Fourier Series

http://www.gap-system.org/~history/PictDisplay/Fourier.html

Joseph Fourier
1768-1830

"In 1822, Joseph Fourier, a French mathematician, discovered that sinusoidal waves can be used as simple building blocks to describe and approximate any periodic waveform including square waves. Fourier used it as an analytical tool in the study of waves and heat flow. It is frequently used in signal processing and the statistical analysis of time series."

http://en.wikipedia.org/wiki/Sine_wave

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WHY DO WE NEED FOURIER ANALYSIS

In communication we send and recieve information laced signal over a medium, the medium and the hardware corrupts the signal. The receiver has to extract the information from the corrupted signal. The transmitted signal have well defined spectral contents, so if the receiver can do spectral analysis of the received signal then it can extract the information.

Fourier analysis - applications

Applications wide ranging and ever present in modern life

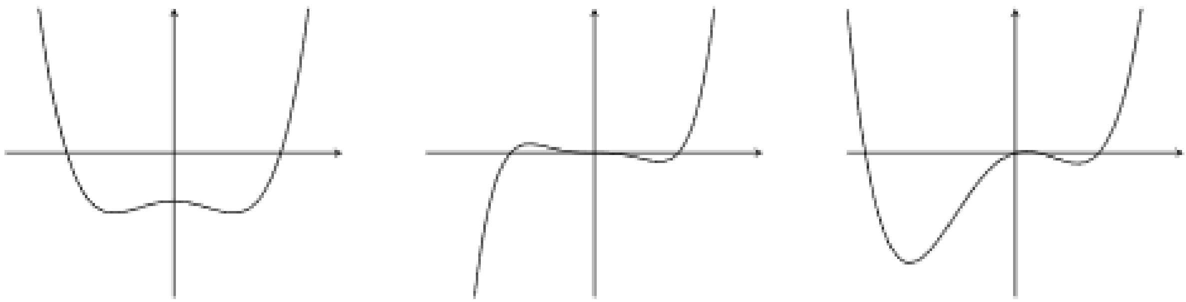
- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated “devices”, biometry, …).

Even and odd functions

Definition

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$.
The function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$.

Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.

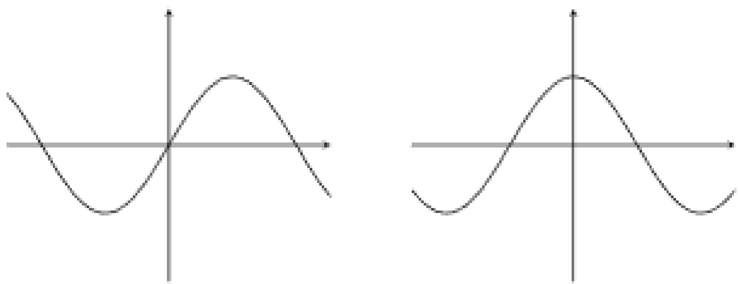


Even Odd Neither

Even and odd functions

Examples:

- Sums of odd powers of x are odd: $5x^3 - 3x$
- Sums of even powers of x are even: $-x^6 + 4x^4 + x^2 - 3$
- $\sin x$ is odd, and $\cos x$ is even



$\sin x$ (odd) $\qquad$ $\cos x$ (even)

- The product of two odd functions is even: $x \sin x$ is even
- The product of two even functions is even: $x^2 \cos x$ is even
- The product of an even function and an odd function is odd: $\sin x \cos x$ is odd

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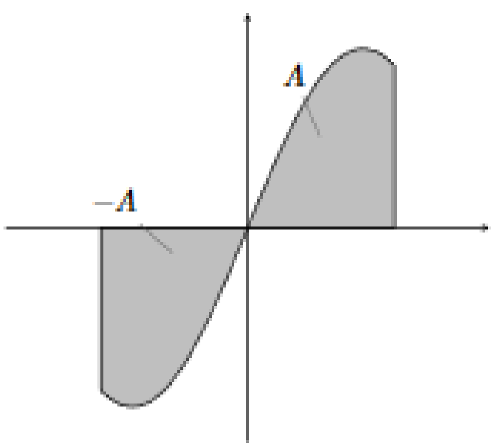
Integrating odd functions over symmetric domains

Let $p > 0$ be any fixed number. If $f(x)$ is an odd function, then

$$\int_{-p}^{p} f(x)\, dx = 0.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but opposite in sign. So, they cancel each other out!



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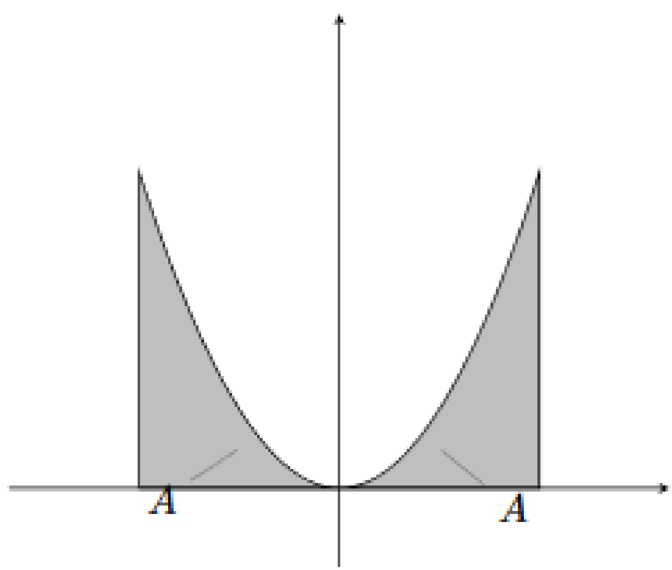
Integrating even functions over symmetric domains

Let $p > 0$ be any fixed number. If $f(x)$ is an even function, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

Intuition: The area beneath the curve on $[-p, 0]$ is the same as the area under the curve on $[0, p]$, but this time with the same sign. So, you can just find the area under the curve on $[0, p]$ and double it!



Periodic functions

Definition

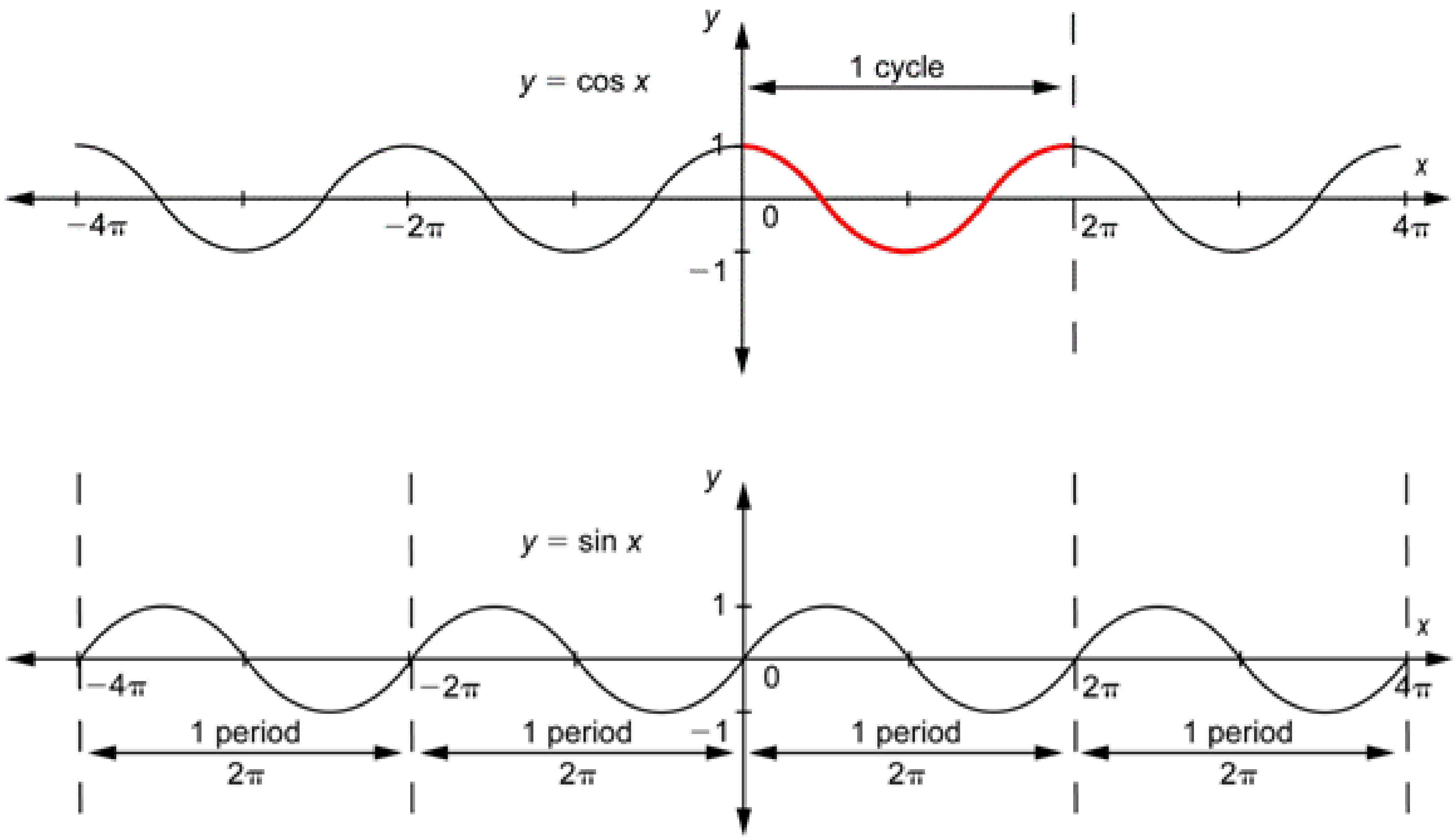
A function $f(x)$ is said to be *periodic* if there exists a number $T > 0$ such that $f(x+T) = f(x)$ for every x. The smallest such T is called the *period* of $f(x)$.

Intuitively, periodic functions have repetitive behavior.
A periodic function can be defined on a finite interval, then copied and pasted so that it repeats itself.
Examples

- $\sin x$ and $\cos x$ are periodic with period 2π
- $\sin(\pi x)$ and $\cos(\pi x)$ are periodic with period 2
- If L is a fixed number, then $\sin(\frac{2\pi x}{L})$ and $\cos(\frac{2\pi x}{L})$ have period L

Sine and cosine are the most "basic" periodic functions!

