $y = r\sin\theta$ in the equation of the circle with center at the origin and radius a *i.e.* in $x^2 + y^2 = a^2$ we get, $r^2 = a^2$

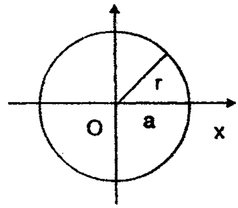


Fig. (7.20)

If we put $x = r\cos\theta$ $y = r\sin\theta$ in the equation of the circle with center $(a, 0)$ and radius a *i.e.* in

$$(x-a)^2 + y^2 = a^2 \text{ i.e. in } x^2 + y^2 - 2ax = 0$$

we get $r^2 - 2a\,r\cos\theta = 0 \therefore\ r = 2a\cos\theta$

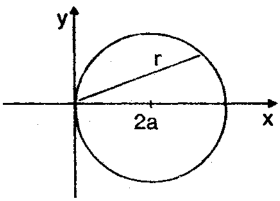


Fig. (7.21)

This is the equation of the circle with center $(a, 0)$ and radius a in polar coordniates.

If we put $x = r\cos\theta$, $y = r\sin\theta$ in the equation of the circle with center $(0, a)$ and radius a *i.e.* in

$$x^2 + (y-a)^2 = a^2 \text{ i.e. } x^2 + y^2 - 2ay = 0$$

we get, $r^2 - 2ar\sin\theta =$ *i.e.* $r = 2a\sin\theta$

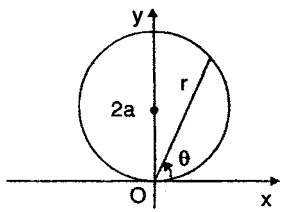


Fig. (7.22)

This is the equation of the circle with center $(0, a)$ and radius a in polar coordniates

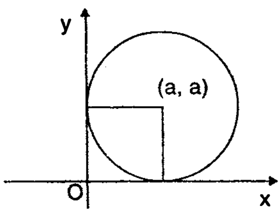


Fig. (7.23)

If we put $x = r\cos\theta$, $y = r\sin\theta$ in the equation of the cricle with center (a, a) and touching both the coordniate axes in the first quadrant

i.e. in $(x-a)^2 + (y-a)^2 = a^2$

i.e. in $x^2 + y^2 - 2ax - 2ay + a^2 = 0$

we get the equation $r^2 - 2ar \cdot (\cos\theta + \sin\theta) + a^2 = 0$ in polar coordniates.

Ex. : Find the equation of the cricle in polar coordinates having centre (3, 0) and radius 3. Also find the polar coordinates of its point of intersection with the line $y = x$.

Sol. : The equation of the circle with centre (3, 0) and radius 3 is $(x-3)^2 + y^2 = 3^2$

i.e. $x^2 + y^2 - 6x = 0$

Putting $x = r\cos\theta$, $y = r\sin\theta$

we get $r^2 - 6r\cos\theta = 0$ i.e. $r = 6\cos\theta$

The equation of the line is $y = x$.

Putting $x = r\cos\theta$, $y = r\sin\theta$, we get the equation in polar coordinates as

$r\sin\theta = r\cos\theta$ i.e. $\tan\theta = 1$ $\therefore$ $\theta = \pi/4$

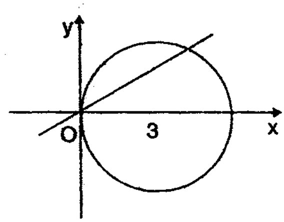


Fig. (7.24)

Putting $\theta = \pi/4$ in $r = 6\cos\theta$, we get $r = 6\cos\pi/4 = 6\cdot\frac{1}{\sqrt{2}} = 3\sqrt{2}$

Hence the polar coordinates of the point of intersection are $(3\sqrt{2}, \pi/4)$

(iii) Parabola :

General equation of parabola is

$$y = ax^2 + bx + c \quad \text{or} \quad x = ay^2 + by + c.$$

By completing the square on x or on y and by shifting the origin the equation can be written in standard form as $y^2 = 4ax$ or $x^2 = 4by$.

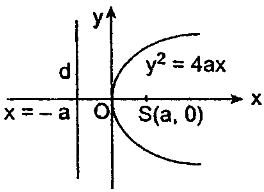


Fig. (7.25)

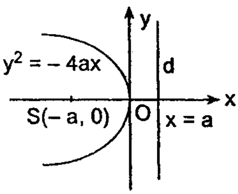


Fig. (7.26)

The equation of the parabola with vertex at (h, k) and

(i) the axis parallel to the x-axis, opening to the right is

$$(y-k)^2 = 4a(x-h)$$

(ii) the axis parallel to the x-axis, opening to the left is

$$(y-k)^2 = -4a(x-h)$$

(iii) the axis parallel to the y-axis, opening upwards is

$$(x-h)^2 = 4a(y-k)$$

(iv) the axis parallel to the y-axis, opening downwards is

$$(x-h)^2 = -4a(y-k)$$

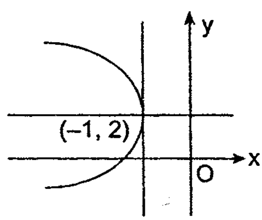


Fig. (7.27)

Ex. 1: Sketch the curve $y^2 + 4x - 4y + 8 = 0$

Sol. : The equation can be written as

$$y^2 - 4y + 4 = -4x - 4 \text{ i.e. } (y-2)^2 = -4(x+1)$$

Putting $y-2=Y, x+1=X$,

Its equation is $Y^2=-4X$.

Its vertex is at $(-1, 2)$ and it opens on the left.

Ex. 2 : Sketch the curve

$$x^2+4x-4y+16=0.$$

Sol. : The equation can be written as

$$(x+2)^2=4(y-3).$$

Putting $x+2=X,\ y-3=Y$,

Its equation is $X^2=4Y$.

Its vertex is $(-2, 3)$ and it opens upwards.

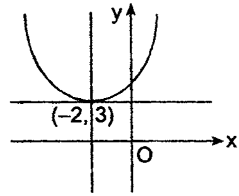


Fig. (7.28)

(iv) Ellipse

The equation of ellipse in standard form is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad(a>b),\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad(a<b)$$

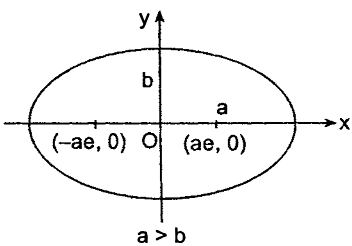


Fig. (7.29)

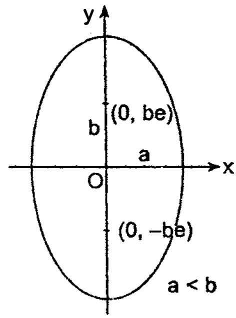


Fig. (7.30)

If the centre of the ellipse is at (h, k) the equation of the ellipse becomes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

If $a>b$, the major axis is parallel to the x-axis and if $a<b$ the major axis is parallel to the y-axis.

(v) Hyperbola

Equation of hyperbola in standard form is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\quad\text{or}\quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

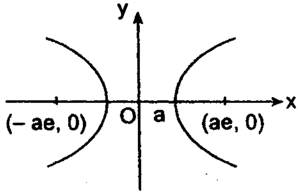


Fig. (7.31)

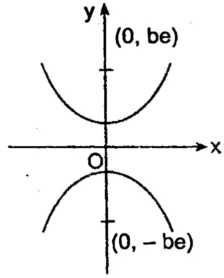


Fig. (7.32)

(vi) Rectangular Hyperbola

If the coordinate axes are the asymptotes the equation of the rectangular hyperbola is $xy = k$ or $xy = -k$.

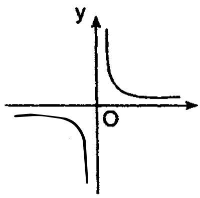


Fig. (7.33)

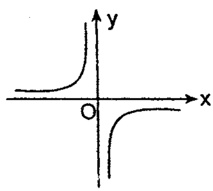


Fig. (7.34)

If the lines $y = +x$ and $y = -x$ are the asymptotes, the equations of the rectangular hyperbola are given by $x^2 - y^2 = a^2$, $y^2 - x^2 = a^2$.

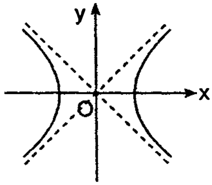


Fig. (7.35)

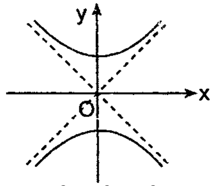


Fig. (7.36)

Polar equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is obtained by putting $x = r\cos\theta$, $y = r\sin\theta$. The equation is $r^2\cos^2\theta - r^2\sin^2\theta = a^2$ *i.e.* $r^2(\cos^2\theta - \sin^2\theta) = a^2$ *i.e.* $r^2\cos 2\theta = a^2$

Similarly the polar equation of the other rectangular hyperbola *i.e.* of $y^2 - x^2 = a^2$ is $r^2\cos 2\theta = -a^2$

Parametric Equations

Ellipse

The parametric equations of the ellipse are $x = a\cos\theta,\ y = b\sin\theta$.

Hyperhola

The parametric equations of the hyperbola are

$$x = a\sec\theta,\ y = b\tan\theta \text{ or } x = a\cosh t,\ y = b\sinh t.$$

Parabola

The parametric equations of the parabola $y^2 = 4ax$ are $x = at^2,\ y = 2at$.

The parametric equations of the parabola $x^2 = 4ay$ are $x = 2at,\ y = at^2$.

Rectangular Hyperbola

The parametric equations of the rectangular hyperbola

$xy = c^2$ are $x = ct$ and $y = c/t$.

Curves of the form $y^m = x^m$ where m and n are positive intigers.

It is interesting to note the shape of the curves whose equations can be expressed in the above form. Some of the curves are known to us and some will be studied in the following pages. We give below the equations and the graphs of these curves.

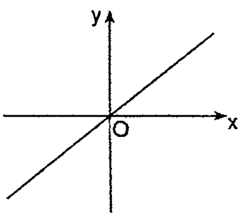


Fig. (7.37)

(1) $y = x$.
The staight line through the origin

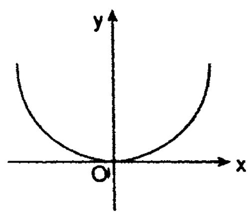


Fig. (7.38)

(2) $y = x^2$.
The parabola through the origin and opening up

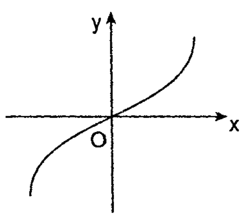


Fig. (7.39)

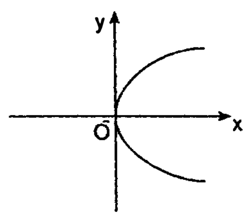


Fig. (7.40)

(3) $y = x^3$.

