25-2-20

$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$

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Solve the DE -

1) $(xy^2 - x)dx = (y + x^2y)\,dy$

$\therefore x(y^2-1)\,dx = y(x^2+1)\,dy$

$\therefore \frac{x}{(x^2+1)}\,dx = \frac{y}{(y^2-1)}\,dy$

$1+x^2 = u \quad , \quad y^2-1 = v$

$2x\,dx = du \quad \quad 2y\,dy = dv$

$x\,dx = \frac{du}{2} \quad , \quad y\,dy = \frac{dv}{2}$

$\frac{1}{2}\frac{du}{u} = \frac{1}{2}\frac{dv}{v}$

$\therefore \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\int \frac{dv}{v} + \log c$

$\therefore \frac{1}{2}\log u = \frac{1}{2}\log v + \log c$

$\frac{1}{2}\log(1+x^2) = \frac{1}{2}\log(y^2-1) + \log c$

$\therefore \log(x^2+1) = \log(y^2-1) + 2\log c$

$\log(x^2+1) = \log(y^2-1) + \log c^2$

Let $c^2 = k$

$\therefore x^2+1 = (y^2-1)(k) \quad \left[\because \log a + \log b = \log ab\right.$

$\therefore 1+x^2 = k(y^2-1)$

$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2} \; ; \; \cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

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2) $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Put $x+y = t$

$1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$\frac{dt}{dx} - 1 = \sin(t) + \cos(t)$

$\therefore \frac{dt}{dx} = \sin t + \cos t + 1$

$\therefore \frac{dt}{\sin t + \cos t + 1} = dx$

$\int \frac{dt}{2\sin\frac{t}{2}\cos\frac{t}{2} + 2\cos^2\frac{t}{2}} = \int dx + c$

$\therefore \int \frac{dt}{2\cos\frac{t}{2}\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)} = x + c$

$\therefore \frac{1}{2}\int \frac{\sec^{t/2}\,dt}{\cos\frac{t}{2}\left(\frac{\sin t/2}{\cos t/2} + 1\right)} = x + c$

$\therefore \frac{1}{2}\int \frac{\sec^2 t/2\,dt}{(\tan t/2 + 1)} = x + c$

$\therefore$ Let $\tan t/2 + 1 = u$

$\therefore \frac{2}{2}\log(\tan t/2 + 1) = x + c$

$\therefore \log\left(1 + \tan\frac{x+y}{2}\right) = x + c$

Q) $(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$

→ Variables can't be separated nor anything can be substituted also each term has same degree Hence Homogenous.

$$\therefore \frac{dy}{dx} = -\frac{(y^4 - 2x^3y)}{(x^4 - 2xy^3)}$$

$$\therefore \frac{dy}{dx} = -\frac{[1 - 2(x/y)^3]}{[(x/y)^4 - 2(x/y)]}$$

$$\therefore \frac{x}{y} = t \qquad \underline{\underline{x = ty}}$$

$$\therefore 1 = t\frac{dy}{dx} + y\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{t} - \frac{y}{t}\frac{dt}{dx}$$

$$\therefore \frac{1}{t} - \frac{y}{t}\frac{dt}{dx} = -\left(\frac{1 - 2t^3}{t^4 - 2t}\right)$$

$$\therefore 1 - \frac{\cancel{x}}{t}\frac{dt}{dx} = -\left(\frac{1 - 2t^3}{t^3 - 2}\right)$$

$$\therefore 1 - \frac{x}{t}\frac{dt}{dx} = \frac{2t^3 - 1}{t^3 - 2}$$

$$1 - \frac{(2t^3 - 1)}{t^3 - 2} = \frac{x}{t}\frac{dt}{dx}$$

$$\therefore \frac{t^3 - 2 - 2t^3 + 1}{t^3 - 2} = \frac{x}{t}\frac{dt}{dx}$$

$$\therefore -\left(\frac{t^3 + 1}{t^3 - 2}\right)t = \frac{x\,dt}{dx}$$

$$\therefore \frac{dx}{x} = \frac{2 - t^3}{t^4 + t}dt$$

$$\therefore \frac{dx}{x} = \frac{2 - t^3}{t(t^3 + 1)}dt$$

$$\text{Let } \frac{2 - t^3}{t(t^3 + 1)} = \frac{A}{t} + \frac{Bx + C}{t^3 + 1}$$