Periodic Function



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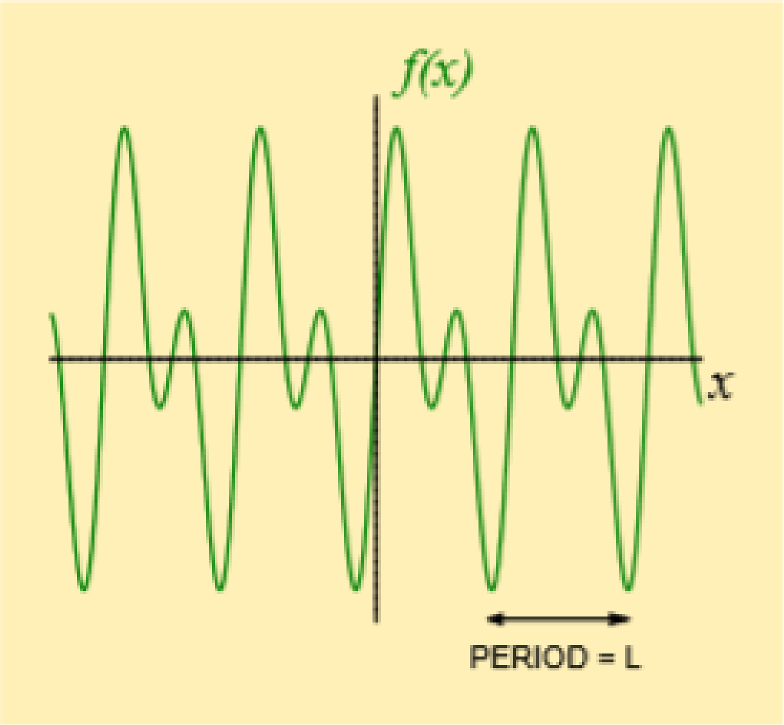
- A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x+L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



Def. A function f is **periodic of period T** $(T > 0)$ if and only if $f(t+T) = f(t)$ for all t.

Therefore the **period** T is defined as the time interval required for one complete fluctuation.
Hence $f(t) = \cos t$ is periodic with period 2π since

$$f(t+2\pi) = \cos(t+2\pi) = \cos t = f(t) \quad \text{for all } t.$$

N.B. If f is periodic with period T, then clearly from the graphs, or from repeated use of the definition, f is also periodic with periods $2T, 3T, \ldots$ – you should choose the *minimum* period of the function to be its period.

Ex 1. Determine whether the following functions are periodic and, if so, determine the periods:-

(i) $f(t) = \sin(2t)$, (ii) $f(t) = \cos(\sqrt{3}\,t)$, (iii) $f(t) = \cos t + \sin(2t)$.

(i) $f(t) = \sin(2t)$ is periodic with period π since

$$f(t+\pi) = \sin 2(t+\pi) = \sin(2t+2\pi) = \sin 2t = f(t) \quad \text{for all } t.$$

(ii) $f(t) = \cos(\sqrt{3}\,t)$ is periodic with period $2\pi/\sqrt{3}$ since

$$f\left(t + \frac{2\pi}{\sqrt{3}}\right) = \cos\left(\sqrt{3}\left(t + \frac{2\pi}{\sqrt{3}}\right)\right) = \cos(\sqrt{3}\,t + 2\pi) = \cos(\sqrt{3}\,t) = f(t) \quad \text{for all } t.$$

(iii) (more complicated)

$$\begin{array}{rll} \cos t & \text{has periods} & 2\pi, 4\pi, 6\pi, \ldots \\ \sin 2t & \text{has periods} & \pi, 2\pi, 3\pi, \ldots \end{array}$$

and clearly the minimum period for the sum of these quantities is the smallest number that appears in both lists of periods. In this example clearly $T = 2\pi$.

The function $f(t) = \cos t + \cos(\sqrt{2}\,t)$ would be even more difficult, since

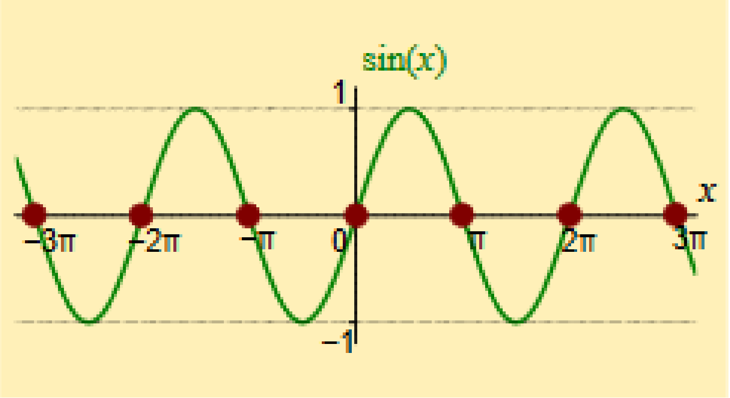
$$\begin{array}{rll} \cos t & \text{has periods} & 2\pi, 4\pi, 6\pi, \ldots \\ \cos(\sqrt{2}\,t) & \text{has periods} & \dfrac{2\pi}{\sqrt{2}}, \dfrac{4\pi}{\sqrt{2}}, \dfrac{6\pi}{\sqrt{2}}, \ldots \end{array}$$

Since the multiplicative factors of π in the periods for $\cos t$ are whole numbers but the corresponding factors for $\cos(\sqrt{2}\,t)$ are irrational (always involving $\sqrt{2}$) there is no number that appears in both lists. Hence the function $f(t) = \cos t + \cos(\sqrt{2}\,t)$ is NOT periodic (it never repeats itself), despite its comparatively simple

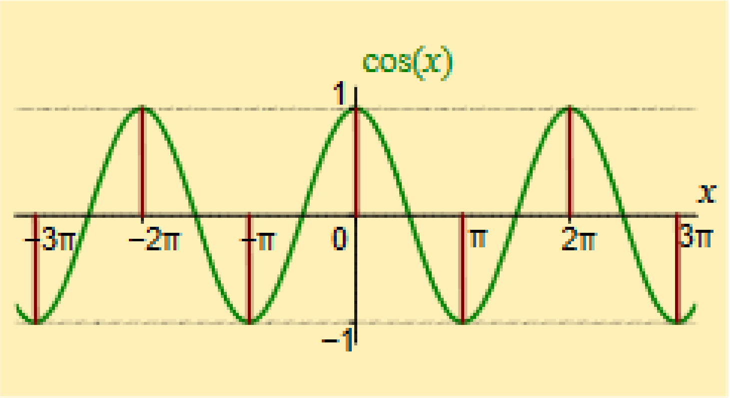
5. Useful trig results

When calculating the Fourier coefficients a_n and b_n, for which $n = 1, 2, 3, \ldots,$ the following trig. results are useful. Each of these results, which are also true for $n = 0, -1, -2, -3, \ldots,$ can be deduced from the graph of $\sin x$ or that of $\cos x$

- $\sin n\pi = 0$



- $\cos n\pi = (-1)^n$



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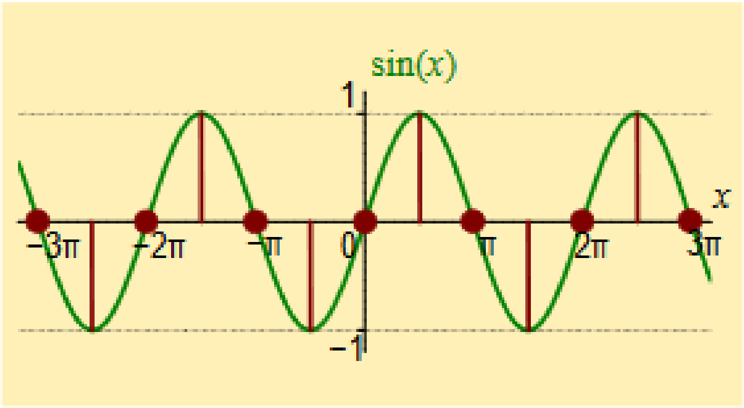
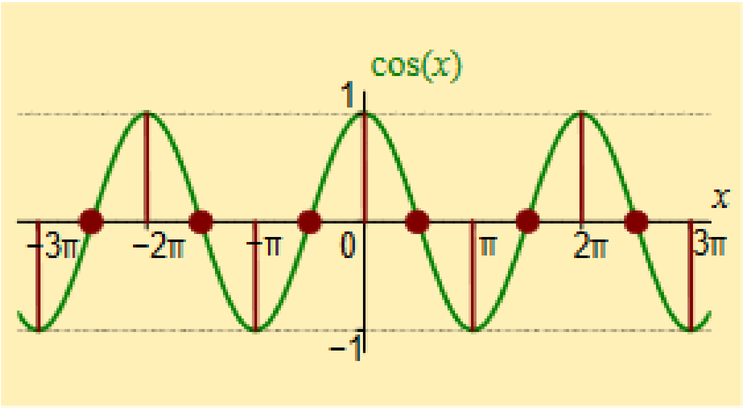
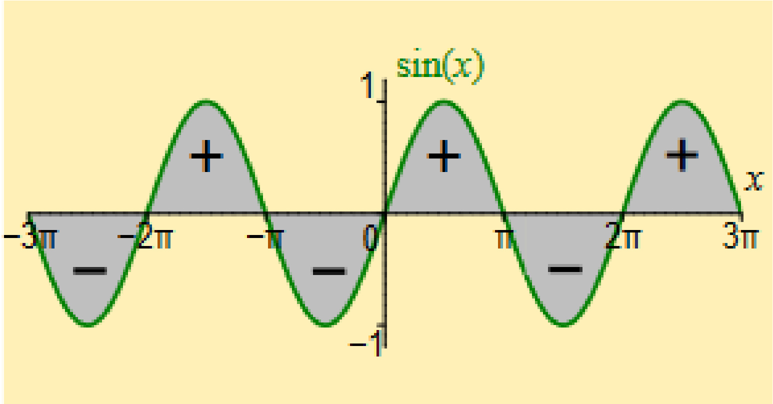




- $\sin n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ even} \\ 1 & , n = 1, 5, 9, \ldots \\ -1 & , n = 3, 7, 11, \ldots \end{cases}$
- $\cos n\frac{\pi}{2} = \begin{cases} 0 & , n \text{ odd} \\ 1 & , n = 0, 4, 8, \ldots \\ -1 & , n = 2, 6, 10, \ldots \end{cases}$