Cubical parabola

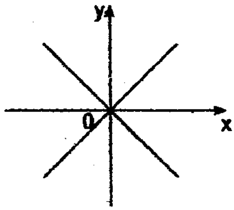


Fig. (7.41)

(4) $y^2 = x$.

The parabola through the origian and opening on the right

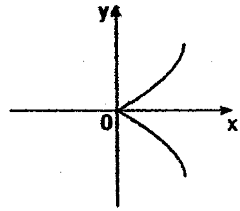


Fig. (7.42)

(5) $y^2 = x^2$.

Pair of two lines passing through the origin

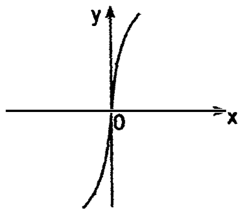


Fig. (7.43)

(6) $y^2 = x^3$.

Semi-cubical parabola

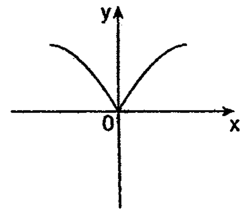


Fig. (7.44)

(7) $y^3 = x$.

Cubical parabola

(8) $y^3 = x^2$.

Semi-cubical parabola

4. Some Well-known Curves

Ex 1 : Cissoid of Diocles : Trace the curve $y^2 (2a - x) = x^3$

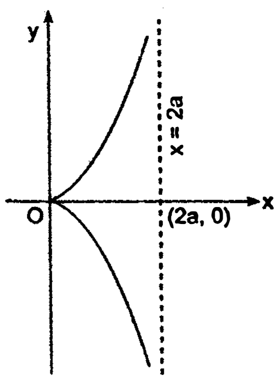


Fig. (7.45)

or $x = \dfrac{2at^2}{1+t^2}$, $y = \dfrac{2at^3}{1+t^2}$ **(S.U. 1986, 90)**

Sol. : (i) The curve is symmetrical about the x - axis.

(ii) The curve passes through the origin and the tangents at the origin are $y^2 = 0$ *i.e.* the x-axis is a double tangent at the origin.

(iii) Since $y^2 = \dfrac{x^3}{2a - x}$ as $x \to 2a$, $y \to \infty$, the line $x = 2a$ is an asymptote.

(iv) When $x > 2a$, y^2 is negative. Hence, the curve does not exist when $x > 2a$.

(**Remark :** By putting $x = r\cos\theta$, $y = r\sin\theta$, show that the polar equation of Cissoid is $r = 2a\tan\theta\sin\theta$).

Ex. 2 : Trace the curve $a^2x^2 = y^3 (2a - y)$ **(S.U. 1981, 86, 90, 2006)**

Sol. : (i) The curve is symmetrical about the y-axis.

(ii) It passes through the origin and the x-axis is a tangent at origin.

(iii) It meets the y-axis at (0,0) and $(0, 2a)$. Further dy/dx is zero at $(0, 2a)$; the tangent at this point is parallel to the x-axis.

(iv) Since $x^2 = y^3 (2a - y) / a^2$ when $y > 2a$ and $y < 0$, x^2 is negative and the curve does not exist for $y > 2a$ and $y < 0$.

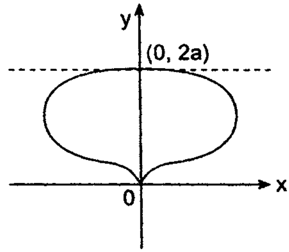


Fig. (7.46)

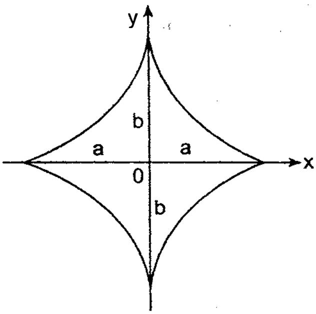


Fig. (7.47)

Ex. 3 : Astroid or Four Cusped Hypocycloid : Trace the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$ or $x = a \cos^3\theta, y = b \sin^3 \theta$ **(S.U. 1985, 2003)**

Sol. : (i) The curve is symmetrical about both axes.

(ii) The curve cuts the x-axis in $(\pm a, 0)$ and the y- axis in $(0, \pm b)$

(iii) Neither x can be greater than a nor y can be greater than b.

Ex. 4 : Witch of Agnesi : Trace the curve $xy^2 = a^2 (a - x)$ **(S.U. 1980, 91, 92, 98, 2003, 04)**

Sol. : (i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the point $(a, 0)$.

(iii) Since $y^2 = a^2 \dfrac{(a-x)}{x}$, x cannot be negative. Also x cannot be greater than a.

(iv) As $x \to 0$, $y \to \infty$, the y-axis is an asymptote.

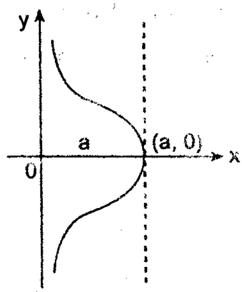


Fig. (7.48)

Ex. 5 : Trace the curve $y^2 = (x - a) (x - b) (x - c)$ where a, b, c are positive.

Sol. : We consider the following cases :

(a) Case I : $a < b < c$

(1) It is symmetrical about the x-axis.

(2) It meets the x-axis in $(a, 0)$, $(b,0)$ and $(c, 0)$.

(3) When $x < a$, y^2 is negative.

when $a < x < b$, $y^2 > 0$

when $b < x < c$, y^2 is negative

when $x > c$, $y^2 > 0$

Hence, there is no curve to the left of $x = a$ and also between $x = b$ and $x = c$.

(4) If $x > c$ and increases then y^2 also increases

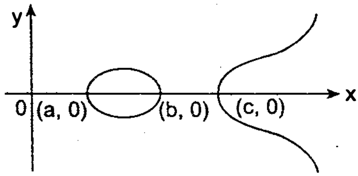


Fig. (7.49)

(b) Case II : $a = b < c$

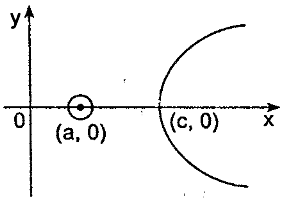


Fig. (7.50)

The equation now becomes

$$y^2 = (x-a)^2 (x-c)$$

(1) As before the curve is symmetrical about the x-axis.

(2) It meets the x-axis in points $(a, 0)$ and $(c, 0)$.

(3) $(a , 0)$ is an isolated point because if $x < a$ (*i.e.* $x < c$), y^2 is negative and if $a < x < c$, y^2 is negative.

(4) If $x > c$ and increases y^2 also increases.

(c) Case III : $a < b = c$

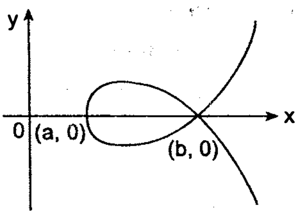


Fig. (7.51)

The equation now becomes

$$y^2 = (x-a)(x-b)^2$$

(1) The cuive is symmetrical about the x-axis.

(2) It meets the x-axis in points $(a, 0)$ and $(b, 0)$.

(3) If $x < a$, y^2 is negative.

(4) If $x > b$ and increases, y^2 also increases.

(d) Case IV : $a = b = c$

(S.U. 2003)

The equation now becomes $y^2 = (x-a)^3$

(1) As before the curve is symmetrical about the x - axis.

(2) The curve meets the x-axis in $(a, 0)$

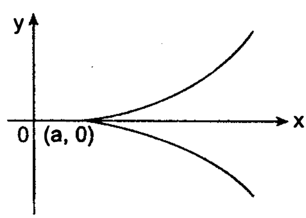


Fig. (7.52)

(3) If $x < a$, y^2 is negative

(4) If $x > a$ and increases, y^2 also increases.

Ex. 6 : Trace the curve $x^{1/2} + y^{1/2} = a^{1/2}$

Sol. : (i) The curve is symmetrical about the line $y = x$.

(ii) If $y = x$ we get $2x^{1/2} = a^{1/2}$

$\therefore$ $4x = a$ *i.e.* $x = a/4$. The line $y = x$ cuts the curve is $(a/4, a/4)$.

(iii) The x-axis is a tangent at $(a, 0)$ and the y - axis is a tangent at $(0, a)$.

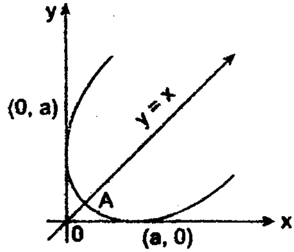


Fig. (7.53)

5. Some Common Loops

Ex. 1 : Strophoid Trace the curvee $y^2 (a + x) = x^2 (b - x)$

Sol. : (i) Curve is symmetrical about the x-axis.

(ii) The curve passes through the origin and the equations of the tangents at the origin are obtained by equating to zero the lowest degree terms.

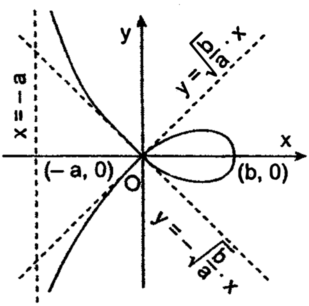


Fig. (7.54)

$\therefore\ ay^2 = bx^2 \qquad \therefore\ y = \pm\sqrt{\frac{b}{a}} \cdot x$

(iii) The curve meets the x-axis in $(0, 0)$ and $(b, 0)$

(iv) Since $y^2 = \dfrac{x^2(b-x)}{(a+x)}$ when $x = -a$, y is infinite.

Hence $x = -a$ is an asymptote.

(v) When $x > b$ and when $x < -a$, y^2 is negative. Hence, the curve does not exist when $x > b$ and $x < -a$.

The curve is shown in the figure.

(a) Trace the curve : $y^2 (a + x) = x^2 (3a - x)$ **(S.U. 1980, 86, 91, 2000)**

When b is replared by $3a$ we get the above curve which is shown on the right.

