Ordinary Differential Equations

1. Form a differential equation whose general solution is

$i)\ y = ae^{-2x} + be^{-3x}$ **(Ans : $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$)**

$ii)\ y = e^x(Acosx + Bsinx)$ **(Ans : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$)**

Solve the differential equations-

1. $\frac{dy}{dx} = \frac{x\, sinx}{2e^y sinhy}$ **(Ans: $\frac{e^{2y}}{2} - y + xcosx - sinx = C$)**
2. $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ **(Ans : log[1 + tan($\frac{x+y}{2}$)] – x = C)**
3. $\frac{dy}{dx} = \frac{x^3 - y^3}{yx^2}$ **(Ans: cy = $e^{\frac{-x^3}{3y^3}}$)**
4. (x + y cot $\frac{x}{y}$)dy – y dx =0 (Ans : y cos $\frac{x}{y}$=C)
5. $\frac{dy}{dx} = \frac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$ (Ans : x.tany – xy – x^2y – tany = C)
6. $ylogy\, dx + (x - \log y)dy = 0$ **(Ans :2xlogy – (logy)^2= C)**
7. $(2y + y^4)dx + (2y^4 + xy^3 - 4x)dy = 0$ **(Ans : $xy + \frac{x}{y^2} + y^2 = C$)**
8. $xcos\, x\, \frac{dy}{dx} + (\cos x - xsin\, x)y = 1$ **(Ans : $xycosx - x = C$)**
9. $\frac{dy}{dx} - xy = -y^3 e^{x^2}$ **(Ans : $\frac{e^{x^2}}{y^2} = 2x + C$)**
10. $(y - 2x^3)dx - x(1 - xy)dx = 0$ **(Ans: $\frac{y}{x} + x^2 - \frac{y^2}{2} = C$)**
11. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$ **Ans : $\frac{x}{y} - 2logx + 3logy = C$**
12. $ye^y dx = (y^3 + 2xe^y)dy$ **(Ans : $\frac{x}{y^2} + e^{-y} = C$)**
13. $sin\, y\, \frac{dy}{dx} - cosx(2cosy - \sin^2 x)\, y = 0$
    (Ans : $4cosy = 2\sin^2 x + 2sinx = 1 = Ce^{-2sinx}$)
14. $\cos y - xsin\, y\, \frac{dy}{dx} = \sec^2 x$ **(Ans: $xcosy = tanx + C$)**
15. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx + x^2\, y\, dy = 0$ **(Ans : $\frac{3y^2}{x^2} - 2e^{-x^{-3}} = C$)**

APPLICATIONS OF DIFFERENTIAL EQUATIONS

Orthogonal Trajectories

13. Find the orthogonal trajectories of the family of $y^2 = 4ax$. **(Ans : $2x^2 + y^2 = C$)**

14. Find the orthogonal trajectory of the family of ellipses $\frac{1}{2}x^2 + y^2 = C$, where C > 0 **(Ans : $x^2 = ky$)**

[**Ref: Kreyszig, page-36**]

15. Find the orthogonal trajectory of the family of r = a cos θ. (Ans : $r = Csin\theta$)

16. Find the orthogonal trajectory of the family of r = a (1- cos θ). (Ans: r = C(1+cosθ))

17. Find the orthogonal trajectory of the family of $e^x + e^{-y} = c$. (Ans: $e^y - e^{-x} = C$)

Electric Circuits

20. An electric circuit contains an inductance of 0.5 henry and a resistance of 10 ohms in series with electro motive force of 20 volts. Find the current at any time t, it is zero at t=0. (Ans : $\frac{1}{5}(1 - e^{-200t})$)

21. A circuit consist of resistance R ohms and condenser C farads connected to constant electromotive force E, if $\frac{q}{C}$ is the voltage of condenser at time t after closing the circuit. Show that the voltage at time t, is $E\left(1 - e^{-\frac{t}{RC}}\right)$. Also find current flowing into the circuit.

22. The charge Q on the plate of a condenser of capacity $'C'$ charged through a resistance $'R'$ by steady voltage $'V'$ satisfies the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} = V$. If $Q = 0$ at $t = 0$ then show that $Q = CV\left[1 - e^{-t/RC}\right]$. Find the current flowing into the plate. (Ans: $i = \frac{V}{R}\, e^{-\frac{t}{RC}}$)

23.Find the current i in the circuit having resistance R and condenser of capacity C in series with emf E sin ωt.

