0.1 What is a double integral?

Recall that a **single integral** is something of the form

$$\int_a^b f(x)\,dx$$

A **double integral** is something of the form

$$\iint_R f(x,y)\,dx\,dy$$

where R is called the **region of integration** and is a region in the (x,y) plane. The double integral gives us the volume under the surface $z=f(x,y)$, just as a single integral gives the area under a curve.

0.2 Evaluation of double integrals

To evaluate a double integral we do it in stages, starting from the inside and working out, using our knowledge of the methods for single integrals. The easiest kind of region R to work with is a rectangle. To evaluate

$$\int\!\!\int_R f(x,y)\,dx\,dy$$

proceed as follows:

- work out the limits of integration if they are not already known
- work out the inner integral for a typical y
- work out the outer integral

Theorem. Let R be the rectangle $a \leq x \leq b$, $c \leq y \leq d$. If $f(x,y)$ is continuous on R then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

0.3 Example

Evaluate

$$\int_{y=1}^{2} \int_{x=0}^{3} (1 + 8xy)\, dx\, dy$$

Solution. In this example the "inner integral" is $\int_{x=0}^{3}(1 + 8xy)\, dx$ with y treated as a constant.

$$\begin{aligned}
\text{integral} &= \int_{y=1}^{2} \left(\underbrace{\int_{x=0}^{3} (1 + 8xy)\, dx}_{\text{work out treating } y \text{ as constant}} \right) dy \\
&= \int_{y=1}^{2} \left[x + \frac{8x^2 y}{2} \right]_{x=0}^{3} dy \\
&= \int^{2} (3 + 36y)\, dy \\
&= \left[3y + \frac{36y^2}{2} \right]_{y=1}^{2} \\
&= (6 + 72) - (3 + 18) \\
&= 57
\end{aligned}$$

0.4 Example

Evaluate

$$\int_0^{\pi/2}\int_0^1 y\sin x\,dy\,dx$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_0^{\pi/2}\Big(\int_0^1 y\sin x\,dy\Big)\,dx \\
&= \int_0^{\pi/2}\left[\frac{y^2}{2}\sin x\right]_{y=0}^{1}\,dx \\
&= \int_0^{\pi/2}\tfrac{1}{2}\sin x\,dx \\
&= \left[-\tfrac{1}{2}\cos x\right]_{x=0}^{\pi/2} = \tfrac{1}{2}
\end{aligned}$$

0.6 Example

Evaluate

$$\int_0^2 \int_{x^2}^{x} y^2 x \, dy \, dx$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_0^2 \int_{x^2}^{x} y^2 x \, dy \, dx \\
&= \int_0^2 \left[\frac{y^3 x}{3} \right]_{y=x^2}^{y=x} dx \\
&= \int_0^2 \left(\frac{x^4}{3} - \frac{x^7}{3} \right) dx = \left[\frac{x^5}{15} - \frac{x^8}{24} \right]_0^2 \\
&= \frac{32}{15} - \frac{256}{24} = -\frac{128}{15}
\end{aligned}$$

0.7 Example

Evaluate

$$\int_{\pi/2}^{\pi}\int_{0}^{x^2}\frac{1}{x}\cos\frac{y}{x}\,dy\,dx$$

Solution. Recall from elementary calculus the integral $\int \cos my\,dy = \frac{1}{m}\sin my$ for m independent of y. Using this result,

$$\begin{aligned}
\text{integral} &= \int_{\pi/2}^{\pi}\left[\frac{1}{x}\,\frac{\sin\frac{y}{x}}{\frac{1}{x}}\right]_{y=0}^{y=x^2} dx \\
&= \int_{\pi/2}^{\pi}\sin x\,dx = [-\cos x]_{x=\pi/2}^{\pi} = 1
\end{aligned}$$

