SYLLABUS:

- Sets, Combinations of sets, Finite and Infinite sets, Uncountable Infinite sets, Principle of Inclusion and Exclusion, Multisets.

OBJECTIVES:

- Notion of set and its properties are the necessary prerequisites for the student to understand more complicated structures.
- To learn the set theory approach for solving problems on counting.

UTILITY:

- As a basic tool to study various discrete structures such as graphs, groups and rings.
- In problems related to combinatorics.

KEY CONCEPTS:

- Subset, universal set, empty set.
- Set operations: Union, Intersection, Complementation.
- Power set.
- Multiset
- Principle of Inclusion – Exclusion.
- Mathematical Induction.

2.0 Introduction

Set is a fundamental concept in the theory of Discrete Structures. Any algebraic structure, be it a 'group' or 'graph', has its 'underlying structure'. Hence, one ought to have a clear understanding of the term **set.**

The theory of sets was first introduced by the German Mathematician G. Cantor (1845 – 1918), who defined a set simply as a collection of objects. Later, contradictions or *paradoxes* were discovered in the definition. We have the famous *Russell's Paradox,* due to Bertrand Russell. Russell asked the following question.

Suppose the universe of discourse in **the set of all sets,** and let S be a set, whose objects are sets, **which are not members of themselves,** then is S a member of itself? If S is not a member of itself, then by the condition imposed on S, S should belong to itself. On the other hand, if S does not belong to itself, then S should be a member of itself, as per the definition. This is the paradox.

The discovery of the paradoxes, however, did not mean that Cantor's original work characterised as 'naive set theory' was to be abandoned. It was found that by suitably redefining the universe of discourse, the paradoxes could be circumvented. Thus, evolved a theory, which ranks today as one of the most important areas of Modern Mathematics.

In the ensuing discussion, we shall adopt the modified version of Cantor's theory. For each separate situation, we shall define a universal set and within its framework, the set under discussion will be a collection of objects, the objects being also members of the universal set.

2.1 Sets

2.1.1 Definition

*A set is a **collection of** objects.*

*An object in the collection is called an **element** or member of the set.*

*The term **class** is also used to denote a set.*

*A set may contain **finite** number of elements or **infinite** number of elements.*

*A set is called an **empty set** or a **null set** if it contains no element. An empty set is denoted by the letter ϕ.*

Examples:

(i) The set of letters forming the word 'PASCAL', is a finite set, whose elements are the five **distinct** letters of the word.

(ii) The set of all telephone numbers in the directory. This is also a finite, though a large, set.

(iii) The set of persons in a moving queue. This is also a finite set, but difficult to list, due to the constant flux (At any instant of time, people are entering as well as leaving the queue).

(iv) The set of whole (natural) numbers greater than 10. This is an infinite set, but the elements in the set can be listed, i.e. 11, 12, 13, ...

(v) The set of all points in the plane. This is also an infinite set, but the elements cannot be listed, as the points are 'dense' not 'discrete.'

(vi) The set consisting of a circle, the number 5, a tree and Bill Gates.
This example shows that the elements in a set can be totally different in character, they need not have a common characteristic.

(vii) The set of real roots of the equation $x^2 + 1 = 0$. This is obviously an empty set.

2.1.2 Notations

A set is generally denoted by capital letters A, B, C, ..., X, Y, Z.

Elements of the set are denoted by small letters a, b, c, ..., x, y, z.

If x is an element of the set A, we express this fact by writing

'$x \in A$' ($\in$ means 'belongs to')

If x is not an element of A, we write

'$x \notin A$'

There are various ways of describing a set.

(a) Listing Method: In this method, the elements are listed within braces.

e.g. (i) $A = \{\text{pencil, byte, } 5\}$

(ii) $B = \{2, 4, 6, 8, \ldots\ldots\}$

(b) Statement Form: A statement describing the set, especially where the elements share a common characteristic. e.g.

(i) The set of all equilateral triangles.

(ii) The set of all Prime Ministers of India.

(c) Set-Builder Notation: It is not always possible or convenient to describe a set by the Listing method or the Statement form. A more concise or compact way of describing the set is to specify the property shared by all the elements of the set. This property is denoted by P(x), where P is a statement concerning an element x of the set. The set is then simply written as

$\{x \mid P(x)\}$ where the braces { } denote the clause " the set of ", and the slash or stroke | denotes "such that" (read as ". A is the set of all x such that x is greater than 10").

Examples: (i) $A = \{x \mid x > 10\}$

(ii) $B = \{x \mid x \text{ is real and } x^8 - 5x^4 + 4 = 0\}$.

2.1.3 Some Special Sets (Number Sets)

The following sets occur frequently in our discussion. We give below the standard notations used to denote these sets.

N	–	the set of all natural numbers {1, 2, 3, ... }.
Z	–	the set of all integers {...... – 2, –1, 0, 1, 2, ... }.
Z^+	–	the set of all positive integers {0, 1, 2, ... }.
Q	–	the set of rational numbers.
Q^+	–	the set of non-negative rational numbers.
$\mathbb{R}$	–	the set of real numbers.
$\mathbb{C}$		the set of complex numbers.

2.2 Subsets

2.2.1 Definition

If every element of a set A is also an element of a set B, then we say A is a ***subset*** *of B, or A is* ***contained*** *in B. This is denoted by writing 'A $\subseteq$ B'. This can be also denoted by 'B $\supseteq$ A'*

If A is not a subset of B, this is indicated by writing 'A $\subseteq$ B'.

Examples:

(i) $N \subseteq Z^+ \subseteq Z \subseteq Q \subseteq \mathbb{R} \subseteq \mathbb{C}$

(ii) A = {1, 3, 6}, B = {–1, 1, 2, 3, 4, 6}

C = {1, 2, 3}

Then A $\subseteq$ B,

But A $\subseteq$ C

It is clear from the definition that **Every set is a subset of itself. The empty set is a subset of any set.**

2.2.2 Universal Set

If all sets, considered during a ***specific discussion*** *are subsets of a given set, then this set is called as the* ***Universal Set****, and is denoted by 'U'.*

Hence, the universal set is a relative concept dependent on the specific discussion. Therefore, it is also referred to as the **universe of discourse.**

2.2.3 Equality of Sets

Two sets A and B are equal if A is a subset of B and B is also a subset of A, i.e. $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

Examples:

(i) If $A = \{\text{BASIC, COBOL, FORTRAN}\}$
and $B = \{\text{FORTRAN, COBOL, BASIC}\}$
then $A = B$.

(ii) If $A = \{x \mid x^2 + 1 = 0\}$
and $B = \{i, -i\}$ $(i = \sqrt{-1})$
then $A = B$

2.2.4 Important Remark

A set itself can be an element of some other set. Hence, one should be able to clearly distinguish between an element of a set and subset of a set.

Examples:

1. Let $A = \{a, b, \{a, b\}, \{\{a, b\}\}\}$.
Identify each of the following statements as true or false. Justify your answers.
(a) $a \in A$, (b) $\{a\} \in A$, (c) $\{a, b\} \in A$ (d) $\{\{a, b\}\} \subseteq A$, (e) $\{a, b\} \subseteq A$ (f) $\{a, \{b\}\} \subseteq A$.

Solution:

(a) True, as a is an element of A.
(b) False, as {a} is not an element but a subset of A.
(c) True, as {a, b} is an element of A, listed third in the set.
(d) True, as a subset containing the single element {a, b} of A.
(e) True, as the subset containing the elements a, b of A.
(f) False, as {b} is not an element of A.

2. Determine whether each of the following statements is true for arbitrary sets A, B, C. Justify your answers.
(a) If $A \in B$ and $B \subseteq C$, then $A \in C$
(b) If $A \in B$ and $B \subseteq C$, then $A \subseteq C$.
(c) If $A \subseteq B$ and $B \in C$, then $A \in C$.
(d) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$.

Solution: (a) True, as A being an element of B, it should also belong to C as B is a subse of C.

(b) False, as A is not a subset but an element of B.

(c) False. Consider A = {a}, B = {a, b}, C = {{a, b}}.

(d) False. Consider the same example as in (c).

2.3 Venn Diagrams

A Venn diagram (named after the British logician John Venn) is a pictorial depiction of a set. A rectangle represents the universal set. The interior of the rectangle represents the elements in the set. A circle drawn within the rectangle depicts an arbitrary set. It is not compulsory to show an arbitrary set, always by a circle. An oval shaped or elliptical curve could also be drawn to represent a set. In fact any closed curve of any shape can be used to depict a set.

Venn Diagram

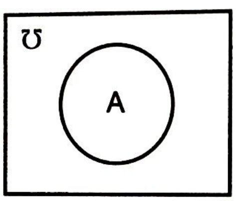
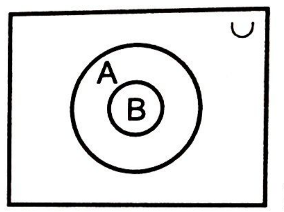


Fig. 2.1



$B \subset A$

Fig. 2.2

2.4 Set Operations

We shall now define various set operations, which will combine the given sets to yield new sets. These operations are analogous to the algebraic operations of addition, multiplication of numbers.

2.4.1 Complement of a Set

Let A be a given set. **Complement** of A, denoted by $\bar{A}$ is defined as

$$\bar{A} = \{x \mid x \notin A\}$$

Examples:

(i)

If $A = \{x \mid x \text{ is a real number and } x \le 7\}$, then

$\bar{A} = \{x \mid x \text{ is a real number and } x > 7\}$

where the universal set $\bar{U} = R$.

(ii) If $U = N = \{1, 2, 3, 4, 5, ...\}$

and $E = \{2, 4, 6, ...\}$

then $\overline{E} = \{1, 3, 5, ...\}$

Note that $\overline{\phi} = U$

and $\overline{U} = \phi$

2.4.2 Union of Sets

The union of two sets A and B is the set consisting of all elements which are in A, or in B, or in both sets A and B. It is denoted by $A \cup B$.

In the set – builder notation,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Examples:

(i) If $A = \{2, 4, 6, 8, 10\}$

$B = \{1, 2, 6, 8, 12, 15\}$

then $A \cup B = \{1, 2, 4, 6, 8, 10, 12, 15\}$.

(ii) If $A = \{n \mid n \in N, 4 < n < 12\}$

$B = \{n \mid n \in N, 8 < n < 15\}$

then $A \cup B = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

(iii) If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \cup B = \{\phi, a, \{\phi\}\}$

$= B.$

This is because $A \subseteq B$

Note that for any set A $A \cup \phi = A$

$A \cup U = U$

$A \cup \overline{A} = U.$

2.4.3 Intersection of Sets

*The intersection of two sets A and B, denoted by $A \cap B$ is the set consisting of elements which are in A **as well as** in B.*

Thus, $A \cap B = \{x \mid x \in A$ **and** $x \in B\}$
→ If $A \cap B = \phi$, the sets are said to be **disjoint**.

Examples:

(i) If $A = \{a, b, c, g\}$
$B = \{d, e, f, g\}$
then $A \cap B = \{g\}$.

(ii) If $A = \{n \mid n \in N, 4 < n < 12\}$
$B = \{n \mid n \in N, 5 < n < 10\}$
then $A \cap B = \{6, 7, 8, 9\} = B$.

(iii) If $A = \{\phi\}$
$B = \{a, \phi, \{\phi\}\}$
then $A \cap B = \{\phi\} = A$.