Areas cancel when
when integrating
over whole periods

- $\int_{2\pi} \sin nx \, dx = 0$
- $\int_{2\pi} \cos nx \, dx = 0$



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Fourier Series

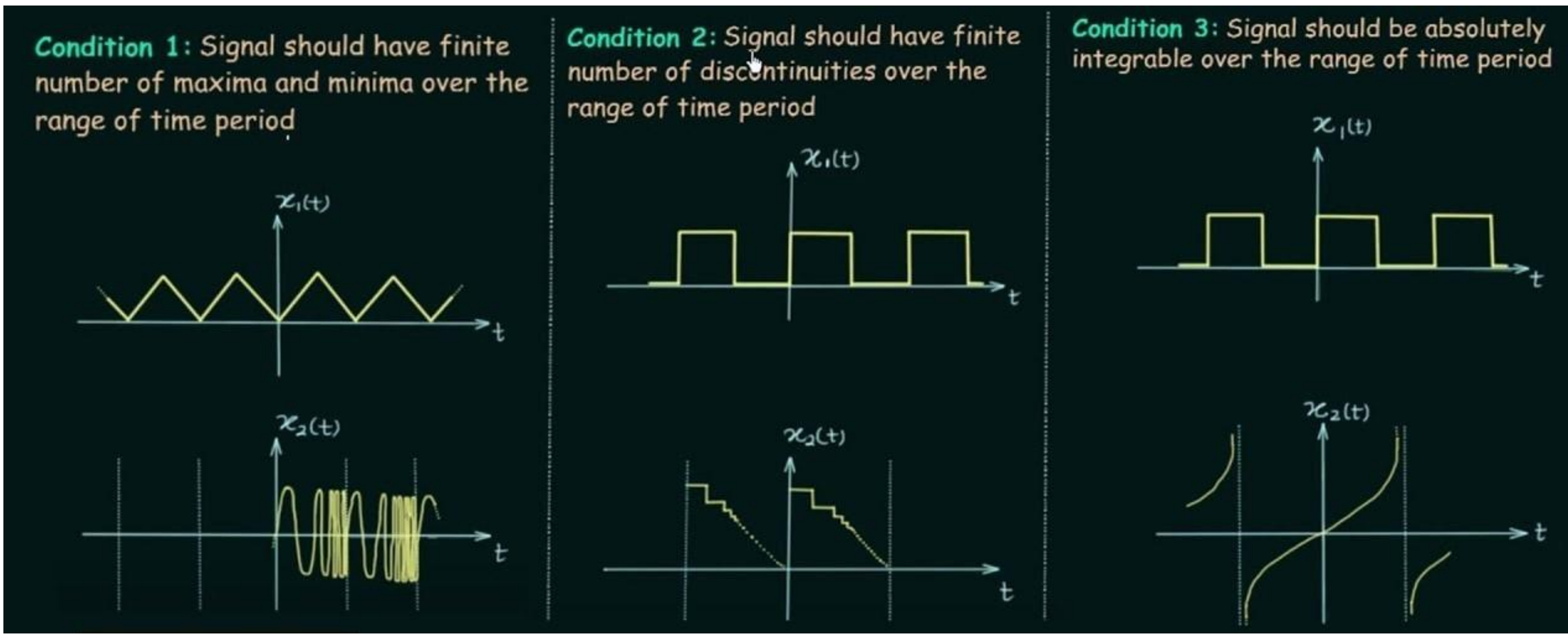
Any periodic power signal → Infinite sum of sines and cosines at different frequencies

The Dirichlet Conditions :

A periodic function $f(x)$ [signal $x(t)$] can be expressed as infinite trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right)$

in the interval $(-C, C+2L)$ with period $2L$ if satisfies following conditions.

1. It must have only finite number of maxima and minima within one period.
2. It must be single valued continuous except possibly at finite number of discontinuities.
3. The integral over one period of function must converge.



Definition: Fourier Series

If a function $f(x)$ satisfies Dirichlet conditions defined in the interval $(C, C+2L)$ with period $2L$ then following trigonometric series is called Fourier series of $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

Where $$a_0 = \frac{1}{L}\int_C^{C+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_C^{C+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Coefficients a_0, a_n, b_n are known as Fourier coefficients.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

Case 1: When $C = 0$

Fourier coefficients are given by

$$a_0 = \frac{1}{L} \int_0^{2L} f(x) dx , \quad a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Case 2: When $C = -L$

Fourier coefficients are given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx , a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

When $f(x)$ is even function : $f(-x) = f(x)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

Fourier Series of even function is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \text{ , } b_n = 0$$

Where

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx \text{ , } a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

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When $f(x)$ is odd function :

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier Series of odd function is given by

$$f(x) = \sum_{n=1}^{n=\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$, $a_0 = 0$, $a_n = 0$

Integration By Parts: General Rule

$$\boxed{\int u\, v \;\; dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots}$$

Where $u' = \frac{d}{dx}u$, $u'' = \frac{d^2}{dx^2}u, u''' = \frac{d^3}{dx^3}\, u$ ...

$$v_1 = \int v\, dx\,,\;\; v_2 = \int v_1\, dx\;\;, v_3 = \int v_2\, dx\;\;, \ldots.$$

Note: Apply this formula when u is polynomial function.

Evaluate

$$\int_0^{\pi} x^2 \, cosnx\, dx = \left[x^2\left(\frac{sinnx}{n}\right) - (2x)\left(-\frac{cosnx}{n^2}\right) + (2)\left(\frac{sinnx}{n^3}\right) - 0\right]_0^{\pi}$$

$$= \left[0 + 2\pi\frac{cosn\pi}{n^2} + 0\right] - [0 - 0 + 0]$$

Using $sinn\pi = 0\,, cosn\pi = (-1)^n$

$$= 2\pi\,\frac{(-1)^n}{n^2}$$

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Case 1: Fourier series in $0 < x < 2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx \,,$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example 1: Find Fourier series of $f(x) = \pi - x\ , 0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$

Solution: Here period of the function is $2L = 2\pi \implies L = \pi$

Fourier series of $f(x)$ in $0 < x < 2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{------(1)}$$

Step 1: To find a_0 $\quad a_0 = \frac{1}{L}\int_0^{2L} f(x)\, dx$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$\implies a_0 = \frac{1}{\pi}[2\pi^2 - 2\pi^2] = 0$$

Step 2: To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)cos\left(\frac{n\pi x}{L}\right)\,dx$$

Substituting $L = \pi$ and $f(x) = \pi - x$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)cos(nx)\,dx$$

$$a_n = \frac{1}{\pi}\left[(\pi - x)\,\frac{sinnx}{n} - (-1)\left(\frac{-cosnx}{n^2}\right) + 0\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\left(0 - \frac{cos2n\pi}{n^2}\right) - \left(0 - \frac{cos0}{n^2}\right)\right]$$

$$a_n = 0$$

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Step 2: To find b_n

$$b_n = \frac{1}{L}\int_0^{2L} f(x) sin\left(\frac{n\pi x}{L}\right)\, dx$$

Substituting $L = \pi$ and $f(x) = \pi - x$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi - x) sin(nx)\, dx$$

$$b_n = \frac{1}{\pi}\left[(\pi - x)\ \frac{-cosnx}{n} - (-1)\left(\frac{-sinnx}{n^2}\right) + 0\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left[\left((-\pi)\frac{-cos2n\pi}{n} - 0\right) - \left((\pi)\frac{-cos0}{n} - 0\right)\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{2}{n}$$

Substituting values of a_0 , a_n and b_n in the equation (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{---(1)}$$

Substituting $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{n}$,we get

$$f(x) = \sum_{n=1}^{n=\infty} \frac{2}{n} \; sinnx$$

$$f(x) = 2\left[\sum_{n=1}^{n=\infty} \frac{sinnx}{n}\right]$$

$$\pi - x = 2\left[\frac{sinx}{1} + \frac{sin2x}{2} + \frac{sin3x}{3} + \cdots\right]$$

Which is required Fourier series expansion of $f(x)$.

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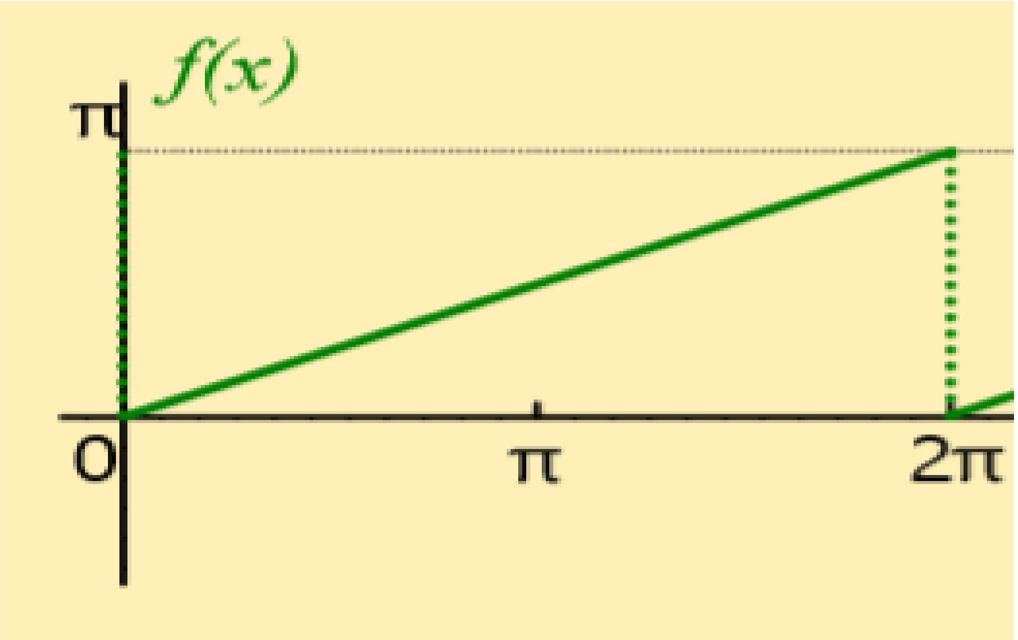
Example 2: Find Fourier series of following function

$f(x) = \frac{x}{2}$, over the interval $0 < x < 2\pi$ and has period 2π



Hence, deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Solution : Given interval is $(0,2\pi)$

Hence period is 2π and $2L = 2\pi \implies L = \pi$

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right]$

STEP ONE

$$a_0 = \frac{1}{L} \int_0^{2L} f(x)\, dx$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x)\, \mathrm{d}x \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2}\, \mathrm{d}x \\ &= \frac{1}{\pi} \left[\frac{x^2}{4}\right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[\frac{(2\pi)^2}{4} - 0\right] \end{aligned}$$

$$\text{i.e. } a_0 = \pi.$$

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Using $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$

STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2}\cos nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\underbrace{\left\{\left[x\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nx\,\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{2\pi}\left\{\left(2\pi\frac{\sin n2\pi}{n} - 0\cdot\frac{\sin n\cdot 0}{n}\right) - \frac{1}{n}\cdot 0\right\} \\
&= \frac{1}{2\pi}\left\{(0-0) - \frac{1}{n}\cdot 0\right\} \quad \text{, see } \textsc{Trig} \\
\text{i.e. } a_n &= 0.
\end{aligned}
$$

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Using $b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$

STEP THREE

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,\mathrm{d}x \\
&= \frac{1}{2\pi}\underbrace{\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right\}}_{\text{using integration by parts}} \\
&= \frac{1}{2\pi}\left\{\frac{1}{n}\left(-2\pi\cos n2\pi + 0\right) + \frac{1}{n}\cdot 0\right\}, \text{ see TRIG} \\
&= \frac{-2\pi}{2\pi n}\cos(n2\pi) \\
&= -\frac{1}{n}\cos(2n\pi) \\
\text{i.e. } b_n &= -\frac{1}{n} \quad \text{, since } 2n \text{ is even (see TRIG)}
\end{aligned}$$

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We have
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

Substituting $L = \pi$ in above equation

We now have
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

where $a_0 = \pi,\ a_n = 0,\ b_n = -\frac{1}{n}$

These Fourier coefficients give

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(0 - \frac{1}{n}\sin nx\right)$$

$$\text{i.e. } f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right\}.$$

Deduction: **We have Fourier series expansion**

$$f(x) = \frac{\pi}{2} - \left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right\}.$$

