IC - Tutorial - 8

Krishnaraj P.T.
109054. I3. Div-9

Q.1. $I_n = \int_0^{\pi/4} \sin^{2n} x \, dx.$

$$= \int_0^{\pi/4} \underbrace{\sin^{2n-1} x}_{u} \cdot \underbrace{\sin x}_{v} \cdot dx.$$

By. uv Rule

$$= \left\{ \sin^{2n-1} x \cdot [-\cos x] \right\}_0^{\pi/4} - \int_0^{\pi/4} (2n-1) \sin^{2n-2} x \cos x \, [-\cos x] \cdot dx$$

$$= \frac{1}{2^n} + (2n-1) \int_0^{\pi/4} \sin^{2(n-1)} x \cos^2 x \cdot dx.$$

$$= \frac{1}{2^n} + (2n-1) \int_0^{\pi/4} \sin^{2(n-1)} (1 - \sin^2 x) \, dx$$

$$= \frac{1}{2^n} + 2n - 1 + I_{n-1} - (2n-1) I_n$$

$$I_n + 2n I_n - I_n = \frac{1}{2^n} + (2n-1) I_{n-1}$$

$$I_n = \frac{1}{n 2^{n+1}} + \left(1 - \frac{1}{2^n}\right) I_{n-1}$$

Hence Proved.

Q.2. $I_n = \int_0^{\pi/3} \cos^n x \cdot dx$

$$= \int_0^{\pi/3} \underset{\uparrow u}{\cos^{n-1} x} \cdot \underset{\uparrow v}{\cos x} \cdot dx$$

$$= \left[\cos^{n-1} x \cdot \sin x\right]_0^{\pi/3} - \int_0^{\pi/3} (n-1)\cos^{n-2} x \, (-\sin x)(\sin x)\,(dx)$$

$$= \left[\frac{1}{2^{n-1}} \cdot \frac{\sqrt{3}}{2}\right] + (n-1)\int_0^{\pi/3} \cos^{n-2} x \cdot \left[1 - \cos^2 x\right] dx$$

$$= \frac{\sqrt{3}}{2^n} + (n-1)\int_0^{\pi/3} \cos^{n-2} x \cdot dx - (n-1)\int_0^{\pi/2} \cos^n x \cdot dx.$$

$$I_n = \frac{\sqrt{3}}{2^n} + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore \quad I_n = \frac{\sqrt{3}}{n 2^n} + \left(1 - \frac{1}{n}\right) I_{n-2}$$

$$\int_0^{\pi/3} \cos^6 x \, dx =$$

$$I_6 = \frac{\sqrt{3}}{6(2)^6} + \left(1 - \frac{1}{6}\right) I_{6-2}$$

$$= \frac{\sqrt{3}}{3 \cdot 2^7} + \frac{5}{6} \cdot I_4$$

$$\boxed{I_6 = \frac{\sqrt{3}}{3(2)^7} + \frac{5}{6}\left[\frac{\sqrt{3}}{2^6} + \frac{3}{4}\left[\frac{\sqrt{3}}{2^3} + \frac{\pi}{6}\right]\right]}$$

$$I_4 = \frac{\sqrt{3}}{4(2)^4} + \left(1 - \frac{1}{4}\right) I_{4-2}$$

$$= \frac{\sqrt{3}}{2^6} + \frac{3}{4} I_2 = \frac{\sqrt{3}}{2^6} + \frac{3}{4}\left[\frac{\sqrt{3}}{2^3} + \frac{\pi}{6}\right]$$

$$I_2 = \frac{\sqrt{3}}{2^3} + \frac{1}{2} I_0 = \frac{\sqrt{3}}{2^3} + \frac{\pi}{6}$$

$$I_0 = \int_0^{\pi/3} 1 \cdot dx = \frac{\pi}{3}$$

(8.3). $I_n = \int_0^{\pi/4} \sin^n x \cdot dx$, we know,

$$I_n = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\pi/4} I_n = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\pi/4} + \frac{n-1}{n} I_{n-2}$$

$$I_n = -\frac{1}{2^n \times n} + \left(1 - \frac{1}{n}\right) I_{n-2}$$

$$\therefore \int_0^{\pi/4} \sin^6 x \, dx = \frac{-1}{2^6 \times 6} + \left(1 - \frac{1}{6}\right) I_4$$

$$= \frac{-1}{2^6} + \left(\frac{5}{6}\right)\left(\frac{3\pi - 4}{2^5}\right)$$

$$= \frac{-1}{2^6 \cdot 6} + \frac{15\pi - 20}{2 \cdot 2^6}$$

$$= \frac{5\pi}{64} - \frac{11}{48}$$

$$I_4 = \frac{-1}{4 \cdot 2^4} + \left(1 - \frac{1}{4}\right) I_2$$

$$= \frac{-1}{2^5} + \frac{3}{4}\left(\frac{\pi - 1}{8}\right) = \frac{3\pi - 3 - 1}{2^5} = \frac{3\pi - 4}{2^5}$$

