Vector Integration :

Line Integral : Indefinite Vector Integrals

Let $\bar{F}(t)$ and $\bar{G}(t)$ are vector functions such that $\frac{d}{dt}\bar{F}(t) = \bar{G}(t)$ then $\bar{F}(t)$ is called an integral of $\bar{G}(t)$ w.r.t 't' and is written as

$$\int \bar{G}(t)\,dt = \bar{F}(t)$$

Note : (i) If $\bar{c}$ is constant vector independent of t, then,

$$\frac{d}{dt}\bar{F}(t) + \bar{c} = \bar{G}(t)$$

In this case, $\int \bar{G}(t)\,dt = \bar{F}(t) + \bar{c}$

$\bar{F}(t)$ is called indefinite integral of $\bar{G}(t)$.
$\bar{c}$ is called constant of integration whose values can be determined if some intial conditions are given.

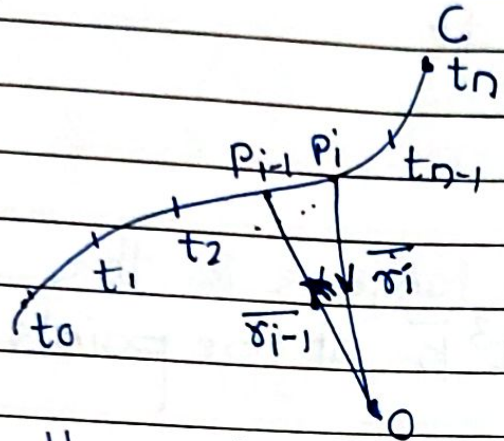
$$\int_a^b \bar{G}(t)\,dt = \left[\bar{F}(t)\right]_{t=a}^{t=b} = \bar{F}(b) - \bar{F}(a)$$

Note : If $\bar{G}(t) = G_1(t)\,\bar{i} + G_2(t)\,\bar{j} + G_3(t)\,\bar{k}$

then $\int \bar{G}(t)\,dt = \bar{i}\int G_1(t)\,dt + \bar{j}\int G_2(t)\,dt + \bar{k}\int G_3(t)\,dt$

Line Integral :

An integral which is to be evaluated along a curve is called line integral.



Let $\bar{F}(t)$ be continous point function defined at every point of curve C in space.

Divide C into n parts by the points $t_0, t_1, \cdots t_n$ with position vectors $\bar{r}_0, \bar{r}_1, \cdots \bar{r}_n$ respectively.

Let Q_i be any point in the interval (P_i, P_{i-1}) i.e $Q_i \in (P_i, P_{i-1})$ then

$\lim\limits_{n \to \infty} \sum\limits_{i=0}^{n} \bar{F}(t_i) \cdot \delta \bar{r}_i$ if exists, is called as line integral of $\bar{F}$ along C and is denoted by $\int_C \bar{F} \cdot d\bar{r}$

Note: In practice, if $\bar{F} = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$ and $d\bar{r} = dx\,i + dy\,\bar{j} + dz\,\bar{k}$ then

$$\int \bar{F} \cdot d\bar{r} = \int F_1(x,y,z)\,dx + \int F_2(x,y,z)\,dy + \int F_3(x,y,z)\,dz.$$

Working rule to evaluate line integral $\int_C \bar{F}\cdot d\bar{r}$ or to find work done Work done W along given curve C.

Step-1: Let $\bar{F} = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$ be given vector function.

Step-2 Let $I = \int_C \bar{F}\cdot d\bar{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$ ——— (1)

By using equation of the curve C, write values of y and z in terms of x
say $y = f_1(x)$, $z = f_2(x)$
$\therefore dy = f_1'(x)\,dx$, $dz = f_2'(x)\,dx$

Write limits of x

Using substitutions, Eqⁿ (1) can be written in terms of x only.
i.e say $I = \int_{x_1}^{x_2} \phi(x)\,dx.$ ——— (2)

Step-3. Evaluate the definite integral in Eqⁿ (2) which is value of given line integral or work done.

Alternate method: If it is possible to write equation of curve C in parametric form say
$x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ then

$dx = f_1'(t)\,dt$, $dy = f_3'(t)\,dt$, $dz = f_3'(t)\,dt$

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Write limits of t.
Use these substitutions in Eqⁿ ①, so that all the terms in integral is expressed in terms of t only.

∴ From eqⁿ ①

$$I = \int_{t_1}^{t_2} \phi(t)\,dt$$

Evaluation of integral gives required value.

Parametric Equations of some standard curves:

1. Equation of straight line:
Let ~~the~~ a straight line passing through the points say (x_1, y_1, z_1) and (x_2, y_2, z_2) then equation of straight line is given by -

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

Equating each term to say t, we get,

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t$$

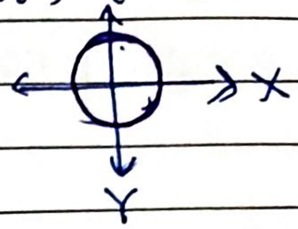
These are parametric equations of straight line.

(2) Equations of circle:
Let equation of circle be, $x^2 + y^2 = a^2$, $z = 0$
Then $x = a\cos t$, $y = a\sin t$, $z = 0$ $\boxed{r = a}$
These are parametric equations of circle.

(i) For complete circle : $t : 0 \longrightarrow 2\pi$

(ii) For upper part of the circle ; $t : 0 \longrightarrow \pi$

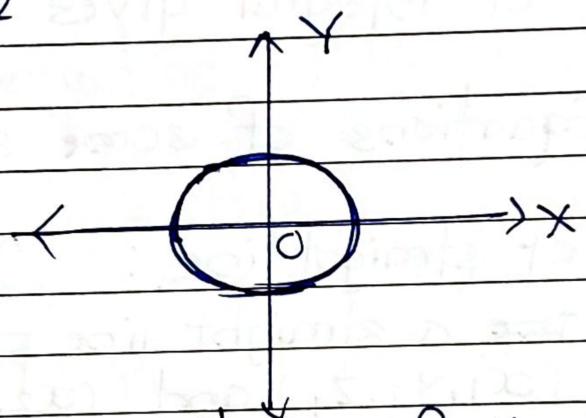
(iii) For arc of circle in first quadrant ; $t : 0 \longrightarrow \pi/2$



(3) Equation of ellipse:

Let the equation of ellipse be,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$



The parametric equations of ellipse are

$x = a\cos t, \quad y = b\sin t, \quad z = 0$

(i) For complete ellipse ; $t : 0 \longrightarrow 2\pi$

(ii) For upper arc of ellipse ; $t : 0 \longrightarrow \pi$

(iii) For arc of ellipse in first quadrant ; $t : 0 \longrightarrow \pi/2$

1. If curve C is a closed curve then line integral of F along C is denoted by $\oint \bar{F} \cdot d\bar{r}$

2. If curve C is a different curve which is combination of different curves say $C_1, C_2, \ldots C_n$ then

$$\oint \bar{F} \cdot d\bar{r} = \oint \bar{F} \cdot d\bar{r} + \oint \bar{F} \cdot d\bar{r} + \cdots + \int \bar{F} \cdot d\bar{r}$$

Note: Line integrals are useful in the calculation of workdone by variable forces along paths in space and also the rate at which fluids flow along path and across boundaries.

Work done:

Work done by a force F in moving a particle along a curve C from point say P_1 to P_2 is given by,

$$W = \int_{C:P_1}^{P_2} \bar{F} \cdot d\bar{r}$$

Note: 1. Work done by a force $\bar{F}$ along C between two points P_1 and P_2 is dependent on path joining P_1 and P_2.

2. If $\bar{F}$ is irrotational i.e if $\nabla \times F = 0$ then there exists scalar potential ϕ such that $\bar{F} = \nabla\phi$.

$$\therefore \int_A^B \bar{F} \cdot d\bar{r} = \int_A^B (\nabla\phi) \cdot d\bar{r} = \int_A^B d\phi = [\phi]_A^B$$

$$= \phi_B - \phi_A$$

Therefore, if $\bar{F}$ is irrotational, then work done by $\bar{F}$ in moving particle from A to B is independent of path joining A and B and depends only on end points.

3. If work done by irrotational force field $\bar{F}$ around a closed curve C is zero then $\bar{F}$ is said to be conservative force field.

4. $$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{(n-1)(n-3)\cdot(n-5)\cdots 2 \text{ or } 1}{n(n-2)(n-4)\cdots 2 \text{ or } 1} \times k$$

where $k = \begin{cases} \pi/2, & \text{if } n \text{ even} \\ 1, & \text{if } n \text{ odd} \end{cases}$

5. $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots 2 \text{ or } 1][(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)(m+n-4)\cdots 2 \text{ or } 1} \times k$$

where $k = \begin{cases} \pi/2, & \text{if both } m, n \text{ are even} \\ 1, & \text{if otherwise} \end{cases}$

6. $$\int_0^{\pi} \sin^m x \cos^n x \, dx = \begin{cases} 2\int_0^{\pi/2} \sin^m x \cos^n x, & \text{if } n \text{ even } (m \text{ any integer}) \\ 0, & \text{if } n \text{ odd } (m \text{ any integer}) \end{cases}$$

7. $$\int_0^{\pi} \sin^m x \, dx = 2\int_0^{\pi/2} \sin^m x \, dx, \text{ for any integer } m$$

8. $$\int_0^{\pi} \cos^n x \, dx = \begin{cases} 2\int_0^{\pi/2} \cos^n x \, dx, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}$$

9. $$\int_0^{2\pi} \sin^n x\,dx = \begin{cases} 4\int_0^{\pi/2} \sin^n x\,dx, & \text{if } n \text{ even integer} \\ 0, & \text{if } n \text{ odd integer} \end{cases}$$

10. $$\int_0^{2\pi} \cos^n x \cdot dx = \begin{cases} 4\int_0^{\pi/2} \cos^n x\,dx, & \text{if } n \text{ even integer} \\ 0, & \text{if } n \text{ odd integer} \end{cases}$$

11. $$\int_0^{2\pi} \sin^m x \cdot \cos^n x\,dx = \begin{cases} 4\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx, & \text{if } m,n \text{ are even integer} \\ 0, & \text{otherwise.} \end{cases}$$

Ex: 1. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ for $\bar{F} = x^2\bar{i} + 2xy\bar{j} + z\bar{k}$ and C is straight line joining (1, 0, 2) and (3, 1, 1)

soln: Let $I = \int_C \bar{F} \cdot d\bar{r}$

$$= \int_C (x^2\bar{i} + 2xy\bar{j} + z\bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$$

$$= \int_C x^2dx + 2xy\,dy + z\,dz \quad \text{--- (I)}$$

Given curve C is straight line joining the points (1, 0, 2) and (3, 1, 1)