Setting $x = \frac{\pi}{2}$ gives $f(x) = \frac{\pi}{4}$ and

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \ldots\right]$$

$$\frac{\pi}{4} = \frac{\pi}{2} - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right]$$

$$\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right] = \frac{\pi}{4}$$

i.e. $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4}.$$

Example 3: Find Fourier series of $f(x) = e^{-x}\,, \quad 0 < x < 2$

Solution: Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right)$$

Here $2L = 2 \quad \Longrightarrow L = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(n\pi x) + \sum_{n=1}^{\infty} b_n sin(n\pi x)$$

Step 1: To find a_0

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \int_0^2 e^{-x}\,dx = [-e^{-x}]_0^2 = -e^{-2} + 1$$

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Step 2: To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^2 f(x)\cos(n\pi x)\,dx$$

$$a_n = \int_0^2 e^{-x}\cos(n\pi x)\,dx = \left[\frac{e^{-x}}{1+n^2\pi^2}(-\cos n\pi x + n\pi\ \sin n\pi x)\right]_0^2$$

Using formula $\int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$a_n = \left[\frac{e^{-2}}{1+n^2\pi^2}(-\cos 2n\pi + 0) - \frac{e^0}{1+n^2\pi^2}(-\cos 0 + 0)\right]$$

$$a_n = \left[\frac{e^{-2}}{1+n^2\pi^2}(-1) - \frac{1}{1+n^2\pi^2}(-1)\right]$$

$$a_n = \left[-\frac{e^{-2}}{1+n^2\pi^2} + \frac{1}{1+n^2\pi^2}\right]$$

$$a_n = \frac{1}{1+n^2\pi^2}\left[1 - e^{-2}\right]$$

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Step 3: To find b_n

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\, sin\left(\frac{n\pi x}{L}\right) dx = \int_0^2 f(x)\, sin(n\pi x)\, dx$$

$$b_n = \int_0^2 e^{-x}\ sin(n\pi x)\, dx = \left[\frac{e^{-x}}{1+n^2\pi^2}\left(-sin n\pi x - n\pi\ cos n\pi x\right)\right]_0^2$$

Using formula $\int e^{ax}\, sinbx\ dx = \frac{e^{ax}}{a^2+b^2}(a\, sinbx - b\, cosbx)$

$$b_n = \left[\frac{e^{-2}}{1+n^2\pi^2}(0 - n\pi cos2n\pi) - \frac{e^{0}}{1+n^2\pi^2}(0 - n\pi cos0)\right]$$

$$b_n = \left[\frac{e^{-2}}{1+n^2\pi^2}(-n\pi) + \frac{1}{1+n^2\pi^2}(n\pi)\right]$$

$$b_n = \left[-\frac{n\pi\, e^{-2}}{1+n^2\pi^2} + \frac{n\pi}{1+n^2\pi^2}\right]$$

$$b_n = \frac{n\pi}{1+n^2\pi^2}\left[1 - e^{-2}\right]$$

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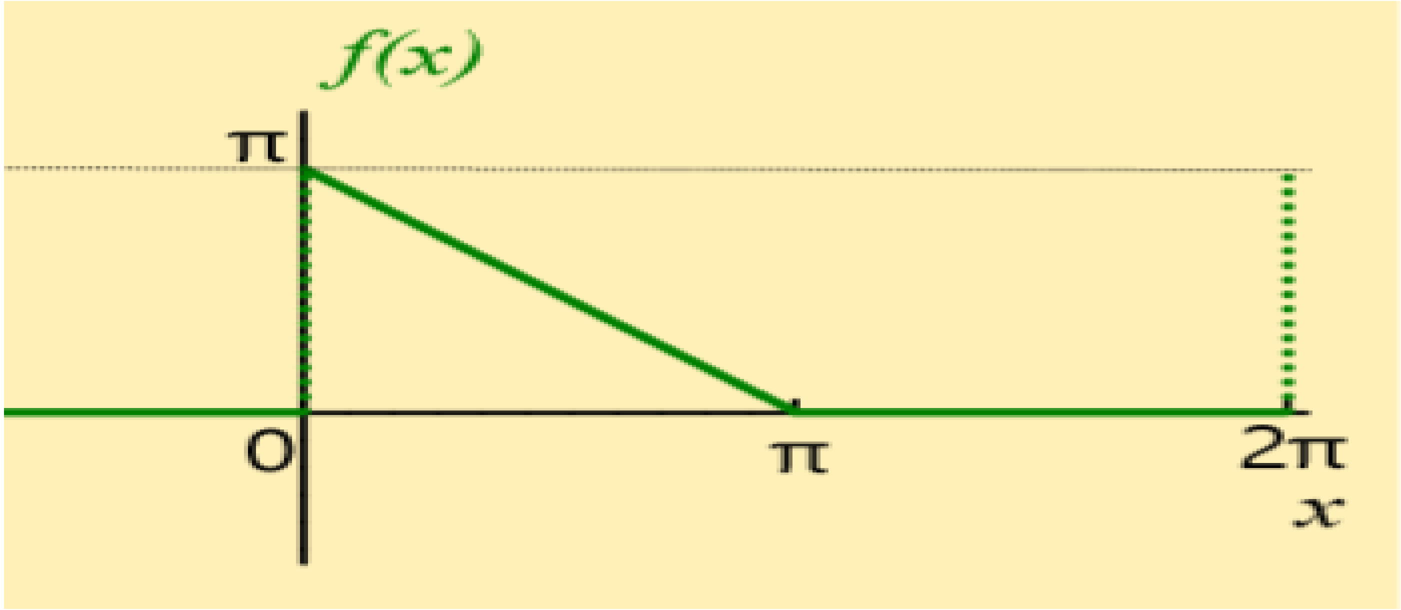
Example 4: Find Fourier series of periodic function

$$f(x) = \begin{cases} \pi - x & , 0 < x < \pi \\ 0 & , \pi < x < 2\pi, \text{ and has period } 2\pi \end{cases}$$



Solution: Again $L = \pi$

Using Fourier series $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

STEP ONE

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} + 0 \\
&= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right] \\
\text{i.e. } a_0 &= \frac{\pi}{2}.
\end{aligned}$$

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STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\
\text{i.e. } a_n &= \frac{1}{\pi}\underbrace{\left\{\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}(-1)\cdot\frac{\sin nx}{n}\,\mathrm{d}x\right\}}_{\text{using integration by parts}} + 0 \\
&= \frac{1}{\pi}\left\{(0-0) + \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right\} \quad \text{, see Trig} \\
&= \frac{1}{\pi n}\left[\frac{-\cos nx}{n}\right]_0^{\pi} \\
&= -\frac{1}{\pi n^2}(\cos n\pi - \cos 0) \\
\text{i.e. } a_n &= -\frac{1}{\pi n^2}((-1)^n - 1) \quad \text{, see Trig}
\end{aligned}
$$

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STEP THREE

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\left\{\left[(\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}(-1)\cdot\left(-\frac{\cos nx}{n}\right)\mathrm{d}x\right\} + 0 \\
&= \frac{1}{\pi}\left\{\left(0 - \left(-\frac{\pi}{n}\right)\right) - \frac{1}{n}\cdot 0\right\}, \text{ see } \textsc{Trig} \\
\text{i.e. } b_n &= \frac{1}{n}.
\end{aligned}
$$

$$\therefore \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\cos x + \frac{2}{\pi}\frac{1}{3^2}\cos 3x + \frac{2}{\pi}\frac{1}{5^2}\cos 5x + \ldots$$

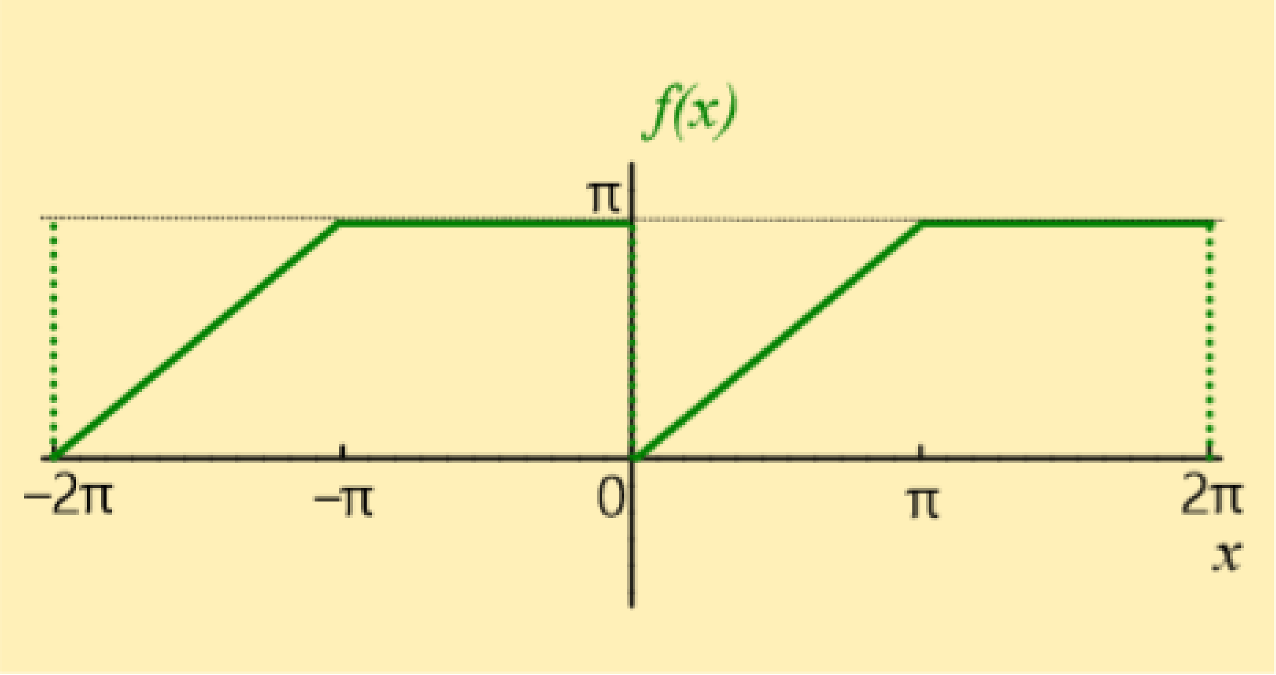
$$+ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots$$

$$\text{i.e. } f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

$$+ \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots$$

Example 5: Obtain Fourier expansion of the function

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \quad \text{and has period } 2\pi$$



STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\
&= \frac{\pi}{2} + \pi \\
\text{i.e. } a_0 &= \frac{3\pi}{2}.
\end{aligned}
$$