$$\therefore 2 - t^3 = At^3 + A + Bxt + Ct$$

By partial fractions -

$$\frac{(t^3 - 2)}{t(t^3 + 1)} = \frac{t^3 - 2}{t(t+1)(t^2 - t + 1)}$$

$$= \frac{A}{t} + \frac{B}{t+1} + \frac{Ct + D}{t^2 - t + 1}$$

$$\therefore t^3 - 2 = A(t+1)(t^2 - t + 1) + Bt(t^2 - t + 1) + (Ct + D)t(t+1)$$

$\therefore$ put $t = 0$; $\underline{-2 = A}$

put $t = -1$; $\underline{B = 1}$

For C, equate t^3 on both side -

$\therefore A + B + C = 1$

$\therefore \underline{C = 2}$

Put $t = 1$, $1 - 2 = 2A + B + 2C + 2D$

$\therefore \underline{D = -1}$

$$\frac{t^3-2}{t(t^3+1)} = \frac{-2}{t} + \frac{1}{t+1} + \frac{2t-1}{t^2-t+1}$$

$$\therefore \int LHS = -2\log t + \log(t+1) + \log(t^2-t+1)$$

$$\therefore \log x = -2\log t + \log(t+1) + \log(t^2-t+1) + \log c$$

$$\therefore x = \left(\frac{1}{t^2}\right)(t+1)(t^2-t+1)c$$

$$\therefore x = \frac{1}{t^2}(t^3+1)c \Rightarrow \left(t + \frac{1}{t^2}\right)c$$

26-2-20

Exact DE

The DE of the form $M\,dx + N\,dy = 0$ is said to be exact if $M\,dx + N\,dy = du$ for some $u = f(x,y)$

$$u \longrightarrow x, y$$

i.e. the DE is exact if the expression $M\,dx + N\,dy$ is Total Derivative of some function u

The necessary and sufficient condition for a D.E. $M\,dx + N\,dy = 0$ to be exact is $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

To find the solution of exact D.E -

Rule 1) $\int_{y\text{-const}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$

OR

Rule 2) $\int_{x\text{-const}} N\,dy + \int (\text{terms of } M \text{ free from } y)\,dx = c$

Solve the DE -

1) $\frac{dy}{dx} = \frac{1+y^2+3x^2y}{1-2xy-x^3}$

→ Convert to $M\,dx + N\,dy$ form

$$(1-2xy-x^3)\,dy = (1+y^2+3x^2y)\,dx$$

$$\therefore (1+y^2+3x^2y)\,dx - (1-2xy-x^3)\,dy = 0$$

$$\therefore M = 1+y^2+3x^2y \quad \& \quad N = -1+2xy+x^3$$

$$\therefore \frac{\partial M}{\partial y} = 0+2y+3x^2 \quad ; \quad \frac{\partial N}{\partial x} = 0+2y+3x^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{It is an exact DE}$$

Hence Sol^n of DE is

$$\int_{y\text{-const}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c$$

$$\Rightarrow \int_{y\,const} (1+y^2+3x^2y)\,dx + \int (-1)\,dy = c$$

$$\therefore x + y^2x + 3y\frac{x^3}{3} - y = c$$

$$\therefore \underline{x(1+y^2) + yx^3 - y = c}$$

2) $(ye^{xy} - \tan x)dx + (xe^{xy} - \sec y)dy = 0$

$\therefore M = ye^{xy} - \tan x$

$N = xe^{xy} - \sec y$

$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} - 0$

$\frac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ It is exact DE

$\therefore$ Sol^n is -

$$\int_{y-const} (ye^{xy} - \tan x)dx + \int (\text{terms of N free from } x)\,dy = c$$

$$\therefore \frac{ye^{xy}}{y} - \log(\sec x) - \log(\sec y + \tan y) = c$$

$$\Rightarrow \underline{e^{xy} - \log\left((\sec x)(\sec y + \tan y)\right) = c}$$

DE reducible to exact forms-

* Consider the DE of the form $Mdx + Ndy = 0$ which is not exact

Rule 1) If DE is homogenous
Then Integrating Factor (IF) = $\frac{1}{xM + yN}$

where $xM + yN \neq 0$

$\boxed{\text{If DE is exact and homogenous Then } xM + yN = c}$

Rule 2) If DE is of the form

$$yf_1(xy)\,dx + xf_2(xy)\,dy = 0$$

then $IF = \frac{1}{xM - yN}$ where $xM - yN \neq 0$

Rule 3) If factor of N appears in numerator
then $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$

then $\boxed{IF = e^{\int f(x)dx}}$

Rule 4) If factor of M appears in numerator
then $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$

then $\boxed{IF = e^{\int g(y)dy}}$

Solve the DE-

1) $x(x-y)\frac{dy}{dx} = y(x+y)$

$\rightarrow$ $\therefore y(x+y)dx - x(x-y)dy = 0$

$\therefore M = y(x+y)$; $N = -x(x-y)$

$\therefore M = xy + y^2$; $N = -x^2 + xy$

$\therefore \frac{\partial M}{\partial y} = x + 2y$; $\frac{\partial N}{\partial x} = -2x + y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ It's not exact.

$\therefore xM + yN = x(xy+y^2) + y(-x^2+xy)$
$= x^2y + xy^2 - x^2y + xy^2$
$= 2xy^2$

$\therefore \frac{1}{xM+yN} = \frac{1}{2xy^2} = IF$

$\therefore$ Multiply through out.