Equation of straight line is —

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}; \quad (x_1, y_1, z_1) = (1, 0, 2), \ (x_2, y_2, z_2) = (3, 1, 1)$$

$$\therefore \frac{x-1}{3-1} = \frac{y-0}{1-0} = \frac{z-2}{1-2}$$

$$\therefore \frac{x-1}{2} = \frac{y}{1} = \frac{z-2}{-1} = t \text{ (say)}$$

$$\therefore x-1 = 2t \quad ; \quad y = t \quad ; \quad z-2 = -t$$
$$\therefore x = 1+2t \quad ; \quad y = t \quad ; \quad z = 2\cancel{+}(-t)$$

$$\therefore dx = 2\,dt \quad ; \quad dy = dt \quad ; \quad dz = -dt$$

Hence equation ① becomes –

In $P(\underline{1}, 0, 2)$ In $Q(\underline{3}, 1, 1)$

x	1	3
$t = \frac{x-1}{2}$	0	1

$$\therefore t : 0 \longrightarrow 1$$

$\therefore$ From equation ①,

$$I = \int_{t=0}^{1} (2t+1)^2 (2\,dt) + 2(2t+1)(t)\,dt + (-t+2)(-dt)$$

$$= \int_{t=0}^{1} (12t^2 + 11t)\,dt$$

$$= \left[12\left(\frac{t^3}{3}\right) + 11\left(\frac{t^2}{2}\right)\right]_0^1$$

$$= 4 + 11/2$$
$$= 19/2$$

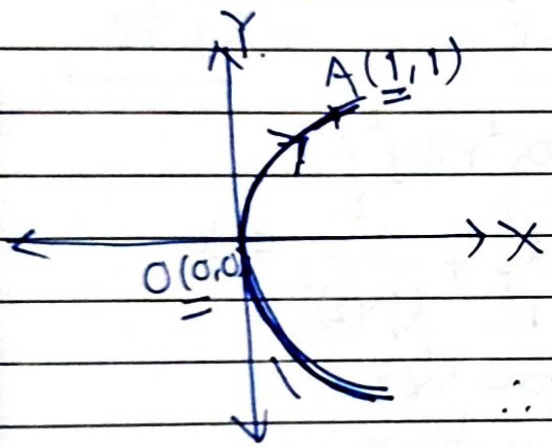
Ex: 2 If $\bar{F} = (2x+y^2)\, i + (3y-4x)\, j$ then evaluate $\int_{C_2} \bar{F}\cdot d\bar{r}$ around the parabolic arc $y^2 = x$ joining $(0,0)$ and $(1,1)$

solⁿ: Let $I = \int_C \bar{F}\cdot d\bar{r}$.

$$I = \int_C (2x+y^2)\,dx + (3y-4x)\,dy \quad \text{—— (1)}$$

Given curve C: $y^2 = x \Rightarrow y = \sqrt{x}$

$$dy = \frac{1}{2\sqrt{x}}\,dx$$



Using these values in the eqⁿ (1)

$$\therefore I = \int_0^1 (2x+x)\,dx + (3\sqrt{x}-4x)\frac{dx}{2\sqrt{x}}$$

$$= \int_0^1 3x\,dx + \tfrac{3}{2} - \frac{4x}{2\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}\,dx$$

$$= \int_0^1 3x\,dx - \frac{2x\sqrt{x}}{x}\,dx + \tfrac{3}{2}$$

$$= \int_0^1 (3x - 2\sqrt{x} + \tfrac{3}{2})\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{2x^{3/2}}{3/2} + \tfrac{3}{2}x\right]_0^1$$

$$= \left(\tfrac{3}{2} - 2\cdot\tfrac{2}{3}\cdot 1 + \tfrac{3}{2}\right)$$

$= 3/2 + 3/2 - 4/3$

$= 10/6$

$I. = 5/3$

Ex: 3. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ for $\bar{F} = 3x^3\bar{i} + (2xz - y)\bar{j} + 2y\bar{k}$

and c is the ellipse $x^2 + 2y^2 = 1$, $z = 0$

Soln: Let $I = \int_C \bar{F} \cdot d\bar{r} = \int_C 3x^3 dx + (2xz - y)dy + 2y\,dz$ —— (1)

Here c is the ellipse $x^2 + 2y^2 = 1$

$\Rightarrow \dfrac{x^2}{1^2} + \dfrac{y^2}{(1/\sqrt{2})^2} = 1.$

$a = 1, \quad b = 1/\sqrt{2}$

Writing parametric equations :

$x = a\cos\theta, \quad y = b\sin\theta, \quad z = 0$

$= 1\cos\theta, \quad y = 1/\sqrt{2}\,\sin\theta, \quad z = 0.$

$\therefore dx = -\sin\theta\,d\theta, \quad dy = 1/\sqrt{2}\,\cos\theta\,d\theta, \quad dz = 0$

and for complete ellipse $\theta : 0 \longrightarrow 2\pi$

$\therefore$ From eqn (1)

$$I = \int_0^{2\pi} 3(\cos\theta)^3(-\sin\theta)d\theta + \left[2\cos\theta\cdot(0) - 1/\sqrt{2}\,\sin\theta\right]\left(1/\sqrt{2}\,\cos\theta\,d\theta\right) + 0$$

$$= \int_0^{2\pi} \left(-3\sin\theta\cos^3\theta - \tfrac{1}{2}\sin\theta\cos\theta\right)d\theta$$

$$= 0$$

Since $\int_0^{2\pi} \sin^m\theta\cos^n\theta\,d\theta = 0$, if n is odd integer.

Ex: 5 · Evaluate $\int_C \bar{F}\cdot d\bar{r}$ for $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the curve $x = 2t^2,\ y = t,\ z = 4t^2 - t$ from $t=0$ to $t=1$

Soln : Let $I = \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C 3x^2dx + (2xz-y)\,dy + z\,dz \quad \text{--- (1)}$$

Given Curve C: $x = 2t^2$, $y = t$, $z = 4t^2 - t$

$\therefore dx = 4t\cdot dt,\ dy = dt,\ dz = (8t-1)dt$

and $t: 0 \longrightarrow 1$

From eqn (1),

$$I = \int_{t=0}^{1} 3(2t^2)^2(4t\,dt) + [2(2t^2)(4t^2-t) - t]\,dt + (4t^2-t)(8t-1)dt$$

$$= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2 + t)\,dt$$

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$$= 48\left(\frac{t^6}{6}\right) + 16\left(\frac{t^5}{5}\right) + 28\left(\frac{t^4}{4}\right) - 12\left(\frac{t^3}{3}\right) + \frac{t^2}{2}$$

$$= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3 + \frac{1}{2}t^2\right]_0^1$$

$$= \left[8 + \frac{16}{5} + 7 - 4 + \frac{1}{2}\right]$$

$$I = \frac{147}{10}$$

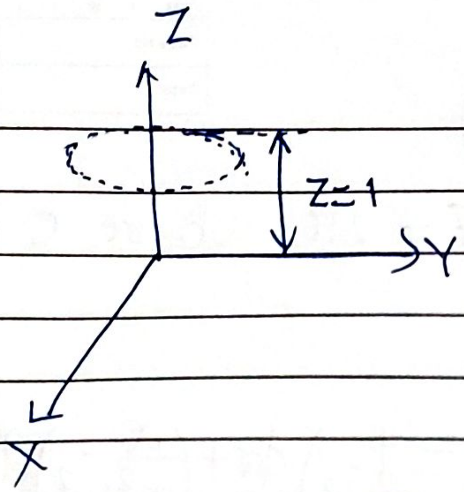
Ex: 6 Evaluate: $\int_C \bar{F}\cdot d\bar{r}$ for $\bar{F} = (2x+y)\bar{i} + (3y-x)\bar{j}$ and C is the straight line joining the points $(0,0)$ and $(3,2)$

Ans: $I = 15$.

Ex: 7. If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $C: x=t,\ y=t^2,\ z=t^3$

Ans $I = 5$

Ex: 8 Evaluate $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = y\bar{i} + xz^3\bar{j} - zy^2\bar{k}$ and C is the circle $x^2+y^2=4$, $z=1$



$$\text{Let } I = \int_C \bar{F}\cdot d\bar{r}$$

$$= \oint_C y\,dx + xz^3 dy - zy^2 dz \qquad \text{--- (1)}$$

For given circle $Z = 1$ ~~on~~

$$\Rightarrow dz = 0$$

Parametric equations of circle are.

$$x = a\cos\theta, \quad y = a\sin\theta \qquad ; a = 2$$

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

$$dx = -2\sin\theta\, d\theta, \quad dy = 2\cos\theta\, d\theta$$

$$\theta : 0 \longrightarrow 2\pi$$

From eqⁿ (1)

$$I = \int_{\theta=0}^{2\pi} (2\sin\theta)(-2\sin\theta\, d\theta) + (2\cos\theta)(2\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} (-4\sin^2\theta + 4\cos^2\theta)\, d\theta$$

$$= -4\int_0^{2\pi} (\sin^2\theta - \cos^2\theta)\, d\theta$$

$$= -4\int_0^{2\pi} \cos 2\theta\, d\theta \qquad , \because \sin^2\theta - \cos^2\theta = \cos 2\theta$$

$$= -4\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi} \qquad \because \sin 4\pi = 0, \ \sin 0 = 0$$

$$= 0$$

Ex: 9. If $\bar{F} = \frac{1}{x^2+y^2}(-y\bar{i} + x\bar{j})$

then show that $\oint_C \bar{F}\cdot d\bar{r} = 2\pi$ where C is any closed curve.

Soln: Consider,

$$I = \oint_C \bar{F}\cdot d\bar{r} = \oint_C \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy$$

$$= \oint_C \frac{-y\,dx + x\,dy}{x^2+y^2} \quad \text{—— (1)}$$

Changing the variables to polar co-ordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\therefore x^2+y^2 = r^2$$

$$\therefore dx = -r\sin\theta\,d\theta, \quad dy = r\cos\theta\,d\theta$$

$\therefore$ For any closed curve C, θ varies from 0 to 2π

$\therefore$ Hence eq^n (1) becomes —

$$I = \int_0^{2\pi} \frac{(-r\sin\theta)(-r\sin\theta\,d\theta) + (r\cos\theta)(r\cos\theta\,d\theta)}{r^2}$$

$$= \int_0^{2\pi} \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2}\,d\theta$$

$$= \int_0^{2\pi} d\theta$$

$$= 2\pi$$

Ex: 10. If $\bar{F} = \frac{1}{(x^2+y^2)}(-y\bar{i} + x\bar{j})$ then show that $\oint_C \bar{F}\cdot d\bar{r} = 2\pi$ where C is circle $x^2+y^2=1$

Refer Ex: 9 (take $r=1$)

Examples on Work done :

Ex: 1. Find the work done by $\bar{F} = x^2 i + yz j + z k$ in moving a particle along the straight line segment from $(1,2,2)$ to $(3,4,4)$

Ans: $W = 100/3$

Ex: 2. Find the work done by $\bar{F} = 2xy^2 i + (2x^2y + y) j$ in taking particle from $(0,0,0)$ to $(2,4,0)$ along the parabola $y = x^2$, $z = 0$.

Soln : We know that,

$$W = \int_C \bar{F} \cdot d\bar{r}$$

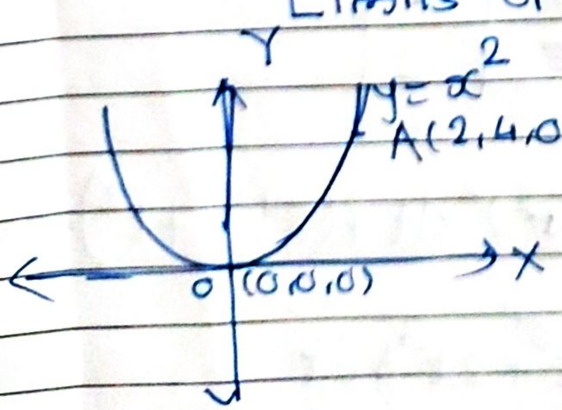
$$= \int_C 2xy^2 dx + (2x^2y + y) dy \quad \text{--- (1)}$$

Here curve C is : $y = x^2$, $z = 0$.