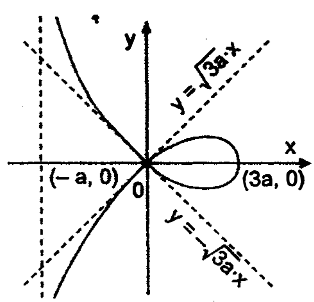


Fig. (7.55)

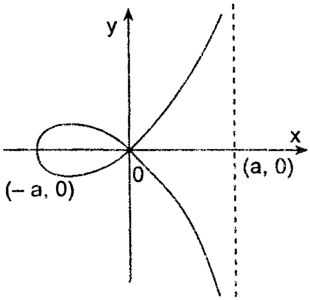


Fig. (7.56)

(b) Trace the curve : $y^2 (a - x) = x^2 (a + x)$

If we replace b by a and x by $-x$ we get the above curve which is shown on the left.

(c) Trace the curve : $y (a + x) = x^2 (a - x)$

Sol. : If we replace b by a in Ex. (1) we get the required curve.

(d) Trace the curve : $x (x^2 + y^2) = a (x^2 - y^2)$

(S.U. 1980, 85, 90)

The equation can the written as

$$x^3 + xy^2 = ax^2 - ay^2$$

$$\therefore\ xy^2 + ay^2 = ax^2 - x^3 \qquad \therefore\ y^2 (a + x) = x^2 (a \quad x)$$

This is the same as the first equation where $b = a$.

Ex. 2 : Trace the curve $xy^2 + (x + a)^2 (x + 2a) = 0$

Sol. : (i) The curve is symmetrical about the x-axis.

(ii) The curve cuts the x-axis in $(-a, 0)$, $(-2a, 0)$

(iii) Since $y^2 = -(x + a)^2 (x + 2a)/x$, x cannot be positive. Also x cannot be less than $-2a$.

(iv) The y - axis is an asymptote.

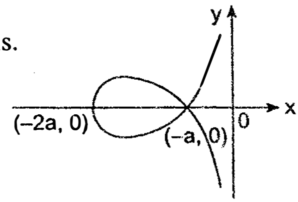


Fig. (7.57)

Ex. 3 : Folium of Decartes. Trace the curve $x^3 + y^3 = 3axy$.

$$\text{or} \quad x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}$$

(S.U. 2003)

Sol. : (i) The curve is symmetrical about the line $x = y$.

(ii) The curve passes through the origin.

(iii) The curve intersects the line $x = y$ in $(0, 0)$ and $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

(iv) The line $x + y + a = 0$ is an asymptote.

(v) If $x < 0$ and $y < 0$, l.h.s. is negative while r.h.s. is positive. Hence, no part of the curve is in the third quadrant.

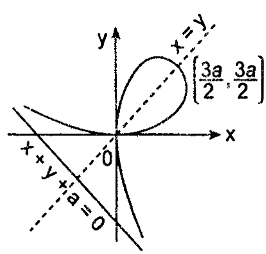


Fig. (7.58)

Ex. 4 : Trace the curve $a^2y^2 = x^2(a^2 - x^2)$

Sol. : (i) The curve is symmetrical about both the axes.

(ii) The points $(0, 0)$ $(a, 0)$ and $(-a, 0)$ lie on the curve.

(iii) If $x > a$, y^2 is negative., Hence there is no curve beyond $x = a$, $x = -a$.

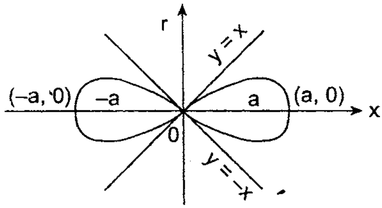


Fig. (7.59)

Ex. 5 : Trace the curvee $y^2 = (x - a)(x - b)^2$

Sol. : We have already dicussed the above curve in Ex. 5 case III on page 7.12.

The shape of the curve will remain unchanged if y we replace y by $\sqrt{c} \cdot y$ *i.e.* the shape of the curve of the equation

$cy^2 = (x - a)(x - b)^2$ will be the same.

(a) Trace the curve : $9ay^2 = x(x - 3a)^2, \ a > 0$. **(S.U. 2004,06)**

Sol. : Replacing c by $9a$, a by zero and b by $3a$, we get the above curve. The curve is shown on the right.

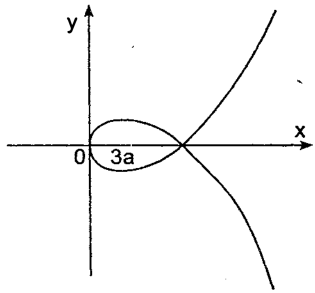


Fig. (7.60)

(b) Trace the curve : $y^2 = x\left(1 - \frac{x}{3}\right)^2$ or $x = t^2, y = t - \frac{t^3}{3}$

Sol. : Replacing a by zero, b by 1/3 and c by 1, we get the above curve. The curve is shown on the right.

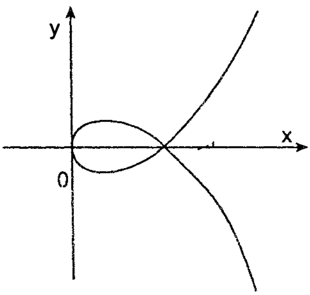


Fig. (7.61)

(c) Trace the curve : $ay^2 = x(x - a)^2$

Sol. : Putting $a = 0$ and $b = a$ we get the above curve. The curve is shown on the right.

When $x < 0$, y is imaginary and there is no curve for $x < 0$

When $x \to \infty$ or $x \to -\infty$, $y \to \infty$.

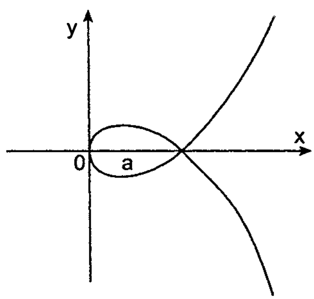


Fig. (7.62)

(d) Trace the curve : $3ay^2 = x^2 (a - x)$ **(S.U. 1980, 85, 89, 97)**

Sol. : Putting $c = 3a$, $a = 0$ and $b = a$ we get the above equation. The curve is shown on the right.

When $0 < x < a$, y is real and when $x > a$, y is imaginary. Hence, there is no curve for $x > a$

When $x \to \infty$ or $x \to -\infty$, $y \to \infty$.

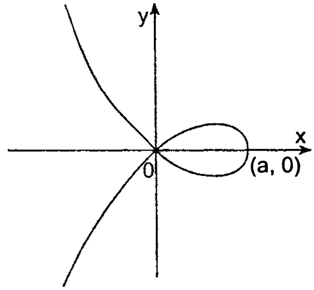


Fig. (7.63)

(e) Trace the curve : $ay^2 = 4x^2 (a - x)$ **(S.U. 1984, 95)**

Sol. : Multiplying by 4 and then putting $a = a/12$ in (d) we get the above curve. Hence, it is similar to it.

(f) Trace the curve : $ay^2 = x^2 (a - x)$ **(S.U. 1985)**

Sol. : Putting $a = a/3$ in (d) we get the above curve.

Ex. 1 : Trace the following curve $y^2 (a - x) = x (x - b)^2$, $a > b$

Sol. : (i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the origin.

(iii) The curve intersects the x-axis at $x = b$, $x = 0$.

(iv) Since $y^2 = \dfrac{x(x-b)^2}{a-x}$, when $x = a$, y is infinite. Hence $x = a$ is an asymptote.

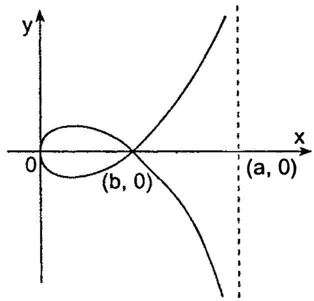


Fig. (7.64)

(a) Trace the curve : $y^2 (4 - x) = x (x - 2)^2$

Sol. : Putting $a = 4$ and $b = 2$ in the above equation, we get, the required curve.

Exercise - I

Trace the following curves

1. $ay^2 = x (a^2 - x^2)$ **(S.U. 1994, 97, 2002, 2005)**
2. $2y^2 = x (4 - x^2)$ **(S.U. 1978, 81, 88)**
3. $a^2y^2 = x^2 (x - a) (2a - x)$ **(S.U. 1979, 84, 2006)**
4. $y (x^2 + 4a^2) = 8a^3$ **(S.U. 1980, 96, 2006)**
5. $y^2 = x^5 (2a - x)$ **(S.U. 1983, 85, 93)**
6. $y^2 (a^2 - x^2) = a^3 x$ **(S.U. 1984, 99, 2004)**

7. $x^2y^2 = a^2 (y^2 - x^2)$

8. $ay^2 = x^2 (x - a)$ **(S.U. 1985)**

9. $x^2 = y^3 (2 - y)$ **(S.U. 1979, 2002)**

10. $ay^2 = x (a^2 + x^2)$

11. $2y^2 = x (4 + x^2)$ **(S.U. 1981, 94)**

12. $y^2 (a - x) = x^3$

13. $a^2y^2 = x^3 (2a - x)$

14. $a^2x^2 = y^2 (a^2 - y^2)$

15. $y^2 = ax^3$

16. $y^2 (x^2+ y^2) = a^2 (y^2 - x^2)$

17. $x^2(x^2 + y^2) = a^2 (x^2 - y^2)$

18. $x^2 (x^2 + y^2) = a^2 (y^2 - x^2)$

19. $y^2 = x^3 - 6x^2 + 11x - 6$ **(Hint :** $y^2 = (x \quad 1) (x - 2) (x - 3)$)

20. $y^2 (x^2 + 4) = x^2 + 2x$

21. $x^5 + y^5 - 5a^2x^2y = 0$

22. $x^2 (x^2 - 4a^2) = y^2 (x^2 - a^2)$

23. $x^4 + y^4 = 2a^2xy$

24. $x^5 + y^5 = 5a^2x^2y^2$

25. $y = \dfrac{x}{1 + x^2}$ **(S.U. 2005)**

26. $x^6 + y^6 = a^2x^2y^2$

27. $(a - x) y^2 = a^2 x$

28. $y^2 (x^2 + a^2) = x^2 (a^2 - x^2)$

29. $a^4y^2 = x^4 (a^2 - x^2)$

30. $a^4y^2 + b^2 x^4 = a^2b^2x^2$

31. $y^2 (a \quad x) (x - b) = x^2$ $(a, b > 0\ a > b)$

[Ans. :

(1)

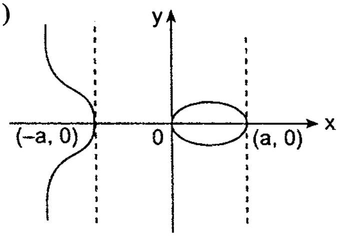


Fig. (7.65)