$\therefore \frac{1}{2xy^2}(y)(x+y)dx - \frac{1}{2xy^2}(x)(x-y)dy = 0$

$y^2(x+y)dx - x(x-y)dy = 0$

$\therefore \frac{(x+y)}{xy}dx - \frac{(x-y)}{y^2}dy = 0$

which is now exact

$\therefore$ Sol^n of DE is

$\int_{y\text{-const}} M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\therefore \int_{y\text{ const}}\left(\frac{1}{y}+\frac{1}{x}\right)dx + \int\frac{1}{y}dy = c$

$\therefore \frac{x}{y} + \log x + \log y = c$

$\therefore \frac{x}{y} + \log(xy) = c$

$\log(xy) = c - \frac{x}{y}$

$\therefore xy = e^{c - x/y} = e^c \cdot e^{-xy}$

$\therefore xy = \frac{e^c}{e^{x/y}} = \frac{k}{e^{x/y}}$

$\therefore xy \cdot e^{x/y} = k$

2-3-20

$y(x^2y^2+5xy+2)dx + x(x^2y^2+4xy+2)dy = 0$

$\rightarrow$ $M = y(x^2y^2+5xy+2)$, $N = x(x^2y^2+4xy+2)$

$M = x^2y^3 + 5xy^2 + 2y$; $N = x^3y^2 + 4x^2y + 2x$

$\Rightarrow \frac{\partial M}{\partial y} = 3x^2y^2 + 10xy + 2$; $\frac{\partial N}{\partial x} = 3x^2y^2 + 8xy + 2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ It's not exact.

But DE is of the form :

$$y f_1(xy)dx + x f_2(xy)dy = 0$$

$$IF = \frac{1}{xM - yN} = \frac{1}{xy(x^2y^2+5xy+2) - yx(x^2y^2+4xy+2)}$$

$$= \frac{1}{xy(\cancel{x^2y^2}+5xy+\cancel{2}-\cancel{x^2y^2}-4xy-\cancel{2})}$$

$$\therefore IF = \frac{1}{x^2y^2}$$

Multiplying IF to DE.

$$\therefore \frac{y(x^2y^2+5xy+2)}{x^2y^2}dx + \frac{x(x^2y^2+4xy+2)}{x^2y^2}dy = 0$$

$$\therefore \left(y + \frac{5}{x} + \frac{2}{x^2y}\right)dx + \left(x + \frac{4}{y} + \frac{2}{xy^2}\right)dy = 0$$

which is now exact DE.

$\therefore$ Sol^n of DE is -

$$\int_{y\,const} M\,dx + \int (\text{Terms of N free from } x)\,dy = c$$

$$\therefore \int_{y\,const}\left(y + \frac{5}{x} + \frac{2}{x^2y}\right)dx + \int \frac{4}{y}dy = c$$

$$\therefore yx + 5\log(x) + \frac{2}{y}\left(\frac{-1}{x}\right) + 4\log y = c$$

$$\therefore xy + 5\log x - \frac{2}{xy} + 4\log y = c$$

Q) $[2x\log(x) - xy]dy + 2y\,dx$

$\rightarrow$ $M = 2y$; $N = 2x\log(x) - xy$

$$\therefore \frac{\partial M}{\partial y} = 2 \; ; \; \frac{\partial N}{\partial x} = \frac{2x}{x} + 2\log(x) - y$$

$\therefore$ It's not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2 - (2\log x + 2 - y)$

$$= \underline{y - 2\log x}$$

Then $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{y - 2\log x}{2x\log x - xy} = \dfrac{\cancel{y-2\log x}}{-x(\cancel{y-2\log x})}$

$$= \frac{-1}{x} = f(x)$$

$$\therefore IF = e^{\int f(x)dx}$$

$$= e^{\int \frac{-1}{x}dx} = e^{-\int \frac{1}{x}dx}$$

$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \underline{+1/x}$$

$\therefore$ Multiplying IF to DE.

$$\frac{2y}{x}dx - \frac{2x\log x - xy}{x}dy = 0$$

$$\Rightarrow 2\frac{y}{x}dx + (2\log x - y)dy = 0$$

$\therefore \int\limits_{x\,const} N\,dy + \int (\text{Terms of M free from } y)\,dx = C$

$\therefore \int\limits_{x\,const} (2\log x - y)\,dy + \int 0\,dx = c$

$\therefore \frac{-y^2}{2} = c$

$\int c\,dy$

$= 2y\log x - \frac{y^2}{2} = C$

$\frac{d}{dy}$

3-3-20

Linear Differential Eq^n

$\frac{dy}{dx} + Py = Q$ (I^{st} order LDE)

$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_0 y = Q$ (II^{nd} order LDE)

$\frac{d^ny}{dx^n} + P_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \dots P_1\frac{dy}{dx} + P_0 y = Q$

where $P_{n-1}, P_{n-2}, P_1, P_0, Q$ are fun^n of x only

DE in which dependent variable and differential coeff has degree one and both are not multiplied together then it is LDE.