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STEP TWO

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin n\pi - 0\cdot\sin n0\right) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\
&\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi)
\end{aligned}
$$

$$\text{i.e. } a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1), \quad \text{see } \textsc{Trig}$$

$$= \frac{1}{n^2\pi}((-1)^n - 1),$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x \\
&= \underbrace{\frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi} \\
&= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right)+\left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\
&= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n}+\left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}(1-(-1)^n) \\
&= \quad -\frac{1}{n}(-1)^n + \qquad 0 \qquad - \frac{1}{n}(1-(-1)^n)
\end{aligned}
$$

$$\text{i.e. } b_n = -\frac{1}{n}(-1)^n - \frac{1}{n} + \frac{1}{n}(-1)^n$$

$$\text{i.e. } b_n = -\frac{1}{n}.$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $a_0 = \frac{3\pi}{2}$, $\quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $\quad b_n = -\frac{1}{n}$

Required Fourier expansion is given by

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right] - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

Additional Examples:

Find Fourier series of following functions

1) $f(x) = \pi x \ ,0 < x < 2 \quad , f(x+2) = f(x)$

2) $f(x) = e^{-x} \quad ,0 < x < 3$

3) $f(x) = 2x - x^2 \ \ in \ (0,\ 3)$

Hence deduce $\dfrac{\pi}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \cdots$

Case 2: Fourier series in $-L < x < L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where

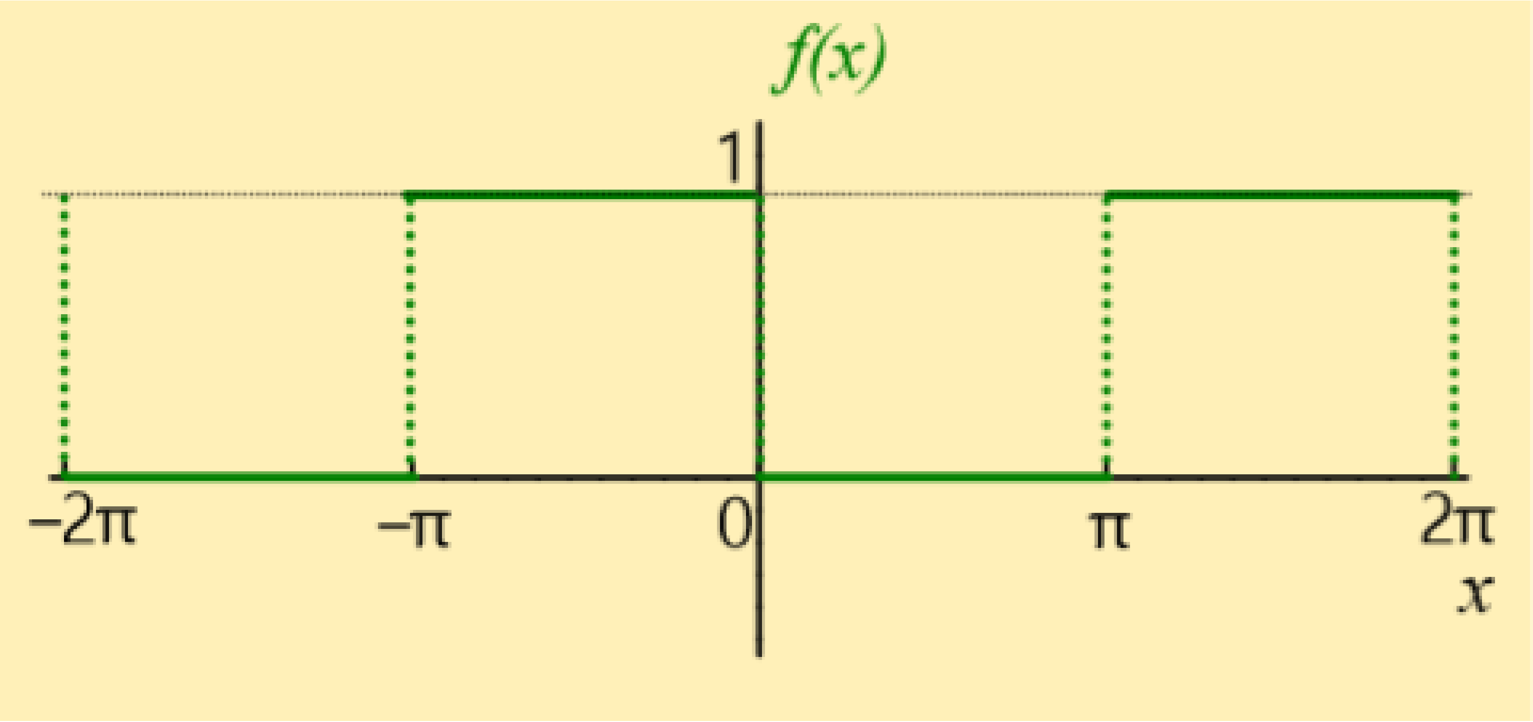
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \ ,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

Example 6: Find Fourier series of following function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi, \end{cases} \quad \text{and has period } 2\pi$$



Solution: We have $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x \\
&= \frac{1}{\pi}\left[x\right]_{-\pi}^{0} \\
&= \frac{1}{\pi}\left(0-(-\pi)\right) \\
&= \frac{1}{\pi}\cdot(\pi) \\
\text{i.e. } a_0 &= 1\,.
\end{aligned}
$$

We have $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

STEP TWO

$$\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}\left[\sin nx\right]_{-\pi}^{0} \\
&= \frac{1}{n\pi}(\sin 0 - \sin(-n\pi)) \\
&= \frac{1}{n\pi}(0 + \sin n\pi) \\
\text{i.e.}\quad a_n &= \frac{1}{n\pi}(0+0) = 0.
\end{aligned}$$

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We have $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)\,dx$

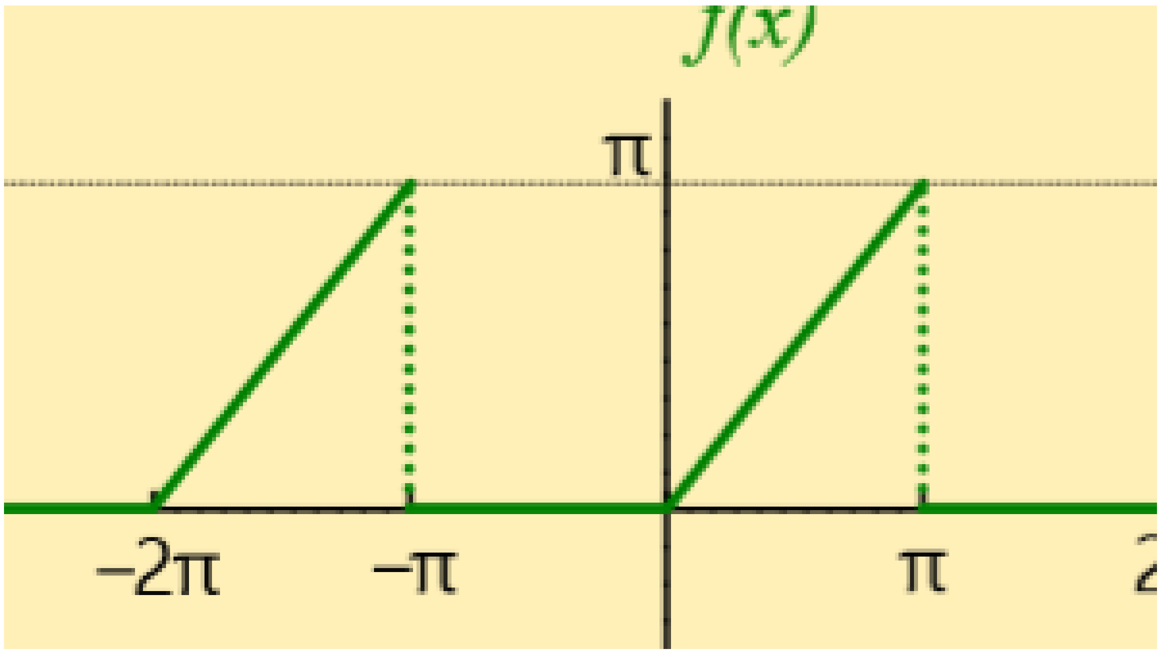
STEP THREE

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} 1\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} 0\cdot\sin nx\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\text{i.e.}\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} \\
&= -\frac{1}{n\pi}[\cos nx]_{-\pi}^{0} = -\frac{1}{n\pi}(\cos 0 - \cos(-n\pi)) \\
&= -\frac{1}{n\pi}(1-\cos n\pi) = -\frac{1}{n\pi}(1-(-1)^n)\,,\quad \text{see } \textsc{Trig} \\
\text{i.e.}\quad b_n &= \begin{cases} 0 & ,\ n \text{ even} \\ -\frac{2}{n\pi} & ,\ n \text{ odd} \end{cases} \qquad ,\ \text{since } (-1)^n = \begin{cases} 1 & ,\ n \text{ even} \\ -1 & ,\ n \text{ odd} \end{cases}
\end{aligned}$$

Example 7 : Find Fourier series of following function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi, \quad \text{and has period } 2\pi \end{cases}$$



Solution: We have $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

STEP ONE

$$
\begin{aligned}
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \quad &= \quad \frac{1}{\pi}\int_{-\pi}^{0} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\mathrm{d}x \\
&= \quad \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\
&= \quad \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \quad \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) \\
\text{i.e.}\quad a_0 \quad &= \quad \frac{\pi}{2}\,.
\end{aligned}
$$

We have $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x$$

i.e. $$a_n = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,\mathrm{d}x\right\}$$

(using integration by parts)

i.e. $$a_n = \frac{1}{\pi}\left\{\left(\pi\frac{\sin n\pi}{n} - 0\right) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(\quad 0 \quad - \; 0) + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi n^2}\left\{\cos n\pi - \cos 0\right\} = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\}$$

i.e. $$a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{\pi n^2} & , n \text{ odd} \end{cases}$$, see TRIG.

We have $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x$$

$$\text{i.e. } b_n = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right)\mathrm{d}x\right\}$$

(using integration by parts)

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,\mathrm{d}x\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left(\pi\cos n\pi - 0\right) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{\pi n^2}(0-0), \text{ see TRIG}$$

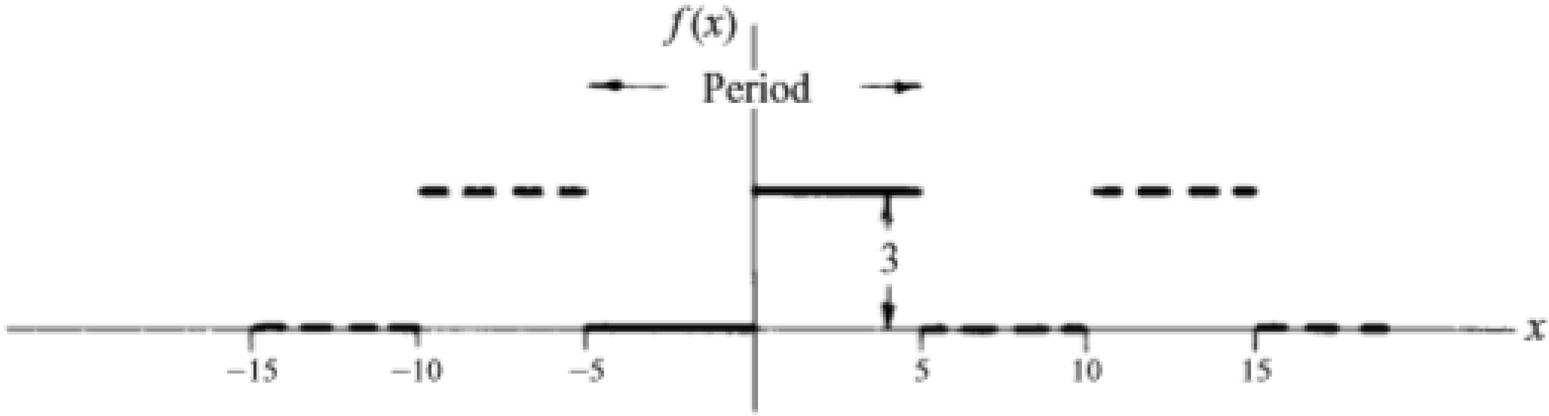
$$= -\frac{1}{n}(-1)^n$$

Example 8:

Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

Write the corresponding Fourier series.