(2) Similar to the previous Ex. 1 with $a = 2$

(3)

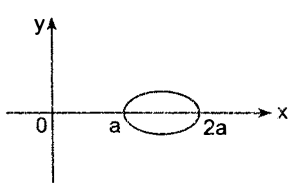


Fig. (7.66)

(4)

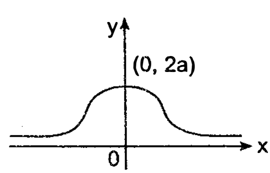


Fig. (7.67)

(5)

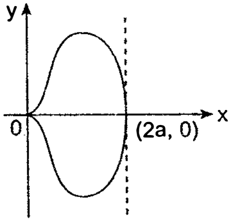


Fig. (7.68)

(6)

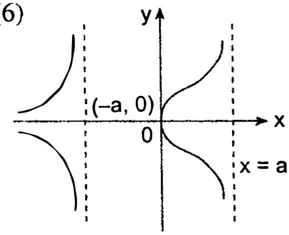


Fig. (7.69)

(7)

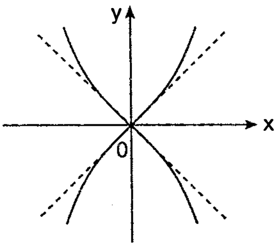


Fig. (7.70)

(8) Compare with solved Ex. 5 Case II (page 7.12)

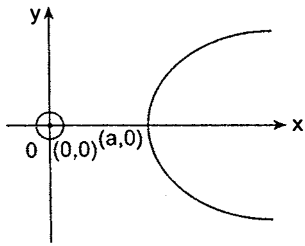


Fig. (7.71)

(9) Similar to solved Ex. 2 (page 7.11) with $a = 1$.

(11) Similar to Ex. 10 above with $a = 2$.

(12) Similar to solved Ex. 1 page (7.10) with $a = a / 2$.

(10)

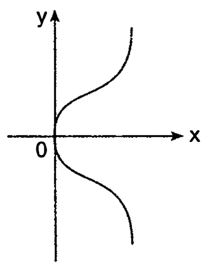


Fig. (7.72)

(14)

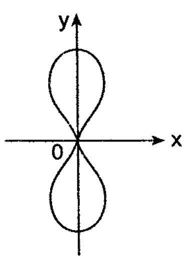


Fig. (7.73)

(13)

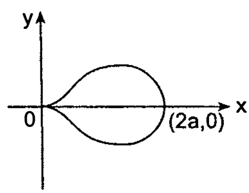


Fig. (7.74)

(15)

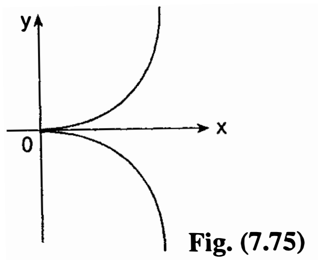


Fig. (7.75)

(16)

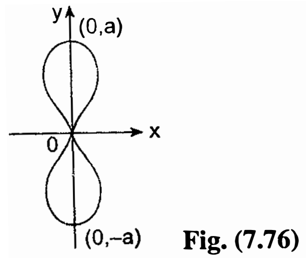


Fig. (7.76)

(17)

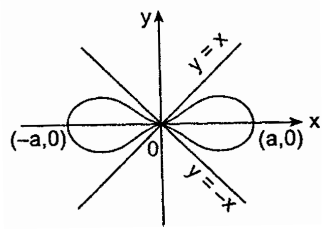


Fig. (7.77)

(18)

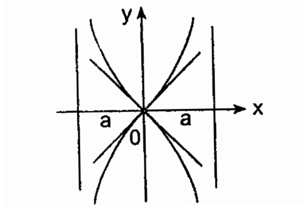


Fig. (7.78)

(19) Similar to solved Ex. 5 Case I page (7.11)

(20)

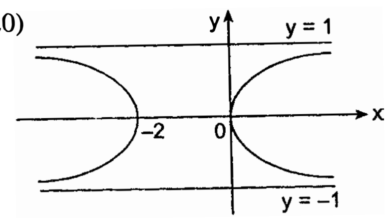


Fig. (7.80)

(21)

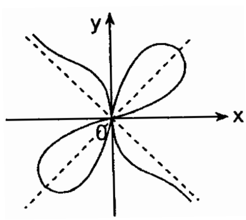


Fig. (7.79)

(22)

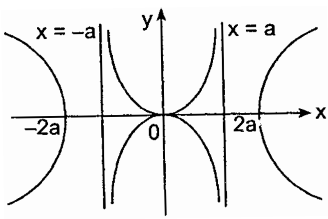


Fig. (7.81)

(23)

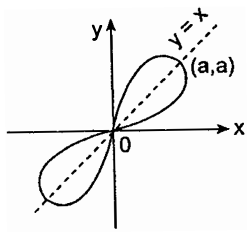


Fig. (7.82)

(24)

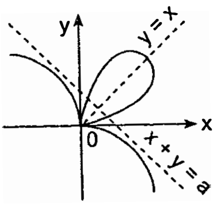


Fig. (7.83)

(25)

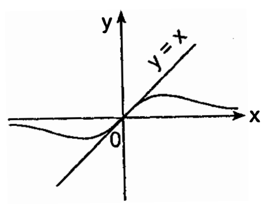


Fig. (7.84)

(26)

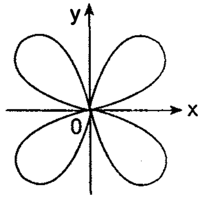


Fig. (7.85)

(27)

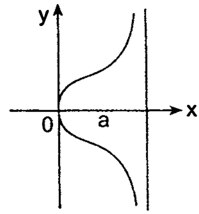


Fig. (7.86)

(28) Same as 17

(29) Similar to Ex. 17.

(30) Similar to Ex. 17.

(31)

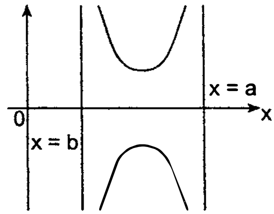


Fig. (7.87)

6. Procedure for tracing curves given in polar equations :

I. Find out the symmetry using the following rules.

(1) If on replacing θ by $-\theta$ the equation of the curve remains unchanged then the curve is symmetrical about the initial line.

(2) If on replacing r by $-r$ *i.e.* if the powers of r are even then the curve is symmetrical about the pole. The pole is then called the centre of the curve.

II. Form the table of values of r for both positive and negtive values of θ. Also find the values of θ which give $r = 0$ and $r = \infty$.

III. Find tan Φ. Also find the points where it is zero or infinity. Find the points at which the tangent coincides with the initial line or is perpendicular to it. (Refer to the explanation and the fig. in § 6 (b) page (2.11))

IV. Find out if the values of r and θ lie between certain limits *i.e.* find the greatest or least value of r so as to see if the curve lies within or without a certain circle.

Ex. 1 : Cardioide

(a) $r = a\,(1 + \cos\theta)$ **(S.U. 1991, 95)**

(b) $r = a\,(1 - \cos\theta)$ **(S.U. 1986, 90)**

(c) $\sqrt{r} = \sqrt{a}\,\cos(\theta/2)$

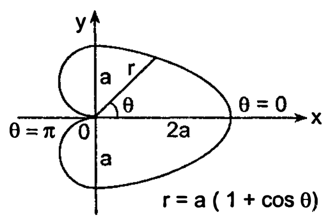


Fig. (7.88)

Sol. : 1 (a) : (i) The curve is symmetrical about the initial line since its equation remains unchanged by replacing θ by $-\theta$.

(ii) When $\theta = 0$, $r = 2a$ and when $\theta = \pi$, $r = 0$.

Also when $\theta = \pi/2$, $r = a$ and when

$\theta = 3\pi/2, \qquad r = a.$

Now $\tan \Phi = \dfrac{r\,d\theta}{dr} = \dfrac{a\,(1 + \cos\theta)}{-a \sin\theta}$

$\therefore \tan \Phi = -\cot\left[\dfrac{\theta}{2}\right] = \tan\left[\dfrac{\pi}{2} + \dfrac{\theta}{2}\right]$

$\therefore \Phi = \dfrac{\pi}{2} + \dfrac{\theta}{2}$ Hence, $\Psi = \theta + \Phi$ gives

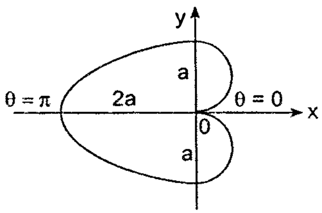
$$\Psi = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{3\theta}{2}$$

When $\theta = 0$, $\Psi = \dfrac{\pi}{2}$ *i.e.* the tangent at this point is perpendicular to the initial line. When $\theta = \pi$, $\Psi = 2\pi$ *i.e.* the tangent at this point coincides with initial line.

(iv) The following table gives some values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$2a$	$3a/2$	a	$a/2$	0

1 (b) : (i) The curve is symmetrical about the initial line.



r = a (1 – cos θ)

Fig. (7.89)

(ii) When $\theta = 0$, $r = 0$;

$\theta = \pi/2, r = a$

when $\theta = \pi, r = 2a$;

$\theta = 3\pi/2, r = a$

(iii) $\tan \Phi = r\dfrac{d\theta}{dr} = \dfrac{a\,(1 - \cos\theta)}{a \sin\theta}$

$= \tan\dfrac{\theta}{2} \qquad \therefore \Phi = \dfrac{\theta}{2}$

Hence, $\Psi = \theta + \Phi$ gives $\Psi = \theta + \dfrac{\theta}{2} = \dfrac{3\theta}{2}$

When $\theta = 0$, $\Psi = 0$ *i.e.* the tangent at this point coincides with the initial line when $\theta = \pi$, $\Psi = 3\pi/2$ *i.e.* the tangent at this point is perpendicular to the initial line.

(iv) The following table gives some values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	0	$a/2$	a	$3a/2$	$2a$

1 (c) : Squaring $\sqrt{r} = \sqrt{a}\cos\left(\dfrac{\theta}{2}\right)$ we get $r = a\cos^2\left(\dfrac{\theta}{2}\right) = \dfrac{a}{2}\,(1 + \cos\theta)$.

This is the cardioide similar to the cardioide shown in 1 (a) with $a/2$ in place of a.

Ex. 2 : Pascal's Limacon $r = a + b\cos\theta$

(a) Case 1 : $a > b$

(i) The curve is symmetrical about the initial line.

(ii) Since $a > b$, r is always positive.