Sol^n of LDE-

Consider homogenous LDE

$\therefore \frac{dy}{dx} + Py = 0$

$\Rightarrow \frac{dy}{dx} = -Py \Rightarrow \int \frac{dy}{y} = -\int P\,dx + c$

$\therefore \log y = -\int P\,dx + c$

$\therefore y = e^{-\int P dx + c}$

$\therefore y_c = e^{-\int P dx} \cdot c_1$

Let $\frac{d}{dx} = D$ (Differential operator)

$\Rightarrow Dy + Py = Q \Rightarrow (D+P)y = Q$

$\Rightarrow y = \frac{1}{(D+P)} Q$

$\Rightarrow y_p = e^{-\int P dx} \int e^{\int P dx} Q\,dx$

$=$ Particular integral

$\therefore y = y_c + y_p$

$y = e^{-\int P dx} c_1 + e^{-\int P dx} \int e^{\int P dx} Q\,dx$

$\Rightarrow y e^{\int P dx} = c_1 + \int e^{\int P dx} Q\,dx$

$\Rightarrow y(IF) = \int (IF) Q\,dx + c_1 \quad \Big| \quad IF = e^{\int P dx}$

Solve the LDE -

i) $(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$

$\Rightarrow \frac{dx}{dy} + \frac{(x - e^{\tan^{-1} y})}{1+y^2} = 0$

$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{\tan^{-1} y}}{1+y^2}$

Its LDE in x

$\therefore P = \frac{1}{1+y^2}$; $Q = \frac{e^{\tan^{-1} y}}{1+y^2}$

$\therefore IF = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = \underline{\underline{e^{\tan^{-1} y}}}$

$\therefore$ Sol^n of LDE

$\therefore x(IF) = \int (IF) Q \, dy + C$

$\therefore x e^{\tan^{-1} y} = \int e^{\tan^{-1} y}\left(\frac{e^{\tan^{-1} y}}{1+y^2}\right) dy + c$

$=$ ~~Let $\tan^{-1} y = t$~~

$\therefore$ ~~$e^{\tan^{-1} y}\left(\frac{1}{1+y^2}\right) dy = dt$~~

$= \cancel{\int t \, dt}$

$\Rightarrow \tan^{-1} y = t$

$\therefore \frac{1}{1+y^2} dy = dt$

$\therefore x e^{\tan^{-1} y} = \int e^{2t} dt + c$

$x e^{\tan^{-1} y} = \frac{e^{2t}}{2} + c$

$\therefore x e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$

$\Rightarrow 2x e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$.

6-3-20 DE Reducible to LDE -

① $f'(y)\frac{dy}{dx} + P f(y) = Q$

Let $f(y) = u \Rightarrow f'(y)\frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{du}{dx} + Pu = Q$

It is LDE in u

② Bernoulli LDE -

$\frac{dy}{dx} + Py = Qy^n$

divide by y^n

$\therefore \frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$

Let $\frac{1}{y^{n-1}} = u$

$\therefore \frac{-(n-1)}{y^n}\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{-1}{(n-1)}\frac{du}{dx}$

$\therefore \frac{-1}{(n-1)}\frac{du}{dx} + Pu = Q$

$\therefore \frac{dy}{dx} + (1-n)Pu = (1-n)Q$

Solve DE -

1) $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

$\rightarrow \frac{dy}{dx} - xy = -e^{-x^2} y^3$

divide by y^3

$\therefore \frac{1}{y^3}\frac{dy}{dx} - \frac{x}{y^2} = -e^{-x^2}$

$\therefore \frac{-1}{y^2} = u \quad \therefore \frac{+2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{1}{2}\frac{du}{dx} + xu = -e^{-x^2}$

$\therefore \frac{du}{dx} + 2xu = -2e^{-x^2}$

It is linear in u

$\therefore P = 2x \; ; \; Q = -2e^{-x^2}$

$\therefore IF = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ General solⁿ

$\therefore u\,IF = \int (IF)Q\,dx + c$

$\therefore \frac{-1}{y^2}(e^{x^2}) = \int e^{x^2}(-2e^{-x^2})\,dx + c$

$\therefore \frac{-e^{x^2}}{y^2} = -2x + c$

$\therefore -e^{x^2} = -2xy^2 + cy^2$

$\therefore 2xy^2 - e^{x^2} - cy^2 = 0$

Q) $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$.

$\rightarrow$ multiply by $\sec y$

$\therefore \sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^3 x$

Put $\sec y = u$

$\therefore \sec y \tan y \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{du}{dx} + \tan x (u) = \cos^3 x$

It is Linear in u.

$\therefore P = \tan x,\ Q = \cos^3 x$

$\therefore IF = e^{\int P dx} = e^{\int \tan x \, dx}$

$\therefore IF = \sec x$

$\therefore UIF = \int (IF) Q \, dx + c$

$\therefore \sec y (\sec x) = \int \sec x \cos^3 x \, dx + c$

$\therefore \sec y (\sec x) = \int \cos^2 x \, dx + c$

$= \int \frac{1 + \cos 2x}{2} dx + c$

$= \int \frac{1}{2} dx + \frac{1}{2} \int \cos 2x \, dx + c$

$= \frac{x}{2} + \frac{1}{2} \frac{\sin 2x}{2} + c$

$= \frac{x}{2} + \frac{\sin 2x}{4} + c$

$\therefore 4 \sec x \sec y = 2x + \sin 2x + C$

1-3-20

Orthogonal Trajectory

A curve which intersects every member of given family of curves at right angle is called orthogonal trajectory to the given family of curves

Q) Find orthogonal trajectory to family of curve given by- $2x^2 + y^2 = cx$

$\rightarrow$ $2x^2 + y^2 = cx$ — (1)

diff wrt x.