$\Rightarrow dy = 2x\,dx$,

Limits of x are 0 to 2

($(0,0,0)$ to $(2,4,0)$)



Equation (1) becomes –

$$\therefore W = \int_0^2 2x(x^2)^2\,dx + \left[2x^2(x^2) + x^2\right] 2x\,dx$$

$$= \int_0^2 (2x^5 + 4x^5 + 2x^3)\,dx$$

$$= \int_0^2 (6x^5 + 2x^3)\,dx$$

$$= \left[\cancel{6}x^6/\cancel{6} + \cancel{2}x^4/\cancel{4}_{2}\right]_0^2$$

$$= 64 + \tfrac{1}{2} \times 16 = 72$$

Alternate method:

$$W = \int_C \bar{F}\cdot d\bar{r} = \oint_C 2xy^2\,dx + (2x^2y + y)\,dy$$

$C : y = x^2,\ z = 0$

We can write C ~~a~~ in parametric form:

$x = t$; $y = t^2$, $z = 0$

$dx = dt$ $\therefore dy = 2t\,dt$

$\therefore$ ~~W=~~

x	0	2
t	0	2

Hence,

$$W = \int_0^2 2t(t^2)^2\,dt + (2t^2t^2 + t^2)\,2t\,dt$$

$$= \int_0^2 (6t^5 + 2t^3)\,dt$$

$$= 72.$$

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Ex: 3. Evaluate $\int_C \frac{x\,dx+y\,dy}{(x^2+y^2)^{3/2}}$ along

$\bar{r}=e^t\cos t\,\bar{i}+e^t\sin t\,\bar{j}$ joining $(1,0)$ and $(e^{2\pi},0)$

Soln: Let $I=\int_C \frac{x\,dx+y\,dy}{(x^2+y^2)^{3/2}}$

Along given curve C:

$\bar{r}=e^t\cos t\,\bar{i}+e^t\sin t\,\bar{j}$

Here, $x=e^t\cos t$; $y=e^t\sin t$

$dx=(e^t(-\sin t)+\cos t\,e^t)\,dt$ $\quad dy=(e^t\cos t+\sin t\,e^t)\,dt$

Consider,

$x\,dx+y\,dy=e^t\cos t\,[-e^t\sin t+e^t\cos t]\,dt+e^t\sin t\,[e^t\cos t+e^t\sin t]\,dt$

$=[\cancel{-e^{2t}\sin t\cos t}+e^{2t}\cos^2 t+\cancel{e^{2t}\sin t\cos t}+e^{2t}\sin^2 t]\,dt$

$=e^{2t}(\sin^2 t+\cos^2 t)\,dt$

$=e^{2t}\,dt$

& $x^2+y^2=e^{2t}\cos^2 t+e^{2t}\sin^2 t$

$=e^{2t}(\sin^2 t+\cos^2 t)$

$=e^{2t}$

Hence $\frac{x\,dx+y\,dy}{(x^2+y^2)^{3/2}}=\frac{e^{2t}\,dt}{(e^{2t})^{3/2}}=\frac{e^{2t}\,dt}{e^{3t}}=e^{-t}\,dt$

∴ x	1	$e^{2\pi}$
t ~~$x = e^t \cos t$~~	0	2π

using $x = e^t \cos t$

$t : 0 \longrightarrow 2\pi$

$$\therefore I = \int_0^{2\pi} e^{-t} dt = \left[-e^{-t}\right]_0^{2\pi}$$

$$= -\left(e^{-2\pi} - 1\right)$$

$$\therefore I = 1 - e^{-2\pi}$$

Examples on Work done when path of integration is not given:

Ex:1. Find the workdone by the force $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ in taking a particle from $(1,1,1)$ to $(3,-5,7)$.

Ex: 2. If $\bar{F} = (2xz^3 + 6y)\bar{i} + (6x - 2yz)\bar{j} + (3x^2z^2 - y^2)\bar{k}$ then evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the ~~line~~ curve joining $(0,0,0)$ and $(1,1,1)$. Is the force $\bar{F}$ conservative?

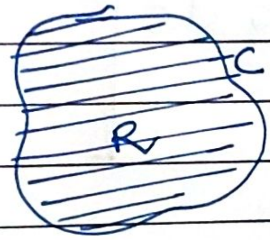
Green's Theorem :

If $u(x,y)$ and $v(x,y)$ continous functions of x and y possesses continous partial derivatives w.r.t x and y

i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continous

in a region R in XY plane bounded by simple closed curve C, then

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



where C is traversed in anticlockwise direction.
R is region bounded by given closed curve C.

Vector form of Green's theorem:

Let $\bar{F} = u(x,y)\,\bar{i} + v(x,y)\,\bar{j}$ be any vector function in XY-plane.

$$\therefore \oint_C u\,dx + v\,dy = \oint_C F \cdot d\bar{r} \quad \text{--- (1)}$$

Now,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix}$$

$$= \bar{i}\left(0 - \frac{\partial v}{\partial z}\right) - \bar{j}\left(0 - \frac{\partial u}{\partial z}\right) + \bar{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= -\frac{\partial v}{\partial z}\bar{i} + \frac{\partial u}{\partial z}\bar{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\bar{k}$$

$$(\nabla \times \bar{F}) \cdot \bar{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

An elementary area in XY plane $dx\,dy = ds$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint (\nabla \times \bar{F}) \cdot \bar{k}\, ds \quad \text{—— (2)}$$

From eq^n (1) & (2).

$$\boxed{\oint_C \bar{F} \cdot d\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \bar{k}\, ds}$$

Example's on Green's Theorem:

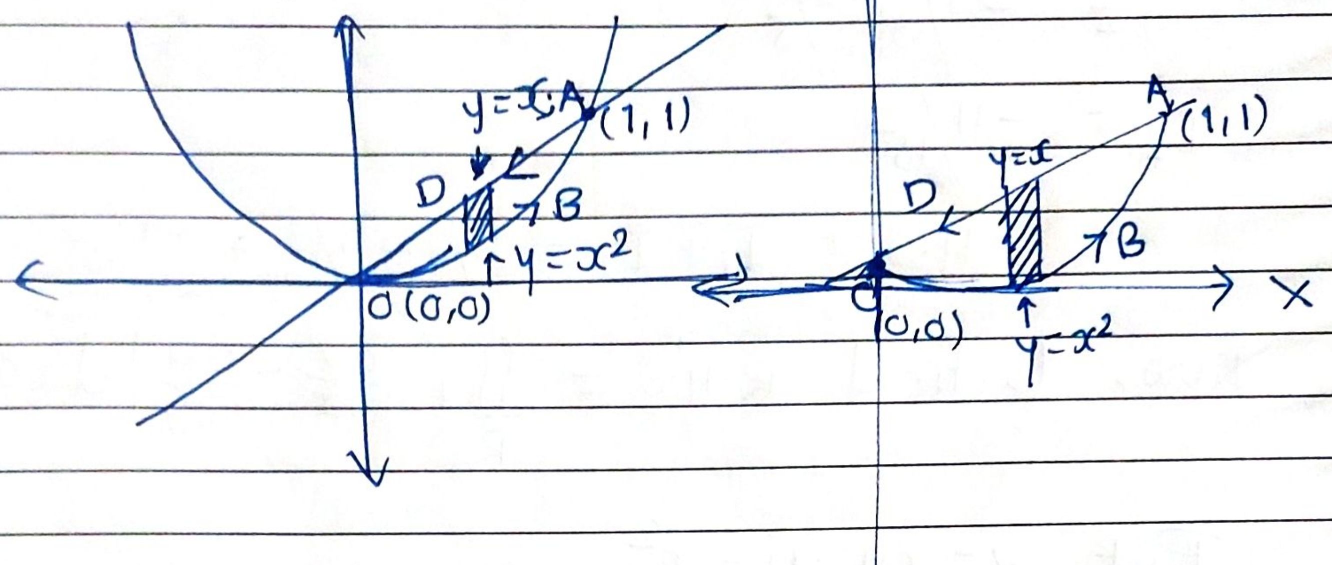
Ex; 1. Verify Green's thm for the field $\bar{F} = x^2\bar{i} + xy\bar{j}$ over the region R enclosed by $y = x^2$ and the line $y = x$

Soln: By Green's thm,

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

where R is the region bounded by the closed curve, C.

Ans: Here closed curve C is made by two parts C_1: OBA & C_2: ADO.



Now, $$\oint_C u\,dx + v\,dy = \int_{C_1: OBA} u\,dx + v\,dy + \int_{C_2: ADO} u\,dx + v\,dy$$

$$= I_1 + I_2$$

To find $I_1 = \int_{C_1: OBA} u\,dx + v\,dy$

Here $u = x^2$, $v = xy$.

$$I_1 = \int_{C_1: OBA} u\,dx + v\,dy = \int_{C_1: OBA} x^2\,dx + xy\,dy$$

Along curve C_1: $y = x^2$

$\Rightarrow dy = 2x\,dx$ & $x: 0 \to 1$

$$\therefore \quad I_1 = \int_0^1 x^2\,dx + x \cdot x^2 (2x\,dx)$$

$$= \int_0^1 (x^3 + 2x^4)\,dx$$

$$= \left[x^3/3 + 2\left(x^5/5\right) \right]_0^1$$

$$I_1 = 1/3 + 2/5 = 11/15$$

To find. $I_2 = \int_{C_2 : ADO} u\,dx + v\,dy.$

Along curve C_2, $y = x$ & $x : 1 \longrightarrow 0$
$\Rightarrow dy = dx$

$$\therefore I_2 = \int_1^0 x^2\,dx + x(x)\,dx$$

$$= \int_1^0 (x^2 + x^2)\,dx$$

$$= 2\int_1^0 x^2\,dx$$

$$= 2\left(x^3/3\right)_1^0$$

$$I_2 = 2(-1/3) = -2/3$$

Hence.

$$L.H.S = \oint_C u\,dx + v\,dy = 11/15 - 2/3 = 1/15 \quad \text{--- (1)}$$

Now to evaluate $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

Here R is region shown by shaded portion in fig.

Consider,

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{y=x} \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2) \right] dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (y - 0)\, dx\, dy$$

$$= \int_0^1 \left(\int_{x^2}^{x} y\, dy \right) dx$$

$$= \int_0^1 y^2/2 \Big|_{x^2}^{x} dx$$

$$= 1/2 \int_0^1 (x^2 - x^4)\, dx$$

$$= 1/2 \left(x^3/3 - x^5/5 \right)\Big|_0^1$$

$$= 1/2 \left(1/3 - 1/5 \right)$$

$$= 1/15$$

Hence R.H.S $= \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = 1/15$ ——— (1)

from (1) & (2)

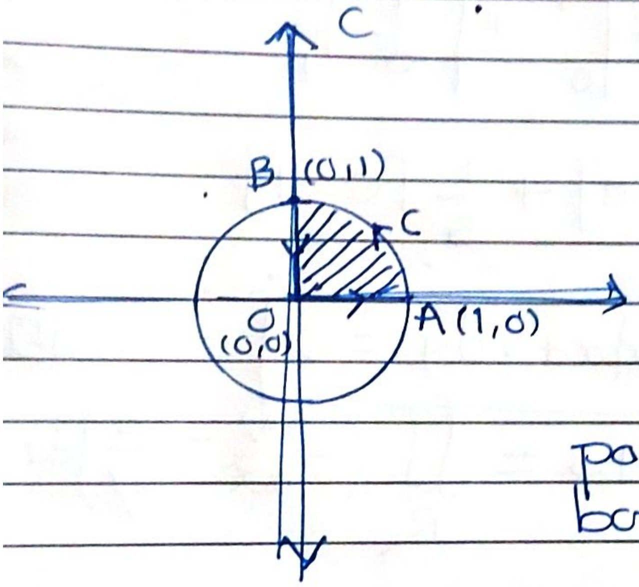
$$\therefore \oint_C u\, dx + v\, dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

Hence Green's thm verified.