0.8 Example

Evaluate

$$\int_{1}^{4}\int_{0}^{\sqrt{y}} e^{x/\sqrt{y}}\,dx\,dy$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_{1}^{4}\left[\frac{e^{x/\sqrt{y}}}{1/\sqrt{y}}\right]_{x=0}^{x=\sqrt{y}} dy \\
&= \int_{1}^{4}(\sqrt{y}e-\sqrt{y})\,dy = (e-1)\int_{1}^{4} y^{1/2}\,dy \\
&= (e-1)\left[\frac{y^{3/2}}{3/2}\right]_{y=1}^{4} = \frac{2}{3}(e-1)(8-1) \\
&= \frac{14}{3}(e-1)
\end{aligned}$$

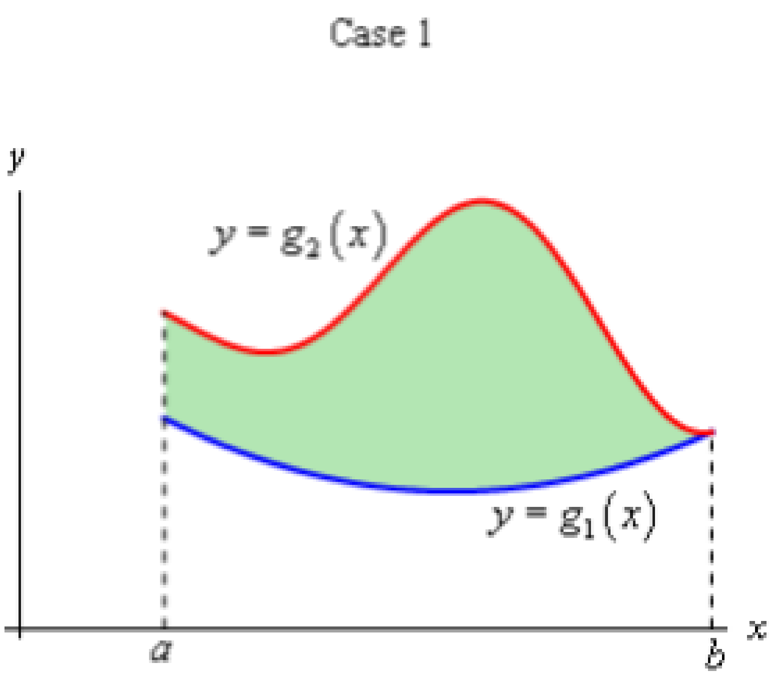
Section 4-3 : Double Integrals over General Regions

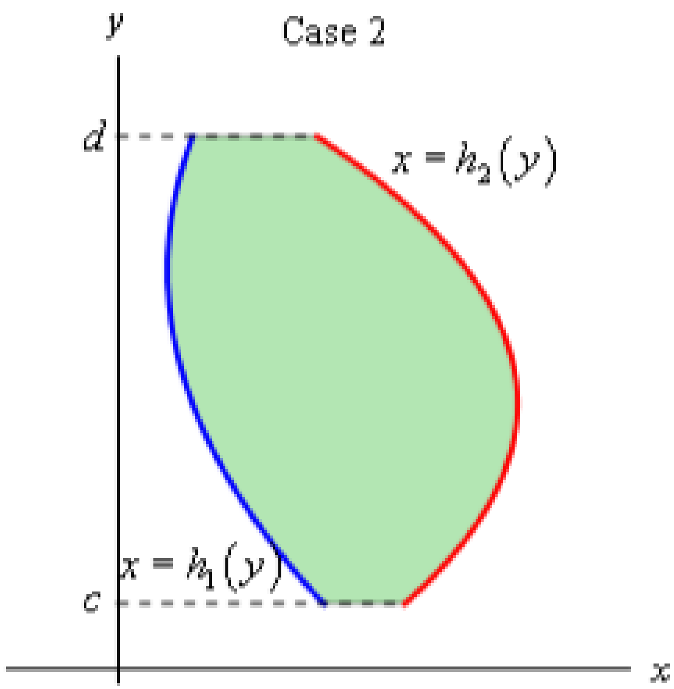
In the previous section we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,

$$\iint\limits_{D} f(x,y)\,dA$$

where *D* is any region.

There are two types of regions that we need to look at. Here is a sketch of both of them.





Case 1

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

and here is the definition for the region in Case 2.

$$D = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$$

This notation is really just a fancy way of saying we are going to use all the points, (x, y), in which both of the coordinates satisfy the two given inequalities.