$$I_2 = \frac{-1}{2 \times 2^2} + \frac{1}{2} \cdot I_0 = \frac{-1}{8} + \frac{\pi}{8} = \frac{\pi - 1}{8}.$$

$$I_0 = \int_0^{\pi/4} \sin^0 x \cdot dx = \pi/4$$

Q.4. $$I_n = \int_0^{\pi/4} \cos^{n} x \cdot dx$$

we know,

$$I_n = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{(n-1)}{n} (I_{n-2})$$

$$I_6 = \frac{1}{2^6 \cdot 6} + \left(1 - \frac{1}{6}\right) \cdot I_n$$

$$= \frac{1}{2^6 \cdot 6} + \frac{5}{6}\left(\frac{3\pi + 4 n}{2^5}\right)$$

$$\boxed{I_6 = \frac{5\pi}{64} + \frac{11}{48}}$$

$$I_4 = \frac{1}{2^4 (4)} + \left(1 - \frac{1}{4}\right) \cdot I_2$$

$$= \frac{1}{2^5} + \frac{3}{2^2}\left(\frac{\pi + 1}{2^3}\right) = \frac{3\pi + 4}{2^5}$$

$$I_2 = \frac{1}{\cancel{2}2^2 \cdot \cancel{4}2} + \frac{1}{2} I_0$$

$$= \frac{1}{8} + \frac{\pi}{8} = \frac{\pi + 1}{8}$$

$$\boxed{I_0 = \pi/4}$$

Q.5 $I_n = \int \tan^n x \cdot dx$

$= \int \tan^{n-2} x \cdot \tan^2 x \cdot dx$

$= \int \tan^{n-2} x \cdot (\sec^2 x - 1) \cdot dx$

$= \int \tan^{n-2} x \cdot \sec^2 x \, dx - \int \tan^{n-2} x \cdot dx$

$\Rightarrow$ Put $\tan x = t$

$\sec^2 x \cdot dx = dt$

$= \int t^{n-2} \cdot dt - \int \tan^{n-2} x \cdot dx$

$\frac{t^{n-1}}{n-1} - \int \tan^{n-2} x \cdot dx$

$$\therefore \boxed{I_n = \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x \cdot dx}$$

Q.6] $I_n = \int \sec^n x \cdot dx$

$= \int \sec^{n-2} x \cdot \sec^2 x \cdot dx$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \cdot \sec x \cdot \tan x \, [\tan x] \cdot dx$

$= \sec^{n-2} x \cdot \tan x - (n-2) \int \sec^{n-2} x \cdot \tan^2 x \cdot dx$

$= \sec^{n-1} x \cdot \tan x - (n-2) I_n + (n-2) I_{n-2}$

$I_n + n I_n - 2 I_n = \sec^{n-1} x \cdot \tan x + (n-2) I_{n-2}$

$I_n (n-1) = \sec^{n-1} x \cdot \tan x + (n-2) \cdot I_{n-2}$

$$I_n = \frac{\sec^{n-1}x \cdot \tan x}{n-1} + \left(\frac{n-2}{n-1}\right) \cdot I_{n-2}$$

$$\int_0^{\pi/4} = \frac{[\sec^{n-1}x \cdot \tan x]_0^{\pi/4}}{(n-1)} + \left(\frac{n-2}{n-1}\right) \cdot I_{n-2}$$

$$= \frac{(\sqrt{2})^{n-1}}{(n-1)} + \left(\frac{n-2}{n-1}\right) \cdot I_{n-2}$$

$$I_6 = \frac{(\sqrt{2})^5}{5} + \frac{4}{5} I_4$$

$$= \frac{(\sqrt{2})^5}{5} + \frac{4}{5}\left(\frac{8+2\sqrt{2}}{3}\right)$$

$$= \frac{4\sqrt{2}}{5} + \frac{8\sqrt{2}}{15} + \frac{32}{15}$$

$$= \frac{20\sqrt{2}}{15} + \frac{32}{15} \Rightarrow$$

$$\boxed{\int_0^{\pi/4} \sec^6 x \cdot dx = \frac{20\sqrt{2} + 33}{5}}$$

$$I_4 = \frac{2^3}{3} + \frac{2}{3} I_2 = \frac{8}{3} + \frac{2\sqrt{2}}{3}$$

$$= \frac{8+2\sqrt{2}}{3}.$$

$$\boxed{I_2 = \frac{\sqrt{2}}{1} + 0 = \sqrt{2}}$$