Ex: 2. Verify Green's thm for $\bar{F} = x\,\bar{i} + y^2\,\bar{j}$ over the first quadrant of the circle $x^2+y^2=1$.

soln: By Green's thm,

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$



To evaluate:

$$\oint_C u\,dx + v\,dy = \int \bar{F}\cdot d\bar{r} \quad \text{(say)}$$

C is closed curve which is part of circle in first quadrant bounded by X and Y axis.

$$\therefore \oint_C u\,dx + v\,dy = \oint_C x\,dx + y^2\,dy$$

$$= \int_{OA} x\,dx + y^2\,dy + \int_{ACB} x\,dx + y\,dy + \int_{BO} x\,dx + y\,dy \quad \text{--- (1)}$$

Along OA: $y=0$, $dy=0$ & $x: 0 \to 1$.

Along ACB: $x = (1)\cos\theta$, $y = (1)\sin\theta$; $r=1$

$dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$.

& $\theta: 0 \to \pi/2$.

Along BO ; $x=0$, $dx=0$ & $y: 1 \to 0$

Hence; Eqⁿ ① becomes, -

$$\int_C x\,dx + y^2\,dy = \int_0^1 x\,dx + \int_0^{\pi/2} (\cos\theta)(-\sin\theta)\,d\theta + (\sin\theta)^2(\cos\theta)\,d\theta + \int_1^0 y^2\,dy$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + \left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} + \left[\frac{y^3}{3}\right]_1^0$$

$$= \left[\tfrac{1}{2} - 0\right] - \tfrac{1}{2}[1-0] + \frac{1}{3}[1-0] + \frac{1}{3}[0-1]$$

$= 0.$

$$\Rightarrow \boxed{L.H.S = \oint_C u\,dx + v\,dy = 0} \quad \text{——— (II)}$$

Now, to evaluate $R.H.S = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

consider.

$$R.H.S = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x)\right) dx\,dy$$

$$= \iint_R (0-0)\,dx\,dy$$

$$R.H.S. = 0. \quad \text{——— (III)}$$

From (II) & (III)

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Ex: 3 Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where C is any square with sides of length 5 and

$$\bar{F} = (2x^2 - y)\bar{i} + (\tan y - e^y + 4x)\bar{j}$$

Soln. By Green's thm,

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Let $I = \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C (2x^2 - y)\,dx + (\tan y - e^y + 4x)\,dy \quad \text{—— (1)}$$

Since C is a square which is closed curve Hence above integral can be evaluated using Green's thm:

Here $u = 2x^2 - y$, $v = \tan y - e^y + 4x$

$\therefore \frac{\partial u}{\partial y} = -1$, $\frac{\partial v}{\partial x} = 4$

Hence.

$$I = \int_C (2x^2 - y)\,dx + (\tan y - e^y + 4x)\,dy$$

$$= \iint_R (4 - (-1))\,dx\,dy$$

$$= 5 \iint_R dx\,dy$$

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$= 5(\text{Area of region } R)$

$= 5\ (\text{Area of square with side } 5)$

$= 5(5)^2$

$$\boxed{I = 125}$$

Ex: 4. Using Green's thm, evaluate $\int \bar{F} \cdot d\bar{r}$ for the field $\bar{F} = x^2\bar{i} + xy\bar{j}$ over the region R bounded by $y = x^2$ and the line $y = x$.

Ans: $1/15$

Ex: 5. A vector field is given by

$$\bar{F} = \cos y\,\bar{i} + x(1-\sin y)\,\bar{j}$$

Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where C is the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1, \quad z = 0$$

Soln: By Green's thm;

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

Let $I = \int_C \bar{F}\cdot d\bar{r} = \int_C \cos y\,dx + x(1-\sin y)\,dy$

Since C is closed curve, we apply Green's thm;

$$\therefore I = \iint_R \left[\frac{\partial}{\partial x}(x(1-\sin y)) - \frac{\partial}{\partial y}\cos y\right] dx\,dy$$

$$= \iint_R [(1-\sin y) - (-\sin y)]\, dx\,dy$$

$$= \iint_R (1-\cancel{\sin y}+\cancel{\sin y})\, dx\,dy$$

$$= \iint_R dx\,dy$$

= Area of region bounded by ellipse.

$$\frac{x^2}{25}+\frac{y^2}{9}=1, \quad z=0.$$

$$= \pi ab \quad , \quad a=5,\ b=3$$

$$= \pi \times 5 \times 3$$

$$= 15\pi.$$

Ex:6. By Green's thm, evaluate.

$\int_C (xy-x^2)\,dx + x^2 dy$ along the curve C formed by $y=0$, $x=1$, $y=x$

Soln: By Green's thm;

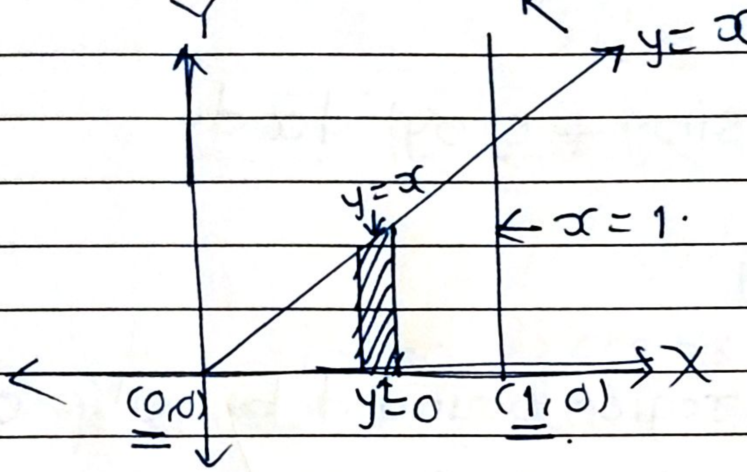
$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Hence $\oint_C (xy-x^2)\,dx + x^2 dy$

$$= \iint_R \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(xy-x^2)\right) dx\,dy$$

$$= \iint_R \left[2x - (x-0)\right] dx\,dy$$

$$= \iint_R x\,dx\,dy$$



Here R is region bounded by the triangle OAB

$$\oint_C (xy - x^2)\,dx + x^2\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=x} x\,dx\,dy$$

$$= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=x} x\,dy\right) dx$$

$$= \int_0^1 \left[xy\right]_0^x dx$$

$$= \int_0^1 x(x-0)\,dx$$

$$= \int_0^1 x^2\,dx$$

$$= x^3/3 \Big|_0^1$$

$$= 1/3$$

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Ex: 7. Use Green's thm to evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = y^3\bar{i} - x^3\bar{j}$ and C is the circle $x^2+y^2=a^2$, $z=0$.

Soln: By Green's thm,

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

Consider $\int_C \bar{F} \cdot d\bar{r} = \int_C y^3 dx - x^3 dy$ — (1)

$u = y^3, \quad v = -x^3$

From eqⁿ (1)

$$\int_C \bar{F} \cdot d\bar{r} = \iint_R \left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y} y^3 \right) dx\,dy$$

$$= \iint_R (-3x^2 - 3y^2)\, dx\,dy \quad \text{— (2)}$$

Here R is region bounded by circle $x^2+y^2=a^2, \quad z=0$

Transforming the variables to polar co-ordinates.

$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$r : 0 \longrightarrow a$ and $\theta : 0 \longrightarrow 2\pi$

from eq^n (2)

$$\oint_C \bar{F}\cdot d\bar{r} = -3 \iint (x^2+y^2)\,dx\,dy$$

$$= -3 \int_0^{2\pi}\int_0^a r^3\,dr\,d\theta$$

$$= -3 \int_0^{2\pi} \left(\frac{r^4}{4}\right)_0^a d\theta$$

$$= -\frac{3a^4}{4}(2\pi)$$

$$\oint_C \bar{F}\cdot d\bar{r} = -\frac{3a^4\pi}{2}.$$

Ex: 8 (a) Using Green's thm, show that the area bounded by simple closed curve C is given by $\frac{1}{2}\int x\,dy - y\,dx$

(b) Hence find the area of the circle $x^2+y^2=a^2$ $z=0$.

Soln: (a) By Green's thm ;

$$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

consider,

$$I = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Since C is simple closed curve, by Green's lemma,

$$I = \frac{1}{2} \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right) dx\,dy$$

$$= \frac{1}{2} \iint_R 2\, dx\,dy$$

$$= \iint_R dx\,dy$$

= Area of region bounded by given closed curve.

(b) We know that, $\text{Area} = \iint_R dx\,dy$

$$\text{Area} = \frac{1}{2} \oint_C x\,dy - y\,dx \quad \text{——— (1)}$$

Here circle C is $x^2+y^2=a^2$, $z=0$

$x = a\cos\theta$, $y = a\sin\theta$

$dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$, $dz = 0$

For complete circle $\theta : 0 \longrightarrow 2\pi$

$\therefore$ From eq^n (1)

$$\text{Area of circle} = \frac{1}{2} \int_0^{2\pi} (a\cos\theta)(a\cos\theta\, d\theta) - (a\sin\theta)(-a\sin\theta\, d\theta)$$

$$= \frac{1}{2} \int_0^{2\pi} a^2 (\cos^2\theta + \sin^2\theta)\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} a^2\, d\theta$$

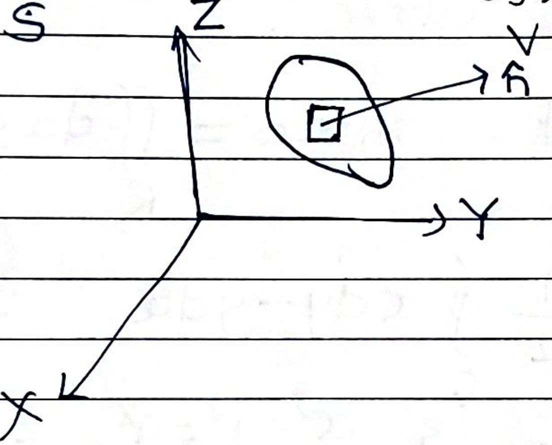
$$= \frac{a^2}{2} \times 2\pi$$

$$\text{Area} = \pi a^2$$

Gauss Divergence Thm :

Let $\bar{F}(x, y, z)$ be continous vector point function. The surface integral of the normal component of a vector point function $\bar{F}$ taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of a vector point function $\bar{F}$ taken over the volume V.

$$\iint_S \bar{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, ds$$



Note : Normally we came across following type of examples.

1. Verify Gauss Divergence thm
2. Evaluate $\iint_S \bar{F}\cdot\hat{n}\, ds$
3. Examples involving vector identities.
4. In evaluation of volume integral, $\iiint_V \nabla\cdot\bar{F}\, ds$

If volume of integral is sphere or some part of sphere $x^2+y^2+z^2=a^2$, it is convinient to express the variables to spherical polar co-ordinates:

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

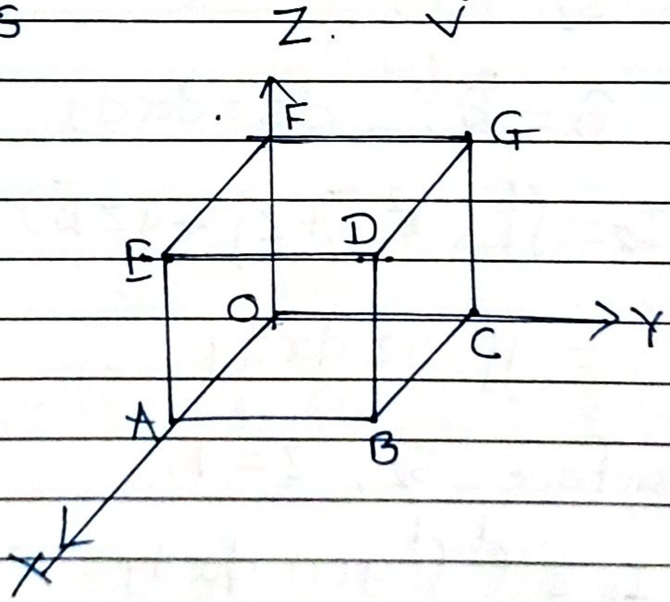
$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

$x^2+y^2+z^2 = r^2$.