The double integral for both of these cases are defined in terms of iterated integrals as follows.

In Case 1 where $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ the integral is defined to be,

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

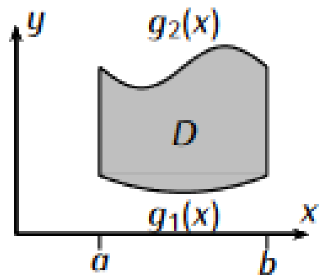
In Case 2 where $D = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$ the integral is defined to be,

$$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Idea: Choose the integration boundaries so that they represent the region.

Case I: Consider **region** of the form

$$D = \left\{ (x, y) : \begin{array}{c} a \le x \le b; \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

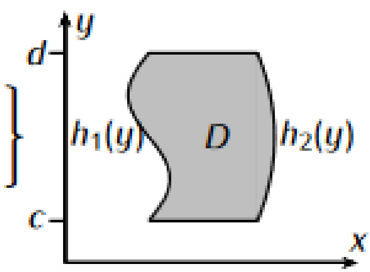


Then the signed volume under f on D is

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Case II: Consider **region** of the form

$$D = \left\{ (x, y) : \begin{array}{c} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$$



Then the signed volume under f on D is

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example 1 Evaluate each of the following integrals over the given region *D*.

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D=\left\{(x,y)\,|\,1\le y\le 2,\ y\le x\le y^3\right\}$

(b) $\displaystyle\iint\limits_D 4xy-y^3\,dA$, *D* is the region bounded by $y=\sqrt{x}$ and $y=x^3$.

(c) $\displaystyle\iint\limits_D 6x^2-40y\,dA$, *D* is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Solution

(a) $\displaystyle\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA$, $D=\left\{(x,y)\,|\,1\le y\le 2,\ y\le x\le y^3\right\}$

Okay, this first one is set up to just use the formula above so let's do that.

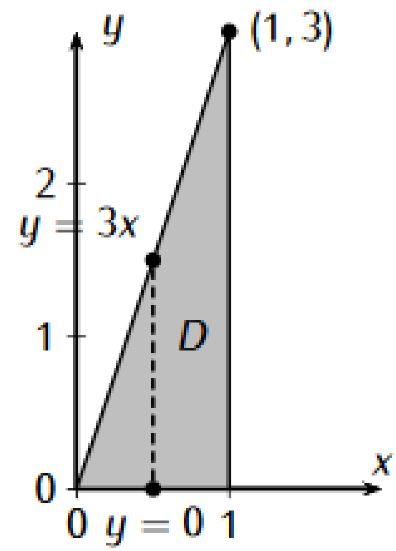
$$\iint\limits_D \mathbf{e}^{\frac{x}{y}}\,dA=\int_1^2\int_y^{y^3}\mathbf{e}^{\frac{x}{y}}\,dx\,dy=\int_1^2 y\,\mathbf{e}^{\frac{x}{y}}\Big|_y^{y^3}\,dy$$

$$=\int_1^2 y\,\mathbf{e}^{y^2}-y\mathbf{e}^1\,dy$$

$$=\left(\frac{1}{2}\mathbf{e}^{y^2}-\frac{1}{2}y^2\mathbf{e}^1\right)\Big|_1^2=\frac{1}{2}\mathbf{e}^4-2\mathbf{e}^1$$

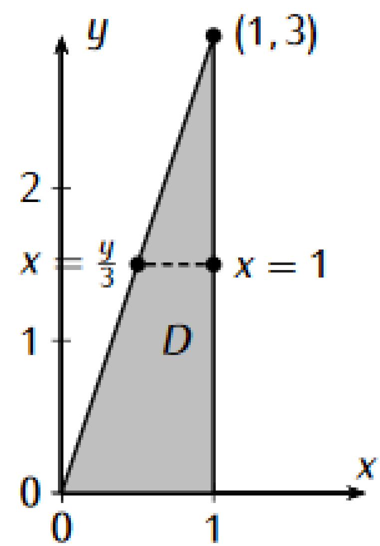
Example: What is the integral of $f(x, y) = 2-3x+xy$ over the triangle R that is spanned by $(0, 0), (1, 0), (1, 3)$?