Note that for any set

$$A \cap \phi = \phi$$
$$A \cap U = A$$
$$A \cap \bar{A} = \phi.$$

2.4.4 Difference of Sets (Relative Complement)

Let A and B be any two sets.
The difference A – B is the set defined as

$A - B = \{x \mid x \in A$ *and* $x \notin B\}$ *is the (relative) complement of B in A.*

Similarly,

$B - A = \{x \mid x \in B$ *and* $x \notin A\}$ *is the complement of A in B.*

Examples:

(i) If $A = \{1, 2, 3, \ldots\ldots, 10\}$
$B = \{1, 3, 5, \ldots 9\}$
then $A - B = \{2, 4, 6, 8, 10\}$
$B - A = \phi$.

(ii) If $A = \{a, b, \{a, c\}, \phi\}$
$A - \{a, b\} = \{\{a, c\}, \phi\}$
$\{a, c\} - A = \{c\}$.

Let A and B be any two sets.

Then (i) $\overline{A} = U - A$

(ii) $A - A = \phi$

(iii) $A - \overline{A} = A,\ \overline{A} - A = \overline{A}$

(iv) $A - \phi = A$

(v) $A - B = A \cap \overline{B}$

(vi) $A - B = B - A$ *if and only if* $A = B$

(vii) $A - B = A$ *if and only if* $A \cap B = \phi$

(viii) $A - B = \phi$ *if and only if* $A \subseteq B$.

Proofs of the properties (i) to (v) are immediate consequences of the definition. We shall prove the remaining properties.

(vi) If $A = B$, then $A - B = \phi = B - A$ by (ii). Conversely let $A - B = B - A$. Let $x \in A$. Assume $x \notin B$, then x should be in $A - B$. But since $A - B = B - A$, it follows that $x \in B - A$ which means $x \in B$, a contradiction. Hence, x should be an element of B. Therefore, $A \subseteq B$. Similarly we can prove $B \subseteq A$. Hence, $A = B$.

(vii) $A - B = A$ implies $A \cap \overline{B} = A$, i.e. $A \subseteq \overline{B}$. Hence, $A \cap B = \phi$. Conversely $A \cap B = \phi$ implies $A \subseteq \overline{B}$, which in turn means that $A \cap \overline{B} = A$, i.e. $A - B = A$.

(viii) If $A - B = \phi$, it implies that $A \cap \overline{B} = \phi$, i.e. $A \subseteq B$. Converse is proved by reversing the steps.

2.4.6 Symmetric Difference

The symmetric difference of two sets A and B, denoted by $A \oplus B$, is defined as

$$A \oplus B = \{x \mid x \in A - B \text{ or } x \in B - A\}$$

In other words, $A \oplus B = (A - B) \cup (B - A)$.

Examples:

(i) If $A = \{a, b, e, g\}$

$B = \{d, e, f, g\}$

then $A \oplus B = \{a, b, d, f\}$.

(ii) If $A = \{2, 4, 5, 9\}$

$B = \{x \in Z^+ \mid x^2 \leq 16\}$

then $A \oplus B = \{0, 1, 3, 5, 9\}$

(iii) If $A = \{\phi\}$,

$B = \{a, \phi, \{\phi\}\}$,

then $A \oplus B = \{a, \{\phi\}\}$.

2.4.7 Properties of Symmetric Difference

(i) $A \oplus A = \phi$

(ii) $A \oplus \phi = A$

(iii) $A \oplus U = \bar{A}$

(iv) $A \oplus \bar{A} = U$

(v) $A \oplus B = A \cup B - A \cap B$.

The properties (i) to (iv) are immediate consequences of the definition.

We shall prove the last property. Let $x \in A \cup B - A \cap B$. Then, $x \in A \cup B$ but $x \notin A \cap B$. This means that if $x \in A$, $x \notin B$. Similarly, if $x \in B$, then $x \notin A$. Hence, $x \in A - B$ or $x \in B - A$, which means that $x \in (A - B) \cup (B - A) = A \oplus B$. Conversely, let $x \in A \oplus B$. Then, $x \in A - B$ or $x \in B - A$. This means that $x \in A \cup B$ but $x \notin A \cap B$, i.e. $x \in (A \cup B) - (A \cap B)$. Hence, the two sets are equal.

2.4.8 Representation of Set Operations on Venn Diagrams

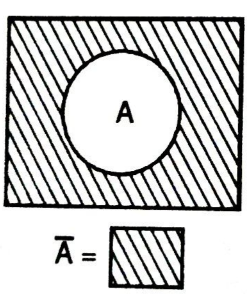


Fig. 2.3

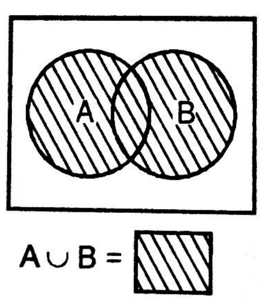


Fig. 2.4

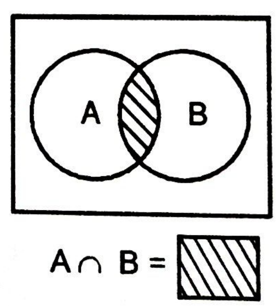


Fig. 2.5

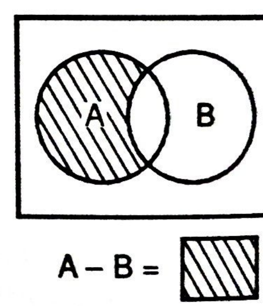


Fig. 2.6

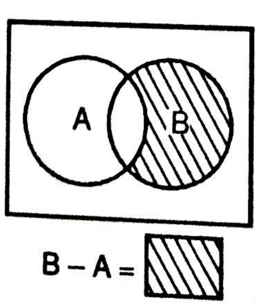


Fig. 2.7

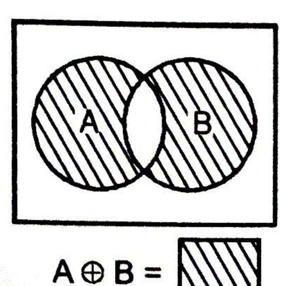


Fig. 2.8

2.5 Algebra of Set Operations

The set operations obey the same rules as those of numbers, such as associativity, commutativity and distributivity. However, the cancellation rule which is true for numbers, is not true for sets in general. In addition, there are rules such as Idempotent laws, Absorption laws, De Morgan's laws, which are true only for sets.

Theorem: The set operations satisfy the following properties, for any sets A, B, C.

1. **Commutativity:**
   (i) $A \cup B = B \cup A$
   (ii) $A \cap B = B \cap A$
2. **Associativity:**
   (i) $A \cup (B \cup C) = (A \cup B) \cup C$, hence written as $A \cup B \cup C$
   (ii) $A \cap (B \cap C) = (A \cap B) \cap C$, hence written as $A \cap B \cap C$
3. **Distributivity:**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **Idempotent Laws:**
   (i) $A \cup A = A$
   (ii) $A \cap A = A$
5. **Absorption Laws:**
   (i) $A \cup (A \cap B) = A$
   (ii) $A \cap (A \cup B) = A$
6. **De Morgan's Laws :**
   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$
   (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
7. **Double Complement:** $\bar{\bar{A}} = A$

Proof: We shall prove properties 3, 6 and 7. The remaining are easy exercises for the reader.

8. **Distributive Laws:**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Let $x \in A \cup (B \cap C)$. Then, $x \in A$ or $x \in B \cap C$. This further implies that $x \in A$ or $(x \in B$

and $x \in C$).

i.e. $(x \in A \text{ or } x \in B)$ and $(x \in A \text{ or } x \in C)$

i.e. $x \in A \cup B$ and $x \in A \cup C$

i.e. $x \in (A \cup B) \cap (A \cup C)$

Hence, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly one can prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can be proved on similar lines.

Refer to the figures below:

9. De Morgan's Laws:

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$\overline{A \cup B} = \{x \mid x \notin A \cup B\}$$
$$= \{x \mid x \notin A \text{ and } x \notin B\}$$
$$= \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\}$$
$$= \bar{A} \cap \bar{B}$$

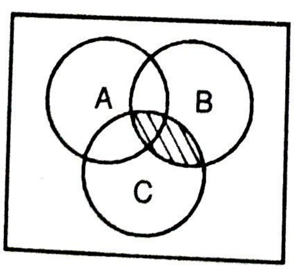
(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$ can be proved in the same way, as above.

10. Double Complement:

$$\bar{\bar{A}} = A$$
$$\bar{\bar{A}} = \{x \mid x \notin \bar{A}\}$$
$$= \{x \mid x \in A\} = A.$$

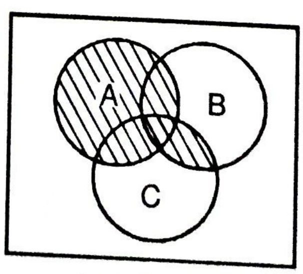
The above properties can also be demonstrated by drawing suitable Venn diagrams, as shown below.

For Distributive Laws:



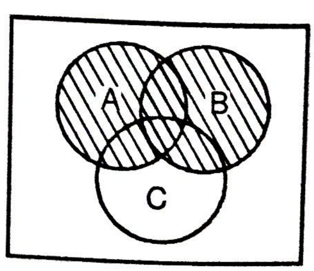
$B \cap C$

Fig. 2.9



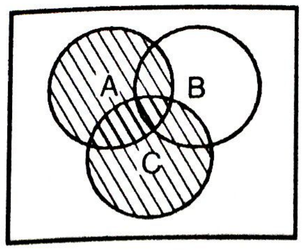
$A \cup (B \cap C)$

Fig. 2.10



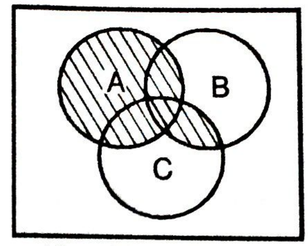
$A \cup B$

Fig. 2.11



$A \cup C$

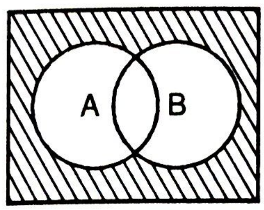
Fig. 2.12



$(A \cup B) \cap (A \cup C)$

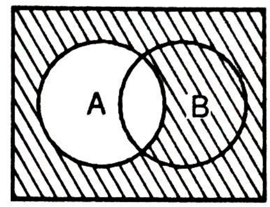
Fig. 2.13

De Morgan's Laws:



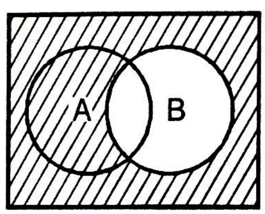
$\overline{A \cup B}$

Fig. 2.14



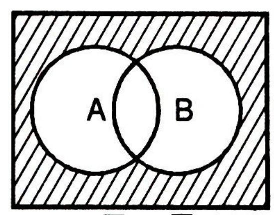
$\overline{A}$

Fig. 2.15



$\overline{B}$

Fig. 2.16



$\overline{A} \cap \overline{B}$

Fig. 2.17

SOLVED EXAMPLES

Example 1: Determine whether the following statements are true or false. Justify your answers.

(i) $\{a, \phi\} \in \{a, \{a, \phi\}\}$.

(ii) $\{a, b\} \subseteq \{a, b, \{a, b\}\}$.

(iii) $\{a, b\} \in \{a, b, \{a, b\}\}$.

(iv) $\{a, c\} \in \{a, b, c, \{a, b, c\}\}$.

Solution: (i) True, since $\{a, \phi\}$ is an element of $\{a, \{a, \phi\}\}$.

(ii) True, since $\{a, b\}$ is a subset of $\{a, b, \{a, b\}\}$ containing the elements a, b.

(iii) True, since $\{a, b\}$ is an element in $\{a, b, \{a, b\}\}$.