Ex: 1. Verify Divergence thm for $\bar{F} = x^2\bar{i} + z\bar{j} + yz\bar{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

soln: To verify Divergence thm, we prove that,

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$



To find $\iint_S \bar{F}\cdot\hat{n}\,ds$

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iint_S (x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot\hat{n}\,ds$$

Surface S of a given cube consists of six faces:

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$S_1 = ABCO$, $S_2 = EDFG$, $S_3 = ABDE$,
$S_4 = OCGF$, $S_5 = BCGD$, $S_6 = AOFE$

① Let $I_1 = \iint\limits_{S_1} \bar{F} \cdot \hat{n}\, ds$

For S_1, $\hat{n} = -\hat{k}$, $ds = dx\,dy$

$$I_1 = \iint (x^2 \bar{i} + z\bar{j} + yz\bar{k}) \cdot (-\bar{k})\, dx\,dy$$

$$= -\iint yz\, dx\,dy$$

But $z = 0$ on $ABCO$, $\therefore \boxed{I_1 = 0.}$

② Let $I_2 = \iint\limits_{S_2} \bar{F} \cdot \hat{n}\, ds$

For S_2, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\therefore I_2 = \iint (x^2 \bar{i} + z\bar{j} + yz\bar{k}) \cdot \hat{k}\, dx\,dy$$

$$= \iint yz\, dx\,dy$$

On surface S_2, $z = 1$.

$$\therefore I_2 = \int_0^1 \int_0^1 y(1)\, dx\,dy$$

$$= \int_0^1 y^2/2 \Big|_0^1 dx$$

$$= 1/2 \int_0^1 dx$$

$$= 1/2\, x \Big|_0^1$$

$$\boxed{I_2 = 1/2}$$

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③ Let $I_3 = \iint_{S_3} \bar{F}\cdot\hat{n}\, ds$

$S_3 = ABDE$

For S_3, $\hat{n} = \bar{i}$, $ds = dy\,dz$

$\bar{F}\cdot\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot\bar{i} = x^2$

For S_3, $x = 1$

$\therefore \bar{F}\cdot\hat{n} = 1.$

Hence,

$$I_3 = \int_0^1\int_0^1 1\, dy\, dz$$

$$= \int_0^1 [y]_0^1\, dz$$

$$= \int_0^1 dz$$

$$= [z]_0^1$$

$$\boxed{I_3 = 1}$$

④ Let $I_4 = \iint_{S_4} \bar{F}\cdot\hat{n}\, ds$, For S_4: OCGF

$\hat{n} = -\bar{i}$, $ds = dy\,dz$

$\therefore \bar{F}\cdot\hat{n} = +(x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot(-\bar{i})$

$= -x^2$

But for S_4, $x = 0$.

$\therefore \boxed{I_4 = 0}$

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(5) Let $I_5 = \iint_{S_5} \bar{F}\cdot\hat{n}\, ds$, For S_5 : BCGD

$\hat{n} = \bar{j}$, $ds = dx\,dz$

$\bar{F}\cdot\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot\bar{j}$

$= z$

$\therefore I_5 = \int_0^1\int_0^1 z\, dx\, dz$

$= \int_0^1 \left(\int_0^1 z\, dz\right) dx$

$= \int_0^1 \left[\frac{z^2}{2}\right]_0^1 dx$

$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$

$\therefore \boxed{I_5 = \frac{1}{2}}$

(6) Let $I_6 = \iint_{S_6} \bar{F}\cdot\hat{n}\, ds$, For S_6 : OAEF

$\hat{n} = -\bar{j}$, $ds = dx\,dz$

$\therefore \bar{F}\cdot\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k})\cdot(-\bar{j})$

$= -z$

$\therefore I_6 = \int_0^1\int_0^1 (-z)\, dx\, dz$

$\boxed{I_6 = -\frac{1}{2}}$

Hence, $\iint_S \bar{F}\cdot\hat{n}\, ds = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$

$= 0 + \frac{1}{2} + 1 + 0 + \frac{1}{2} - \frac{1}{2}$

$= \frac{3}{2}$ ——— (1)

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To find $\quad R.H.S = \iiint_V \nabla\cdot\bar{F}\,dv$

$$\nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\bar{i} + z\bar{j} + yz\bar{k}\right)$$

$$= 2x + 0 + y$$

$$= 2x + y.$$

$$\therefore \iiint_V \nabla\cdot\bar{F}\,dv = \int_0^1\int_0^1\int_0^1 (2x+y)\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^1 (2xz + yz)_0^1\,dx\,dy$$

$$= \int_0^1\int_0^1 (2x+y)\,dx\,dy$$

$$= \int_0^1 \left(2xy + y^2/2\right)_0^1 dx$$

$$= \int_0^1 \left(2x + 1/2\right) dx$$

$$= \left(2x^2/2 + 1/2\right)_0^1$$

$$= 1 + 1/2$$

$$= 3/2. \qquad \text{(2)}$$

From eq^n (1) & (2).

Divergence thm verified.

Ex: 2. Verify Divergence thm for $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$ and S, the surface of the cube bounded by the planes $x=0, x=2, y=0, y=2, z=0, z=2$.

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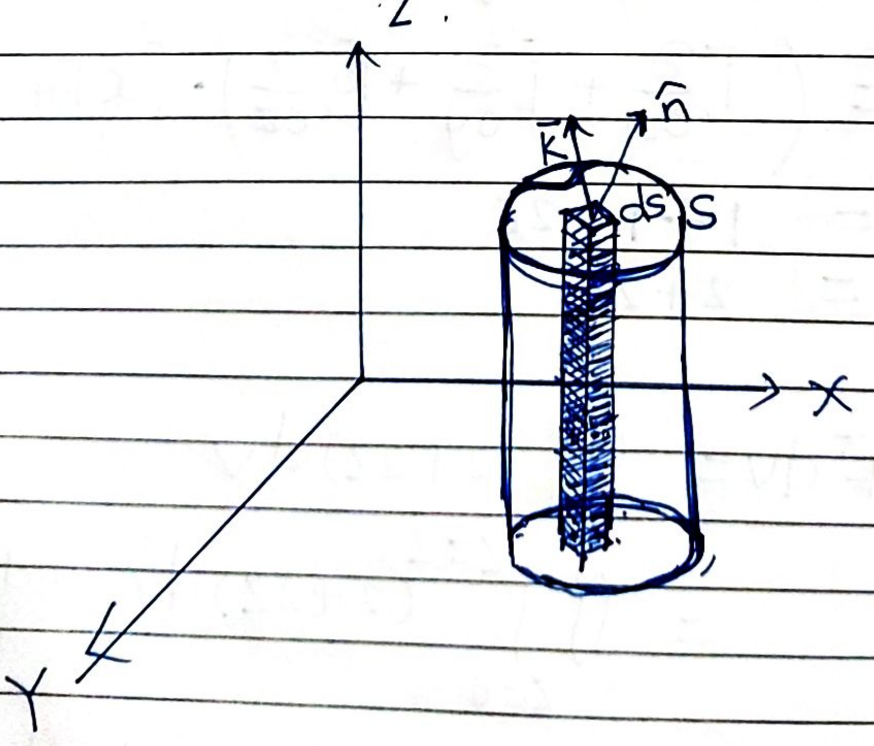
Note: To evaluate the surface integral over S, it is convinient to express it into double integral which is evaluated over the region which is orthogonal projection of surface S. Let the region R be orthogonal projection of S on XY plane.

then $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$; $\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_R \bar{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Similarly, if we take projection of S on YZ plane then $ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$; $\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_R \bar{F}\cdot\hat{n}\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

If we take projection of S on XZ plane, then

$\therefore ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$ $\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_R \bar{F}\cdot\hat{n}\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$



Ex: 3. Verify divergence thm for:

$$\bar{F} = (x+y^2)\,\bar{i} - 2x\,\bar{j} + 2yz\,k$$

and the volume of a tetrahedron bounded by co-ordinate planes and the plane $2x+y+2z=6$

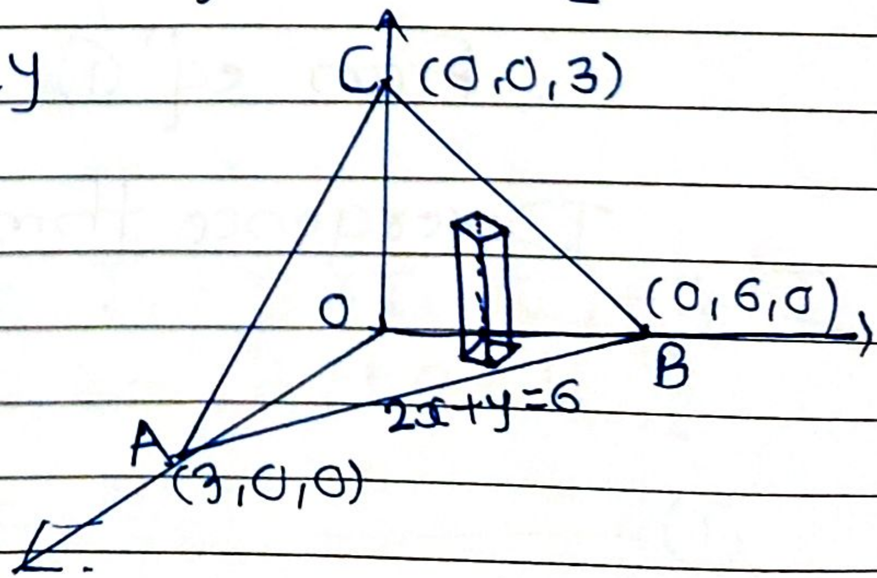
Soln: Let us first evaluate the volume integral.

Given plane cuts off intercepts 3, 6, 3 on x, y and z axes respectively.

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x+y^2) + \frac{\partial}{\partial y}(-2x) + \frac{\partial}{\partial z}(2yz)$$

$$= 1+0+2y$$

$$= 1+2y$$



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$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_{x=0}^{3}\int_{y=0}^{y=6-2x}\int_{z=0}^{z=(6-2x-y)/2} (1+2y)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^{6-2x} (1+2y)\,[z]_0^{(6-2x-y)/2}\,dx\,dy$$

$$= \int_0^3\int_0^{6-2x} (1+2y)\left[(6-2x-y)/2\right]dx\,dy$$

$$= \frac{1}{2}\int_0^3\left(\int_{y=0}^{y=6-2x} (6-2x)(1+2y) - y - 2y^2\,dy\right)dx$$

$$= \frac{1}{2}\int_0^3\left[(6-2x)(y+2y^2/2) - y^2/2 - 2y^3/3\right]_0^{6-2x} dx$$

$$= \frac{1}{2}\int_0^3\left[(6-2x)\left((6-2x)+(6-2x)^2\right) - \frac{(6-2x)^2}{2} - \frac{2}{3}(6-2x)^3\right]dx$$

$$= \frac{1}{2}\int_0^3\left[(6-2x)^2 + (6-2x)^3 - \frac{(6-2x)^2}{2} - \frac{2}{3}(6-2x)^3\right]dx$$

$$= \frac{1}{2}\int_0^3\left[\frac{(6-2x)^2}{2} + \frac{(6-2x)^3}{3}\right]dx$$

$$= \frac{1}{2}\left[\frac{(6-2x)^3}{2\times -6} + \frac{(6-2x)^4}{3\times(-8)}\right]_0^3$$

$$= \frac{1}{2}\left\{\left[\frac{(6-2\times 3)^3}{-12} + \frac{(6-2\times 3)^4}{-24}\right]\right.$$

$$-\left(\frac{(6-0)^3}{-12}+\frac{(6)^4}{-24}\right)\Bigg\}$$

$$=\frac{1}{2}\left(\frac{\cancel{6}\times\cancel{6}\times 6}{\cancel{12}_{2}}+\frac{\cancel{6}\times\cancel{6}\times\cancel{6}\times 6}{\cancel{24}_{4\cancel{2}}}\right)$$

$$=\frac{1}{2}(18+54)=\frac{72}{2}=36$$

To evaluate the surface integrals, consider the four surfaces ~~S_1 [plane ABC], (S_2)~~

$S_1 \equiv$ plane ABC

$S_2 \equiv$ plane $z=0$

$S_3 \equiv$ plane $y=0$

$S_4 \equiv$ plane $x=0$

1. Consider the surface S_1 whose equation is $2x+y+2z-6=0$.

Let $\phi = 2x+y+2z-6$

$$\frac{\partial\phi}{\partial x}=2,\quad \frac{\partial\phi}{\partial y}=1,\quad \frac{\partial\phi}{\partial z}=2$$

$$\nabla\phi = 2\bar{i}+\bar{j}+2\bar{k}$$

$$\hat{n}=\frac{2i+j+2\bar{k}}{\sqrt{4+1+4}}=\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}$$

$$\bar{F}\cdot\hat{n}=\left\{(x+y^2)\,\bar{i}-2x\bar{j}+2yz\bar{k}\right\}\cdot\left\{\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\right\}$$

$$=\frac{1}{3}\left\{2(x+y^2)-2x+4yz\right\}$$

Let ds be an element of area in plane ABC

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Taking it's projection in XOY plane, we get $ds \cos\theta = dx\,dy$, where θ is angle between normals to the surfaces S_1 and XOY plane respectively.