Integration order 1:



$$\int_0^1 \Big(\int_0^{3x} 2-3x+xy\,dy \Big) dx$$
$$= \int_0^1 2y-3xy+\frac{1}{2}xy^2 \Big|_{y=0}^{3x} dx$$
$$= \int_0^1 6x-9x^2+\frac{9}{2}x^3 dx = \frac{9}{8}$$

Integration order 2:



$$\int_0^3 \Big(\int_{y/3}^{1} 2-3x+xy\,dx \Big) dy$$
$$= \int_0^3 2x-\frac{3}{2}x^2+\frac{1}{2}x^2y \Big|_{x=y/3}^{1} dy$$
$$= \int_0^3 \Big(-\frac{1}{18}y^3+\frac{1}{6}y^2-\frac{1}{6}y+\frac{1}{2} \Big) dy = \frac{9}{8}$$

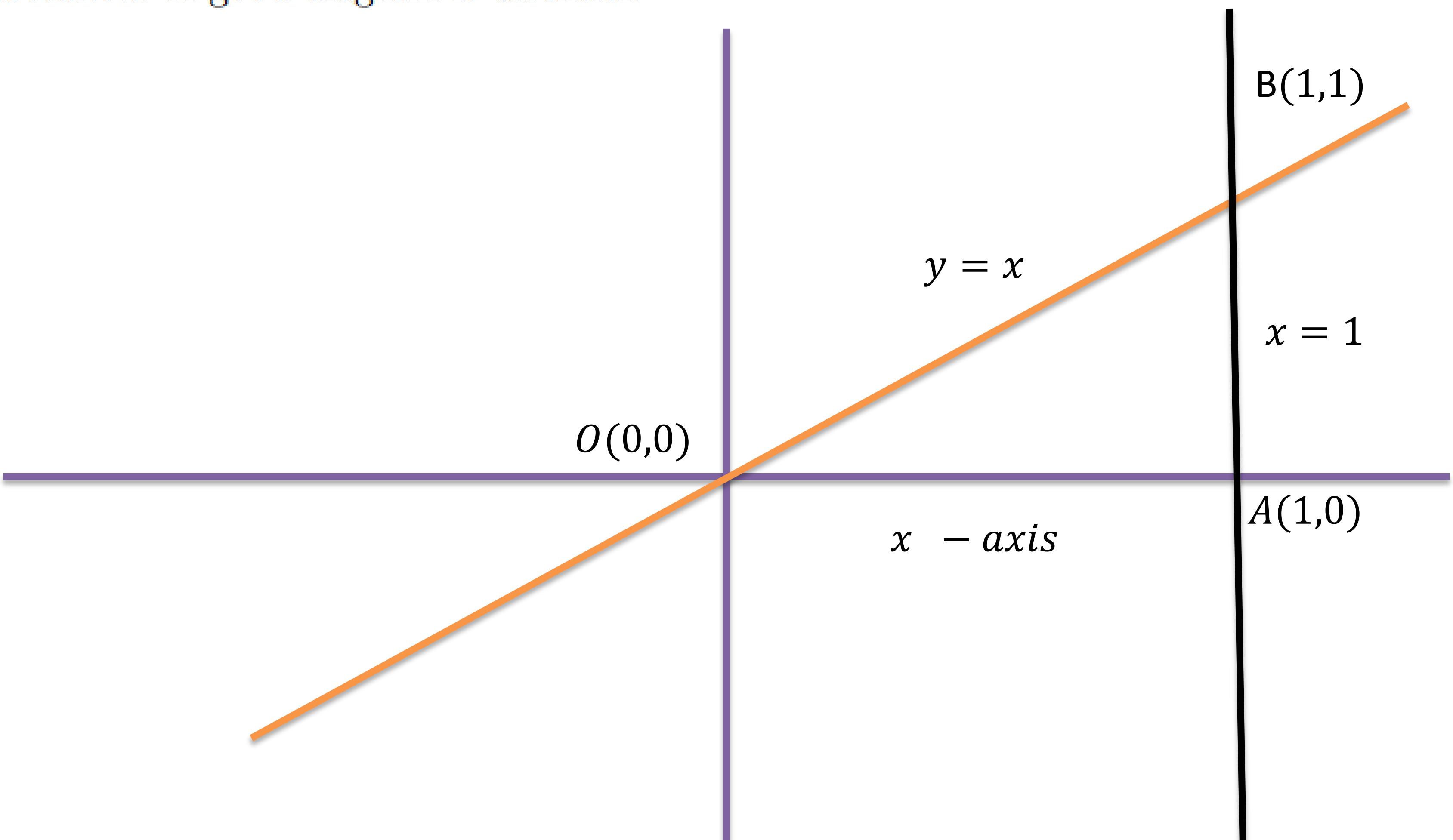
0.10 Example

Evaluate

$$\iint_D (3 - x - y)\, dA \qquad [dA \text{ means } dxdy \text{ or } dydx]$$

where D is the triangle in the (x, y) plane bounded by the x-axis and the lines $y = x$ and $x = 1$.

Solution. A good diagram is essential.



Method 1:

$$\begin{aligned}
\iint_D (3-x-y)\,dA &= \int_0^1 \int_y^1 (3-x-y)\,dx\,dy \\
&= \int_0^1 \left[3x - \frac{x^2}{2} - yx\right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(\left(3 - \tfrac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right)\right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5y}{2} - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} \\
&= \frac{5}{2} - 2 + \frac{1}{2} = 1
\end{aligned}$$

Method 2:

$$
\begin{aligned}
\iint_D (3-x-y)\,dA &= \int_0^1 \int_0^x (3-x-y)\,dy\,dx \\
&= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx \\
&= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx \\
&= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{1} = 1
\end{aligned}
$$

0.11 Example

Evaluate

$$\iint_D (4x + 2)\, dA$$

where D is the region enclosed by the curves $y = x^2$ and $y = 2x$.

Solution. Again we will carry out the integration both ways, x first then y, and then vice versa, to ensure the same answer is obtained by both methods.