24.The equation of L-R circuit is given by L$\frac{dI}{dt}$. + RI = 10 sin t .If I=0, at t = 0 , express I as a function of t . (Ans: $I = \frac{10}{\sqrt{R^2+L^2}}[\sin(t - \emptyset) + sin\emptyset\, e^{\frac{-Rt}{L}}$)

Heat Conduction

23. A steam pipe 20 cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity is k = 0.0003 cal/cm in steady state. Find the heat lost per hour through a meter length of the pipe, if the inner surface of the pipe is at 200 ℃ and the outer surface of the covering is at 30 ℃. (Ans : q=245443.3861)

24. A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm the inner surface is kept at 200 ℃ and outer surface at 50 ℃. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 m long and also find temperature at distance x=7.5 cm from the centre of pipe. (Ans: T=113 C)

Tracing of Curve

Trace the following curves

1) $y^2(2a-x)=x^3$
2) $(x^2+y^2)x=(x^2-y^2)$
3) $xy^2=a^2(a-x)$
4) $x^2y^2=a^2(y^2-x^2)$
5) $(x^2+a^2)y^2=a^2x^2$
6) $(x^2+4a^2)y=8a^3$
7) $x=a(t+sint)\,,y=a(1-cost)$
8) $x=a(t-sint)\,,y=a(1-cost)$
9) $x=a(t+sint)\,,y=a(1+cost)$
10) $r^2=a^2cos2\theta$
11) $r=a\,cos2\theta$
12) $r=a\,cos5\theta$
13) $r=a\,(1-cos\theta)$
14) $r=a\,sin2\theta$
15) $r=2\,sin5\theta$

Reduction Formulae, Beta and Gamma

1. Evaluate $\int_0^{\pi} x\sin^5 x\cos^8 x\,dx$ **Ans.** $\frac{8\pi}{1287}$

2. Evaluate $\int_0^{2a} x^3(2ax-x^2)^{\frac{3}{2}}\,dx$ **Ans.** $\frac{9\pi a^7}{16}$

3. Find the reduction formula for $\int_0^{\frac{\pi}{3}} cos^n x\,dx$ and using it evaluate $\int_0^{\frac{\pi}{3}} \cos^6 x\,dx$.

   Ans. $\boldsymbol{I_n = \frac{\sqrt{3}}{n2^n} + \frac{n-1}{n} I_{n-2}}$, $\boldsymbol{\frac{3\sqrt{3}}{32} + \frac{5\pi}{48}}$

4. If $I_n = \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\,dx$ then prove that $n(I_{n+1} - I_n) = \sin\frac{n\pi}{2}$ and hence find I_3.

   Ans. $1 + \frac{\pi}{4}$

5. If $I_n = \int_0^{\infty} e^{-x}\sin^n x\,dx$, Obtain the relation between I_n *and* I_{n-2} .

   Ans. $I_n = \frac{n(n-1)}{n^2+1} I_{n-2}$

6. Evaluate $\int_0^{\infty} x^7 e^{-2x^2}\,dx$ **Ans.** $3/16$

7. Evaluate $\int_0^{\infty} 3^{-4x^2}\,dx$ **Ans.** $\frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

8. Evaluate $\int_0^{\infty} \frac{x^4}{4^x}\,dx$ **Ans.** $\frac{24}{(\log 4)^5}$

9. Evaluate $\int_0^{1} \frac{dx}{\sqrt{-\log x}}$ **Ans.** $\sqrt{\pi}$

10. Evaluate $\int_0^{1} x^3(\log x)^4\,dx$ **Ans.** $\frac{3}{128}$

11. Show that $\int_0^{1} \frac{dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}}$

12. Show that $\int_0^{\infty} \frac{x^6-x^3}{(1+x^3)^5} x^2\,dx = 0$

13. Evaluate $\int_3^{7} (x-3)^{\frac{1}{4}}(7-x)^{\frac{1}{4}}\,dx$ **Ans.** $\frac{2}{3\sqrt{3}}\left(\gamma\left(\frac{1}{4}\right)\right)^2$

14. Prove that $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

15. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\,d\theta = \frac{\pi^2}{2}$.