Solution:

Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$.

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0$$

If $n = 0$, $a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

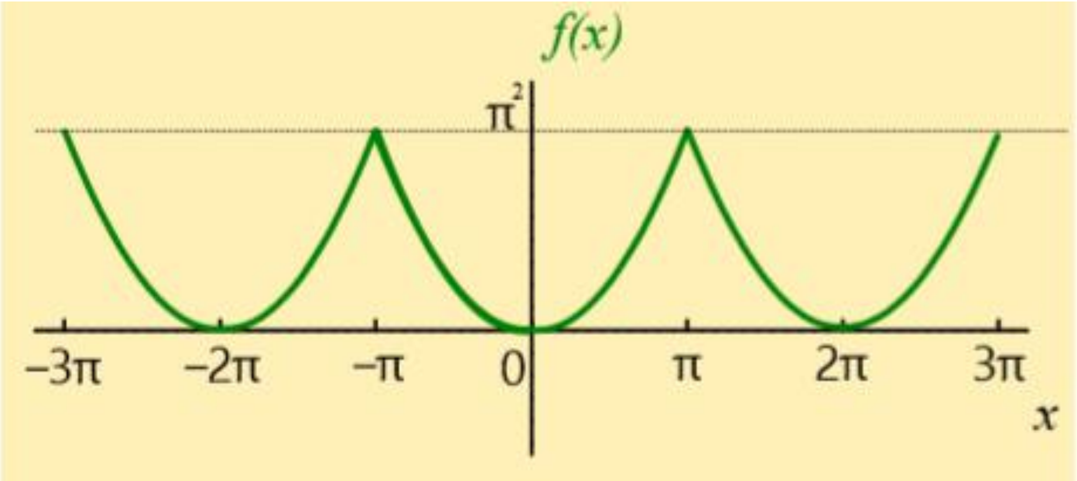
The corresponding Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right)$$

Example 9: Find Fourier series of following function

$$\boxed{f(x) = x^2, \text{ over the interval } -\pi < x < \pi \text{ and has period } 2\pi}$$



Answer:

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where $\quad a_0 = \frac{2\pi^2}{3}, \qquad a_n = \begin{cases} \frac{4}{n^2} & , n \text{ even} \\ \frac{-4}{n^2} & , n \text{ odd} \end{cases}, \qquad b_n = 0$

Additional Examples:

Find Fourier series of following functions

1) $f(x) = x + x^2 \quad , -1 < x < 1 \quad , f(x+2) = f(x)$

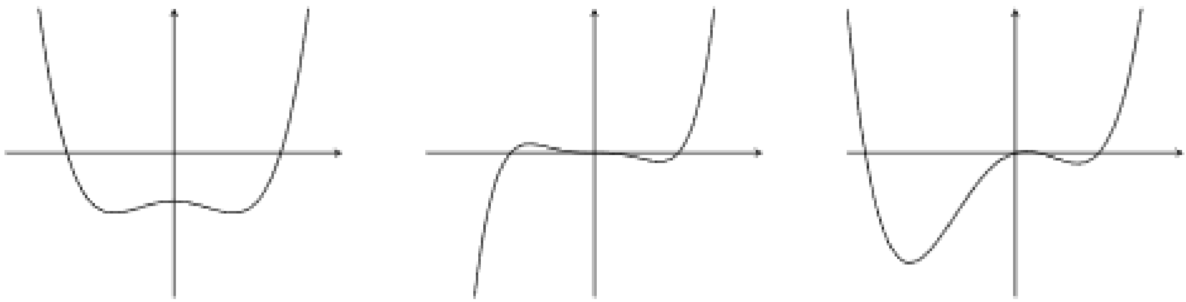
2) $f(x) = e^{-x} \quad , -\pi < x < \pi$

Even and odd functions

Definition

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$.
The function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$.

Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.



Even Odd Neither

$f(x)$ **is even function if** $f(-x) = f(x)$

Example :

$f(x) = x^2$, $f(x) = |x|$, $f(x) = a - x^2$ **are even functions.**

Fourier Series of even function in $-L < x < L$ **is given by**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad , \qquad b_n = 0$$

Ex.10 Find Fourier series of $f(x) = x^2$ in $-2 < x < 2$

Solution: Here $f(x)$ is defined in $-L < x < L$

Hence period is $2L = 4 \implies L = 2$

$f(-x) = (-x)^2 = x^2 = f(x)$

Hence $f(x) = x^2$ is even function.

Fourier series of even function given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\implies f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad \text{---------------------(1)}$$

Step 1: To find a_0

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

Step 2: To find a_n

$$\boldsymbol{a_n} = \frac{\boldsymbol{2}}{\boldsymbol{L}} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad \Longrightarrow \quad a_n = \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\Longrightarrow \quad a_n = \int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\Longrightarrow \quad a_n = \left[x^2 \frac{sin\frac{(n\pi x)}{2}}{\frac{n\pi}{2}} - (2x)\left(\frac{-cos\frac{(n\pi x)}{2}}{\frac{n^2\pi^2}{4}}\right) + (2)\left(\frac{-sin\frac{(n\pi x)}{2}}{\frac{n^3\pi^3}{8}}\right) - 0 \right]_0^2$$

$$\Longrightarrow a_n = \left[0 + 4\left(\frac{cos\frac{(n\pi 2)}{2}}{\frac{n^2\pi^2}{4}}\right) - 0 \right] - [0 - 0 + 0]$$

$$\Longrightarrow a_n = \frac{16}{n^2\pi^2} cosn\pi = \frac{16}{n^2\pi^2}(-1)^n$$

equation (1) becomes

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{2}\right) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{2}\right)}$$

Ex.11 Find Fourier series of $f(x) = |x|\,,\ -\pi < x < \pi$ and $f(x+2\pi) = f(x)$

Solution: Period of given function is $2L = 2\pi \implies L = \pi$

Here $f(-x) = |-x| = |x| = f(x)$

$\implies f(x)$ **is even function.**

Fourier series of even function given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\implies f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(nx) \qquad \text{-------------(1)}$$

Step 1: To find a_0

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{\pi}\int_0^{\pi} |x|\, dx$$

$$\implies a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^2 = \frac{4}{\pi}$$

Step 2: To find a_n

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \Longrightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} |x| \cos(nx)\, dx$$

$$\Longrightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} x \;\cos(nx)\, dx$$

$$\Longrightarrow \quad a_n = \frac{2}{\pi}\left[x\,\frac{sinnx}{n} - (1)\left(-\frac{cosnx}{n^2}\right) + 0\right]_0^{\pi}$$

$$\Longrightarrow a_n = \frac{2}{\pi}\left[x\,\frac{sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{\pi}$$

$$\Longrightarrow a_n = \frac{2}{\pi}\left\{\left[0 + \frac{cosn\pi}{n^2}\right] - \left[0 + \frac{cos0}{n^2}\right]\right\}$$

$$\Longrightarrow a_n = \frac{2}{\pi}\left\{\frac{1}{n^2}cosn\pi - \frac{1}{n^2}\right\} = \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}$$

equation (1) becomes

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\frac{n\pi x}{2}\right) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(nx)}$$

Additional Examples:

Find Fourier series of following functions

1) $f(x) = 4 - x^2 \ , -2 < x < 2$

2) $f(x) = cosx \ , -\pi < x < \pi$

3) $f(x) = \pi^2 - x^2 \ in \ -\pi \leq x \leq \pi$

Hence deduce a) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

b) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

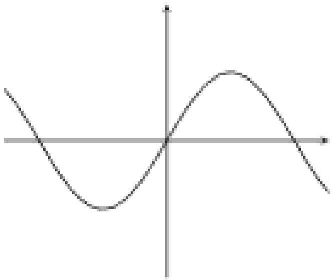
4) $f(x) = \begin{cases} -x + 1, & -\pi \leq x \leq 0 \\ x + 1, & 0 \leq x \leq \pi \end{cases}$

Deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

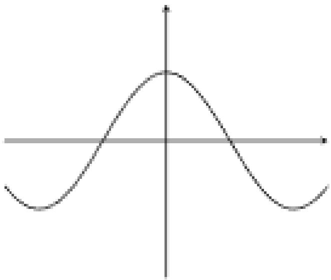
Even and odd functions

Examples:

- Sums of odd powers of x are odd: $5x^3 - 3x$
- Sums of even powers of x are even: $-x^6 + 4x^4 + x^2 - 3$
- $\sin x$ is odd, and $\cos x$ is even



$\sin x$ (odd)



$\cos x$ (even)

- The product of two odd functions is even: $x \sin x$ is even
- The product of two even functions is even: $x^2 \cos x$ is even
- The product of an even function and an odd function is odd: $\sin x \cos x$ is odd

$f(x)$ is odd function if $f(-x) = -f(x)$

Ex. $f(x) = x$, $f(x) = x^3$, $f(x) = sinx$ are odd functions.

Fourier Series of odd function in $-L < x < L$ is given by

$$f(x) = \sum_{n=1}^{n=\infty} b_n \, sin\left(\frac{n\pi x}{L}\right)$$

Where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad , \quad a_0 = 0 \ , \ a_n = 0$$

Ex.12 Obtain Fourier series of $f(x) = x^3 \quad , -1 < x < 1$

Solution: Here $f(x)$ is defined in $-L < x < L$

Hence period is $2L = 2 \implies L = 1$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

Hence $f(x) = \mathrm{x}^3$ is odd function.