(iii) The following table gives some values of r and θ.

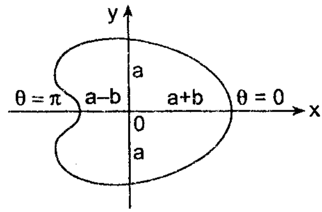


Fig. (7.90)

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$a+b$	$a+b/2$	a	$a-b/2$	$a-b$

(b) Case II : $a = b$

When $a = b$ we get $r = a(1+\cos\theta)$, the cardioide shown in the ex. 1 (i)

(c) Case III : $a < b$ **(S.U. 1992)**

(i) The curve is symmetrical about the initial line.

(ii) $r = 0$ when $a + b\cos\theta = 0$

i.e. $\theta = \cos^{-1}\left(-\frac{a}{b}\right)$.

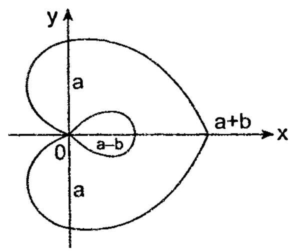


Fig. (7.91)

(iii) When $\theta = 0$, $r = a + b$ and is maximum.

(iv) When $\theta = \cos^{-1}(-a/b)$, $r = 0$ *i.e.* the curve passes through the origin.

(v) When $\cos^{-1}\left(-\frac{a}{b}\right) < \theta < \pi$, r is negative and at $\theta = \pi$, $r = a - b$.

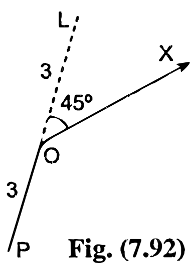


Fig. (7.92)

It may be noted that to plot a point $(-r, \theta)$, we rotate the radius vector through θ and measure r in opposite direction in this position. For example, to get the point $(-3, 45^\circ)$, we rotate the radius vector from OX through 45° into the position OL. We have to measure along OL a distance -3 *i.e.* we have to measure a distance 3 not along OL, but in opposite direction. Producing LO to P so that $OP = 3$ units we get the required point P.

(vi) The following table gives some values of r and θ

θ	0	$\pi/2$	$\cos^{-1}(-a/b)$	$\cos^{-1}(-a/b) < \theta < \pi$	π
r	$a+b$	a	0	negative	$a-b$

Note that we get a loop inside another toop as shown in the figure.

Ex. : Trace the curve $r = 2 + 3\cos\theta$

(S.U. 2007)

Sol. : Following the above line we see that the curve $r = 2 + 3\cos\theta$ can be shown as on the right.

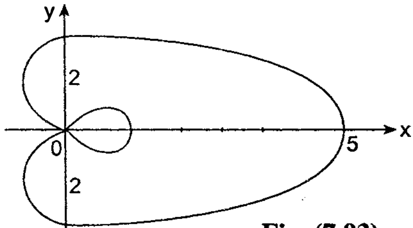


Fig. (7.93)

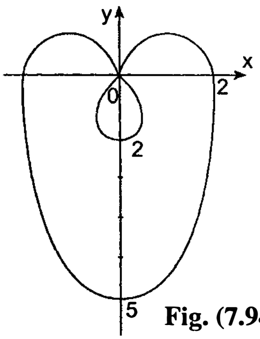


Fig. (7.94)

(d) Pascal's Limacon is also given by $r = a + b\sin\theta$ where the y-axis becomes the initial line.

For example, the curve $r = 2 - 3\sin\theta$ is shown in the neighbouring figure.

Ex. 3 : Bernoulli's Lemniscate $r^2 = a^2\cos 2\theta$ or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

(S.U. 1995, 97, 98, 2003)

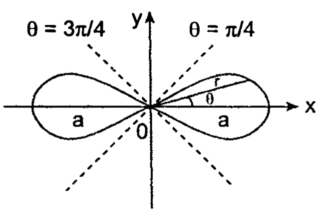


Fig. (7.95)

Sol. : (i) Since on changing r by $-r$ the equation remains unchanged, the curve is symmetrical about the initial line.

(ii) Since on changing θ by $-\theta$ the equation remains unchanged the curve is symmetrical about the pole.

(iii) Considering only positive values of r we get the following table.

θ	0	$\pi/6$	$\pi/4$	$\pi/4 < \theta < 3\pi/4$	$3\pi/4$	π
r^2	a^2	$a^2/2$	0	negative	0	a^2
r	a	$a/\sqrt{2}$	0	imaginary	0	a

(iv) Consider the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. If we put from $x^2 - y^2 = 0$ *i.e.* $y = \pm x$ *i.e.* $\theta = \pi/4$ and $\theta = 3\pi/4$, we see that $\theta = \pi/4$ and $\theta = 3\pi/4$ are tangents at the origin.

7. Curves of the form

$r = a\sin n\theta$ or $r = a\cos n\theta$

Sol. : (1) Since $\sin n\theta$ or $\cos n\theta$ cannot be greater than one in both the cases r cannot be greater than a. Hence, the curve wholly lies within the circle of radius a.

(2) To find the curve $r = a \sin n\theta$ put $r = 0$. Then $n\theta = 0, \pi, 2\pi, ...$

$$\therefore \theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}$$

Draw these lines. If n is odd there are n loops in alternate divisions and if n is even there are $2n$ loops one in each division.

(3) To trace the curve $r = a \cos n\theta$ put $r = 0$ Then,

$$n\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$$

$$\therefore \theta = -\frac{\pi}{2n}, \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, ...$$

Draw these lines. If n is odd there are n loops in alternate divisions and if n is even there are $2n$ loops one in each division.

Ex. 1 : Three-leaved Rose

(i) $r = a \sin 3\theta$ **(S.U. 1996, 97, 99, 2004)**

(ii) $r = a \cos 3\theta$. **(S.U. 1999)**

Sol. : (i) The curve consists of three loops lying within the circle $r = a$.

Put $r = 0$.

$\therefore \sin 3\theta = 0$

$\therefore 3\theta = 0, \pi, 2\pi, ...$

$$\therefore \theta = 0, \frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

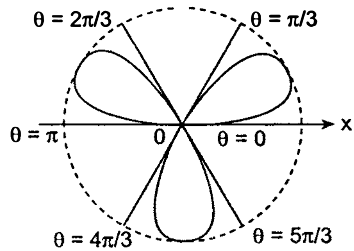


Fig. (7.96)

Draw these lines and place equal loops in alternate divisions.

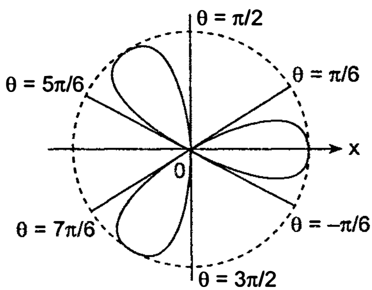


Fig. (7.97)

(ii) The curve consists of three loops lying within the circle $r = a$. Put $r = 0$.

$\therefore \cos 3\theta = 0$

$$\therefore 3\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} .....$$

$$\theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$$

Draw these lines and place equal loops in alternate divisions.

Ex. 2 : Four leaved Rose

(i) $r = a \sin 2\theta$ **(S.U. 1978, 1980)** (ii) $r = a \cos 2\theta$ **(S.U. 1997, 2003)**

Sol. : (i) The curve consists of four (2×2) loops lying within the circle $r = a$. Put $r = 0$, $\sin 2\theta = 0$

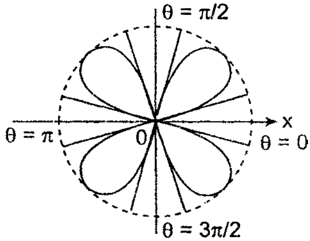


Fig. (7.98)

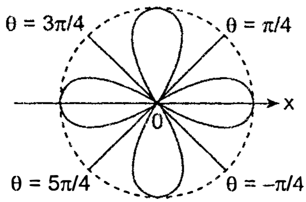


Fig. (7.99)

$\therefore\ 2\theta = 0, \pi, 2\pi \ldots.$

$\therefore\ \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Draw these lines and place equal loops in all divisions.

(ii) The curve consists of four (2×2) loops lying within the circle $r = a$. Put $r = 0$.

$\therefore\ \cos 2\theta = 0$

$\therefore\ 2\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots.$

$\therefore\ \theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots.$

Draw these lines and place equal loops in all divisions.

Ex. 3 : Spiral of Archimedes $r = a\,\theta$

Sol. : Let us consider first only positive values of θ. When $\theta = 0$, $r = 0$, so the curve passes through the pole. As θ increases through $\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ .. and so on we see that r increases and the point tracing the curve goes away from the pole as shown in the neighbouring figure. When $\theta \to \infty$, $r \to \infty$.

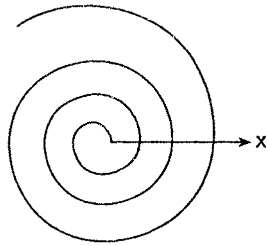


Fig. (7.100)

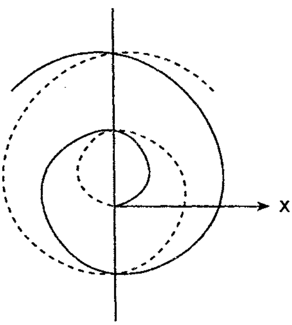


Fig. (7.101)

If we consider negative values of θ also we get a curve shown by dotted line. (Fig. 7.101)

Ex. 4 : Reciprocal Spiral $r\,\theta = a$ (S.U. 1985)

Sol. : Let us first consider only positive values of θ. From the values considered below we see that as θ increases r decreases. This means as θ increases the point tracing the curve goes nearer and nearer to the pole and when $\theta \to \infty$, $r \to 0$.

$\theta = \ldots\ 2\pi, \frac{3\pi}{2}, \pi, \frac{\pi}{2} \ldots.$

$r = \frac{a}{\theta} \ldots. \frac{a}{2\pi}, \frac{2a}{3\pi}, \frac{a}{\pi}, \frac{2a}{\pi} \ldots.$

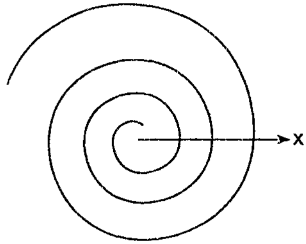


Fig. (7.102)

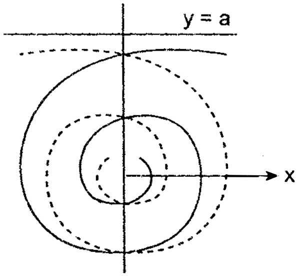


Fig. (7.103)

Further from the above values we note that as $\theta \to 0$, $r \to \infty$. If we consider negative values of θ also we get a curve shown by dotted line.