(iv) False, since $\{a, c\}$ is not an element but a subset of $\{a, b, c, \{a, b, c\}\}$ containing the elements a, c.

(Dec. 200…)

Example 2 : If $A = \{a, b, \{a, c\}, \phi\}$, determine the following sets

(i) $A - \{a, c\}$

(ii) $\{\{a, c\}\} - A$

(iii) $A - \{\{a, b\}\}$

(iv) $\{a, c\} - A$.

Solution : (i) $A - \{a, c\} = \{b, \{a, c\}, \phi\}$

(ii) $\{\{a, c\}\} - A = \phi$

(iii) $A - \{\{a, b\}\} = A$

(iv) $\{a, c\} - A = \{c\}$.

Example 3: If

$$U = \{n \mid n \in N, n \le 15\},$$
$$A = \{n \mid n \in N, 4 < n < 12\},$$
$$B = \{n \mid n \in N, 8 < n < 15\},$$
$$C = \{n \mid n \in N, 5 < n < 10\},$$

find $\bar{A} - \bar{B}$ and $\bar{C} - \bar{A}$.

Solution:

$$\bar{A} = \{1, 2, 3, 4, 12, 13, 14, 15\}$$
$$\bar{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 15\}$$
$$\bar{C} = \{1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15\}$$
$$\therefore \quad \bar{A} - \bar{B} = \{12, 13, 14\}$$
$$\bar{C} - \bar{A} = \{5, 10, 11\}.$$

Example 4: Let A, B, C be subsets of the universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer.

Solution: Yes, $B = C$.

We can express B as

$$B = B \cap U = B \cap (A \cup \bar{A})$$
$$= (B \cap A) \cup (B \cap \bar{A}) \qquad \text{(Distributive law)}$$
$$= (A \cap B) \cup (\bar{A} \cap B) \qquad \text{(Commutative law)}$$

$= (A \cap C) \cup (\bar{A} \cap C)$ (Given condition)

$= (A \cup \bar{A}) \cap C$ (Distributive law)

$= U \cap C$

$= C.$

Example 5 : (i) Given that $A \cup B = A \cup C$, is it necessary that $B = C$?

(ii) Given that $A \cap B = A \cap C$, is it necessary that $B = C$?

Solution: (i) No. Let

$A = \{1, 2, 3\}$

$B = \{1\}$

$C = \{3\}$

$A \cup B = \{1, 2, 3\} = A \cup C.$

but $B \neq C.$

(ii) No. Let

$A = \{1, 2\}$

$B = \{2, 3, 4, 5\}$

$C = \{2, 6, 7\}$

then $A \cap B = \{2\} = A \cap C.$

but $B \neq C$

Example 6: If $A \oplus B = A \oplus C$, is $B = C$?

Solution: Yes. Consider any element $x \in B$. This element is then in A or not in A. Suppose $x \in A$. Then, $x \in A \cap B$ which implies that $x \notin A \oplus B$ and hence $\notin A \oplus C$. Now $A \oplus C = A \cup C - A \cap C$. Therefore, it follows that $x \in A \cap C$ which means that $x \in C$. Hence, if $x \in A$, then $B \subseteq C$.

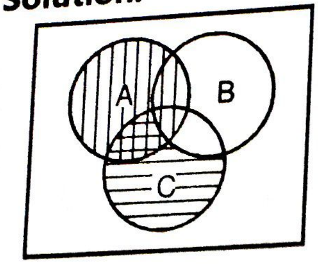
Suppose we have the other possibility that $x \notin A$. Then $x \notin A \cap B$ so that $x \in A \oplus B$ which in turn implies that $x \in A \oplus C$. This means that $x \in A \cup C$. Therefore, $x \in C$. Hence, in this case also $B \subseteq C$.

Similarly we can show that $C \subseteq B$. Hence, $B = C$.

Problems involving Venn Diagrams:

Example 7: Show that $A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$, using Venn diagram.

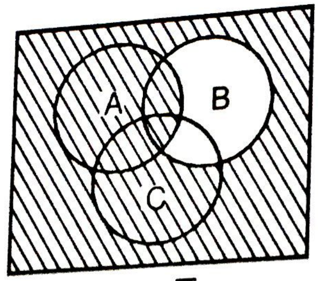
Solution:



$\overline{B} \cap C$ =

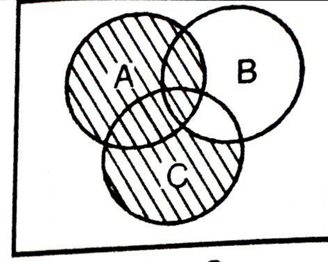
$A \cup (\overline{B} \cap C)$ = &

Fig. 2.18



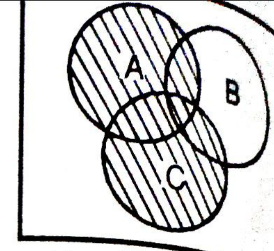
$A \cup \overline{B}$

Fig. 2.19



$A \cup C$

Fig. 2.20

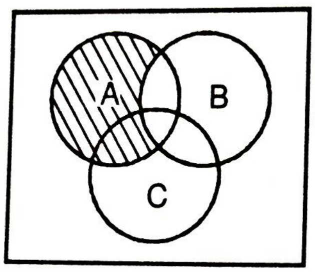


$(A \cup \overline{B}) \cap (A \cup C)$

Fig. 2.21

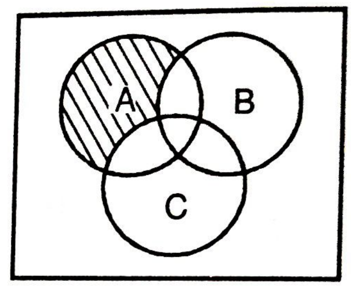
Example 8: Show that $(A - B) - C = A - (B \cup C)$ using Venn diagram.

Solution:



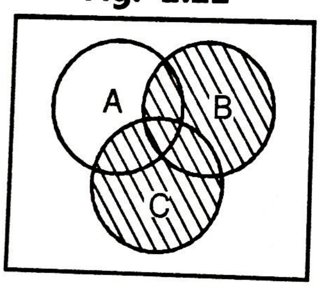
$A - B$

Fig. 2.22



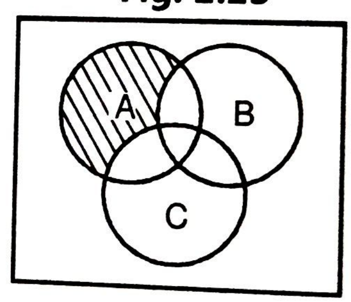
$(A - B) - C$

Fig. 2.23



$B \cup C$

Fig. 2.24

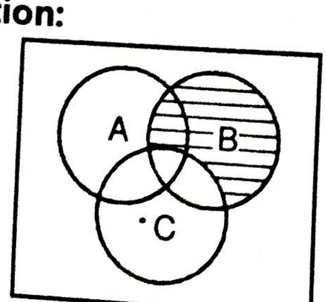


$A - (B \cup C)$

Fig. 2.25

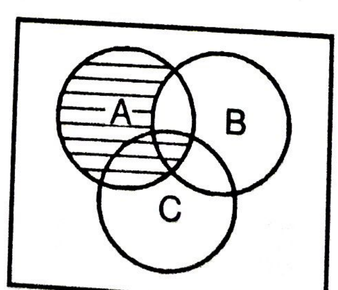
Example 9: Drawing Venn diagram, prove that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

Solution:



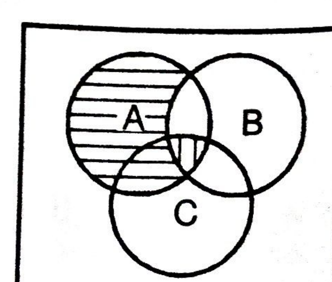
$B - C$

Fig. 2.26



$A - (B - C)$

Fig. 2.27



$A - B$ =

$A \cap B \cap C$ =

Fig. 2.28

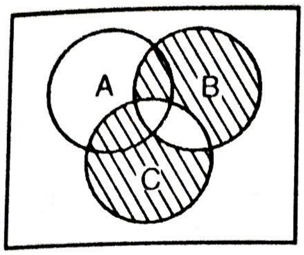
Example 10: Using Venn diagram, prove or disprove

(i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$

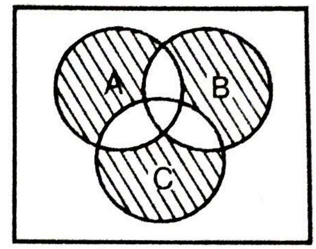
Solution:

(i)



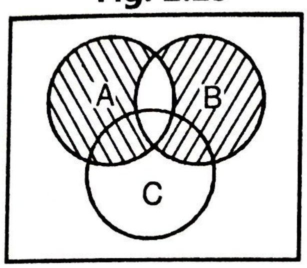
$B \oplus C$

Fig. 2.29



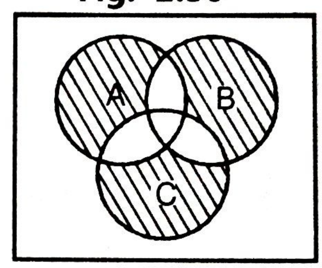
$A \oplus (B \oplus C)$

Fig. 2.30



$A \oplus B$

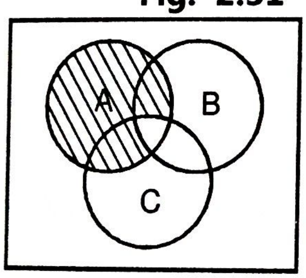
Fig. 2.31



$(A \oplus B) \oplus C)$

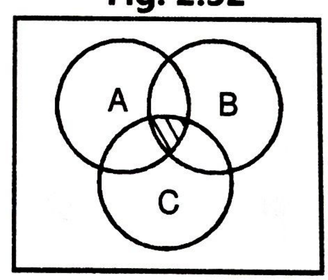
Fig. 2.32

(ii)



$(A - B) \cup (A - C)$

Fig. 2.33



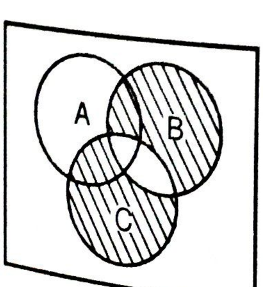
$A - [(A - B) \cup (A - C)]$
$= A \cap B \cap C$

Fig. 2.34

Example 11: Using Venn diagram, prove or disprove

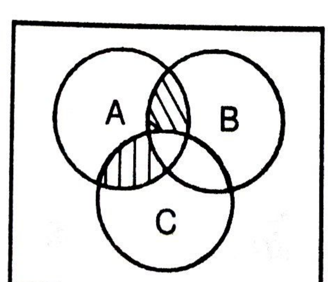
$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

Solution:



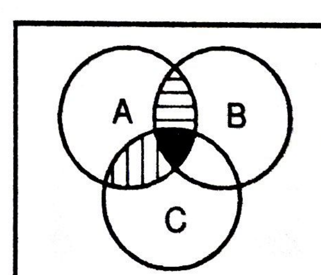
$B \oplus C$

Fig. 2.35



$A \cap (B \oplus C)$

Fig. 2.36



$(A \cap B) \oplus (A \cap C) =$ +

Fig. 2.37

Hence, the equation is true.

Example 12: Let A denote the set of students who study Data Structures, B denote the set of students who study Discrete Mathematics, C denote the set of students who study Assembly Language Programming, D denote the set of students studying Theory of Computer Science. Let E denote the set of students who are staying in the Hostel and F denote the set of students who went to watch a cricket match last Monday. Express the following statements in set theoretic notation.