Differentiation Under Integral Sign (DUIS)

1. Show that $\int_0^1 \frac{x^a-1}{logx} = \log(a+1)\,,\ \ a \geq 0$
2. Show that $\int_0^\infty \frac{e^{-x}-e^{-ax}}{xsecx}dx = \frac{1}{2}\log(\frac{a^2+1}{2})$
3. Find $\int_0^\infty \frac{e^{-ax}sinmx}{x}dx$ and hence evaluate $\int_0^\infty \frac{sinx}{x}dx$
4. Prove that $\int_0^\infty \frac{1-cosax}{x^2}dx = \frac{\pi a}{2}$
5. If $y = \int_0^x f(t)sina(x-t)dt$ then show that $\frac{d^2y}{dx^2} + a^2y = af(x)$
6. If $\emptyset(a) = \int_a^{a^2} \frac{sinax}{x}dx$ then find $\frac{d\emptyset}{da}$
7. Verify the DUIS rule for the $\int_a^{a^2} logaxdx$

Error Function

1. Prove that $erfc(-x) + erfc(x) = 2$
2. Show that $\int_a^b e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}[\mathrm{erf}(b) - \mathrm{erf}(a)]$
3. Find $\frac{d}{dx}erfc(ax^n)$
4. Prove that $\mathrm{erf}(x)$ is an odd function. Deduce that $erfc(-x) - \mathrm{erf}(x) = 1$
5. Show that $\int_0^\infty e^{-st}\,\mathrm{erf}(\sqrt{t})\,dt = \frac{1}{s\sqrt{s+1}}$
6. Show that $\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}[1 - \mathrm{erf}(a)]$
7. Show that $\frac{d}{dt}\mathrm{erf}(\sqrt{t}) = \frac{e^{-t}}{\sqrt{\pi t}}$ and hence evaluate $\int_0^\infty e^{-t}\,\mathrm{erf}(\sqrt{t})\,dt$.
8. Show that $\int_0^t \mathrm{erf}(ax)\,dx + \int_0^t erfc(ax)dx = t$.

Double Integral and Applications

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2 dx\, dy$ (Ans: $\frac{856}{945}$)
2. $\iint \sqrt{4x^2 - y^2}\ \ dx\, dy$ over the area of triangle $y = 0\,, y = x\ \&\ x = 1$

   Ans: $\frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$
3. $\iint_R xy\sqrt{1 - x - y}\, dxdy$ over the region $x \geq 0\,, y \geq 0\ \&\ x + y \leq 1$ (Ans: $\frac{16}{945}$)
4. Evaluate $\iint_R x^2 + y^2\ dxdy$ over area of triangle whose vertices are $(0,1)$ $(1\,,1)$ & $(1,\,2)$. (Ans: $\frac{7}{6}$)
5. Show that $\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xyiog(x+a)}{(x-a)^2} dxdy = \frac{a^2}{8}(2loga + 1)$
6. Evaluate by changing the order

   I) $\int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ (Ans: $\frac{3}{8}$)

   II) $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dxdy$ (Ans : $\frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$
7. Express the following integral as a single integral

   $\int_0^1 \int_0^y f(x,y)dxdy + \int_1^\infty \int_0^{\frac{1}{y}} f(x,y)dxdy$ (Ans:$\int_0^1 \int_x^{\frac{1}{x}} f(x,y)dxdy$)
8. Evaluate

   I) $\int_0^{\frac{a}{\sqrt{2}}} \int_0^{\sqrt{a^2-y^2}} \ln(x^2 + y^2)\ dxdy$ (Ans: $\frac{\pi}{2}\left[\frac{a^2}{2} loga - \frac{a^2}{4}\right]$)

   II) $\int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dxdy$ (Ans:$\frac{a^2}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$)
9. Evaluate over one loop of $r^2 = a^2 cos2\theta$ $\iint_R \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ (Ans : $2a(1 - \frac{\pi}{4})$)
10. Find area bounded by curve $y^2\ (2a - x) = x^3$ & its Asymptote (Ans :$3\pi a^2$)
11. Find area of cardioid $r = a(1 + cos\theta)$ (Ans: $\frac{3\pi a^2}{2}$)
12. Find area bounded by curve $y^2 x = 16(4 - x)$ & its Asymptote. (Ans:16π)
13. Find area bounded by curves $y^2 = 4x\ \ \&\ \ 2x - y - 4 = 0$ (Ans: 9)
14. Find area bounded by curves $y^2 = x\ \ \&\ \ x^2 = -8y$ (Ans: $\frac{8}{3}$)
15. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^y cos2y\sqrt{1 - a^2 sin^2 x}\ dxdy$ (Ans: $\frac{1}{3a^2}[(1 - a^2)^{\frac{3}{2}} - 1$)