It may be noted that the line $y = a$ is an asymptote.

Ex. 5 : Equiangular Spiral $r = ae^{b\theta}$

(S.U. 1985)

Sol. : When $\theta = 0$, $r = a$. As θ increases r also increases. When $\theta \to \infty$, $r \to \infty$.

When $\theta \to -\infty$, $r \to 0$

$$\text{Further , } \tan \Phi = r \frac{d\theta}{dr}$$

$$\therefore \tan \Phi = \frac{r}{abe^{\theta}} = \frac{r}{br} = \frac{1}{b}$$

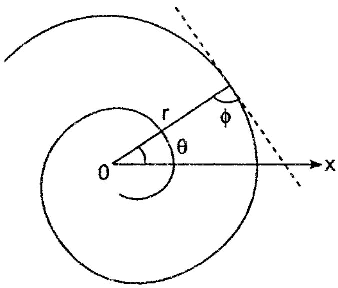


Fig. (7.104)

Thus, Φ *i.e.* the angle between the tangent and the radius vector is always constant. Hence, the name equiangular.

Ex. 6 : $r = a (\sec \theta + \cos \theta)$

(S.U. 1979, 2004)

Sol. : The equaition can be written as

$$r^2 = a (r \sec \theta + r \cos \theta)$$

$$= a \left[\frac{r^2}{r \cos \theta} + r \cos \theta\right]$$

$$\therefore x^2 + y^2 = a \left[\frac{x^2 + y^2}{x} + x\right]$$

$$\therefore y^2 (x - a) = x^2 (2a \quad x)$$

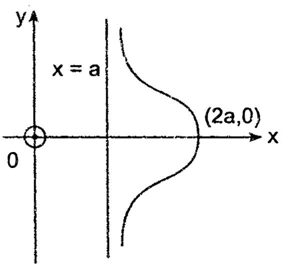


Fig. (7.105)

(i) The curve is symetrical about x-axis.

(ii) When $x = 0$, $y = 0$. Further $y^2 + 2x^2 = 0$ are the tangents at the origin. The tangents at the origin are imaginary. The origin is an isolated point.

(iii) $x = a$ is an asymptote.

Exercise II

1. (a) $r = 2 (1 + \cos \theta)$, (b) $r = 1 + \cos \theta$ **(S.U. 1978, 80, 93, 94, 97)**

2. $r = \left[\frac{a}{2}\right] (1 + \cos \theta)$ **(S.U. 1989)**

3. $r = 3 + 2 \cos \theta$ **(S.U. 1978, 2000)**

4. $r = 1 + 2\cos\theta$ **(S.U. 1980)**

5. $r^2 = 4\cos 2\theta$ **(S.U. 1980, 99)**

6. $r(1 + \cos\theta) = 2a$ **(S.U. 1981)**

7. $r(1 - \cos\theta) = 2a$ **8.** $r = 2a\cos\theta$

9. $r = 3\cos 2\theta$ **(S.U. 1988)**

10. $r = a(1 + \sin\theta)$ **(S.U. 1981, 2003)**

11. $r^2\cos 2\theta = a^2$ or $r^2 = a^2\sec 2\theta$

12. $r^2 = a^2\sin 2\theta$ **(S.U. 2005, 06)**

13. $r = a(1 - \sin\theta)$ **(S.U. 2003)**

14. $r = 2a\tan\theta\sin\theta$

[Ans. :

1. Similar to solved Ex.1 (1) Fig. (7.88). **2.** Similar to solved Ex. 1(1).

3. Similar to solved Ex. 2 (a) Fig. (7.90). **4.** Similar to solved Ex. 2 (c) Fig. (7.91)

5. Similar to solved Ex. 3. Fig. (7.95).

6. Parabola opening on the left. **7.** Parabola opening on the right.

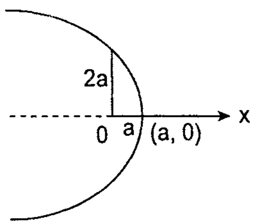


Fig. (7.106)

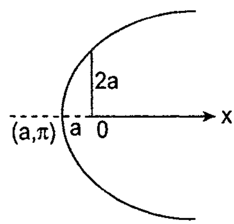


Fig. (7.107)

8. Circle

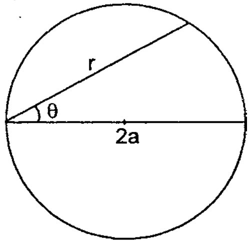


Fig. (7.108)

9. Cricle similar to the previous one with $a = 3/2$

10.

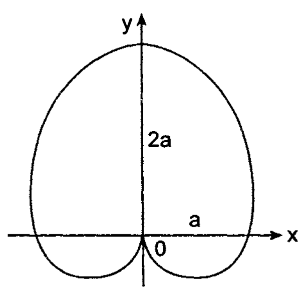


Fig. (7.109)

11.

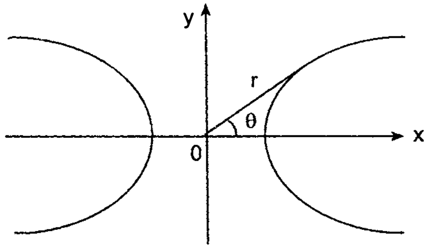


Fig. (7.110)

12.

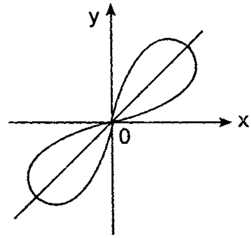


Fig. (7.111)

13.

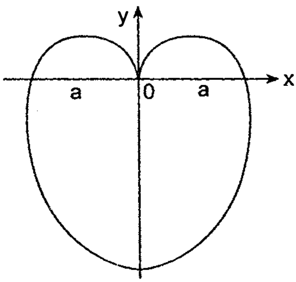


Fig. (7.112)

14. See remark under Ex. 1 page (7.10) See Fig. (7.45).

8. Some More Curves :

(1) Cycloids

Because of its graceful form and beautiful properties, mathematicians call this curve the Helen of Geometry. (After all, mathematicians are not dry as they are sometimes called !) When a circle rolls, on a straight line without sliding any fixed point on its circumference traces a cycloid.

The curve traced by a fixed point on the cricumference of a cricle, which rolls without sliding on the cricumference of another fixed cricle is called an epicycloid or hypocycloid.

If the rolling circle is outside the fixed cricle the curve is called the **epicycloid** and if the rolling cricle is inside the fixed cricle, the curve is called the **hypocycloid.** We have seen one particular hypocycloid in Ex. (3) page 7.11.

If the radius of the rolling circle is equal to the radius of the fixed cricle, the epicycloid is called cardioide because of its heart like shape (Cardio = Heart, eidos = shape). We have studied this curve Ex. 1 on page 7.20.

The cyloid is generally given in one of the following forms.

(a) $x = a\,(t + \sin t),\ y = a\,(1 + \cos t)$ **(b)** $x = a\,(t - \sin t),\ y = a\,(1 + \cos t)$

(c) $x = a\,(t + \sin t),\ y = a\,(1 - \cos t)$ **(S.U. 2004, 07)**

(d) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ **(S.U. 1987)**

Sol. : (a) $\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = -a \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 + \cos t)} = -\tan\left[\frac{t}{2}\right]$$

Some of the values of x, y and dy/dx are

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	$-a\pi$	$-a(\pi/2 + 1)$	0	$a(\pi/2 + 1)$	$a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	$-\infty$

From the above table we see that at $t = -\pi$ and at $t = \pi$, the tangents are parallel to the y-axis. These points are called cusps. Further at $t = 0$, the tangent is parallel to the x-axis.

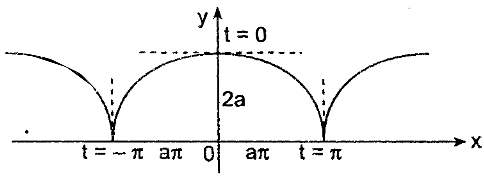


Fig. (7.113)

(b) $\frac{dx}{dt} = a(1 - \cos t)$, $\frac{dy}{dt} = -a \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 - \cos t)} = -\cot\left[\frac{t}{2}\right]$$

Some of the values of x, y, dy/dx are

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a(\pi/2 - 1)$	$a\pi$	$a(3\pi/2 + 1)$	$2a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

From the above table we see that at $t = 0$ and $t = 2\pi$ the tangents are parallel to the y-axis. These are cusps. Further at $t = \pi$, $y = 0$ and $dy/dx = 0$ *i.e.* the x- axis is the tangent.

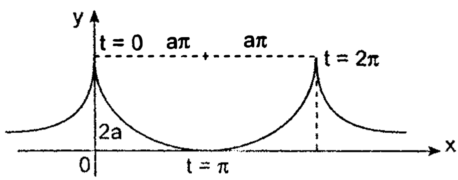


Fig. (7.114)

(c) $\frac{dx}{dt} = a\,(1 + \cos t\,)\,,\ \frac{dy}{dt} = a \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a\,(1 + \cos t)} = \tan\left[\frac{t}{2}\right]$$

Some of the values of x, y, dy/dx are

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	$-a\pi$	$-a\,(\pi/2 + 1)$	0	$a\,(\pi/2 + 1)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

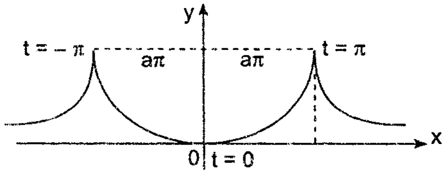


Fig. (7.115)

(d) $\frac{dx}{dt} = a\,(1 - \cos t\,)\,,\ \frac{dy}{dt} = a \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a\,(1 - \cos t)} = \cot\left[\frac{t}{2}\right]$$

Some of the values of x, y, dy/dx are

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a\,(\pi/2 - 1)$	$a\pi$	$a\,(3\pi/2 + 1)$	$2a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	$-\infty$

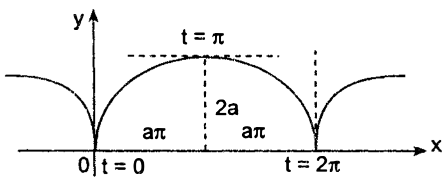


Fig. (7.116)

(2) Catenary **(S.U. 2003)**

If a heavy uniform string is allowed to hang freely under gravity, it hangs in the form a curve called catenary. Its equation can be shown to be

$$y = c \cos h\left(\frac{x}{e}\right)$$

Its shape is as shown in the figure

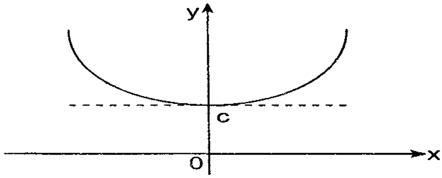


Fig. (7.117)

Its lowest point is at a dirtance c from the origin because,

when $x = 0$, $y = c \cosh 0 = c$.