$4x + 2y \frac{dy}{dx} = c$ — (2)

From (1) & (2)

$2x^2 + y^2 = \left[4x + 2y \frac{dy}{dx} \right] x$

$\rightarrow 2x^2 + y^2 = 4x^2 + 2xy \frac{dy}{dx}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\Rightarrow 2x^2 + y^2 = 4x^2 - 2xy \frac{dx}{dy}$

$\Rightarrow 2x^2 - y^2 - 2xy \frac{dx}{dy} = 0$

Replace $x = vy$

$\therefore \frac{dx}{dy} = v + y \frac{dv}{dx}$

$\therefore 2(vy)^2 - y^2 - 2(vy) y \left(v + y \frac{dv}{dy} \right) = 0$

$2v^2y^2 - y^2 - 2v^2y^2 - 2vy^3 \frac{dv}{dy} = 0$

$\therefore -y^2 = 2vy^3 \frac{dv}{dy}$

$-dy = 2vy \, dv$

$\frac{dy}{y} = -2v \, dv$

$\therefore \int \frac{dy}{y} = -2\int v\,dv + c$

$\therefore \log y = -\frac{2v^2}{2} + c$

$\therefore \log y = -\left(\frac{x}{y}\right)^2 + c$

$\therefore \log y + \left(\frac{x}{y}\right)^2 = c$

or $c = \log k$

$\therefore e^{x^2/y^2} y = k$

Q) Find the orthogonal trajectory to family of curves given by $ay^2 = x^3$

→ $ay^2 = x^3$ — ①

$\therefore$ Diff w.r.t. x

$\Rightarrow 2ay\frac{dy}{dx} = 3x^2$ — ②

$\therefore$ ①/② $\Rightarrow$

$3ax^2y^2 = 2ax^3y\frac{dy}{dx}$

$\therefore 3y = 2x\frac{dy}{dx}$

$\therefore \cancel{\frac{1}{2}\frac{dx}{x} = \frac{1}{3}\frac{dy}{y}}$

On integrating

$\cancel{\frac{1}{2}\log x = \frac{1}{3}\log y + \log c}$

$\therefore \log x^{1/2} = \log y^{1/3} + \log c$

$\therefore \cancel{x^{1/2} = y^{1/3}.c}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\therefore 3y = -2x\frac{dx}{dy}$

$\therefore 3y\,dy = -2x\,dx$

on integrating

$\int 3y\,dy = -2\int x\,dx + c$

$\therefore \frac{3y^2}{2} = -\frac{2x^2}{2} + c$

$\therefore x^2 + \frac{3y^2}{2} = c$

or

$2x^2 + 3y^2 = k$

In Polar form –

$f(r,\theta) = c$

$\therefore$ slope $= \boxed{\frac{1}{r}\frac{dr}{d\theta}}$

then replace $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

Q) Find orthogonal trajectory to the family of curve given by

$r = a(1-\sin\theta)$ — ①

→ $r = a - a\sin\theta$

$\frac{dr}{d\theta} = -a\cos\theta$ — ②

$\therefore r\cos\theta = \frac{dr}{d\theta}(\sin\theta - 1)$

replace $\frac{dr}{d\theta} \rightarrow \frac{-1}{r^2}\frac{d\theta}{dr}$

~~$r\cos\theta = \frac{-1}{r^2}\frac{d\theta}{dr}(\sin\theta - 1)$~~

~~$\therefore r^3\cos\theta = \frac{d\theta}{dr}(1-\sin\theta)\frac{d\theta}{dr}$~~

~~$\therefore r^3 dr = \frac{1-\sin\theta}{\cos\theta}$~~

replace $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$\therefore r\cos\theta = -r^2\frac{d\theta}{dr}(\sin\theta - 1)$

$\therefore \frac{dr}{r} = \left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta$

$\therefore$ On integrating

$\int \frac{dr}{r} = \int \sec x\,d\theta - \int \tan\theta\,d\theta + c$

$\log r = \log|\sec\theta + \tan\theta| - \log|\sec\theta| + \log c$

$r = \left(\frac{\sec\theta + \tan\theta}{\sec\theta}\right)c$

$\therefore \frac{r}{c} = kr = 1 + \sin\theta$

$\therefore 1 + \sin\theta = kr$

Q) $r = a(1+\cos\theta)$ a) $r(1-\cos\theta) = 2a$

Q) $r = 2 + 3\sin\theta$ a) $r(1+\sin\theta) = c$

Q) Find orthogonal trajectory of family of curves given by $r = a\cos^2\theta$

$\rightarrow$ $r = a\cos^2\theta$ — ①

$\frac{dr}{d\theta} = -2a\cos\theta\sin\theta$ — ②

$-2a\cos\theta\sin\theta\, r = \frac{dr}{d\theta}a\cos^2\theta$

$-2r\sin\theta = \cos\theta\frac{dr}{d\theta}$

replace $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$\therefore 2r\sin\theta = r^2\cos\theta\frac{d\theta}{dr}$

