IC Tutorial 7

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Q1.

$$I_n = \int \cos^n x \, dx$$

$$= \int \underbrace{\cos^{n-1} x}_{u} \cdot \underbrace{\cos x}_{v} \, dx$$

$$= \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} \sin x \cdot \sin x \cdot dx$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} \sin^2 x \cdot dx$$

$$= \cos^{n-1} \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \cdot dx -$$
$$(n-1) \int \cos^n x \cdot dx$$

$$\therefore I_n = \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1)(I_n)$$

$$\therefore I_n = \cos^{n-1} x \cdot \sin x + (n-1)(I_n - 2)$$

$$\therefore I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} (I_n - 2)$$

Q.2. $$I_n = \int_0^{\pi/2} \cos^n x \cdot dx$$

we know,

$$I_n = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} I_n - 2$$

$$\therefore \int_0^{\pi/2} I_n = \left[\frac{\cos^{n-1} x \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \cdot dx$$

$$\therefore \quad I_n = \left[\frac{\cos^{n-1} x \cdot \sin x}{n} \right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \cdot dx$$

$$\therefore \quad I_n = \frac{n-1}{n} (I_{n-2}) \quad - (2)$$

Now replace n by $(n-2)$ in eqn. (1)

$$I_{n-2} = \frac{n-3}{n-2} (I_{n-4})$$

Now replace n by $(n-4)$ in eq. (1)

$$I_{n-4} = \frac{n-5}{n-4} (I_{n-6})$$

If we keep reducing, we get

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \cdots I_0 \quad (n = \text{even})$$

$$I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \cdots I_1 \quad (n = \text{odd})$$

Now,

$$I_0 = \int_0^{\pi/2} \cos^0 x \cdot dx = \pi/2$$

$$I_1 = \int_0^{\pi/2} \cos x \cdot dx = [\sin x]_0^{\pi/2} = 1.$$

So Finally;

$$\boxed{I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \pi/2}$$

$(n = \text{even})$

$$\boxed{I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \cdots 1}$$

$(n = \text{odd})$

Q.3. $I_n = \int_0^{\pi/2} x \cdot \sin^n x \cdot dx = \int_0^{\pi/2} \underbrace{x \cdot \sin^{n-1} x}_{u} \cdot \underbrace{\sin x}_{v} \cdot dx$

$= \left[x \cdot \sin^{n-1} x \, (-\cos x) \right]_0^{\pi/2} - \int_0^{\pi/2} \left[(n-1)\, x \cdot \sin^{n-2} \cos x + \sin^{n-1} x \right] (-\cos x) \cdot dx$

$= 0 + (n-1) \int_0^{\pi/2} x \cdot \sin^{n-2} \cos^2 x \cdot dx + (n-1) \int_0^{\pi/2} \sin^{n-1} x \cdot \cos x \cdot dx$

$= (n-1) \int_0^{\pi/2} x \cdot \sin^{n-2} x \cdot (1 - \sin^2 x) \cdot dx + (n-1) \int_0^{\pi/2} \sin^{n-1} x \cdot \cos x \cdot dx$

$= (n-1) \int_0^{\pi/2} x \cdot \sin^{n-2} x \; dx - (n-1) \int_0^{\pi/2} x \cdot \sin^n x \cdot dx + (n-1) \int_0^{\pi/2} \sin^{n-1} x \cdot \cos x \; dx$

$\therefore \; I_n = (n-1)\, I_{n-2} - (n-1)\, I_n + \int_0^{\pi/2} \sin^{n-1} x \; \cos x \cdot dx$

Put $\sin x = t \Rightarrow x = 0 \Rightarrow t = 0$

$\Rightarrow \cos x \cdot dx = dt \qquad x = \pi/2 \Rightarrow t = 1$

$I_n = (n-1)\, I_{n-2} - n-1\, (I_n) + \int_0^1 t^{n-1} dt$

$= (n-1)\, I_{n-2} - (n-1)\, I_n + \left[\frac{t^n}{n} \right]_0^1$

$I_n = (n-1)\, I_{n-2} - (n-1)\, I_n + 1/n$

$\therefore \; I_n + n I_n - I_n = (n-1)\, I_{n-2} + I_n$

$\therefore \; \boxed{I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}}$

$$\therefore \; I_4 = \frac{4-1}{4} \cdot I_{4-2} + \frac{1}{4^2}$$

$$= \frac{3}{4} I_2 + \cancel{\;} \frac{1}{16} = \frac{3}{4} \times \left(\frac{\pi^2}{16} + \frac{1}{4}\right) + \frac{1}{16}$$

$$\boxed{I_4 = \frac{3\pi^2}{64} + \frac{1}{4}}$$

$$I_2 = \frac{2-1}{2} \cdot I_0 + \cancel{\;} \frac{1}{2^2}$$

$$= I_0 \cdot \frac{1}{2} + \frac{1}{4}$$

$$\boxed{I_2 = \frac{\pi^2}{16} + \frac{1}{4}}$$

$$I_0 = \int_0^{\pi/2} x \cdot dx = \left[\frac{x^2}{2}\right]_0^{\pi/2}$$

$$\boxed{I_0 = \frac{\pi^2}{4 \times 2} = \frac{\pi^2}{8}}$$