Method 1 : We do the integration first with respect to x and then with respect to y. We shall need to know where the two curves $y = x^2$ and $y = 2x$ intersect. They intersect when $x^2 = 2x$, i.e. when $x = 0, 2$. So they intersect at the points $(0, 0)$ and $(2, 4)$.

For a typical y, the horizontal line will enter D at $x = y/2$ and leave at $x = \sqrt{y}$. Then we need to let y go from 0 to 4 so that the horizontal line sweeps the entire region. Thus

$$\begin{aligned}
\iint_D (4x+2)\,dA &= \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} (4x+2)\,dx\,dy \\
&= \int_0^4 \left[2x^2+2x\right]_{x=y/2}^{x=\sqrt{y}} dy = \int_0^4 \left((2y+2\sqrt{y}) - \left(\frac{y^2}{2}+y\right)\right) dy \\
&= \int_0^4 \left(y + 2y^{1/2} - \frac{y^2}{2}\right) dy = \left[\frac{y^2}{2} + \frac{2y^{3/2}}{3/2} - \frac{y^3}{6}\right]_0^4 = 8
\end{aligned}$$

Method 2 : Integrate first with respect to y and then x, i.e. draw a vertical line across D at a typical x value. Such a line enters D at $y = x^2$ and leaves at $y = 2x$. The integral becomes

$$\begin{aligned}\iint_D (4x+2)\,dA &= \int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx \\ &= \int_0^2 [4xy + 2y]_{y=x^2}^{y=2x}\,dx \\ &= \int_0^2 \left(\left(8x^2+4x\right) - \left(4x^3+2x^2\right)\right)\,dx \\ &= \int_0^2 (6x^2 - 4x^3 + 4x)\,dx = \left[2x^3 - x^4 + 2x^2\right]_0^2 = 8\end{aligned}$$

0.12 Example

Evaluate

$$\iint_D (xy - y^3)\, dA$$

where D is the region consisting of the square $\{(x, y) : \ -1 \le x \le 0,\ 0 \le y \le 1\}$ together with the triangle $\{(x, y) : \ x \le y \le 1,\ 0 \le x \le 1\}$.