Further, we see that $y = c \cosh \dfrac{x}{c} = c\left(\dfrac{e^{x/c} + e^{-x/c}}{2}\right)$

Hence, if x is replaced by $-x$ the equation remains the same. Hence, the curve is symmetrical about the y-axis

Since $\dfrac{dy}{dx} = \dfrac{e^{x/c} - e^{-x/c}}{2}$ is zero. when $x = c$, the tangent at $x = c$ is parallel to the x-axis.

(3) Tractrix

$$x = a \cos t + \frac{1}{2} a \log \tan^2\left(\frac{t}{2}\right), \quad y = a \sin t$$

Sol. : $$\frac{dx}{dt} = -a \sin t + \frac{a}{2} \cdot \frac{2 \tan\left(\frac{t}{2}\right) \sec^2\left(\frac{t}{2}\right) \cdot \frac{1}{2}}{\tan^2\left(\frac{t}{2}\right)}$$

$$= -a \sin t + \frac{a}{2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)}$$

$$= -a \sin t + \frac{a}{\sin t} = \frac{a}{\sin t}(1 - \sin^2 t)$$

$$= \frac{a \cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a \cos t \cdot \frac{\sin t}{a \cos^2 t} = \tan t$$

Some of the values of t, x, y, $\dfrac{dy}{dx}$ are ,

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	∞	0	∞	0	∞
y	0	$-a$	0	a	0
dy/dx	0	$-\infty$	0	∞	0

From the above data we see that as $t \to -\pi$ the point on the curve is $(\infty, 0)$ and as $t \to 0$ the point is $(-\infty, 0)$.

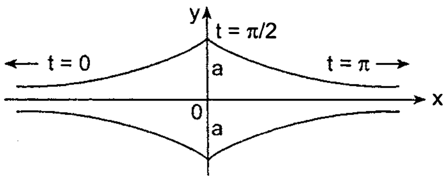


Fig. (7.118)

9. Some Solids

You have studied to some extent plane and straight line in three dimensions. We shall here get acquainted with some more three dimensional solids such as sphere, cylinder, cone, paraboloid etc.

1. Plane : The general equation of a plane is linear of the form $a'x + b'y + c'z + d' = 0$ which can be written as

$$\frac{a'}{d'}x + \frac{b'}{-d'}y + \frac{c'}{-d'}z = 1$$

i.e. $ax + by + cz = 1$

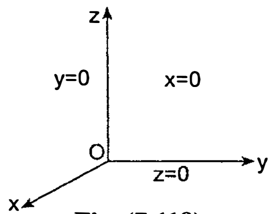


Fig. (7.119)

Simplest planes are the yz plane whose equation is $x = 0$; the zx plane whose equation is $y = 0$; the xy plane whose equation is $z = 0$.

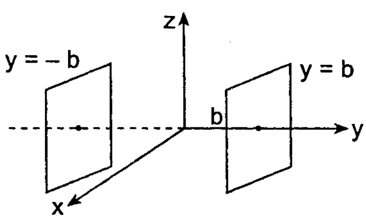


Fig. (7.120)

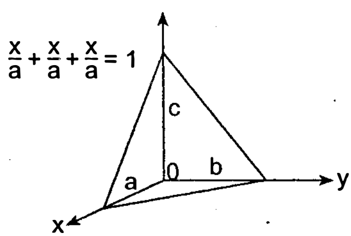


Fig. (7.121)

The planes parallel to the coordinate planes are $x = \pm a$, $y = \pm b$, $z = \pm c$.

The plane $ax + by + cz = 1$ cuts off intercepts $1/a$, $1/b$, $1/c$ on the coordinate axes. Still more common form of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This plane cuts off intercepts a, b, c on the coordinate axes.

2. Cylinders : An equation involving only two variables represents a cylinder in three dimensional geometry. Thus, $f(x, y) = 0, f(y, z) = 0, y(z, x) = 0$ represent cylinders in three dimensions.

(a) Right Circular Cylinders :
$x^2 + y^2 = a^2$ is a right circular cylinder, whose generator is parallel to the z-axis. Similarly, $y^2 + z^2 = b^2$ is a cylinder whose generators are paralel to the x-axis. $z^2 + x^2 = c^2$ is a cylinder whose generators are parallel to the y-axis.

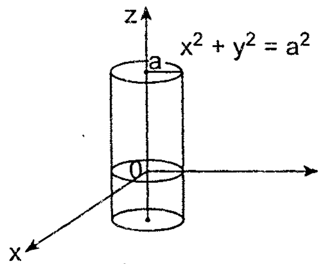


Fig. (7.122)

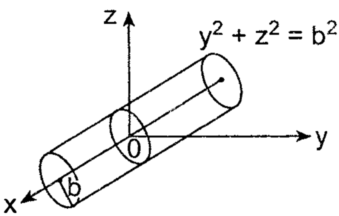


Fig. (7.123)

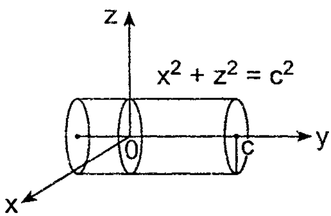


Fig. (7.124)

(b) Parabolic Cylinders : The equation $y^2 = 4ax$ represents a paraola in two dimensions. But in three dimensions it represents a parabolic cylinder.

The equations $z^2 = 4by$, $x^2 = 4cz$ also represent parabolic cylinders as shown in the following figures.

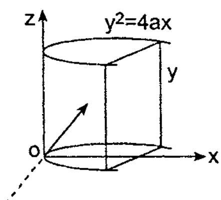


Fig. (7.125)

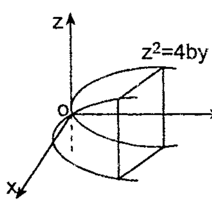


Fig. (7.126)

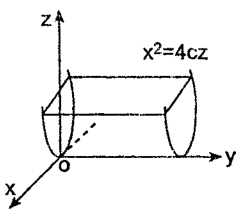


Fig. (7.127)

3. Sphere : The equation of sphere in standard form *i.e.* with center at the origin and radius a is $x^2 + y^2 + z^2 = a^2$.

4. Cone : The right circular cone given by $x^2 + y^2 = z^2$ is shown in the following figure 7.129.

The other two cones $y^2 + z^2 = x^2$ and $x^2 + z^2 = y^2$ are shown in the following figures 7.130 and 7.31.

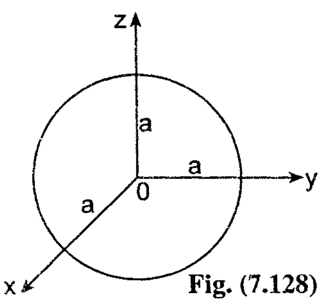


Fig. (7.128)

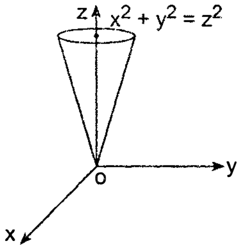


Fig. (7.129)

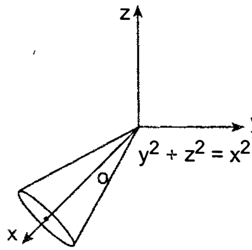


Fig. (7.130)

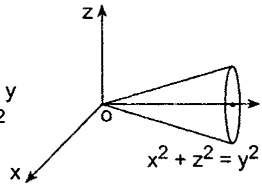


Fig. (7.131)

5. Ellipsoid : The equation of ellipsoid in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The sections of the ellipsoid by planes parallel to the coordinate planes are ellipses.

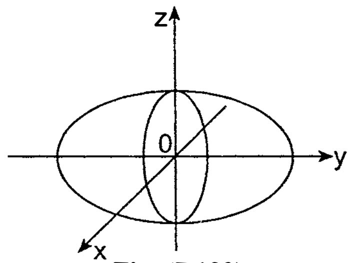


Fig. (7.132)

6. Hyperboloid : The equations of hyperboloid of one sheet and two sheet are respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$$

The hyperboloid of one sheet is shown on the right. Sections of the hyperboloid of one sheet parallel to the *xy*-planes are ellipses and sections parallel to *yz*-plane or *zx*-plane are hyperbolas.

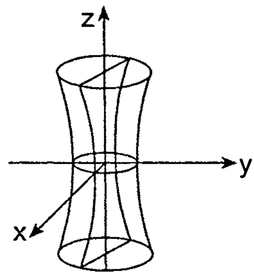


Fig. (7.133)

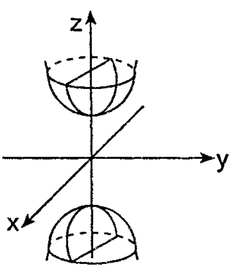


Fig. (7.134)

The hyperboloid of two sheets is shown on the left.

Sections of this hyperboloid by planes paralles to the *xy*-plane *i.e.* $z = k$, $|k| > c$ are real ellipses.

Sections of this hyperboloid by planes parallel to the *zx*-plane *i.e.* $y = k$ are hyperbolas.

7. Paraboloids : We shall discuss here only one elliptic paraboloid given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}.$$

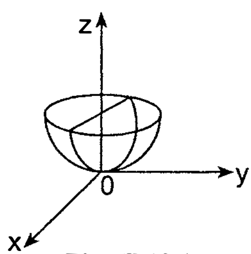


Fig. (7.135)

Sections of this paraboloid parallel to the xy-planes are ellipse. Sections parallel to the yz-plane or the zx-plane are parabolas.

Exercise - III

Sketch the following figures.