$2\sin\theta = r\cos\theta\frac{d\theta}{dr}$

$\therefore 2\frac{dr}{r} = \cot\theta\,d\theta$

$\therefore$ On integrating

~~$2\log r = \log|\operatorname{cosec}\theta - \cot\theta|$~~

$2\log r = \log|\sin\theta| + \log c$

$\therefore r^2 = c\sin\theta$

HW

1) $r = a(1+\cos\theta)$ — ①

$\therefore \frac{dr}{d\theta} = -a\sin\theta$ — ②

$\therefore r\sin\theta = -\frac{dr}{d\theta}(1+\cos\theta)$

$\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$\therefore r\sin\theta = r^2\frac{d\theta}{dr}(1+\cos\theta)$

$\frac{dr}{r} = \frac{1+\cos\theta}{\sin\theta}\,d\theta$

~~On~~ integrating

$\int \frac{dr}{r} = \int \frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} + c$

$\therefore \log r = \int \cot\theta/2 + c$

$\log r = \log|\sin\theta/2| + c$

$\therefore \frac{dr}{r} = \operatorname{cosec}\theta + \cot\theta$

on integrating

$\log r = \log|\operatorname{cosec}\theta - \cot\theta| + \log|\sin\theta| + \log$

$\therefore r = (\sin\theta\operatorname{cosec}\theta - \frac{\sin\theta\cos\theta}{\sin\theta})c$

$r = c(1-\cos\theta)$

2) $r = 2 + 3\sin\theta$ — ①

$\therefore \frac{dr}{d\theta} = +3\cos\theta$ — ②

$\therefore (r-2)(\cos\theta) = 3\sin\theta\frac{dr}{d\theta}$

$\cot\theta\, d\theta = \frac{dr}{r-2}$

$\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$\therefore (r-2)(\cos\theta) = -r^2\sin\theta\frac{d\theta}{dr}$

$\therefore \frac{r-2}{r^2}\,dr = -\tan\theta\, d\theta$

Integrating both sides -

$r^2 = t$
$\therefore 2r\,dr = dt$

$\therefore \int\left(\frac{1}{r} - \frac{2}{r^2}\right)dr = -\log|\sec\theta| + \log c$

$\log r - 2\left(\frac{-1}{r}\right) = \log c - \log\sec\theta$

$\log r + \log\sec\theta - \log c = -\frac{2}{r}$

$\therefore \frac{r\sec\theta}{c} = \frac{c}{r\sec\theta} = e^{2/r}$

$\therefore re^{2/r}\sec\theta = c$

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3) $r(1-\cos 2\theta) = 2a$

$\therefore r = \dfrac{2a}{1-\cos 2\theta} \Rightarrow \dfrac{\cancel{2}a}{\cancel{2}\sin^2\theta} \Rightarrow a\,\text{cosec}^2\theta$

$\therefore r = a\,\text{cosec}^2\theta \quad -①$

$\therefore \dfrac{dr}{d\theta} = -2a\,\text{cosec}^2\theta \cot x \quad -②$

$\therefore 2r\, \cancel{\text{cosec}\,x} \cot x = -\dfrac{dr}{d\theta}$

$\therefore \dfrac{dr}{d\theta} \rightarrow -r^2\dfrac{d\theta}{dr}$

$\therefore 2r\cot\theta = r^2\dfrac{d\theta}{dr}$

$\dfrac{dr}{r} = \dfrac{d\theta}{2\cot\theta}$

$\therefore \dfrac{dr}{r} = \dfrac{1}{2}\tan\theta\, d\theta$

$\therefore$ Integrating we get -

$\log r = \dfrac{1}{2}\log|\sec\theta| + \log c$

$r = \sec\theta^{1/2} c$

4) $r(1+\sin\theta) = c$

$\therefore r + r\sin\theta = c$

$\therefore$ On diff -

$\therefore \dfrac{dr}{d\theta} + r\cos\theta + \sin\theta\dfrac{dr}{d\theta} = 0$

$\therefore (1+\sin\theta)\dfrac{dr}{d\theta} = -r\cos\theta$

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$\therefore \dfrac{dr}{d\theta} \rightarrow -r^2\dfrac{d\theta}{dr}$

$\therefore +(1+\sin\theta)r^2\dfrac{d\theta}{dr} = -r\cos\theta$

$\therefore \left(\dfrac{1+\sin\theta}{\cos\theta}\right)d\theta = \dfrac{dr}{r}$

$\therefore (\sec\theta + \tan\theta)\, d\theta = \dfrac{dr}{r}$

On integrating -

$\log|\sec\theta + \tan\theta| + \log|\sec\theta| = \log r + \log c$

$\sec^2\theta + \tan\theta\sec\theta = rc$

11-3-20 **Simple Electric Circuit**

① Current is defined as the rate of flow of electric changes.

$$i = \frac{dq}{dt} \quad \text{or} \quad q = \int i\,dt$$

② Voltage drop across the resistance R is iR
voltage drop across capacitance C is q/c
Votage drop across inductor L is $L\frac{di}{dt}$

Kirchoff's Law -

The algebraic sum of voltage drop across any closed circuit is equal to the resultant EMF in the circuit.