Method 1 : (easy). integrate with respect to x first. A diagram will show that x goes from -1 to y, and then y goes from 0 to 1. The integral becomes

$$\begin{aligned}
\iint_D (xy - y^3)\, dA &= \int_0^1 \int_{-1}^y (xy - y^3)\, dx\, dy \\
&= \int_0^1 \left[\frac{x^2}{2} y - xy^3 \right]_{x=-1}^{x=y} dy \\
&= \int_0^1 \left(\left(\frac{y^3}{2} - y^4 \right) - \left(\tfrac{1}{2} y + y^3 \right) \right) dy \\
&= \int_0^1 \left(-\frac{y^3}{2} - y^4 - \tfrac{1}{2} y \right) dy = \left[-\frac{y^4}{8} - \frac{y^5}{5} - \frac{y^2}{4} \right]_{y=0}^1 = -\frac{23}{40}
\end{aligned}$$

0.13 Example

Evaluate

$$\iint_D \frac{\sin x}{x}\, dA$$

where D is the triangle $\{(x, y) : \ 0 \le y \le x, \ 0 \le x \le \pi\}$.

Double Integral by changing the order of integration

Change the Order and evaluate $\displaystyle\int_0^{\pi}\int_y^{\pi}\frac{\sin x}{x}\,dx\,dy$

Compute the double integral

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 e^{x^3}\, dx\, dy$$

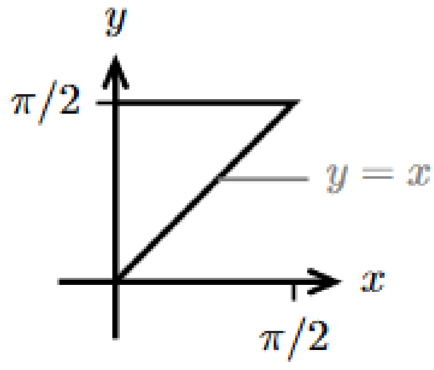
$$
\begin{aligned}
\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 e^{x^3}\, dx\, dy &= \int_0^2 \Big(\int_0^{\sqrt{x}} y^3 e^{x^3}\, dy \Big) dx \\
&= \int_0^2 e^{x^3} \left(\frac{1}{4} y^4\right)\Big|_{y=0}^{y=\sqrt{x}} dx \\
&= \frac{1}{4} \int_0^2 e^{x^3} x^2\, dx \\
&\overset{(*)}{=} \frac{1}{12} \cdot e^{x^3}\Big|_{x=0}^{2} = \frac{1}{12}(e^8 - 1)
\end{aligned}
$$

1. Evaluate

$$I = \int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$

by changing the order of integration.

Answer:
The given limits are (inner) y from x to $\pi/2$; (outer) x from 0 to $\pi/2$.
We use these to sketch the region of integration.



The given limits have inner variable y. To reverse the order of integration we use horizontal stripes. The limits in this order are

(inner) x from 0 to y; (outer) y from 0 to $\pi/2$.

So the integral becomes

$$I = \int_0^{\pi/2} \int_0^y \frac{\sin y}{y}\, dx\, dy$$

We compute the inner, then the outer integrals.

Inner: $\left.\dfrac{\sin y}{y} x\right|_0^y = \sin y.$ Outer: $-\cos y|_0^{\pi/2} = 1.$

Problem 1. Evaluate the integral by first reversing the order of integration,

$$\int_{x=0}^{x=3} \int_{y=x^2}^{y=9} x^3 e^{y^3} \, dy \, dx.$$

$$
\begin{aligned}
\int_{x=0}^{x=3}\int_{y=x^2}^{y=9} x^3 e^{y^3}\,dy\,dx &= \int_{y=0}^{y=9}\int_{x=0}^{x=\sqrt{y}} x^3 e^{y^3}\,dx\,dy \\
&= \int_{y=0}^{y=9}\left(\frac{1}{4}x^4 e^{y^3}\right)\Bigg|_{x=0}^{x=\sqrt{y}} dy \\
&= \int_{y=0}^{y=9}\frac{1}{4}y^2 e^{y^3}\,dy \\
&= \frac{1}{12}e^{y^3}\Bigg|_{y=0}^{y=9} \\
&= \frac{1}{12}\left(e^{729}-1\right).
\end{aligned}
$$

Evaluate the integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{-y}\, dy\, dx$.