1. $y = x^2, x^2 + y^2 = 2$
2. $x^2 + y^2 = b^2, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \quad 1$
3. $y^2 = x + 6, x = y$
4. $y = x^2, y = 1, x = 1, x = 2$
5. $y = x^2, y = x + 2, x = -1, x = 2$
6. $x^2 + y^2 = ax, ax = by$
7. $x^2 + y^2 = by, ax = by$
8. $x^2 - y^2 = 1, x^2 - y^2 = 2, xy = 4, xy = 2$
9. $xy = 2 - y$
10. $y = x^2 - 3x, \; y = 2x$
11. $y^2 = -4(x-1), \; y^2 = -2(x-2)$
12. $a^2 x^2 = 4y^2 (a^2 - y^2)$ or $x = a \sin 2t, y = a \sin t$
13. $r(\cos\theta + \sin\theta) = 2a \sin\theta \cos\theta$ or $(x+y)(x^2+y^2) = 2axy$
14. $x^4 + y^4 = 2a^2 xy$ or $r^2 = \dfrac{2a^2 \sin\theta \cos\theta}{\sin^4\theta + \cos^4\theta}$
15. $x^6 + y^6 = a^2 x^2 y^2$ or $r^2 \quad a^2 \cdot \dfrac{\sin^2\theta \cos^2\theta}{\sin^6\theta + \cos^6\theta}$
16. $x^4 - 2xya^2 + a^2 y^2 = 0$ or $r^2 = 2a \tan\theta \sec^2\theta - a^2 \tan^2\theta \sec^2\theta$
17. $y = x^2 - 6x + 3, \; y = 2x - 9$
18. $y = 3x^2 - x - 3, \; y = -2x^2 + 4x + 7$
19. $y^2 = 4x, \; y = 2x - 4$
20. $y = 4x - x^2, y = x$
21. A cylindrical hole in a sphere.
22. A cone and paraboloid given by $z^2 = x^2 + y^2$ and $z = x^2 + y^2$
23. A paraboloid $x^2 + y^2 = az$, and the cylinder $x^2 + y^2 = a^2$
24. The paraboloids $z \quad 4 - x^2 - \dfrac{y^2}{4}, \; z = 3x^2 + \dfrac{y^2}{4}$
25. $y = 4x(1-x)$

[Ans. :

(1)

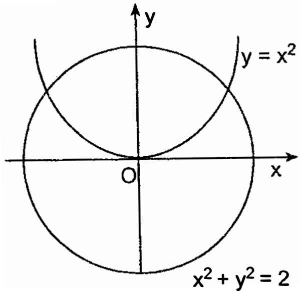


Fig. (7.136)

(2)

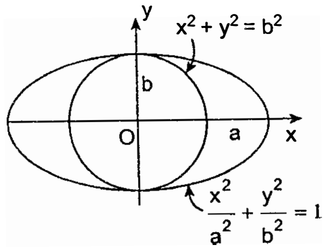


Fig. (7.137)

(3)

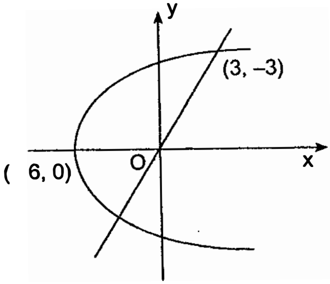


Fig. (7.138)

(4)

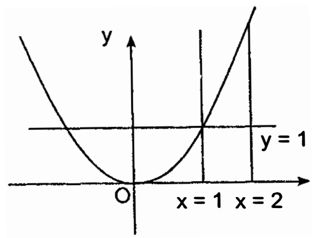


Fig. (7.139)

(5)

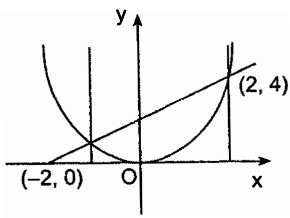


Fig. (7.140)

(6)

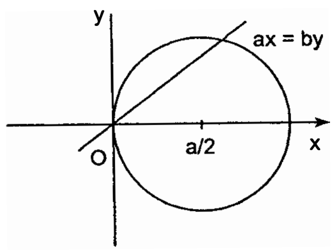


Fig. (7.141)

(7)

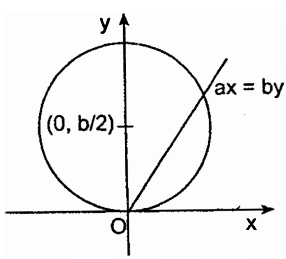


Fig. (7.142)

(8)

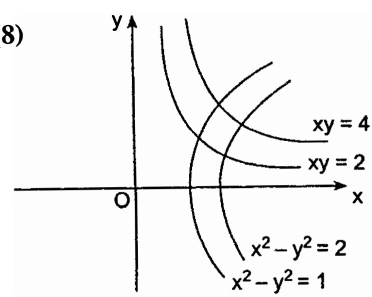


Fig. (7.143)

(9)

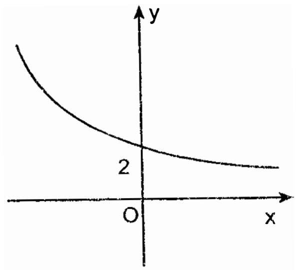


Fig. (7.144)

(10)

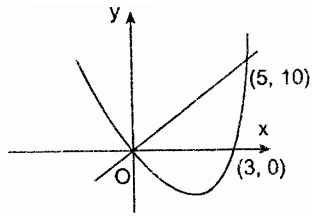


Fig. (7.145)

(11)

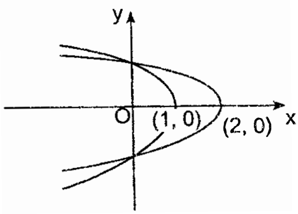


Fig. (7.146)

(12)

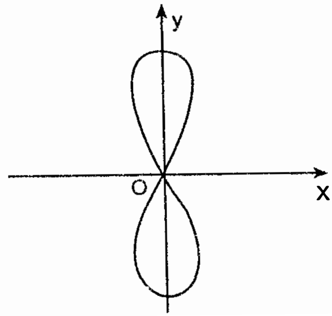


Fig. (7.147)

(13)

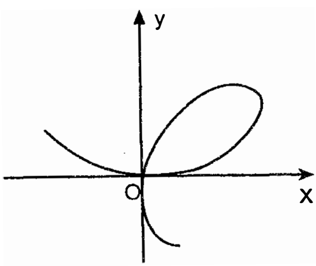


Fig. (7.148)

(14)

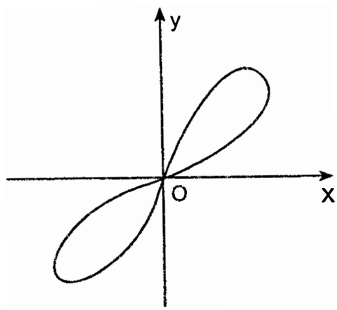


Fig. (7.149)

(15)

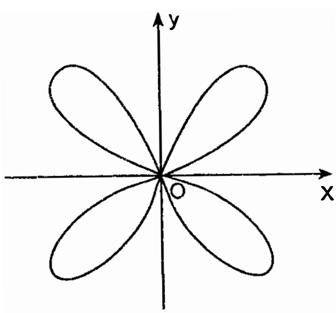


Fig. (7.150)

(16)

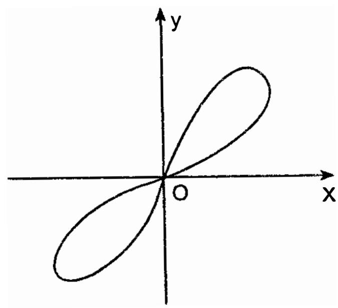


Fig. (7.151)

(17)

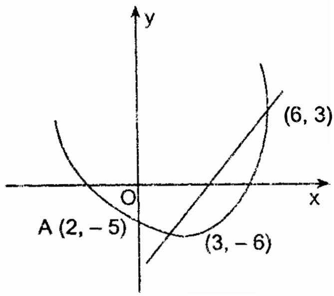


Fig. (7.152)

(18)

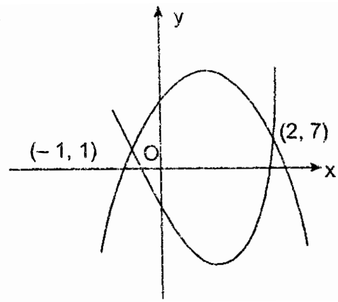


Fig. (7.153)

(19)

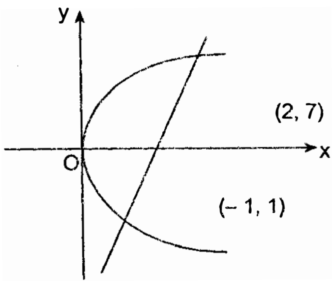


Fig. (7.154)

(20)

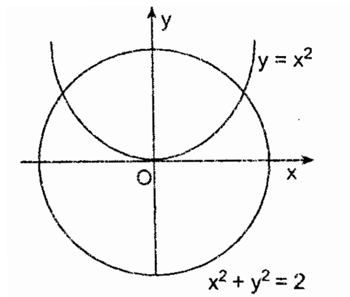


Fig. (7.136)

(21)

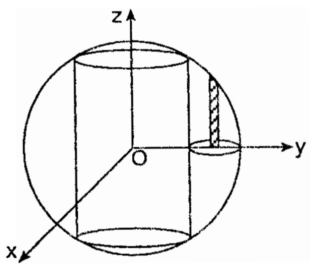


Fig. (7.156)

(22)

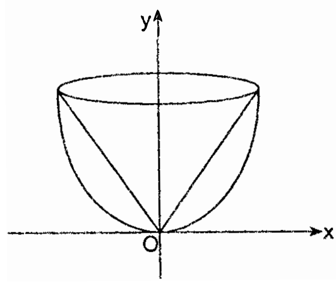


Fig. (7.157)

(23)

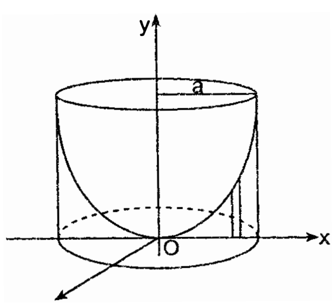


Fig. (7.158)

(24)

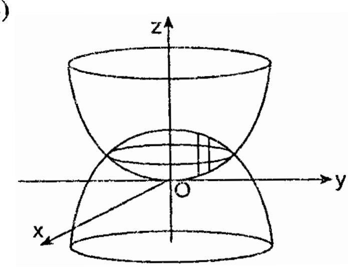


Fig. (7.159)

(25)

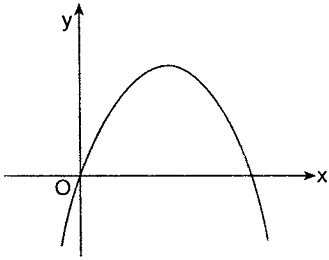


Fig. (7.160)

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