Unit normal to the XOY plane is $\bar{k}$

$$\cos\theta = \hat{n}\cdot\bar{k}$$

Hence, $$ds = \frac{dx\,dy}{\cos\theta} = \frac{dx\,dy}{\hat{n}\cdot\bar{k}}$$

Here. $$\hat{n} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{3}$$

$$\hat{n}\cdot\bar{k} = \frac{(2\bar{i}+\bar{j}+2\bar{k})\cdot\bar{k}}{3} = \frac{2}{3}$$

$$\therefore ds = \frac{dx\,dy}{2/3} = \frac{3}{2}\,dx\,dy.$$

$$I_1 = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \frac{1}{3}\left\{2(x+y^2)-2x+4yz\right\}\frac{3}{2}\,dx\,dy$$

putting $z = \dfrac{6-2x-y}{2}$

$$I_1 = \frac{1}{2}\int_0^3 \int_0^{6-2x} \left\{2(x+y^2)-2x+4y\left(\frac{6-2x-y}{2}\right)\right\}dx\,dy$$

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$$= \frac{1}{2}\int_0^3\int_0^{6-2x}\left\{\cancel{2x} + 2y^2 - \cancel{2x} + \frac{\overset{2}{\cancel{4}}y(6-2x)}{\cancel{2}} - \frac{\overset{2}{\cancel{4}}y^2}{\cancel{2}}\right\}dx\,dy$$

$$= \frac{1}{2}\int_0^3\left(\int_0^{6-2x}(2y^2 + 2y(6-2x) - 2y^2)\,dy\right)dx$$

$$= \frac{1}{2}\int_0^3\left[\cancel{2}y^3/\cancel{3} + \cancel{2}y^2/\cancel{2}(6-2x) - 2y^3/3\right]_0^{6-2x} dx$$

$$= \frac{1}{2}\int_0^3 \frac{2(6-2x)^3}{\cancel{3}} + (6-2x)^3(6-2x) - \cancel{\frac{2}{3}(6-2x)^3}\, dx$$

$$= \frac{1}{2}\int_0^3 (6-2x)^3\, dx$$

$$= \frac{1}{2}\left[\frac{(6-2x)^4}{-8}\right]_0^3 = \frac{1}{2}\,\frac{6^4}{+8} = \frac{\cancel{6}^{3}\times\cancel{6}^{3}\times\cancel{6}^{3}\times\cancel{6}^{3}}{\cancel{16}\ \cancel{8}\ \cancel{4}\ \cancel{2}}$$

$$= 81$$

Now for surface S_2 (plane $z=0$)

$\hat{n} = -\bar{k}$, $\bar{F}\cdot\hat{n} = \bar{F}\cdot(-\bar{k})$
$= -2yz$

$ds = dx \cdot dy$

$I_2 = \iint -2yz\,dx\,dy = 0$ as $z=0$.

For surface S_3 (plane $y=0$)

$\hat{n} = -\bar{j}$ $\bar{F}\cdot(-\bar{j}) = 2x$

$I_3 = \int_0^3 \int_0^{3-x} 2x\,dx\,dz = \int_0^3 2x\,[z]_0^{3-x}\,dx$

$= \int_0^3 2x(3-x)\,dx$

$= \left(6\cdot x^2/2 - 2x^3/3\right)_0^3$

$= 27-18 = 9$

Lastly, consider the surface S_4 (plane $x=0$)

$\hat{n} = -\bar{i}$, $\bar{F}\cdot(-\bar{i}) = -(x+y^2)$

$I_4 = \iint_{S_4} -(x+y^2)\,dy\,dz$

$= \int_0^6 \int_0^{(6-y)/2} (-y^2)\,dz = -\int_0^6 y^2[z]_0^{(6-y)/2}\,dy$

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$$= -\frac{1}{2}\int_0^6 y^2(6-y)\,dy$$

$$= -\frac{1}{2}\left[\frac{6y^3}{3} - y^4/4\right]_0^6$$

$$= -\frac{1}{2}\left[2 \times 216 - 324\right]$$

$$= -\frac{108}{2} = -54$$

Surface Integral $= I_1 + I_2 + I_3 + I_4$

$= 81 + 0 + 9 - 54$

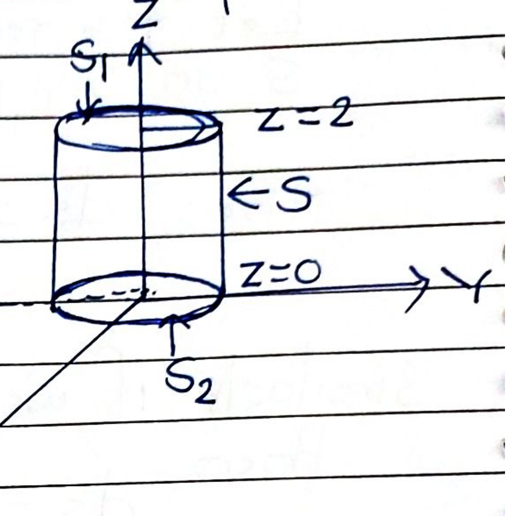
$= 36$

$=$ Volume integral

Hence the divergence thm verified.

Ex: 3. Evaluate $\iint_S (x\bar{i}+y\bar{j}+z^2\bar{k})\cdot d\bar{s}$

where S is curved surface of cylinder $x^2+y^2=4$ bounded by the ~~surface~~ planes $z=0$ and $z=2$.



soln: The surface S, S_1, S_2 constitutes a cylinder which is closed surface.

$\therefore$ By Gauss divergence thm

$$\iint_{S_1} \bar{F}\cdot\hat{n}\,ds + \iint_{S_2} \bar{F}\cdot\hat{n}\,ds + \iint_{S} \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

where V is volume of cylinder.

Here $\bar{F} = x\bar{i}+y\bar{j}+z^2\bar{k}$

$$\therefore \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\left[x\bar{i}+y\bar{j}+z^2\bar{k}\right]$$

$$= 1+1+2z$$

$$= 2+2z.$$

$$\iiint \nabla\cdot\bar{F}\,dv = \iiint (2+2z)\,dv$$

$$= \iint\int_{z=0}^{z=2} (2+2z)\,dx\,dy\,dz$$

$$= \iint \left[2z+2\frac{z^2}{2}\right]_0^2 dx\,dy$$

$$= \iint (4+4)\, dx\, dy$$

$$= 8 \iint dx\, dy$$

$$= 8\,(\text{Area of base circle } x^2+y^2=4)$$

$$\therefore \iiint_V \nabla\cdot\bar{F}\, dv = 8\pi(2)^2$$

$$= 32\pi \qquad \text{—— (2)}$$

Now, for surface S_1, $\hat{n} = \bar{k}$

$$\therefore \bar{F}\cdot\hat{n} = (x\bar{i}+y\bar{j}+z^2\bar{k})\cdot\bar{k}$$

$$= z^2$$

On surface S_1, $z=2$.

$$\therefore \bar{F}\cdot\hat{n} = (2)^2 = 4.$$

$$\iint_{S_1} \bar{F}\cdot\hat{n}\, ds = \iint_{S_1} 4\, ds = 4\iint dx\, dy$$

$$= 4\,(\text{Area of circle } S_1)$$

$$= 4\pi(2)^2$$

$$= 16\pi \qquad \text{—— (3)}$$

For surface S_2, $\hat{n} = -\bar{k}$

$\bar{F}\cdot\hat{n} = -z^2$, But for S_2, $z=0$

$$\therefore \bar{F}\cdot\hat{n} = 0$$

$$\therefore \iint_{S_2} \bar{F}\cdot\hat{n}\, ds = 0 \qquad \text{—— (4)}$$

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Using values from eq^n (2), (3), (4) in (1) we get.

$$16\pi + 0 + \iint_S \bar{F}\cdot\hat{n}\,ds = 32\pi$$

$$\therefore \boxed{\iint_S \bar{F}\cdot\hat{n}\,ds = 16\pi}$$

Stoke's Theorem :

Stokes thm is relation between surface integral and line integral.

Statement: Surface integral of normal component of curl of a vector point function $\bar{F}$ taken over an open surface S is equal to line integral of tangential component of $\bar{F}$ around perimeter of curve C bounding S.

i.e $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla \times \bar{F})\cdot \hat{n}\, ds$

Ex: Verify stoke's thm in the plane $z=0$ for $\bar{F} = (x-y^2)\,\bar{i} + 2xy\,\bar{j}$ for the region bounded by $y=0$, $x=2$, $y=x$

soln: To verify stoke's thm, we prove that,

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla \times \bar{F})\cdot \hat{n}\, ds$$