(i) All hostelites who study neither Data Structures nor Discrete Mathematics went to watch cricket match last Monday.

(ii) The students who went to see cricket match are only those who study Assembly Language Programming or Data Structures.

(iii) No student who is studying Data Structures went to see cricket match.

(iv) Those and only those students who are studying Theory of Computer Science and Discrete Mathematics went for a cricket match.

(v) All went to see cricket match.

Solution:

(i) $E \cap \overline{A} \cap \overline{B} \subseteq F$ or $(E - A) - B \subseteq F$

(ii) $F \subseteq C \cup A$

(iii) $A \cap F = \phi$

(iv) $F \subseteq D \cap B$

(v) If the universal set is $A \cup B \cup C \cup D \cup E$, then $A \cup B \cup C \cup D \cup E = F$. Otherwise $A \cup B \cup C \cup D \cup E \subset F$.

Example 13: Let A denote the set of all automobiles that are manufactured domestically. Let B denote the set of all imported automobiles. Let C denote the set of all automobiles manufactured before 1977. Let D denote the set of all automobiles with a current market value of less than 2000 \$. Let E denote the set of all automobiles owned by students at the University.

Express the following in set theoretic notation.

(i) The automobiles owned by the students at the University are either domestically manufactured or imported.

(ii) All domestic automobiles manufactured before 1977 have a market value of less than 2000 \$.

(iii) All imported automobiles manufactured after 1977 have a market value of more than 200 \$.

Solution:

(i) $E \subseteq A \cup B$.

(ii) $A \cap C \subseteq D$.

(iii) $B \cap \bar{C} \subseteq \bar{D}$.

where the universal set U = set of all automobiles = $A \cup B$.

i.e. $B \cap ((A \cup B) - C) \subseteq (A \cup B) - D$.

Example 14: Tony, Mike and John belong to the Alpine Club. Every Club member is either a skier or mountain climber or both. No mountain climber likes rain and all skiers like snow. Mike dislikes whatever Tony likes and likes whatever Tony dislikes. Tony likes rain and snow. Is there a member of the Alpine Club who is a mountain climber but not skier? **(Dec. 2004)**

Solution: Let A denote the set of all members of the Alpine Club. Let S denote the set of skiers and M the set of all mountain climbers. Then $A \subset M \cup S$. If $x \in M$, x does not like rain and if $y \in S$, y likes snow. Since, Tony likes both rain and snow, Tony $\in$ S-M. Since, Mike dislikes whatever Tony likes and likes what Tony dislikes, it follows that Mike $\in$ M-S. Hence, there is a member (that is Mike), of the Alpine Club, who is a mountain climber but not skier.

Example 15: Consider the following assumptions.

S_1 : Poets are happy people.

S_2 : Every doctor is wealthy.

S_3 : No one who is happy is also wealthy.

Determine the validity of the following arguments, using Venn diagram.

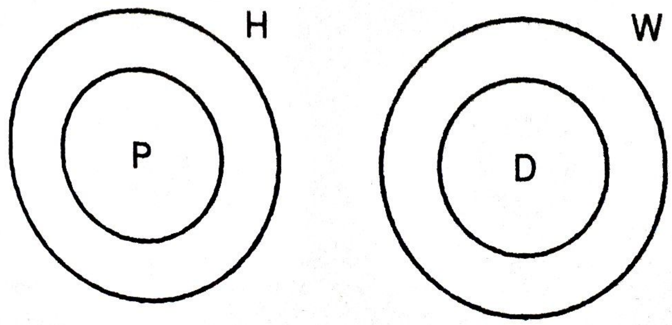
(1) No poet is wealthy.

(2) Doctors are happy people.

(3) No one can be both a poet and a doctor. **(Dec. 2007)**

Solution: Let H be the set of happy people, P set of poets, W set of wealthy people and D set of doctors. By S_1, $P \subseteq H$, S_2 implies $D \subseteq W$, S_3 implies $H \cap W = \phi$.

We have the Venn diagram.



From the Venn diagram, we observe that $P \cap W = \phi$, i.e. no poet is wealthy. Hence argument (1) is valid $D \cap H = \phi$, here doctors are not happy people so that (2) is invalid. $P \cap D = \phi$, hence no-one can be both a poet and doctor. Hence, (3) is valid.

2.6 Cardinality of Finite Set

A very important problem in Discrete Structures is that of determining the number of objects in a finite set. In the analysis of computer algorithms, one is often required to count the number of operations executed by various algorithms. This is necessary to estimate the cost effectiveness of a particular algorithm. In the study of data structures of files, determining the average and maximum lengths of searches for items stored in a data structure, also involve counting. Hence, in this section, we shall introduce the concept of **cardinality** of a finite set, and study its properties.

2.6.1 Definition

Let A be a finite set. The cardinality of A, denoted by $|A|$ is the number of elements in the set.

If $A = \phi$, then $|A| = 0$.

If $A \subseteq B$, where B is a finite set, then $|A| \leq |B|$.

The following theorem enables us to find the cardinality of disjoint union of two sets.

2.6.2 Theorem (The Addition Principle)

Theorem (The Addition Principle): Let A and B be finite sets which are disjoint. Then $|A \cup B| = |A| + |B|$.

Proof:

If A or B is the empty set, the proof is trivial.

Hence, let us assume that $A \neq \phi$, $B \neq \phi$.

Since, A and B are finite disjoint sets, let $A = \{a_1, a_2, \ldots a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $a_i \neq b_j$ for $1 \leq i \leq m$, $1 \leq j \leq n$. $|A| = m$ and $|B| = n$.

Then $A \cup B = \{a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_n\}$, i.e. $A \cup B$ contains exactly $m + n$ elements.

Hence, $|A \cup B| = m + n = |A| + |B|$. Thus, the theorem is proved.

The above theorem can be extended to a finite collection of finite mutually disjoint sets.

2.6.3 Corollary

Let $A_1, A_2, \ldots A_n$ be a finite collection of mutually disjoint finite sets.

Then $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$.

Proof is left as an exercise.

2.6.4 Theorem

Let A be a finite set and let B be any set (not necessarily finite).

Then $|A - B| = |A| - |A \cap B|$.

Proof:

Consider the Venn diagram.

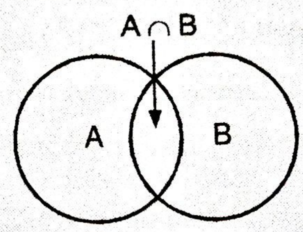


Fig. 2.39

From the Venn diagram, it is clear that $A = (A - B) \cup (A \cap B)$ (Disjoint union of two sets)

Hence, by the addition principle,

$$|A| = |A - B| + |A \cap B|,$$

so that $$|A - B| = |A| - |A \cap B|.$$

2.6.5 Theorem (Principle of Inclusion–Exclusion)

Theorem: Let A and B be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$

Proof:

Consider the Venn diagram.

A – B is the shaded portion.

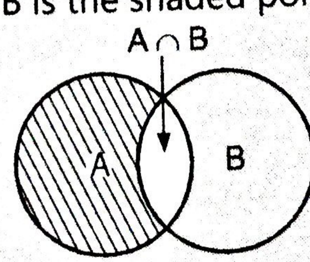


Fig. 2.40

We may express $A \cup B$ as disjoint union of two sets, by writing

$A \cup B = (A - B) \cup B.$

Hence, by the addition principle,

$$|A \cup B| = |A - B| + |B|$$

$$= |A| - |A \cap B| + |B| \text{ (by the previous theorem)}$$

Hence, $|A \cup B| = |A| + |B| - |A \cap B|$.

SOLVED EXAMPLES

Example 1: In a survey, 2000 people were asked whether they read India Today or Business Times. It was found that 1200 read India Today, 900 read Business Times and 400 read both. Find how many read at least one magazine and how many read none.

Solution: Let A denote the set of people who read India Today, B denote the set of people who read Business Times.

Now $|A| = 1200, \ |B| = 900$

and $|A \cap B| = 400.$

By the mutual inclusion-exclusion principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 1200 + 900 - 400 = 1700$$

and $|U - (A \cup B)| = |U| - |A \cup B|$

$$= 2000 - 1700 = 300.$$

Hence, 1700 read at least one magazine and 300 read neither.

Example 2: Among the integers 1 to 300, find how many are not divisible by 3, nor by 5. Find also, how many are divisible by 3, but not by 7.

Solution: Let A denote the set of integers 1 – 300, divisible by 3; B, the set of integers divisible by 5; C, the set of integers divisible by 7. We have to find $\left|\bar{A} \cap \bar{B}\right|$ and $|A - C|$.

By De Morgan's laws, $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

Hence, $\left|\overline{A \cup B}\right| = |U| - |A \cup B|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A| = \left[\frac{300}{3}\right] = 100.$$

$$|B| = \left[\frac{300}{5}\right] = 60.$$

$$|A \cap B| = \left[\frac{300}{15}\right] = 20.$$

$\therefore$ $$|A \cup B| = 100 + 60 - 20 = 140.$$

$\therefore$ $$\left|\overline{A \cup B}\right| = 300 - 140 = 160.$$

$\therefore$

Hence, 160 integers between 1 to 300 are not divisible by 3, nor by 5.

$$|A - C| = |A| - |A \cap C|$$

Now

$$|A \cap C| = \left[\frac{300}{21}\right] = 14.$$

$$|A - C| = 100 - 14 = 86.$$

Hence,

Hence, 86 integers between 1 – 300 are divisible by 3, but not by 7.

2.6.6 Theorem (Mutual Inclusion – Exclusion Principle for Three Sets)

Theorem: Let A, B, C be finite sets.

Then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Proof:

Let D denote the union set $B \cup C$. Then $A \cup B \cup C = A \cup D$.

$|A \cup D| = |A| + |D| - |A \cap D|$ (by the previous theorem) ... (1)

$|D| = |B \cup C| = |B| + |C| - |B \cap C|$ ... (2)

$|A \cap D| = |A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$

$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$ ... (3)

Substituting equations (2) and (3) in (1), we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Thus, the principle is proved for three sets. We now have the general theorem for a finite collection of finite sets.

Theorem: Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sets.

Then $$\left|A_1 \cup A_2 \cup \ldots\ldots \cup A_n\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots\ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

Proof is by induction on n.

We have already proved the theorem for n = 2, 3.

Hence, let us assume the theorem for (n – 1) numbers of sets and prove it for n sets.

Regarding $A_1 \cup A_2 \cup \ldots\ldots \cup A_n$ as $(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n$, we have

$$\left|A_1 \cup A_2 \ldots \cup A_n\right| = \left|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n\right|$$

$$= \left|A_1 \cup A_2 \cdot \cup \ldots \cup A_{n-1}\right| + |A_n| - \left|(A_1 \cup A_2 \cup \cdot \cup A_{n-1}) \cap A_n\right| \quad \ldots (1)$$

By induction hypothesis,

$$\left|A_1 \cup A_2 \cup \ldots \cup A_{n-1}\right| = \sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| - \ldots\ldots\ldots + (-1)^{n-2} \left|A_1 \cap A_2 \cap \ldots\ldots \cap A_{n-1}\right| \quad \ldots (2)$$

Now $$\left|(A_1 \cup A_2 \cup \ldots\ldots \cup A_{n-1}) \cap A_n\right| = \left|(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots\ldots \cup (A_{n-1} \cap A_n)\right|$$

$$= \sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| + \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k \cap A_n| -$$

$$\ldots + (-1)^{n-2} \left|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n\right| \quad \ldots (3)$$

Substituting equations (2) and (3) in (1) we obtain the equation

$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j < n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

SOLVED EXAMPLES

Example 1: In a computer laboratory out of 6 computers:

(i) 2 have floating point arithmetic unit.