Fourier series of odd function given by

$$f(x) = \sum_{n=1}^{n=\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies f(x) = \sum_{n=1}^{n=\infty} b_n \sin(n\pi x) \quad \text{--------------------(1)}$$

Step 1: To find b_n

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)dx$$

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$$\boldsymbol{b_n = 2\int_0^1 f(x) sin(n\pi x) dx}$$

$$b_n = 2\int_0^1 x^3 \ \sin(n\pi x) \ dx$$

$$b_n = 2\left[x^3\left(\frac{-cosn\pi x}{n\pi}\right) - (3x^2)\left(\frac{-sinn\pi x}{n^2\pi^2}\right) + (6x)\left(\frac{cosn\pi x}{n^3\pi^3}\right) - (6)\left(\frac{sinn\pi x}{n^4\pi^4}\right) + 0\right]_0^1$$

$$\Rightarrow b_n = 2\left[\left(\frac{-x^3 \ cosn\pi x}{n\pi}\right) + (3x^2)\left(\frac{sinn\pi x}{n^2\pi^2}\right) + (6x)\left(\frac{cosn\pi x}{n^3\pi^3}\right) - (6)\left(\frac{sinn\pi x}{n^4\pi^4}\right) + 0\right]_0^1$$

$$\Rightarrow b_n = 2\left[-\frac{cosn\pi}{n\pi} + 0 + 6\frac{cosn\pi}{n^3\pi^3} - 0\right] - [0 + 0 + 0 - 0]$$

$$\Rightarrow b_n = 2\left[\frac{-(-1)^n}{n\pi} + 6\frac{(-1)^n}{n^3\pi^3}\right]$$

Equation (1) becomes

$$\boldsymbol{\Rightarrow f(x) = \sum_{n=1}^{n=\infty} b_n \ sin(n\pi x)}$$

$$\boldsymbol{\Rightarrow f(x) = \sum_{n=1}^{n=\infty} 2\left[\frac{-(-1)^n}{n\pi} + 6\frac{(-1)^n}{n^3\pi^3}\right] \quad sin(n\pi x)}$$

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Ex.13 Find Fourier series of $f(x) = sinax$, $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

Solution: Here $f(x)$ is defined in $-L < x < L$

Hence period is $2L = 2\pi \implies L = \pi$

Hence $f(x)$ is odd function.

Fourier series of odd function given by

$$f(x) = \sum_{n=1}^{n=\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\implies f(x) = \sum_{n=1}^{n=\infty} b_n \sin(nx) \quad \text{--------------------(1)}$$

Step 1: To find b_n

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} sinax \ \ sinnx \ \ dx$$

Using $\boldsymbol{sinA\, sinB = \frac{1}{2}[\cos(A-B) - cos(A+B)}$

$$b_n = \frac{1}{\pi}\int_0^{\pi} [\cos(ax-nx) - \ \cos(ax+nx)] \ \ dx$$

$\boldsymbol{\sin(\theta \pm n\pi) = (-1)^n \, sin\theta}$

$$\Longrightarrow b_n = \frac{1}{\pi}\left[\frac{\sin(ax-nx)}{a-n} - \frac{\sin(ax+nx)}{a+n}\right]_0^{\pi}$$

$$\Longrightarrow b_n = \frac{1}{\pi}\left[\frac{\sin(a\pi-n\pi)}{a-n} - \frac{\sin(a\pi+n\pi)}{a+n} - 0 - 0\right] \ \ , \ a \neq n$$

$$\Longrightarrow b_n = \frac{1}{\pi}\left[\frac{(-1)^n \ \ sina\pi}{a-n} - \frac{(-1)^n \ \ sina\pi}{a+n}\right]$$

$$\Longrightarrow b_n = \frac{(-1)^n}{\pi} sina\pi \left[\frac{1}{a-n} - \frac{1}{a+n}\right]$$

$$\boldsymbol{\Longrightarrow b_n = \frac{(-1)^n}{\pi} sina\pi \left[\frac{2n}{a^2-n^2}\right], \ \ \ a \neq n}$$

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Additional Examples:

Find Fourier series of following functions

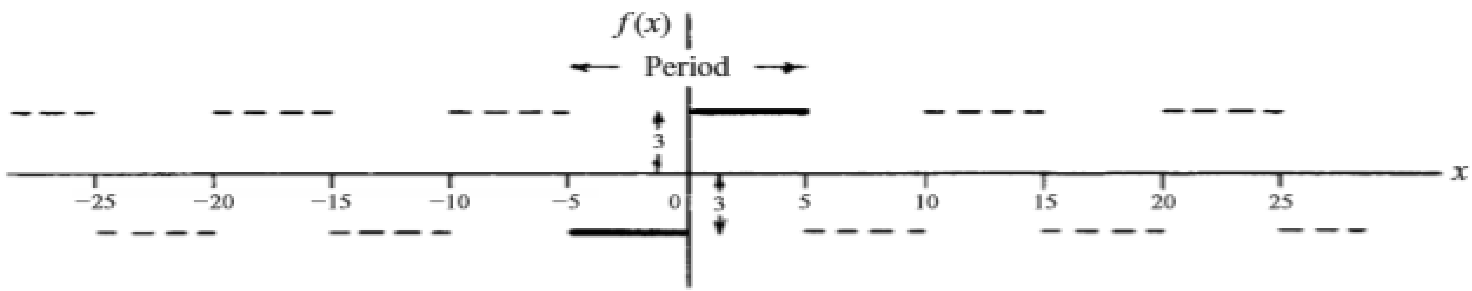
1) $f(x) = x - x^3 \ \ , -1 < x < 1$
2) $f(x) = sinx \ \ , -\pi < x < \pi$
3) $f(x) = \frac{x}{2} \ \ in \ \ -2 \leq x \leq 2$

…

Additional Examples:

Graph each of the following functions.

(a) $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases}$ Period = 10



(b) $f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

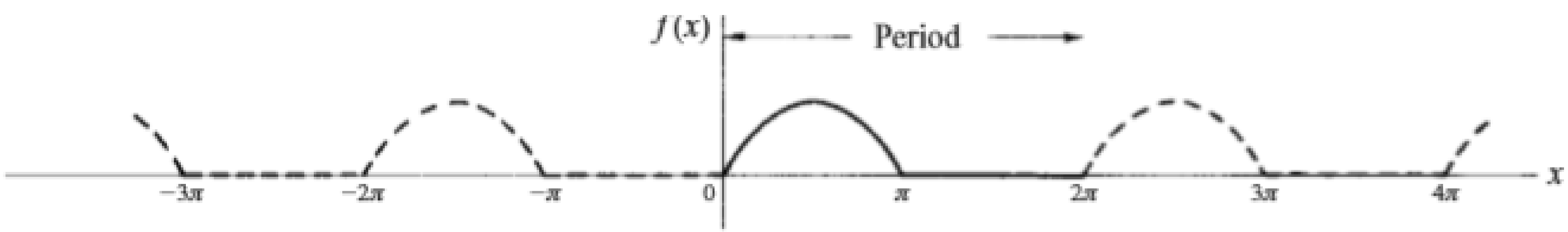


Fig. 13-4

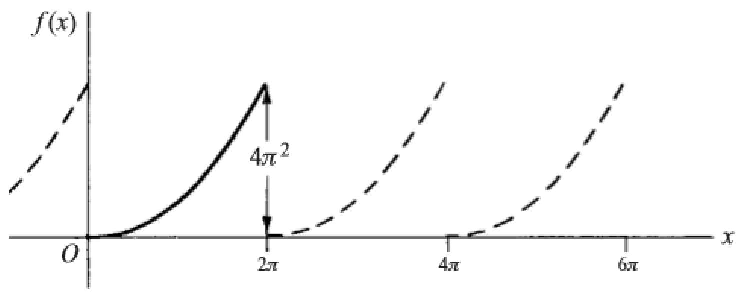
Classify each of the following functions according as they are even, odd, or neither even nor odd.

(a) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$ Period $= 6$

From Fig. 13-8 below it is seen that $f(-x) = -f(x)$, so that the function is odd.

(b) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

Expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series

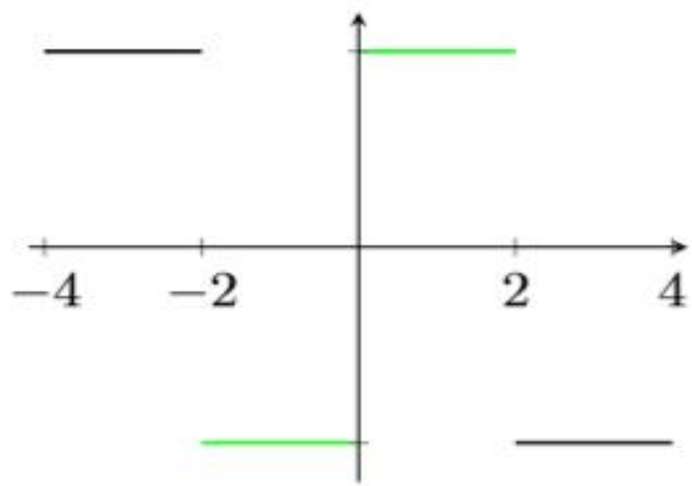


Beautiful Fourier Series Representation

Let $f(x)$ be periodic and defined on one period by the formula

$$f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$$

Graph of $f(x)$ (original part in green):



Since $f(x)$ is an odd function, we conclude that $a_0 = a_n = 0$ for each n. A bit of computation reveals

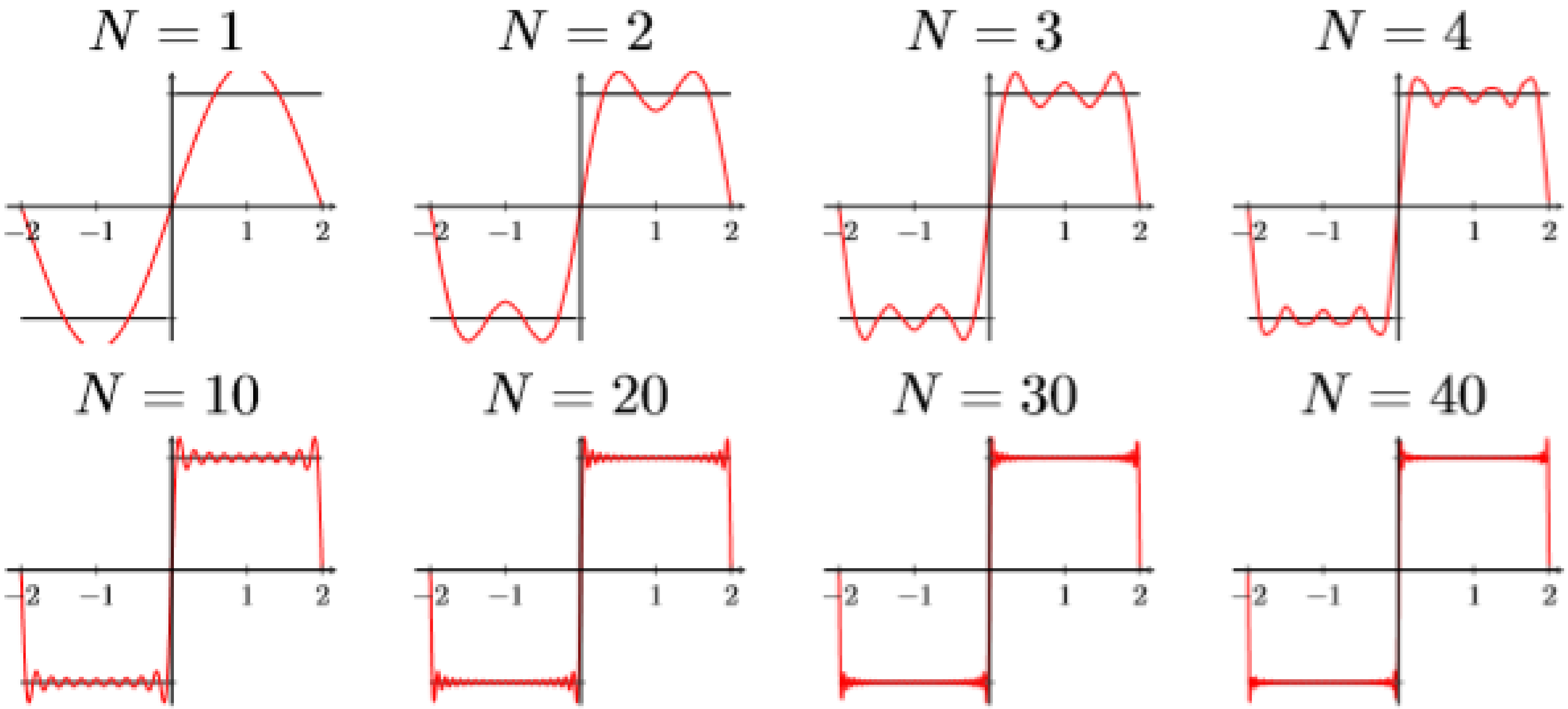
$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})\,dx = \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n)$$