The algebraic sum of the currents flowing into (or from) any node is 0.

Formation of DE for given circuit

① RC circuit - Consider an electric circuit containing a resistance R and a capacitance C along with a voltage source E, all in series, if current i is flowing in the circuit by Kirchoff's law

$$Ri + \frac{q}{c} = E$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\Rightarrow \frac{dq}{dt} + \frac{1}{Rc}q = \frac{E}{R}$$

which is LDE in q

$$IF = e^{\int \frac{1}{Rc}dt} = e^{t/Rc}$$

$$\therefore q(IF) = \int (IF)Q\,dt + k$$

② RL circuit - Consider the circuit containing a resistance R and an inductance L along with an EMF E all in series. If the current is flowing in the circuit by Kirchoff's law -

$$L\frac{di}{dt} + Ri = E$$

$$\therefore \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

It's linear in i

$$\therefore IF = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

$$\therefore i(IF) = \int (IF)Q\,dt + k$$

Q) In an electric circuit containing an inductance L = 640 H, a resistance R = 250 Ω and voltage E = 500V. current is flowing. Find the current at any time 't'. Also find the time that ellapses before the current reaches 90%

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of its max value.

→ $\cancel{L\frac{di}{dt} + Ri = E}$

$\therefore \cancel{\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}}$

The given circuit is RL circuit by Kirchoff's Law.

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

$$\therefore \frac{di}{dt} + \frac{250}{640} i = \frac{500}{640}$$

$$\therefore \cancel{\frac{di}{dt}} + \frac{di}{dt} + \frac{25}{64} i = \frac{25}{32}$$

$\therefore$ Its LDE in i

$$\therefore IF = e^{\int \frac{25}{64} dt} = e^{\frac{25t}{64}}$$

$$\therefore i\left(e^{\frac{25t}{64}}\right) = \int \left(e^{\frac{25t}{64}}\right) \frac{25}{32} dt + k$$

$$\therefore i e^{\frac{25t}{64}} = \frac{25}{32} e^{\frac{25t}{64}} \times \frac{64^2}{25} + k$$

$$\therefore i e^{\frac{25t}{64}} = 2 e^{\frac{25t}{64}} + k$$

$$\therefore i = 2 + k e^{-25t/64}$$

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At $t=0$; current $i=0$

$$\therefore 0 = 2 + k e^{-0}$$

$$\therefore \underline{\underline{K = -2}} \quad \left(\text{Also } K = -\frac{E}{R}\right)$$

$$\therefore i = 2 - 2e^{-25t/64}$$

$$\therefore i = 2(1 - e^{-25t/64}) \quad -①$$

$\therefore i_{max}$ is obtained as $t \to \infty$

$$\therefore i_{max} = 2(1 - e^{-\infty})$$

$$\therefore \underline{\underline{i_{max} = 2}} \quad \left(\text{Also } i_{max} = \frac{E}{R}\right)$$

$$\therefore 90\% \text{ of } i_{max} = \frac{90}{100} \times 2 = \underline{\underline{1.8}}$$

$\therefore$ Putting in ① we get

$$1.8 = 2(1 - e^{-25t/64})$$

$$\Rightarrow 0.9 = 1 - e^{-25t/64}$$

$$\Rightarrow e^{-25t/64} = 1 - 0.9$$

$$\therefore -\frac{25t}{64} = \ln(0.1)$$

$$\therefore t = 5.89 \text{ sec}$$

Q) A change q on plate of a condenser of capacity C is charged through a resistance R by a steady voltage V and satisfies the DE $R\frac{dq}{dt} + \frac{q}{C} = V$.

Initially if charge is 0, show that $q = CV(1 - e^{-t/RC})$. Also find the current flowing into the plate at time 't'.