(ii) 5 have magnetic disk memory.

(iii) 3 have graphics display.

(iv) 2 have both floating point arithmetic unit and magnetic disk memory.

(v) 3 have both magnetic disk memory and graphics display.

(vi) 1 has both floating point arithmetic unit and graphics display.

(vii) 1 has floating point arithmetic, magnetic disk memory and graphics display.

How many have at least one specification?

Solution: Let A be the set of computers having floating point arithmetic unit, B having magnetic disk memory and C having graphics display.

Then
$$|A| = 2, \quad |B| = 5, \quad |C| = 3,$$
$$|A \cap B| = 2, \quad |B \cap C| = 3,$$
$$|A \cap C| = 1, \quad |A \cap B \cap C| = 1$$

We have to determine $|A \cup B \cup C|$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| \\ &\quad - |B \cap C| - |A \cap C| + |A \cap B \cap C| \\ &= 2 + 5 + 3 - 2 - 3 - 1 + 1 = 5. \end{aligned}$$

Hence, 5 computers out of 6, have at least one specification.

Example 2: Among the integers 1 to 1000:

(i) How many of them are not divisible by 3, nor by 5, nor by 7?

(ii) How many are not divisible by 5 and 7 but divisible by 3? **(May 2006)**

Solution: (i) Let A, B, C denote respectively the set of integers from 1 to 1000 divisible by 3, by 5 and by 7. Then $\bar{A} \cap \bar{B} \cap \bar{C}$ denote the set of integers not divisible by 3, nor by 5, nor by 7.

By De Morgan's laws, $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{(A \cup B \cup C)}$

Hence,
$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 1000 - |A \cup B \cup C|$$

$$|A| = \left[\frac{1000}{3}\right] = 333,\ |B| = \left[\frac{1000}{5}\right] = 200,\ |C| = \left[\frac{1000}{7}\right] = 14$$

$$|A \cap B| = \left[\frac{1000}{15}\right] = 66$$

$$|B \cap C| = \left[\frac{1000}{35}\right] = 28$$

$$|A \cap C| = \left[\frac{1000}{21}\right] = 47$$

$$|A \cap B \cap C| = \left[\frac{1000}{105}\right] = 9.$$

Hence, $|A \cup B \cup C| = |A| + |B| + |C|$

$$= |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 333 + 200 + 142 - 66 - 28 - 47 + 9$$

$$= 543.$$

Hence $\left|\bar{A} \cap \bar{B} \cap \bar{C}\right| = 1000 - 543 = 457.$

(ii) Consider the Venn diagram

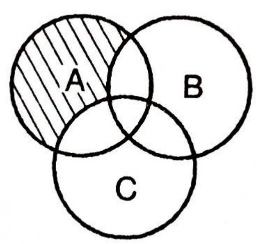


Fig. 2.41

The set of integers not divisible by 5 and 7 but divisible by 3 is the set $A \cap \bar{B} \cap \bar{C}$.

$A \cap \bar{B} \cap \bar{C} = A \cap (\overline{B \cup C}) = A - (B \cup C)$, the shaded portion shown in the Venn diagram. It is clear from the diagram that

$$|A - (B \cup C)| = |A| - |(A \cap B) \cup (A \cap C)|$$

Now $|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$

$$= 66 + 47 - 9 = 104$$

$\therefore$ $|A - (B \cup C)| = 333 - 104 = 229.$

Hence, 229 integers from 1 to 1000 are not divisible by 5 and 7 but divisible by 3.

Example 3: How many integers between 1 – 1000 are divisible by 2, 3, 5 or 7?

Solution: Let A, B, C, D denote respectively the set of integers from 1 to 1000 divisible by 2, 3, 5 or 7.

∴

$$|A| = \left[\frac{1000}{2}\right] = 500$$

$$|B| = \left[\frac{1000}{3}\right] = 333$$

$$|C| = \left[\frac{1000}{5}\right] = 200$$

$$|D| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{6}\right] = 166$$

$$|A \cap C| = \left[\frac{1000}{10}\right] = 100$$

$$|A \cap D| = \left[\frac{1000}{14}\right] = 71$$

$$|B \cap C| = \left[\frac{1000}{15}\right] = 66$$

$$|B \cap D| = \left[\frac{1000}{21}\right] = 47$$

$$|C \cap D| = \left[\frac{1000}{35}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{30}\right] = 33$$

$$|B \cap C \cap D| = \left[\frac{1000}{105}\right] = 9$$

$$|A \cap C \cap D| = \left[\frac{1000}{70}\right] = 14$$

$$|A \cap B \cap D| = \left[\frac{1000}{42}\right] = 23$$

$$|A \cap B \cap C \cap D| = \left[\frac{1000}{210}\right] = 4$$

∴ $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$

$- |B \cap D| - |C \cap D| \; |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$

$= 772$

Example 4: An investigator interviewed 100 students to determine their preferences for the three drinks – Milk (M), Coffee (C) and Tea (T). He reported the following:
10 students had all the three drinks, 20 had 'M' and 'C', 30 had 'C' and 'T', 25 had 'M' and 'T', 12 had 'M' only, 5 had 'C' only and 8 had 'T' only.

(i) How many did not take any of the three drinks?

(ii) How many take milk but not coffee?

(iii) How many take tea and coffee but not milk?

Solution: Consider the Venn diagram, incorporating the given data.

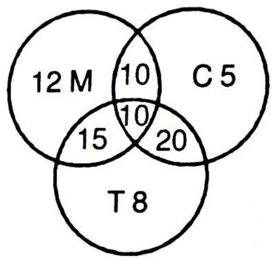


Fig. 2.42

(i) Taking the cardinalities of the disjoint sets into account,

$$\begin{aligned}\left|\bar{M} \cap \bar{C} \cap \bar{T}\right| &= 100 - |M \cup C \cup T| \\ &= 100 - [12 + 10 + 10 + 15 + 20 + 8 + 5] \\ &= 100 - 80 = 20.\end{aligned}$$

Hence, 20 students did not take any drink.

(ii) The set of students taking milk but not coffee is $M - C$.

$$|M - C| = 12 + 15 = 27$$

(iii) The set of students taking tea and coffee, but not milk is $(T \cap C) - M$.

$$\begin{aligned}|(T \cap C) - M| &= |T \cap C| - |T \cap C \cap M| \\ &= 30 - 10 = 20.\end{aligned}$$

Example 5: (i) Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both examinations?

(ii) If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination is 40 and if 4 students did not get an A in either examination, then determine the number of students who got an A in the first examination only, who got an A in the second examination only, and who got an A in both the examinations.

Solution: (i) Let F denote the set of students who got an A in the first examination, S that of students who got an A in the second examination.

The set of students who did not get an A in either is $\overline{F \cup S}$.

$$\left|\overline{F \cup S}\right| = 50 - |F \cup S| = 17$$

$\therefore$ $$|F \cup S| = 50 - 17 = 33.$$

$\therefore$ $$|F \cap S| = |F| + |S| - |F \cup S|$$

$$= 26 + 21 - 33 = 14.$$

(ii) $|F| = |S|$, by given condition.

The set of students who got an A in exactly one examination is $(F - S) \cup (S - F) = F \oplus S$.

$$|F \oplus S| = 40, \quad \left|\overline{F \cup S}\right| = 4 \text{ (given)}$$

$\therefore$ $$|F \cup S| = 50 - 4 = 46$$

$$|F \oplus S| = |F \cup S| - |F \cap S|$$

$\Rightarrow$ $$40 = 46 - |F \cap S|$$

$\therefore$ $$|F \cap S| = 46 - 40 = 6.$$

$$|F \cup S| = |F| + |S| - |F \cap S|$$

$$= 2|F| - |F \cap S|$$

i.e. $$46 = 2|F| - 6$$

$\therefore$ $$|F| = \frac{52}{2} = 26 = |S|.$$

The set of students who got an A in the first examination only is $F - S$.

$\therefore$ $$|F - S| = |F| - |F \cap S| = 26 - 6 = 20.$$

Similarly, the number of students who got an A in the second examination only is $|S - F| = |S| - |S \cap F| = 20.$

Example 6: In a survey, it is reported that of 1000 programmers, 650 habitually flowchart their programs, 788 are skilled COBOL programmers, 675 are men, 278 of the women are skilled COBOL programmers, 440 programmers both habitually flowchart and are skilled in COBOL, 210 women habitually flowchart and 166 women are both skilled in COBOL and habitually flowchart. Would you accept these data as being accurately reported? Justify your answer.

Solution: Let F denote the set of programs (both men and women) who habitually flow-chart their programs, and let C denote the set of all skilled COBOL programmers. Let M and W denote the set of men and women programmers respectively.

$$|M| = 675, \quad \therefore \quad |W| = 1000 - 675 = 325.$$

$$|F| = 650, \quad |C| = 788$$

$$|W \cap C| = 278, \quad |W \cap F| = 210, \quad |F \cap C| = 440$$

$$|W \cap F \cap C| = 166.$$

$$\therefore \quad |M \cap C| = |C| - |W \cap C| = 788 - 278 = 510.$$

$$|M \cap F| = |F| - |W \cap F| = 650 - 210 = 440.$$

$$|M \cap F \cap C| = |F \cap C| -$$

$$= 440 - 166 = 274.$$

The set of $M \cap (F \cup C)$ is the set of male programmers who habitually flowchart their programs or are skilled COBOL programmers.

$$\therefore \quad |M \cap (F \cup C)| = |M \cap F| + |M \cap C| - |M \cap F \cap C|$$

$$= 510 + 440 - 274$$

$$= 676.$$

Hence, there should be at least 676 men programmers. But this contradicts the given data that there are in all only 675 men programmers.

Hence, the data is inaccurately reported.

Example 7: 75 children went to an amusement park, where they can ride on the merry-go-round, roller coaster, and the Ferris wheel. It is known that 20 of them have taken all three rides, and 55 of them have taken at least 2. Each ride costs 5 rupees and the total collection of the park was 700 rupees. Determine the number of children who did not try any of the rides.

Solution: The total number of rides $= \frac{700}{5} = 140$.

The number of children who have taken exactly 2 rides $= 55 - 20 = 35$.

The number of children who have taken only one ride $= 140 - 2 \times 35 - 3 \times 20$

$$= 140 - 70 - 60 = 10$$

Hence, the number of children who have not taken any ride

$$= 75 - (35 + 20 + 10) = 75 - 65 = 10.$$

Example 8: It was found that in first year of computer science of 80 students 50 know Cobol, 55 know 'C', 46 know Pascal. It was also known that 37 know 'C' and Cobol, 28 know 'C' and Pascal, 25 know Pascal and Cobol. 7 students, however, know none of the languages.

Find:

(i) How many know all the three languages?

(ii) How many know exactly two languages?

(iii) How many know exactly one language?

Solution: Let B, C and P denote the set of students who know Cobol, 'C' and Pascal respectively.

Then $$|B \cup C \cup P| = 80 - 7 = 73$$

is the number of students who know at least one of the languages.