1. Consider,

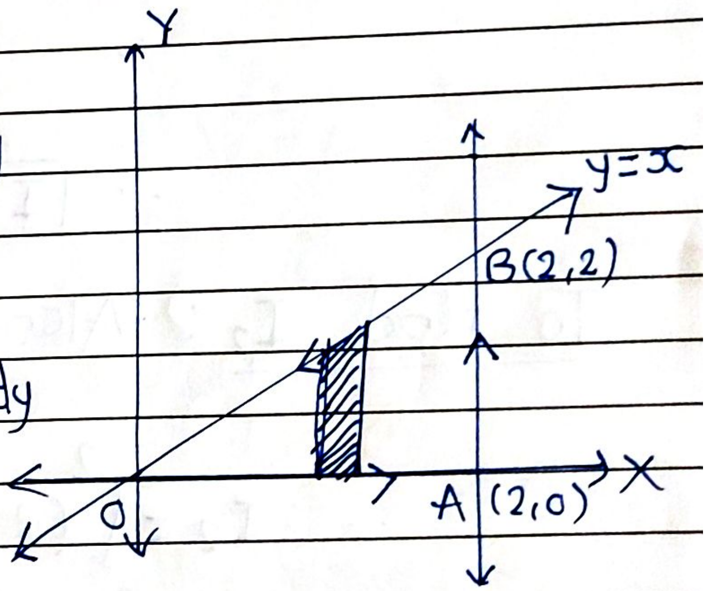
$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y)^2 & 2xy & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}[2y-(-2y)]$$

$$= 4y\bar{k}$$

$\hat{n}$ is unit normal to plane of ΔOAB

$\therefore \hat{n} = \bar{k}$



$(\nabla \times \bar{F})\cdot\hat{n} = (4y\bar{k})\cdot\bar{k} = 4y$

and $ds = dx\,dy$

$$R.H.S = \iint_S (\nabla \times \bar{F})\cdot\hat{n}\, ds = \int_0^2 \int_{y=0}^{x} 4y\,dx\,dy$$

$$= 4\int_0^2 \left(\frac{y^2}{2}\right)_0^x dx$$

$$= 2\int_0^2 x^2\,dx$$

$$= 2\left(x^3/3\right)_0^2$$

$$R.H.S = 16/3 \quad \text{———} \quad (1)$$

Now, to find LHS $= \oint_C \bar{F}\cdot d\bar{r}$

$$= \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BO} \bar{F}\cdot d\bar{r}$$

$$= I_1 + I_2 + I_3 \quad (2)$$

Here, $\oint_C \bar{F}\cdot d\bar{r} = \oint_C (x-y^2)dx + 2xy\,dy$

To find I_1 : Along OA. $y=0$. and $x: 0 \to 2$

$\therefore dy=0$

$$\therefore I_1 = \int_0^2 x\,dx + 0 = \left[x^2/2\right]_0^2$$

$$= 4/2 = 2$$

$$\therefore \boxed{I_1 = 2}$$

To find I_2 : Along AB, $x=2$ & $y: 0 \to 2$

$dx=0$

$$\therefore I_2 = \int_0^2 (2-y^2)(0) + 2(2)y\,dy$$

$$= 4\int_0^2 y\,dy$$

$$I_2 = 4\left(y^2/2\right)_0^2 = 8$$

$$\boxed{I_2 = 8}$$

To find I_3 : Along BO, $y=x$ & $y: 2 \to 0$

$dy=dx$

$$\therefore I_3 = \int_2^0 (x-x^2)dx + 2(x)(x)dx$$

$$= \int_2^0 (x+x^2)\,dx$$

$$= \left(\frac{x^2}{2} + \frac{x^3}{3}\right)_2^0$$

$$= 0 - \left(\frac{2^2}{2} + \frac{2^3}{3}\right)$$

$$\boxed{I_3 = -\frac{14}{3}}$$

Hence eq^n (2) becomes;

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = 2+8-\frac{14}{3} = \frac{16}{3} \quad \text{--- (3)}$$

From eq^n (1) & (3).

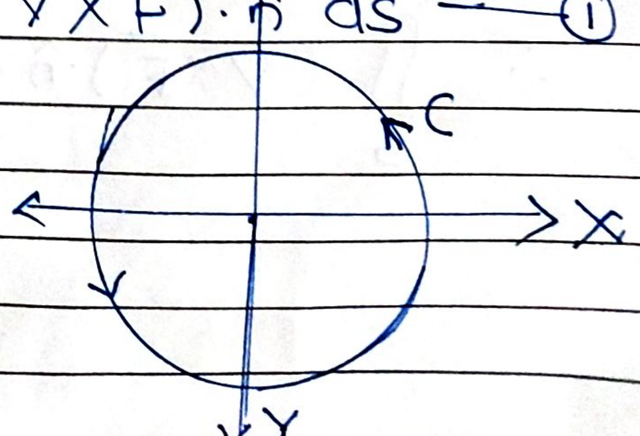
$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds.$$

Hence Stoke's thm verified.

Ex: 2. Verify Stoke's thm for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ and the closed curve C is the boundary of circle $x^2+y^2=1$.

Sol^n: To verify Stoke's thm, we have to prove tha

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds \quad \text{--- (1)}$$



Consider,

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

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$$\nabla \times \bar{F} = \bar{i}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^3)\right] - \bar{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y^3)\right]$$
$$+ \bar{k}\left[\frac{\partial}{\partial x}x^3 - \frac{\partial}{\partial y}(-y^3)\right]$$
$$= (3x^2 + 3y^2)\,\bar{k}$$

$\hat{n}$ is unit outward normal to surface.
For the surface circle $\hat{n} = \bar{k}$

Hence,
$$(\nabla \times \bar{F}) \cdot \hat{n} = \left[(3x^2 + 3y^2)\,\bar{k}\right] \cdot \bar{k}$$
$$= 3x^2 + 3y^2$$

Also for the circle, $x^2 + y^2 = 1$ on XY plane
$$ds = dx\,dy$$

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_R (3x^2 + 3y^2)\, dx\, dy$$

R is region bounded by circle $x^2 + y^2 = 1$
$\therefore$ Transforming the variables to polar co-ordinates.

$$x = r\cos\theta, \quad y = r\sin\theta,$$
$$dx\,dy = r\,dr\,d\theta$$

For complete circle, $r : 0 \longrightarrow 1$
$\theta : 0 \longrightarrow 2\pi$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^2 (r\,dr\,d\theta)$$
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^3\, dr\, d\theta$$

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$$= 3\int_0^{2\pi} d\theta \int_0^1 r^3\, dr$$

$$= 3\,[\theta]_0^{2\pi} \left[\frac{r^4}{4}\right]_0^1$$

$$= 3(2\pi) \times \frac{1}{4}$$

$$R.H.S = \frac{3\pi}{2}$$

$$\Rightarrow \boxed{\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \frac{3\pi}{2}} \quad \text{--- (1)}$$

To evaluate; $\oint_C \bar{F} \cdot d\bar{r}$

Hence, $\oint_C \bar{F}\cdot d\bar{r} = \oint_C -y^3 dx + x^3 dy$

Hence C is closed curve which is circle $x^2+y^2=1$.

Put $x = 1\cos t$, $y = 1\sin t$.
$dx = -\sin t\, dt$, $dy = \cos t\, dt$.

For complete circle $t: 0 \longrightarrow 2\pi$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{2\pi} -(\sin t)^3(-\sin t\, dt) + (\cos t)^3 \cos t\, dt$$

$$= \int_0^{2\pi} (\sin^4 t + \cos^4 t)\, dt$$

$$= \int_0^{2\pi} \sin^4 t\, dt + \int_0^{2\pi} \cos^4 t\, dt$$

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$$\oint_C \bar{F}\cdot d\bar{r} = 4\int_0^{\pi/2} \sin^4 t\, dt + 4\int_0^{\pi/2} \cos^4 t\, dt$$

since $\int_0^{2\pi} \sin^n x\, dx$ ~~$\int_0^{2\pi} \cos^n x\, dx$~~

$= 4\int_0^{\pi/2} \sin^n x\, dx$, n is even

& $\int_0^{2\pi} \cos^n x\, dx = 4\int_0^{\pi/2} \cos^n x\, dx$, n is even

$$\Rightarrow \oint_C \bar{F}\cdot d\bar{r} = 4\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) + 4\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4}$$

$$= \frac{3\pi}{2}$$

As $\int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)(n-3)\cdots 2 \text{ or } 1}{n(n-2)(n-4)\cdots 2 \text{ or } 1} \times K$

$K = \pi/2$, n is even

$= 0$, n is odd

& $\int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)\cdots 2 \text{ or } 1}{n(n-2)\cdots 2 \text{ or } 1} \times K$

$K = \pi/2$, n is even

$= 0$, n is odd

$$\therefore \boxed{\oint_C \bar{F}\cdot d\bar{r} = 3\pi/2}$$

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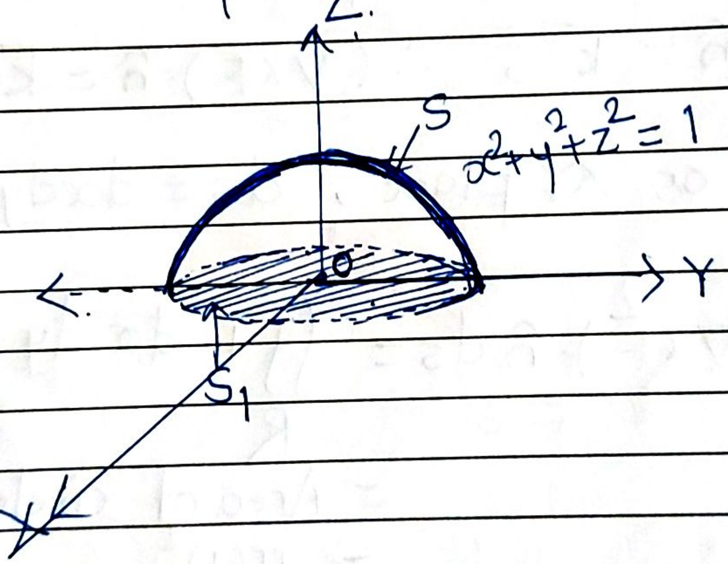
From ① & ②.

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \oint_C \bar{F} \cdot d\bar{r}$$

Hence stokes thm verified.

Ex: 3. Verify stoke's thm for the field $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ and S is the surface of the hemisphere $x^2+y^2+z^2 = 1$ above XOY plane.



Soln: To verify Stoke's thm, we have to ~~verify~~ prove that,

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

Since $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_{S_1} (\nabla \times \bar{F}) \cdot \hat{n}\, ds$

where S is surface of hemisphere $x^2+y^2+z^2=1$ above XY plane and

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S_1 is plane circular lamina on XY plane.

Consider,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \bar{i}(-2yz + 2yz) - \bar{j}(0-0) + \bar{k}(0-(-1))$$

$$= \bar{k}$$

For surface S_1 on XY plane

$$\hat{n} = \bar{k} \; ; \quad \therefore (\nabla \times \bar{F}) \cdot \hat{n} = \bar{k} \cdot \bar{k} = 1$$

Also on XY plane, $ds = dx\,dy$

Now,
$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_R 1\, dx\, dy$$

$$= \text{Area of circle } x^2+y^2=1$$

$$= \pi(1)^2$$

$$= \pi \quad \text{——— (1)}$$

To evaluate $\oint_C \bar{F} \cdot d\bar{r}$

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$$

Here curve C is circle, $x^2+y^2=1$

$\therefore x = 1\cos\theta, \quad y = 1\sin\theta, \quad z = 0$

$\therefore dx = -\sin\theta \, d\theta, \; dy = \cos\theta \, d\theta, \; dz = 0$

and $\theta : 0 \longrightarrow 2\pi$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} [2(\cos\theta) - \sin\theta](-\sin\theta \, d\theta) - 0 - 0$$

$$= \int_0^{2\pi} -2\sin\theta\cos\theta \, d\theta + \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= 0 + 4\int_0^{\pi/2} \sin^2\theta \, d\theta \quad , \quad n \text{ is even}$$

$$= 4 \frac{(2-1)}{2} \times \pi/2$$

$$= 4 \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\oint_C \bar{F} \cdot d\bar{r} = \pi \qquad \text{——— (2)}$$

from eq^n (1) & (2)

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n} \, ds = \oint_C \bar{F} \cdot d\bar{r}$$

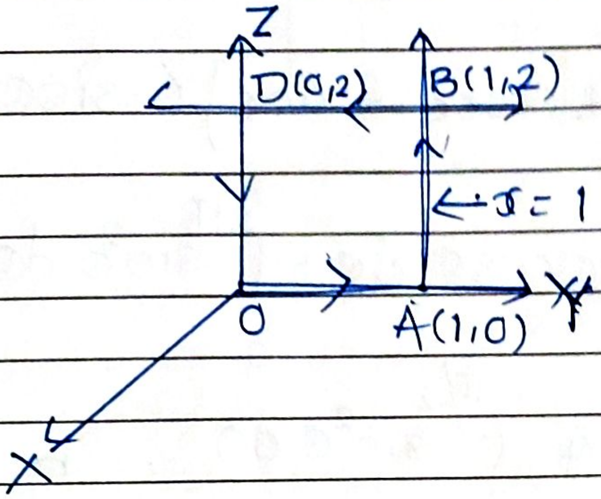
Ex. 4. Verify stoke's thm for $\bar{F} = xy^2 \bar{i} + y\bar{j} + xz^2 \bar{k}$ for the surface of rectangular lamina bounded by $x = 0, y = 0, x = 1, y = 2, z = 0$

Soln: To verify Stoke's thm, we have to prove that,

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n} \, ds$$

Step-1 : Calculate $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -2$ —— ①

Ans = −2



Step II : Evaluate $\oint_C \bar{F} \cdot d\bar{r}$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BD} \bar{F}\cdot d\bar{r} + \int_{DO} \bar{F}\cdot d\bar{r}$$

$$= I_1 + I_2 + I_3 + I_4$$

Along OA $y=0, z=0$; $x: 0 \to 1$ $\Rightarrow \boxed{I_1 = 0}$
$dy=0, dz=0$.