Therefore

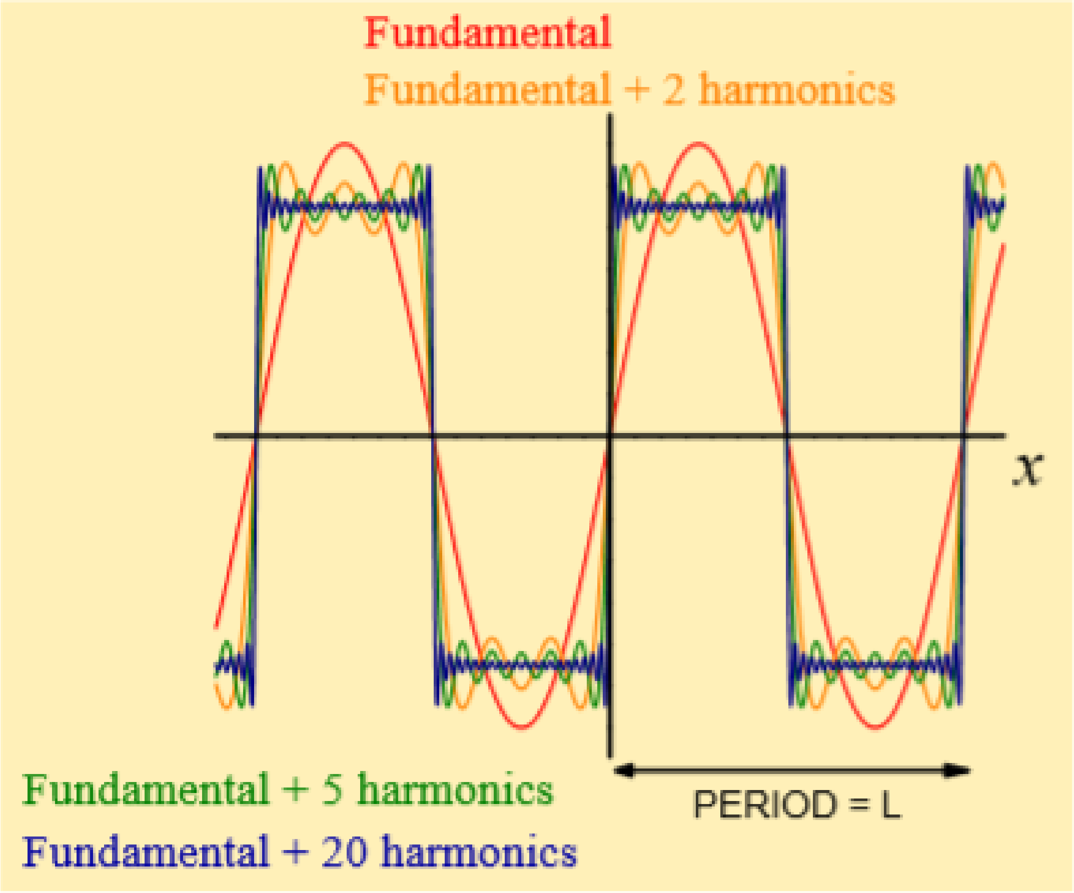
$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-(-1)^n)\sin(\frac{n\pi x}{2}) \\ &= \underbrace{\frac{4}{\pi}\sin(\frac{\pi x}{2})}_{n=1} + \underbrace{\frac{4}{3\pi}\sin(\frac{3\pi x}{2})}_{n=3} + \cdots \end{aligned}$$

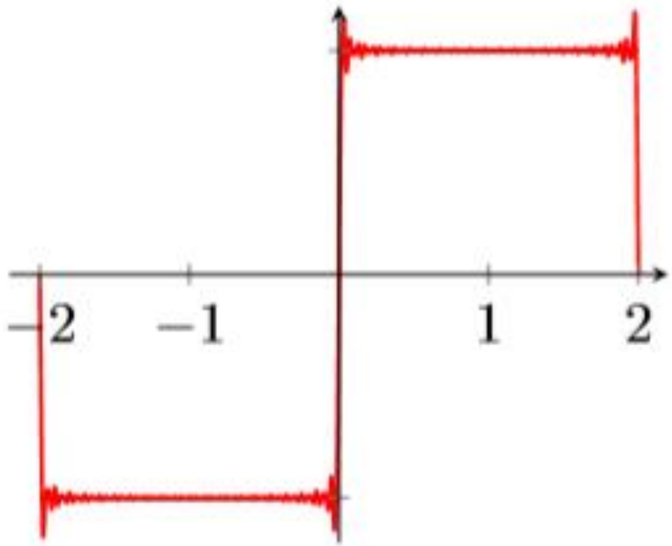
Notice: The even b_n terms are all 0 since $1-(-1)^n = 1-1 = 0$ when n is even.

If we plot the first N non-zero terms, we get approximations of $f(x)$:



One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**





Observations:

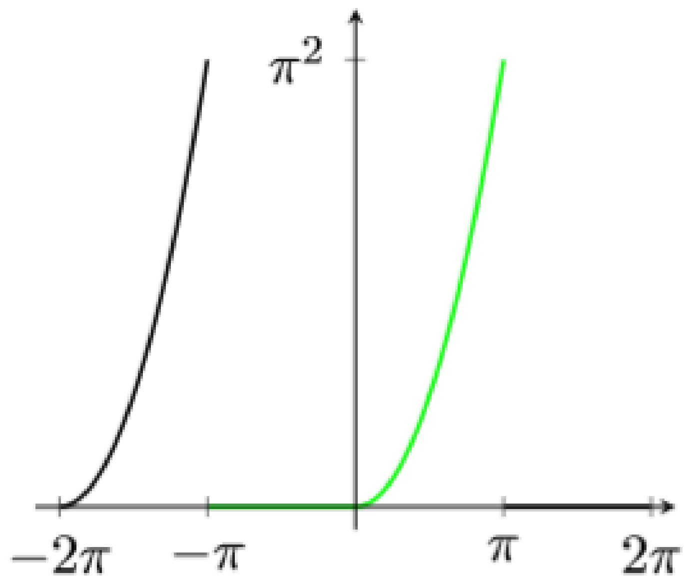
- As the number of terms used increases, the approximation gets closer and closer to the original function
- The original function has a discontinuity at $x = 0$. The approximation converges to 0 there, which is the average of the right- and left-hand limits as $x \to 0$.

In general, if $f(x)$ has a discontinuity at x_0, then the Fourier series converges to the average of $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$.

Let $f(x)$ be periodic and defined on one period by the formula

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$$

Graph of $f(x)$ (original part in green):



The function is neither even nor odd since it has no symmetry.

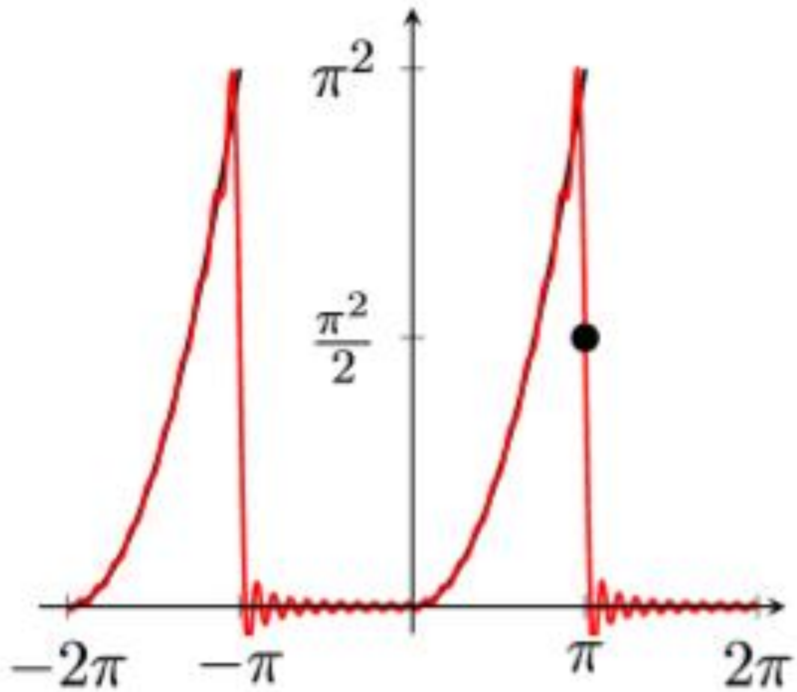
After some calculations (which are very tedious and involve lots of IBP),

$$a_0 = \frac{1}{3}\pi^2, \qquad a_n = \frac{2(-1)^n}{n^2}, \qquad b_n = \frac{(-1)^n(2 - \pi^2 n^2) - 2}{n^3 \pi}$$

Thus,

$$f(x) = \underbrace{\frac{1}{6}\pi^2}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \left(\underbrace{\frac{2(-1)^n}{n^2}}_{a_n} \cos(nx) + \underbrace{\frac{(-1)^n(2 - n^2\pi^2) - 2}{n^3\pi}}_{b_n} \sin(nx) \right)$$

Plot of Fourier series (first 20 terms):



Notice: At $x = \pi$, the series converges to $\frac{1}{2}(\pi^2 + 0) = \frac{\pi^2}{2}$.

By plugging in $x = \pi$ into the Fourier series for $f(x)$ and using the fact that the series converges to $\frac{\pi^2}{2}$,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2(-1)^n}{n^2} \cos(n\pi) + \frac{(-1)^n(2 - \pi^2 n^2) - 2}{n^3 \pi} \sin(n\pi) \right)$$

Because $\sin(n\pi) = 0$ and $(-1)^n \cos(n\pi) = (-1)^n(-1)^n = 1$, one can derive the following formula (c.f. example from lecture 14)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

QUESTIONS/ANSWERS