(i) $|B \cup C \cup P| = |B| + |C| + |P| - |B \cap C| - |B \cap P| - |C \cap P| + |B \cap C \cap P|$

Hence, the number of students who know all the three languages is

$$|B \cap C \cap P| = 73 - 50 - 55 - 46 + 37 + 28 + 25 = 12$$

(ii) The number of students who know Cobol and 'C' but not Pascal is

$$|B \cap C \cap \bar{P}| = |B \cap C| - |B \cap C \cap P|$$
$$= 37 - 12 = 25$$

Similarly, the number of students who know Cobol and Pascal but not 'C' is

$$|B \cap P \cap \bar{C}| = 25 - 12 = 13$$

and the number of students who know Pascal and 'C' but not Cobol is

$$|\bar{B} \cap P \cap C| = 28 - 12 = 16$$

Hence, the number of students who known exactly two languages is

$$25 + 13 + 16 = 54$$

(iii) The number of students who know only Cobol (i.e. neither 'C' nor Pascal) is

$$|B| - |B \cap C| - |B \cap P| + |B \cap P \cap C| = 50 - 37 - 25 + 12 = 0$$

Similarly the number of students who know only 'C' is 55 – 37 – 28 + 12 = 2 and the numb of students knowing only Pascal is 46 – 28 – 25 + 12 = 5.

Hence, the number of students who know exactly one language is 0 + 2 + 5 = 7

Example 9: How many elements are in the union of five sets if the sets contain 10,00 elements each, each pair of sets has 1000 common elements, each triple of sets has 10 common elements, every four of the sets has 10 common elements, and there is 1 elemen common in all five sets?

Solution: Let A_1, A_2, A_3, A_4, A_5 denote the five sets.

Then $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$

$$= \sum_{1 \le i \le 5} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$- \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$$

Taking two sets at a time there are $5C_2 = 10$ such sets; taking three sets at a time, there are $5C_3 = 10$ such sets. Taking 4 sets at a time, there are $5C_4 = 5$ such sets.

$\therefore |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = 5 \times 10{,}000 - 10 \times 1000 + 10 \times 100 - 5 \times 10 + 1 = 40{,}951$

Example 10: Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.

Solution: Let A be the set of odd integers between 1 and 100, B the set of integers between 1 and 100, that are squares of an integer.

$$B = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 50 + 10 - 5 = 55$$

Example 11: A college record gives the following information: 119 students enrolled in Introductory Computer Science; of these 96 took Data Structures, 53 took Foundations, 39 took Assembly Language, 31 took both Foundations and Assembly Language, 32 took both Data Structures and Assembly Language, 38 took Data Structures and Foundations and 22 took all the three courses.

Is the information correct? Why?

Solution: Let D, F and A denote the set of students who took Data Structure, Foundations and Assembly Language respectively.

Given: $|D| = 96$, $|F| = 53$, $|A| = 39$, $|F \cap A| = 31$, $|D \cap A| = 32$, $|D \cap F| = 38$ and $|F \cap D \cap A| = 22$.

$$\therefore \quad |F \cup D \cup A| = |F| + |D| + |A| - |F \cap A| - |D \cap A| - |D \cap F| + |F \cap D \cap A|$$
$$= 53 + 96 + 39 - 31 - 32 - 38 + 22$$
$$= 109 \text{ which is less than } 119.$$

Since, there were 119 students enrolled for the course, assuming that all these students had taken at least one course, the given information is not correct.

Example 12: A software company writes a new package which integrates a word processing program with a spread sheet program and they wish it to run on a 64 K machine. The word processor requires 40 K for program and data and the spread sheet requires 32 K for the same. If 16 K must be reserved for the code integrator, what is the minimum amount of overlapping space that will be necessary?

Solution: Let A denote the memory space reserved for word processor and B that for spread sheet.

$|A| = 40$, $|B| = 32$.

Available memory is $64 - 16 = 48$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$\therefore \quad |A \cup B| \le 48$$
$$\therefore \quad |A| + |B| - |A \cap B| \le 48$$
$$\text{i.e.} \quad |A \cap B| \ge |A| + |B| - 48$$
$$= 40 + 32 - 48 = 24$$

Hence, the minimum amount of overlapping space that will be necessary is 24 K.

Example 13: Among 130 students, 60 study Mathematics, 51 study Physics and 30 study both Mathematics and Physics. Out of 54 students studying Chemistry, 26 study Mathematics, 21 study Physics and 12 study both Mathematics and Physics. All the students studying neither Mathematics nor Physics are studying Biology.

Find:

1. How many are studying Biology?
2. How many not studying Chemistry are studying Mathematics but not Physics?
3. How many students are studying neither Mathematics nor Physics nor Chemistry?

Solution: 1. $|M \cup P| = |M| + |P| - |M \cap P|$

$= 60 + 51 - 30 = 81$

$\therefore$ Number of students studying neither Mathematics nor Physics

$= 130 - |M \cup P|$

$= 130 - 81 = 49$

Hence, the number of students studying Biology is 49.

2. The set of students studying Mathematics but neither Chemistry nor Physics is $M - M \cap (C \cup P)$.

$\therefore \quad |M - [M \cap (C \cup P)]| = |M| - |M \cap C| - |M \cap P| + |M \cap C \cap P|$

$= 60 - 26 - 30 + 12 = 16$

3. Set of students studying neither Mathematics nor Physics nor Chemistry is the complement of the set $M \cup P \cup C$, i.e. $\overline{M \cup P \cup C}$.

$\therefore \quad |\overline{M \cup P \cup C}| = 130 - |M \cup P \cup C|$

$= 130 - |M| - |P| - |C| + |M \cap P| + |M \cap C| + |P \cap C| - |M \cap P \cap C|$

$= 130 - 60 - 51 - 54 + 30 + 26 + 21 - 12$

$= 30$

2.6.7 Infinite Sets

We have seen that if a set is finite; its elements can be counted or listed and this counting ceases in finite time. On the other hand if the counting is interminable or impossible, then such a set is said to be infinite. Familiar examples of infinite sets are:

(i) N = {1, 2, 3, ...} the set of natural numbers.

(ii) The set of prime positive integers {2, 3, 5, 7, ...}.

(iii) The set of all points in the first quadrant of the plane, whose x and y co-ordinates are integers.

(iv) The set of all binary strings of odd length.

The above examples are of infinite sets, whose elements are although infinitely many in number can be listed or 'counted', in other words these elements are put into one-to-one correspondence with the set of natural numbers. The cardinality of such a set is denoted by $\aleph_0$ (pronounced as aleph nought).

If a set is not countable, then it is called "uncountable" set. The set of real numbers, denoted by IR, is uncountable. The open interval (0, 1), as a set is uncountable. Another important example of an uncountably infinite set is the power set of N, the set of natural numbers. The cardinality of this set is 2^{N_0} denoted by C and is called the 'continuum'.

We shall discuss countably infinite and uncountably infinite sets, more in detail, in the chapter on functions.

2.7 Multiset

Multiset is generalization of a set. A set, we know, is a collection of distinct objects. In a multiset however, an object can occur more than once. For example, the collection of books, in a library can contain multiple copies of the same book; such a collection is a multiset. Similarly names of persons, birth months of individuals, account numbers of transactions of a bank on a given day (the same account number may have more than one transaction), are practical examples of multisets. A multiset is also called as "bag", "heap", "bunch", "weighted set". In our discussion, we will use the term "multiset" (first coined by N.G. de Bruijn) or briefly "mset".

To distinguish a set and a multiset, we denote the latter by enclosing the elements within square brackets. For example, [a, b, a, a] is an mset, whereas the underlying or "generic" set is {a, b}. A multiset containing no elements is denoted by [ ], corresponding to the empty set ϕ.

The multiplicity of an element in an mset is defined as the number of times the element appears in the mset. Thus, in the mset [a, b, a, a] multiplicity of a is 3 whereas multiplicity of b is 1. If an element dsoes not belong to the multiset, its multiplicity is zero.

Hence, it follows that sets are special cases of msets, in which multiplicity of the elements is either zero or one. In fact we can characterize a multiset as a pair (A, μ), where A is the generic set and μ is the multiplicity function defined as

$$\mu \; : \; A \rightarrow \{1, 2, 3, \ldots\}$$

so that $\mu(a) = k$ where k is the number of times the element a occurs in the mset.

For example, if [a, b, c, c, a, c] is the mset, $\mu(a) = 2$, $\mu(b) = 1$, $\mu(c) = 3$.

2.7.1 Equality of Msets

If the number of occurrences of each element is the same in both the msets, then the msets are equal.

Example, $[a, b, a, a] = [a, a, b, a]$

However, $[a, b, a] \neq [a, b]$

Multisubset (or msubset): *A multiset A is said to be a multisubset of B if multiplicity of each element in A is less or equal to its multiplicity in B.*

Example: $[1, 2, 2, 3] \subseteq [1, 1, 1, 2, 2, 3]$

2.7.2 Union of Msets

If A and B are two msets, then $A \cup B$ is the mset such that for each element $x \in A \cup B$, $\mu(x) = \max. \{\mu_A(x), \mu_B(x)\}$

Example: $A = [a, b, b, c], B = [b, c, c, d]$

Then $A \cup B = [a, b, b, c, c, d]$

2.7.3 Intersection of Msets

If A and B are msets, then $A \cap B$ is defined as the mset such that for each element $x \in A \cap B$, $\mu(x) = \min. (\mu_A(x), \mu_B(x))$.

Example:

$$A = [1, 1, 1, 2, 2, 3]$$
$$B = [1, 2, 2, 2, 3, 3]$$

Then $A \cap B = [1, 2, 2, 3]$

2.7.4 Difference of Msets

For multisets A and B, the difference $A - B$ is an mset such that for each $x \in A - B$,

$\mu(x) = \mu_A(x) - \mu_B(x)$, if the difference is greater than zero.

$\mu(x) = 0$ if difference is zero or negative.

From the above definition it follows that

$$A - A = \phi[\ ]$$

Example:

$$A = [a, b, c, c, c]$$
$$B = [b, c, d, d]$$

Then $A - B = [a, c, c]$

Sum of Msets:

This concept is not defined for ordinary sets. However, for multisets A and B, we define A + B as follows:

For each element $x \in A + B$,

$$\mu(x) = \mu_A(x) + \mu_B(x)$$

Example:

$$A = [1, 1, 2, 3]$$
$$B = [2, 3, 3, 3]$$
$$A + A = [1, 1, 1, 1, 2, 2, 3, 3]$$
$$A + B = [1, 1, 2, 2, 3, 3, 3, 3]$$

An interesting observation = $A + A \neq A$.

(i.e. idempotent law is not true for sum).

The above definitions (excluding sum) are consistent with those defined for sets. Hence, one can easily see that the laws of associativity, commutativity, distributivity adsorption and idempotent are satisfied for union and intersections. It is also an easy exercise to verify that

$$(A + B) \cup C = A \cup C + B \cup C$$
$$(A + B) \cap C = A \cap C + B \cap C$$
$$A \cup (B + C) = A \cup B + A \cup C$$
$$A \cap (B + C) = A \cap B + A \cap C$$

The concept of symmetric difference, however, cannot be carried over to that of multisets. Recall that we define symmetric difference of two sets as:

$$A \oplus B = (A \cup B) - (A \cap B)$$

Symmetric difference satisfies the associative law for sets. For multisets, this law is not valid. Consider, for example,

$$A = [2, 2, 3, 3]$$
$$B = [1, 1, 2]$$
$$C = [3, 3, 2]$$

Then, if $A \oplus B = A \cup B - A \cap B$, this is equal to $[1, 1, 2, 2, 3, 3] - [2]$.

$$= [1, 1, 2, 3, 3]$$

$\therefore$ $(A \oplus B) \oplus C = [1, 1, 2, 3, 3] \cup [3, 3, 2] - [1, 1, 2, 3, 3] \cap [3, 3, 2]$

$$= [1, 1, 2, 3, 3] - [2, 3, 3]$$
$$= [1, 1]$$

On the other hand consider,

$A \oplus (B \oplus C)$ $B \oplus C = [1, 1, 2] \cup [3, 3, 2] - [1, 1, 2] \cap [3, 3, 2]$

$$= [1, 1, 2, 3, 3] - [2]$$
$$= [1, 1, 3, 3]$$

$\therefore$ $A \oplus (B \oplus C) = [2, 2, 3, 3] \cup [1, 1, 3, 3] - [2, 2, 3, 3] \cap [1, 1, 3, 3]$

$$= [1, 1, 2, 2, 3, 3] - [3, 3]$$
$$= [1, 1, 2, 2]$$

Thus, we see that $(A \oplus B) \oplus C \neq A \oplus (B + C)$.