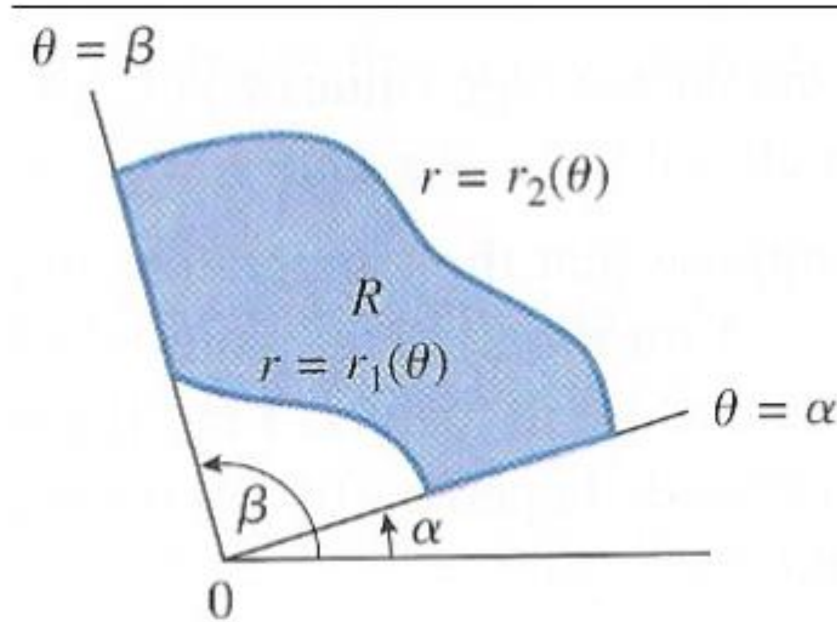
A simple polar region

is a region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two continuous polar curves $r = r_1(\theta)$, $r = r_2(\theta)$ which satisfy

(i) $\alpha \leq \beta$,

(ii) $\beta - \alpha \leq 2\pi$,

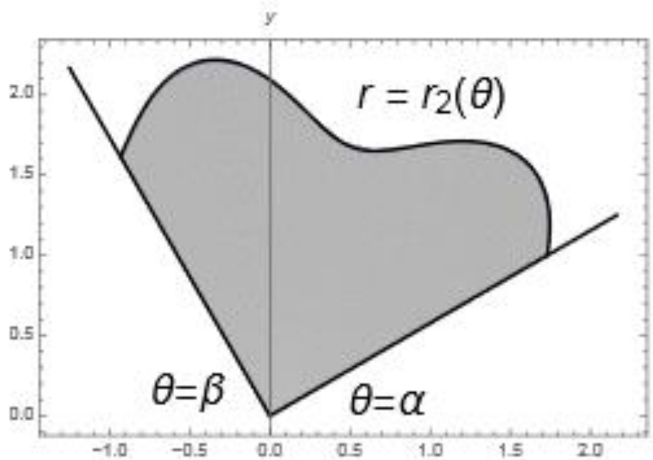
(iii) $0 \leq r_1(\theta) \leq r_2(\theta)$.



Theorem. If R is a simple polar region enclosed between two rays, $\theta = \alpha$, $\theta = \beta$, and two continuous polar curves $r = r_1(\theta)$, $r = r_2(\theta)$, and if $f(r, \theta$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Examples.



$$r_1(\theta) = 0 \text{ and } \beta - \alpha < 2\pi$$

5.4 EVALUATION OF DOUBLE INTEGRAL IN POLAR COORDINATES

To evaluate $\int_{\theta=\alpha}^{\theta=\beta} \int_{r=\varphi(\theta)}^{r=\Psi(\theta)} f(r, \theta)\, dr\, d\theta$, we first integrate with respect to r between the limits $r = \varphi(\theta)$ to $r = \psi(\theta)$ keeping θ as a constant and then the resulting expression is integrated with respect to θ from $\theta = \alpha$ to $\theta = \beta$.