→

$$\therefore \frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}$$

$$\therefore \frac{dq}{dt} + \frac{1}{RC} q = \frac{V}{R}$$

: Its LDE in q.

$$\therefore IF = e^{\int \frac{1}{RC} dt} = \underline{e^{t/RC}}$$

$$\therefore q(e^{t/RC}) = \int \frac{V}{R} (e^{t/RC})\, dt + K$$

$$\therefore q e^{t/RC} = \frac{V}{\cancel{R}} e^{t/RC} \times \cancel{R}C + k$$

$$\therefore q e^{t/RC} = CV e^{t/RC} + k$$

$$\therefore q = CV + k e^{-t/RC}$$

$\therefore$ at $t=0$; $q=0$

$$\therefore 0 = CV + k e^{-0}$$

$$\therefore \underline{K = -CV}$$

$$\therefore q = CV - CV e^{-t/RC}$$

$$\therefore \underline{q = CV(1 - e^{-t/RC})} \quad -①$$

$\therefore$ diff w.r.t t

$$\therefore \frac{dq}{dt} = i = 0 - \cancel{C}V e^{-t/RC} \times \frac{-1}{R\cancel{C}}$$

$$\therefore i = \underline{\frac{V}{R} e^{-t/RC}}$$

H.W

Q) A voltage Ee^{-at} is applied at time $t=0$ to a circuit containing an inductance L and a resistance R. Find the current flowing in the circuit at any time t.

Q) Solve the DE - $L\frac{di}{dt} + Ri = 150\cos(200t)$.
Find the current at any time t if initial current is 0 and $L = 0.2H$, $R = 10\Omega$

Q) A const. EMF 'E' volts is applied to a circuit containing a const. resistance R ohms and a const. inductance L Henrys in series. If initial current is 0, show that the current builts up to half of its theoritical maximum in $\frac{L}{R}\log_e 2$ sec

Heat Flow -

The fundamental principle of heat flow are -

1) Heat flows from higher temp to lower temp.
2) The quantity of heat in a body is proportional to its mass and temperature.
3) Fourier Law to heat conduction -
The rate of heat flow across an area is proportional to the area and rate of change of temp wrt distance normal to the area.

Let q cal/sec heat flows from a slab of area A in cm^2 and the thickness δx, then -

$$q = -KA\frac{dT}{dx}$$

where K is called thermal conductivity and -ve sign indicates that T decreases as x is increased.

Q) A pipe 20 cm in diameter contains steam at 150°c and is protected with covering 5 cm thick whose thermal conductivity $K = 0.0025$. If the temp of outer surface of covering is 40°c, find the temp half way through the covering under steady state condⁿ

→ Let A be the area of unit length of the pipe of radius x

$\therefore A = 2\pi x$ (i)

Then by Fourier law -

$$q = -KA\frac{dT}{dx}$$

$$\Rightarrow q = -K2\pi x\frac{dT}{dx}$$

$$\frac{dx}{x} = -\frac{2\pi K x}{q}\,dT$$

OR

$$\frac{q}{2\pi K}\int\frac{dx}{x} = -\int dT \quad -(1)$$

10 cm	150°c
10+5 cm	40°c
10.2.5 cm	?

To find q

$$\therefore \frac{q}{2K\pi}\left[\log_e x\right]_{10}^{15} = -\left[T\right]_{150}^{40}$$

$$\therefore q = \frac{(150-40)}{\log\frac{3}{2}} \times 2\times 0.0025\pi$$

$$\therefore q = 4.261$$

$\therefore \frac{4.2614}{2h\pi}[\ln x]_{10}^{12.5} = -[T]_{150}^{T}$

$\therefore 271.2\left[\ln\left(\frac{12.5}{10}\right)\right] = 150 - T$

$60.516 = 150 - T$

$\therefore \underline{\underline{T = 89.5}}$

Q) A pipe 10cm in diameter contains steam at $100°c$. It is covered with a 5cm thick covering whose thermal conductivity $K = 0.0006$. The outside surface is maintained at $30°c$, find the amount of heat loss per hour from a meter long pipe.

→ $A = 2\pi x(100)$

$A = 200\pi x$

$\therefore$ By Fourier Law -

$q = -KA\frac{dT}{dx}$

$\therefore q = -K\,200\pi x\,\frac{dT}{dx}$

$\therefore \frac{q}{200K\pi}\int\frac{dx}{x} = -\int dT$

$\therefore$ To find q

$\frac{q}{200K\pi}[\ln x]_{5}^{10} = -[T]_{100}^{30}$

$\therefore q = 38.07$ cal/sec

$\therefore q = 38.07 \times 60 \times 60 = 137058.37$ cal/hr

<u>HW</u>

Q) $L\frac{di}{dt} + iR = Ee^{-at}$

$\therefore \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$

$\therefore$ It is LDE in i

$IF = e^{\int\frac{R}{L}dt} = e^{\frac{Rt}{L}}$

$\therefore i\left(e^{\frac{Rt}{L}}\right) = \int\frac{E}{L}e^{-at}\times e^{\frac{Rt}{L}}dt + A$

$\therefore ie^{\frac{Rt}{L}} = \frac{E}{L}e^{\frac{Rt}{L}-at}\times\frac{L}{(R-aL)} + A$

Also at $t=0, i=0$

$\therefore 0 = \frac{E(1)}{R-aL} + A \quad \therefore A = -\frac{E}{R-aL}$

$\therefore i = \frac{Ee^{-at}}{R-aL} - \frac{E}{R-aL}$

$\therefore i = \frac{E}{R-aL}\left(e^{-at}-1\right)$