Cardinality of a multiset containing finitely many elements is defined as the total number of elements in the multiset, i.e. it is the sum of multiplicities of each element in the set. For example,

If $\quad A = [a, a, b, b, c], \quad |A| = 2 + 2 + 1 = 5$

Then extending the principle of mutual inclusion – exclusion to multisets, it follows that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For example, $\quad A = [1, 1, 1, 1], \; B = [1, 2, 2, 3]$

then $\quad |A \cup B| = 4 + 4 - 1 = 7$

This is immediate because

$$A \cup B = [1, 1, 1, 1, 2, 2, 3] \text{ and } A \cap B = [1]$$

2.7.5 Polynomial Representation of Multisets

A multiset can be represented by a monomial. The empty multiset [ ] corresponds to $x^0 = 1$, [x] corresponds to x^1, [x, x] corresponds to x^2 and [x, y] to xy. In fact the multiset of submultisets (equivalent to power set) can be represented by a suitable polynomial expression. For example, [[ ], [x], [x], [x, x]] correspond to the polynomial $1 + 2x + x^2$. Similarly $(1 + x)(1 + y) = 1 + x + y + xy$ correspond, s to [[ ], [x], [y], [x, y]]. In general the polynomial $(1 + x)^n = \sum_{0}^{n} {}^nC_k x^{n-k}$ represents an mset consisting of ${}^nC_k$ subsets of cardinality, k varying from 0 to n.

An interesting observation is that $(1 - x)^{-1} = 1 + x + x^2 \ldots$ corresponds to the infinite mset [[ ], [x], [x, x], [x, x, x], ...]. Similarly $(1 - x)^{-2} = 1 + 2x + 3x^2 \ldots$ corresponds to the mset [[ ], [x], [x], [x, x], [x, x], [x, x] ...].

Such polynomials are called as **Cumulant generating functions.**

Given a set of n (distinct) elements, one can determine how many multisets, consisting of k elements, can be formed, where $0 \leq k \leq n$. This is nothing but the combinatorial problem of distributing k identical objects in n distinct boxes, whose solution is given by $\binom{n+k-1}{k-1}$. (Refer to the chapter on combinatories). We denote this number by the symbol $\left(\!\binom{n}{k}\!\right)$, which is called as **n multichoose k**. For example, if A = {a, b, c} is the generic set, all multisets of cardinality 2 are given by [a, b], [a, c], [b, c], [a, a], [b, b], [c, c]. These total 6 msets are given by the number $\left(\!\binom{3}{2}\!\right) = \binom{3+2-1}{3-1} = \binom{4}{2} = 6$.

2.7.6 Application of Multisets

In mathematics, the prime factorization of every non-negative integer is n > 0, corresponds to a multset. Hence, there is a one-to-one correspondence between the prime factors of the integer and the corresponding multiset. For example, if $n = 2^3 \cdot 3^2$ 5, the corresponding multiset is [2, 2, 2, 3, 3, 5].

The roots of an algebraic equation also form a multiset. For example the roots of the equation $x^3 - 4x^2 + 5x - 2 = 0$, given rise to the multiset [1, 1, 2]. Hence, multisets are useful in representing the zeros and poles of meromorphic (analytic) functions, invariants of a matrix (eigenvalues) in canonical forms.

In Computer Science, multisets are applied in a variety of search and sort procedures, for fast retrieval of keys (of the same key value with multiple copies), which allows fast access to stored key values.

SOLVED EXAMPLES

Example 1: Find the union and intersection of each of the following multisets:

(a) [a, b] and [a, b, c]

(b) [a, b, b] and [a, b, a, b]

(c) [a, a, a, b] and [a, a, b, b, c]

(d) [1, 1, 3, 3, 3, 4] and [1, 2, 2, 4, 5, 5]

(e) [a, a, (b, b), (b, b)] and [a, a, b, b]

(f) [a, a, (b, b), [a, (b)]] and [a, a, (b), (b)]

Solution: (a) [a, b] ∪ [a, b, c] = [a, b, c]

[a, b] ∩ [a, b, c] = [a, b]

(b) [a, b, b] ∪ [a, b, a, b] = [a, a, b, b]

[a, b, b] ∩ [a, b, a, b] = [a, b, b]

(c) [a, a, a, b] ∪ [a, a, b, b, c] = [a, a, a, b, b, c]

[a, a, a, b] ∩ [a, a, b, b, c] = [a, a, b]

(d) $[1, 1, 3, 3, 3, 4] \cup [1, 2, 2, 4, 5, 5] = [1, 1, 2, 2, 3, 3, 3, 4, 5, 5]$

$[1, 1, 3, 3, 3, 4] \cap [1, 2, 2, 4, 5, 5] = [1, 4]$

(e) $[a, a, (b, b), (b, b)] \cup [a, a, b, b] = [a, a, (b, b), (b, b), b, b]$

$[a, a, (b, b), (b, b)] \cap [a, a, b, b] = [a, a]$

(f) $[a, a, (b, b), [a, (b)]] \cup [a, a, (b), (b)] = [a, a, (b, b), [a, (b)], (b), (b)]$

$[a, a, (b, b), [a, (b)]] \cap [a, a, (b), (b)] = [a, a]$

Example 2: Find a multiset that solves the equation

$$A \cup [a, b, b, c] = [a, a, b, b, c, c, d]$$

$$A \cap [a, b, b, c, d] = [a, b, c, d]$$

Solution: Maximum multiplicity of each element is as follows

$$\mu(a) = 2, \quad \mu(b) = 2, \quad \mu(c) = 2, \quad \mu(d) = 1.$$

Minimum multiplicity of each element is as follows:

$$\mu(a) = 1, \quad \mu(b) = 1, \quad \mu(c) = 1, \quad \mu(d) = 1.$$

$$\therefore \quad A = [a, a, b, c, c, d]$$

2.9 Power Set

2.9.1 Definition

Let A be any set. The power set of A, denoted by P(A) is the set of all subsets of A.

Examples:

(i) If $A = \{a\}$, then $P(A) = \{A, \phi\}$.

(ii) If $A = \{a, b\}$, then $P(A) = \{\phi, \{a\}, \{b\}, A\}$.

(iii) If $A = \{a, \{a\}\}$, then
$$P(A) = \{\{a\}, \{\{a\}\}, A, \phi\}.$$

The following theorem determines the size of the power set.

2.9.2 Theorem (Cardinality of a Power Set)

Let A be a finite set containing n elements. Then the power set of A has exactly 2^n elements.

Proof:

We prove the theorem by mathematical induction.

For $n = 1$, $A = \{a\}$, so that $P(A) = \{A, \phi\}$.

Hence, $|P(A)| = 2^1$ elements.

Assume that if $|A| = k$, $|P(A)| = 2^k$.

Let $|A| = k + 1$. For an element $a \in A$, consider the subset $B = A - \{a\}$. Since, $|B| = k$, by induction hypothesis $|P(B)| = 2^k$, i.e. there are exactly 2^k subsets of B.

Since, every subset of B is also a subset of A, it follows that A contains at least 2^k subsets.

In addition, for each subset of B, say C, we have another subset $C \cup \{a\}$ of A.

Hence, the total number of subsets of A is $2^k + 2^k = 2.2^k = 2^{k+1}$ subsets.

Hence, by induction, it follows that if $|A| = n$, $|P(A)| = 2^n$.

SOLVED EXAMPLES

Example 1: If $A = \{\phi, a\}$, then construct the sets $A \cup P(A)$, $A \cap P(A)$.

Solution :

$$P(A) = \{\phi, \{\phi\}, \{a\}, A\}$$

$\therefore$
$$A \cup P(A) = \{\phi, a, \{\phi\}, \{a\}, A\}.$$
$$A \cap P(A) = \{\phi\}.$$

Example 2: Let $A = \{\phi\}$. Let $B = P(P(A))$.

(i) Is $\phi \in B$? $\phi \subseteq B$?

(ii) Is $\{\phi\} \in B$? $\{\phi\} \subseteq B$?

(iii) Is $\{\{\phi\}\} \in B$? $\{\{\phi\}\} \subseteq B$?

Solution:

$$P(A) = \{\phi, \{\phi\}\}$$

$$B = P(P(A)) = \{\phi, \{\phi\}, \{\{\phi\}\}, P(A)\}.$$

(i) The element $\phi \in B$. The empty set ϕ is always a subset of B.

(ii) Both are true, one as element and the other as subset containing the single element ϕ.

(iii) Both are true, the first as element and the second as a single to n subset containing the element $\{\phi\}$.

Example 3: If $A \subseteq B$, then $P(A) \subseteq P(B)$.

Solution: Let $C \in P(A)$. Then $C \subseteq A$ which implies that $C \subseteq B$.

Hence, $C \in P(B)$. $\therefore$ $P(A) \subseteq P(B)$.

Example 4: Let A and B be two arbitrary sets.

(i) Show that $P(A \cap B) = P(A) \cap P(B)$ or give a counter example.

(ii) Show that $P(A \cup B) = P(A) \cup P(B)$ or give a counter example.

Solution: (i) Let $C \in P(A \cap B)$. Then $C \subseteq A \cap B$

$\Rightarrow$ $C \subseteq A$ and $C \subseteq B \Rightarrow C \in P(A)$ and $C \in P(B) \Rightarrow C \in P(A) \cap P(B)$

$\therefore$ $P(A \cap B) \subseteq P(A) \cap P(B)$.

Conversely, let $C \in P(A) \cap P(B)$.

This implies $C \in P(A)$ and $C \in P(B)$

$\Rightarrow$ $C \subseteq A$ and $C \subseteq B$

$\Rightarrow$ $C \subseteq A \cap B$, i.e. $C \in P(A \cap B)$.

Hence, $P(A) \cap P(B) \subseteq P(A \cap B)$

Hence, $P(A \cap B) = P(A) \cap P(B)$.

(ii) Equality is not true.