Geometrical Illustration: Let AB and CD be the two continuous curves $r = \varphi(\theta)$ and $r = \Psi(\theta)$ bounded between the lines $\theta = \alpha$ and $\theta = \beta$ so that $ABDC$ is the required region of integration.

Let PQ be a radial strip of angular thickness $\delta\theta$ when OP makes an angle θ with the initial line.

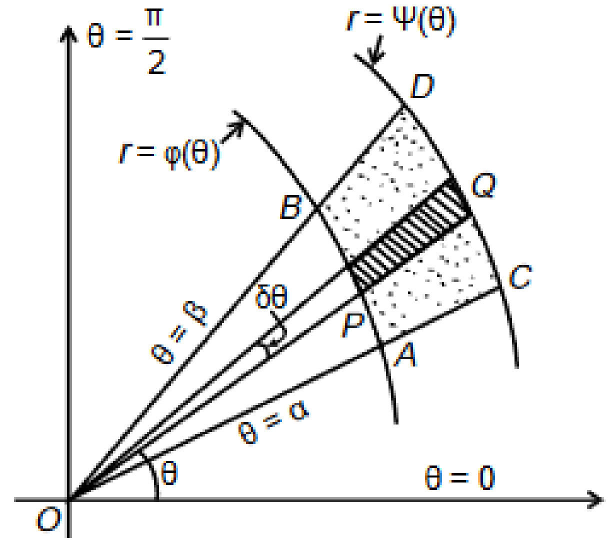


Fig. 5.23

Here $\int_{r=\varphi(\theta)}^{r=\Psi(\theta)} f(r, \theta)\, dr$ refers to the integration with respect to r along the radial strip PQ and then integration with respect to θ means rotation of this strip PQ from AC to CD.

0.16 Transforming a double integral into polars

A very commonly used substitution is conversion into polars. This substitution is particularly suitable when the region of integration D is a circle or an annulus (i.e. region between two concentric circles). Polar coordinates r and θ are defined by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

The variables u and v in the general description above are r and θ in the polar coordinates context and the Jacobian for polar coordinates is

$$\begin{aligned} J &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

So $|J| = r$ and the change of variables rule (0.1) becomes

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

0.17 Example

Use polar coordinates to evaluate

$$\iint_D xy\,dx\,dy$$

where D is the portion of the circle centre 0, radius 1, that lies in the first quadrant.
Solution. For the portion in the first quadrant we need $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. These inequalities give us the limits of integration in the r and θ variables, and these limits will all be constants.
With $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\begin{aligned}
\iint_D xy\,dx\,dy &= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\, r\,dr\,d\theta \\
&= \int_0^{\pi/2}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^1 d\theta \\
&= \int_0^{\pi/2}\frac{1}{4}\sin\theta\cos\theta\,d\theta = \int_0^{\pi/2}\frac{1}{8}\sin 2\theta\,d\theta \\
&= \frac{1}{8}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{8}
\end{aligned}$$

0.18 Example

Evaluate

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy$$

where D is the region between the two circles $x^2+y^2=1$ and $x^2+y^2=4$.
Solution. It is not feasible to attempt this integral by any method other than transforming into polars.
Let $x = r\cos\theta$, $y = r\sin\theta$. In terms of r and θ the region D between the two circles is described by $1 \le r \le 2$, $0 \le \theta \le 2\pi$, and so the integral becomes

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_1^2 e^{-r^2}\,r\,dr\,d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi}\left[-\tfrac{1}{2}e^{-r^2}\right]_{r=1}^{2} d\theta \\
&= \int_0^{2\pi}\left(-\tfrac{1}{2}e^{-4}+\tfrac{1}{2}e^{-1}\right) d\theta \\
&= \pi(e^{-1}-e^{-4})
\end{aligned}$$

Evaluate $\iint r \sin\theta \, dr \, d\theta$ over the cardiod $r = a\,(1 - \cos\theta)$ above the initial line.