Triple Integral and Applications

1. Evaluate $\iiint xyz \; dx\,dy\,dz$ Over positive octant of a sphere $x^2+y^2+z^2=a^2$.
   Ans: $\frac{a^6}{48}$
2. Evaluate $\int_0^2\int_0^y\int_{x-y}^{x+y}(x+y+z)dzdx\,dy$ Ans: 16
3. Evaluate $\iiint x^2yz \; dxdydz$ throughout the volume bounded by planes $x=0, y=0, z=0$ and $\frac{x}{2}-y+z=1$. Ans: $\frac{8}{2520}$
4. Evaluate $\iiint \frac{z^2dxdydz}{x^2+y^2+z^2}$ over the volume of sphere $x^2+y^2+z^2=2$ Ans: $\frac{8\pi\sqrt{2}}{9}$
5. Evaluate $\iiint z^2dxdydz$ over the volume bounded by the cylinder $x^2+y^2=a^2$ and the paraboloid $x^2+y^2=z$ and the plane $z=0$. Ans: $\frac{\pi a^8}{12}$
6. Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} rdzdrd\theta$ Ans: $\frac{5a^3}{64}$
7. Evaluate $\iiint\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}-\frac{z^2}{64}}\,dxdydz$ throughout the volume of Ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{64}=1$. Ans: $12\pi^2$
8. Evaluate $\iiint(x^2+y^2+z^2)^{3/2}\;dx\,dy\,dz$ Over the hemisphere defined by $x^2+y^2+z^2=9 \quad z\geq 0$. Ans: 243π
9. Calculate the volume of the solid bounded by the following surfaces $z=0,\; x^2+y^2=1,\; x+y+z=3$. Ans: 3π
10. Find by triple integration the volume of region bounded by paraboloid $az=x^2+y^2$ and the cylinder $x^2+y^2=r^2$. Ans: $\frac{\pi r^4}{2a}$
11. A cylindrical hole of radius 4 is bored through a sphere of radius 6. Find the volume of the remaining solid. Ans: $\frac{4\pi}{3}(20)^{3/2}$
12. Find the volume bounded by coordinate planes and the plane $lx+my+nz=1$. Ans: $\frac{1}{6mln}$
13. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$. Ans: 16π
14. Find the volume bounded by $y^2=x, x^2=y$ and the planes $z=0, x+y+z=1$. Ans: $\frac{1}{30}$
15. Find the volume of the solid bounded by the cylinder $x^2+y^2=4y$, the paraboloid $x^2+y^2=2z$ and the plane $z=0$ Ans: 12π

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Mathematics-II (SCI105A)
Practice Problems

Fourier series

Q.1) Find the Fourier series expansion for $f(x) = a(2 - x)$ in the interval $0 \leq x \leq 2$

Q.2) Find Fourier series for $f(x) = x + x^2$ in the interval $-\pi < x < \pi$

and $f(x) = f(x + 2\pi)$ hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \ldots$

Q.3) Find Fourier series for $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4} in\ (-\pi, \pi)$.

Q.4) Obtain Fourier series expansion for $f(x) = 2 - \frac{x^2}{2}, 0 \leq x \leq 2$.

Q.5) Find the Fourier series expansion for $f(x) = \frac{1}{2}(\pi - x)$ in the interval $0 \leq x \leq 2\pi$.

Q.8) Find the Fourier series of the function $f(x) = \begin{cases} -1; & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

where $(x) = f(x + 2\pi)$.

Q.9) Determine the Fourier series for the following function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

Q.10) Expand the function $f(x) = x \sin x$, as a Fourier series

in the interval $-\pi < x < \pi$. Hence deduce that $\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots$

Q.11) Expand the function $f(x) = x \cos x$, as a Fourier series

in the interval $-\pi < x < \pi$.

Harmonic Analysis

Q.12) Determine the first two harmonics of the Fourier series for the following values:

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Express T as a Fourier series up to first two harmonics.

Q.15) Obtain the constant term and the coefficient of the first harmonic in the

Fourier series of $f(x)$ as given in the following table

x	0	1	2	3	4	5
f(x)	9	18	24	28	26	20