Consider

$$A = \{1\}, \quad B = \{2\}.$$

$$A \cup B = \{1, 2\}$$

$$P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$$

$\therefore$

$$P(A) = \{\phi, \{1\}\}$$

$$P(B) = \{\phi, \{2\}\}$$

$$P(A) \cup P(B) = \{\phi, \{1\}, \{2\}\}$$

$\therefore$

$$\neq P(A \cup B).$$

ample 5: Let $A = \{\phi, b\}$; construct the following sets: **(Dec. 2004)**

(i) $A - \phi$

(ii) $\{\phi\} - A$

(iii) $A \cup P(A)$

(iv) $A \cap P(A)$

where P(A) is power set of A.

lution: (i) $A - \phi = A$

(ii) $\{\phi\} - A = \phi$

(iii) $A \cup P(A) = \{\phi, b, \{\phi\}, \{b\}, A\}$

(iv) $A \cap P(A) = \{\phi\}$

EXERCISE - 2.1

1. If $A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$, determine whether the following statements are true or false. Justify your answer.
   (i) $\phi \in A$
   (ii) $\{\phi\} \subseteq A$
   (iii) $\{\phi\} \in A$
   (iv) $\{\phi, \{\phi\}\} \subseteq A$
   (v) $\{\{\phi\}\} \in A$
2. If $u = \{n \in N \mid 1 \leq n \leq 9\}$,
   $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$, $C = \{x \in Z^+ \mid x^2 \leq 16\}$ and $D = \{7, 8\}$,
   find (i) $A \oplus B$, $B \oplus C$, $C \oplus D$
   (ii) $A - B$, $B - A$, $C - D$
   (iii) $\overline{A \cup B}$, $\overline{A \cap B}$
   (iv) $A \cap (\bar{C} \cup D)$

3. For $A = \{a, b, \{b, c\}, \phi\}$ determine the following sets:
   (i) $A - \{a\}$, (ii) $A - \{b, c\}$, (iii) $\{\{b, c\}\} - A$, (iv) $A - \{c, \phi\}$, (v) $\{a\} - \{A\}$.
4. Give an example of sets A, B, C such that $A \in B$, $B \in C$ and $A \notin C$.
5. Draw Venn diagrams for the following situations.
   (i) A, B, C are sets such that $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$ and $A \subseteq (B \cap C)$.
   (ii) $(A \cap B \cap C) = \phi$, $A \cap B \neq \phi$, $B \cap C \neq \phi$, $A \cap C \neq \phi$.
6. Using Venn diagrams, prove or disprove the following:
   (i) $(A - B) - C = (A - C) - B$
   (ii) $(A - B) - C = (A - C) - (B - C)$
   (iii) $(A - B) \cap (A - C) = A - (B \cup C)$
   (iv) $(A - C) \cup (B - C) = (A \cup B) - C$
   (v) $A - (B - C) = (A - B) \cup (A \cap C)$
   (vi) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (vii) $(A \cap B) - C = (A - C) \cap (B - C)$
   (viii) $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$
   (ix) $A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$. **(May 2005)**
7. Using the rules of set operations, simplify the following:
   (i) $(\overline{A \cup B}) \cup (\bar{A} \cap B)$
   (ii) $[(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)] \cap B$
   (iii) $((A \cup B) \cap \bar{A}) \cup (\overline{B \cap A})$
   (iv) $\overline{[(A \cap B) \cup C]} \cap \bar{B}$.
8. What can you say about sets A and B, if
   (i) $A - B = B$?
   (ii) $A - B = B - A$?
   (iii) $A \oplus B = A$?
9. It is known that at the University, 60 percent of the professors play tennis, 50 percent of them play bridge, 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog and 40 percent play bridge and jog. If someone claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim? Why?
10. A survey was conducted among 1000 people. Of these 595 are graduates, 595 wear glasses and 550 like ice cream, 395 of them are graduates who wear glasses, 350 of

them are graduates who like ice cream and 400 of them wear glasses and like ice cream; 250 of them are graduates who wear glasses and like ice cream. How many of them who are not graduates do not wear glasses and do not like ice cream? How many of them are graduates who do not wear glasses and do not like ice cream?

11. Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7? Also indicate how many are divisible by 3 or 7 but not by 5.
12. How many integers between 1 and 2000 are divisible by 2, 3, 5 or 7?
13. A college record gives the following information: 119 students enrolled in Introductory Computer Science; of these, 96 took Data Structures, 53 took Foundations, 39 took Assembly Language. Also 38 took both Data Structures and Foundations, 31 took both Foundations and Assembly Language, 32 took both Data Structures and Assembly language and 22 took all the three courses. Is the information correct? Why?
14. A survey of 100 students of the Management Programme shows that 70 read India Today, 31 read Fortune and 54 read Business India. Also the people who read Business India do not read Fortune. Draw a Venn diagram to represent the situation.
15. A software company writes a new package which integrates a word processing program with a spreadsheet program, and they wish it to run on a 64 K machine. The word processor requires 40 K for program and data and the spreadsheet requires 32 K for the same. If 16 K must be reserved for the code integrator, what is the minimum amount of overlapping space that will be necessary?
16. Consider a set of integers 1 to 500. Find how many of these numbers are divisible by 3 or by 5 or by 11?
    (i) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.
    (ii) How many are divisible by 3 or 11 but not by 5? **(May 2005)**
17. It was found that in first year of computer engineering out of 80 students, 50 know 'C' language, 55 know 'basic' and 25 know 'C++', while 8 did not know any language. Find,
    (i) How many know all the three languages?
    (ii) How many know exactly two languages? **(May 2005)**
18. In the survey of 60 people, it was found that 25 read Newsweek Magazine, 26 read time, 26 read fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune and 8 read no magazine at all.
    (i) Find out the number of people who read all the three magazines.

(ii) Fill in the correct numbers in all the regions of the Venn diagram.

(iii) Determine number of people who reads exactly one magazine. **(Dec. 2005)**

19. Among 130 students, 60 study Mathematics, 51 Physics and 30 both Mathematics and Physics. Of the 54 students studying Chemistry, 26 study Mathematics, 21 Physics and 12 both Mathematics and Physics. All the students studying neither Mathematics nor Physics are studying Biology.

    (i) How many students are studying Biology?

    (ii) How many students not studying Chemistry are studying Mathematics but not Physics?

    (iii) How many students are studying neither Mathematics nor Physics nor Chemistry. **(May 2006)**

20. It was found that in first year of computer science of 80 students, 50 know COBOL 55 know C language and 46 know Pascal. It was also known that 37 know C and COBOL, 28 know C and Pascal, and 25 know Pascal and COBOL. 7 students however know none of the language. Find:

    (i) How many know all the three languages?

    (ii) How many know exactly two languages?

    (iii) How many know exactly one language?

21. A survey has been taken on methods of computer travels. Each respondent was asked to check BUS, TRAIN or AUTOMOBILE as a major method of traveling to work. More than one answer was permitted. The results reported were as follows: BUS – 30 people, TRAIN – 35 people, AUTOMOBILE – 15 people, TRAIN and AUTOMOBILE – 20 people and all three methods-5 people. How many people completed the survey form? **(Dec. 2008)**

22. A survey of 500 television watchers produced the following information. 285 watch football, 195 watch hockey, 115 watch basket ball. 45 watch football and basket ball, 70 watch football and hockey, 50 watch hockey and basketball and 50 do not watch any of the three games.

    (i) How many people in the survey watch all the three games?

    (ii) How many people watch exactly one game? **(May 2008)**

23. 100 of the 120 engineering students in a college take part in at least one of the activities: group discussion, debate and quiz. Also 65 participate in group discussion 45 participate in debate, 42 participate in quiz, 20 participate in group discussion and

debate, 25 participate in group discussion and quiz, 15 participate in debate and quiz. Find the number of students:

(i) Who participate in all the three activities

(ii) Who participate in exactly one of the activities.

24. In a class of 55 students, the number of students studying different subjects are as follows: Maths 23, Physics – 24, Chemistry 19, Maths + Physics – 12, Maths + Chemistry – 9, Physics + Chemistry – 7, all three subjects - 4. Find the numbers of students who have taken: (i) At least one subject, (ii) Exactly one subject, (iii) Exactly two subjects. **(May 2007)**

25. In a survey of 100 new cars, it is found that 60 had Air Conditioner (AC), 48 had Power-Steering (PS), 44 had Power Windows (PW), 36 had AC + PW, 20 had AC + PS, 16 had PW + PS, 12 had all three. Find the number of cars that had: (i) Only PW, (ii) PS and PW but not AC, (iii) AC and PS but not PW. **(Dec. 2006)**

Problems on Power Sets:

38. Let A = {a, {a}}. Determine which of the following statements are true or false.

    (i) $\phi \in P(A)$

    (ii) $\phi \subseteq P(A)$

    (iii) $\{a\} \in P(A)$

    (iv) $\{a, \{a\}\} \in P(A)$

    (v) $\{\{\{a\}\}\} \subseteq P(A)$

39. Determine whether the following statements are true or false. Justify your answer.

    (i) $A \cup P(A) = P(A)$

    (ii) $\{A\} \cup P(A) = P(A)$

    (iii) $A - P(A) = A$

    (iv) $P(A) - \{A\} = P(A)$

    (v) $\{A\} \cap P(A) = A$.

40. For multisets, define in brief: **(May 2010)**

    (i) Multisets.

    (ii) Multiplicity of an element in a multiset.

    (iii) Cardinality of mulstiset.

    (iv) Union of multiset.

    (v) Intersection of multiset.

    (vi) Difference of multiset.

41. A survey has been taken on methods of computer travel. Each respondent was asked to check bus, train or automobile as a major method of travelling to work. More than one answer was permitted. The results reported were as follows:

    Bus - 30 people, train - 35 people, automobile - 100 people, bus and train - 15 people, bus and automobile - 15 people, train and automobile - 20 people and all three methods - 5 people. How many people completed a survey form? **(May 2010)**

42. In a survey of 260 college students, the following data were obtained: 64 had taken a Mathematical course, 94 has taken a Computer Science course, 58 had taken a Business course, 28 had taken both Mathematic and Business courses, 26 had taken both Mathematical and Computer Science course, 22 had taken both Computer Science and Business course and 14 had taken all 3 types of courses.

(1) How many students were surveyed who had taken none of the three types of courses?

(2) Of the students surveyed, how many had taken only Computer Science course?

ANSWERS - 2.1

1. (i) True, (ii) True, (iii) True, (iv) True, (v) False.

2. (i) $A \oplus B = \{1, 5, 6, 8, 9\}$

   (ii) $A - B = \{1, 6, 8\}$

   (iii) $\overline{A \cup B} = \{3, 7\}$.

3. (i) $A - \{a\} = \{b, \{b, c\}, \phi\}$ (v) $\{a\} - \{A\} = \{a\}$.

7. (i) $\bar{A}$, (iii) $\bar{A} \cup \bar{B}$.

9. Claim is false.

10. 155, 100

11. 86 numbers between 1 to 250 are divisible by 3 or 7 but not by 5.

12. 1499

19. (i) 49, (ii) 16, (iii) 30.

20. (i) 12, (ii) 66, (iii) 7.

22. (i) 20

Points to Remember

- A set is a **collection of** objects.
- An object in the collection is called an **element** or member of the set.
- The term **class** is also used to denote a set.
- A set may contain **finite** number of elements or **infinite** number of elements.
- A set is called an **empty set** or a **null set** if it contains no element. An empty set is denoted by the letter ϕ.

- If every element of a set A is also an element of a set B, then we say A is a **subset** of B, or A is **contained** in B. This is denoted by '$A \subseteq B$'. This can be also denoted by '$B \supseteq A$'. If A is not a subset of B, this is indicated by '$A \subseteq B$'.
- If all sets, considered during a **specific discussion** are subsets of a given set, then this set is called as the **Universal Set**, and is denoted by 'U'.
- A Venn diagram (named after the British logician John Venn) is a pictorial depiction of a set.
- Let A be a given set. **Complement** of A, denoted by $\bar{A}$ is defined as

$$\bar{A} = \{x \mid x \notin A\}$$

- The union of two sets A and B is the set consisting of all elements which are in A, or in B, or in both sets A and B. It is denoted by $A \cup B$.
- The intersection of two sets A and B, denoted by $A \cap B$ is the set consisting of elements which are in A **as well as** in B.
- If the counting of the elements of a set is interminable or impossible, then such a set is said to be infinite.
- If a set **contains multiple occurrences of an object then such set is called 'multiset'**
- **Multisubset (or msubset):** A multiset A is said to be a multisubset of B if multiplicity of each element in A is less or equal to its multiplicity in B.