Along AB, $x=1, z=0$, $y: 0 \to 2$
$dx=0, dz=0$

$\Rightarrow \boxed{I_2 = 2}$

Along BD, $y=2, z=0$
$dy=0, dz=0$; $x: 1 \to 0$

$\Rightarrow \boxed{I_3 = -2}$

Along DO, $x=0, z=0$ $y: 2 \to 0$
$dx=0, dz=0$

$\boxed{I_4 = -2}$

$$I = 0 + 2 - 2 - 2 = -2 \quad \text{—— (2)}$$

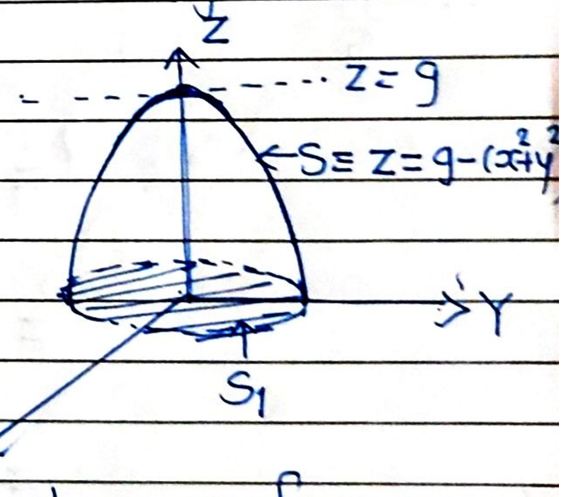
From (1) & (2)

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \oint_C \bar{F} \cdot d\bar{r}$$

—— Hence Stokes thm verified.

Evaluation of $\iint (\nabla \times F) \cdot \hat{n}\, ds$

Ex: 1. Evaluate $\iint \text{curl}\,\bar{F} \cdot \hat{n}\, ds$ for the surface of paraboloid $z = 9 - (x^2 + y^2)$ above the plane $z = 0$ and $\bar{F} = (x^2 + y - 4)\,i + 3xy\,\bar{j} + (2xz + z^2)\,k$



Soln: From fig,

$$\iint_S \text{curl}\,\bar{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{curl}\,\bar{F} \cdot \hat{n}\, ds$$

where S is curved surface of paraboloid $z = 9 - (x^2 + y^2)$ above XY plane and S_1 is plane surface on XY plane.

Consider,

$$\text{curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= \bar{i}(0 - 0) - \bar{j}(2z - 0) + \bar{k}(3y - 1)$$

$$= -2z\,\bar{i} + (3y - 1)\,\bar{k}$$

For surface S_1, $\hat{n} = -\bar{k}$ and $ds = dx\,dy$

$$\therefore (\nabla \times \bar{F}) \cdot \hat{n} = \left[-2z\bar{j} + (3y-1)\bar{k}\right] \cdot (-\bar{k})$$
$$= -(3y-1)$$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -\iint_S (3y-1)\, ds$$

R is the region bounded by circle
$x^2+y^2=9, \quad z=0$
$\therefore x = r\cos\theta, \; y = r\sin\theta, \; z=0.$
$dx\,dy = r\,dr\,d\theta$

For complete circle $r: 0 \to 3$ & $\theta: 0 \to 2\pi$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -\int_{\theta=0}^{2\pi}\int_{r=0}^{3} \left[3(r\sin\theta) - 1\right] r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^3 \left[3r^2\sin\theta - r\right] dr\,d\theta.$$

$$= -\int_0^{2\pi} \left[3\sin\theta\, r^3/3 - r^2/2\right]_0^3 d\theta$$

$$= -\int_0^{2\pi} \left[27\sin\theta - 9/2\right] d\theta.$$

$$= -27\int_0^{2\pi} \sin\theta\, d\theta - 9/2\int_0^{2\pi} d\theta$$

$$= 0 - 9/2\left[\theta\right]_0^{2\pi}$$

$$\iint (\nabla \times \bar{F}) \cdot \hat{n}\, ds = -9\pi$$

$$\because \int_0^{2\pi} \sin^n\theta\, d\theta = 0, \text{ if } n \text{ is odd}$$

Alternate method:

By stoke's thm $\iint (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \oint_C F \cdot d\bar{r}$

$\therefore \oint_C \bar{F} \cdot d\bar{r} = \oint_C (x^2+y-4)\,dx + 3xy\,dy + (2xz+z^2)\,dz$

where C is circle $x^2+y^2=9,\ z=0$

$\therefore x = 3\cos\theta,\ y = 3\sin\theta,\ z=0$

$dx = -3\sin\theta\, d\theta,\ dy = 3\cos\theta\, d\theta,\ dz = 0$

For complete circle: $\theta : 0 \longrightarrow 2\pi$

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} [(3\cos\theta)^2 + 3\sin\theta - 4](-3\sin\theta\, d\theta) + 3(3\sin\theta)(3\cos\theta\, d\theta) + 0$$

$$= \int_0^{2\pi} -27\cos^2\theta \sin\theta\, d\theta - \int_0^{2\pi} 9\sin^2\theta\, d\theta + \int_0^{2\pi} 12\sin\theta\, d\theta + \int_0^{2\pi} 81\cos^2\theta \cdot \sin\theta\, d\theta$$

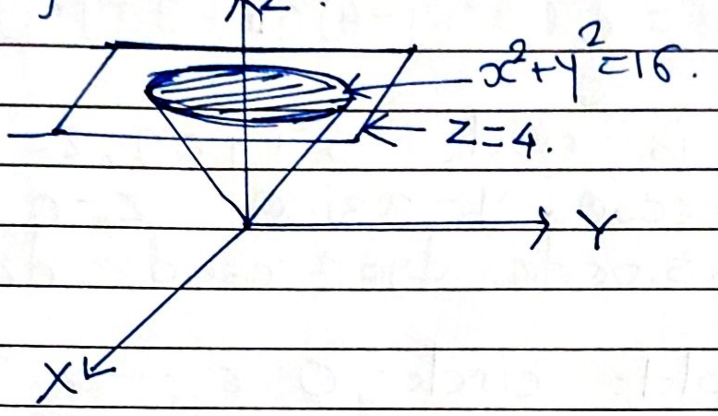
$$= 0 - 9\times 4\int_0^{\pi/2} \sin^2\theta\, d\theta + 0 + 0$$

$$= -36\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= -9\pi$$

Ex: 2. Evaluate $\iint_S (\nabla \times \bar{F}) \cdot d\bar{s}$

where $\bar{F} = (x^2 - yz)\bar{i} - (x-y)\bar{j} + 3x^2y^2\bar{k}$ and S is curved surface of the cone $z^2 = x^2+y^2$ bounded by $z = 4$.



By Stoke's thm,

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \oint_C \bar{F} \cdot d\bar{r} \quad \text{——— (1)}$$

where C is circle $x^2+y^2=16$, $z=4$.

Consider, $\oint_C \bar{F} \cdot d\bar{r} = \oint_C (x^2 - yz)dx - (x-y)dy + 3x^2y^2 dz$

put $x = 4\cos\theta$, $y = 4\sin\theta$, $z = 4$ ——— (2)

$\therefore dx = -4\sin\theta d\theta$, $dy = 4\cos\theta d\theta$, $dz = 0$.

For complete circle $\theta : 0 \longrightarrow 2\pi$.

$$\int \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \left[(4\cos\theta)^2 - 4(4\sin\theta)\right](-4\sin\theta\, d\theta)$$

$$- \left[4\cos\theta - 4\sin\theta\right](4\cos\theta d\theta) + 0$$

$$= \int_0^{2\pi} \left[-64\cos^2\theta \sin\theta d\theta + 64 \int_0^{2\pi} \sin^2\theta\, d\theta\right.$$

$$-16\int_0^{2\pi}\cos^2\theta\,d\theta+16\int_0^{2\pi}\sin\theta\cos\theta\,d\theta$$

$$=0+64\times 4\int_0^{\pi/2}\sin^2\theta\,d\theta-16\times 4\int_0^{\pi/2}\cos^2\theta\,d\theta+0$$

$$=256\left(\frac{1}{2}\frac{\pi}{2}\right)-64\left(\frac{1}{2}\frac{\pi}{2}\right)$$

$$=48\pi.$$

From eqⁿ ①

$$\iint_S(\nabla\times\bar{F})\cdot\hat{n}\,ds=\oint_C\bar{F}\cdot d\bar{r}=48\pi$$

Ex: 3. Evaluate $\iint_S(\nabla\times\bar{F})\cdot d\bar{s}$ for $\bar{F}=y\bar{i}+z\bar{j}+x\bar{k}$ over the surface $x^2+y^2=\cancel{1-z}$, $z\geq 0$
$=1-z.$

Soln: We have,

$$z=1-(x^2+y^2)$$

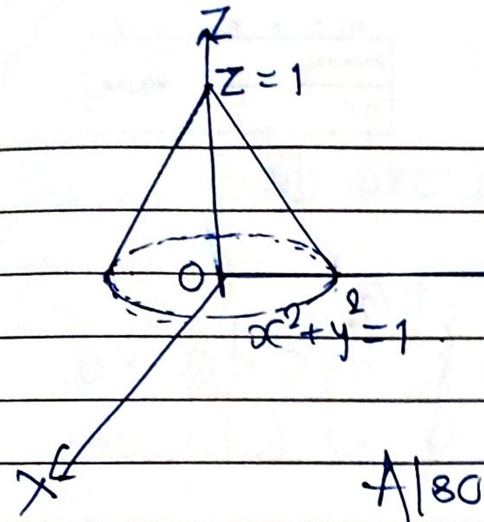
$$I=\iint_S(\nabla\times\bar{F})\cdot d\bar{s} \quad \text{——— ①}$$

Here, $\bar{F}=y\bar{i}+z\bar{j}+x\bar{k}$

Consider,

$$\nabla\times\bar{F}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}$$

$$=-\bar{i}-\bar{j}-\bar{k} \quad \text{——— ②}$$



$\therefore$ For given surface, $\hat{n} = -\bar{k}$

$\therefore (\nabla \times \bar{F}) \cdot \hat{n} = (-\bar{i} - \bar{j} - \bar{k}) \cdot (-\bar{k})$
$= 1.$

Also, $ds = dx\,dy$

$\therefore (\nabla \times \bar{F}) \cdot \hat{n}\, ds = (\nabla \times \bar{F}) \cdot \bar{ds} = dx\,dy$ — (3)

Using values from eqⁿ (2) & (3) in (1), we get.

$$I = \iint_S dx\,dy$$

$=$ Area bounded by given circle $x^2+y^2=1$

$= \pi(1)^2$

$I = \pi$

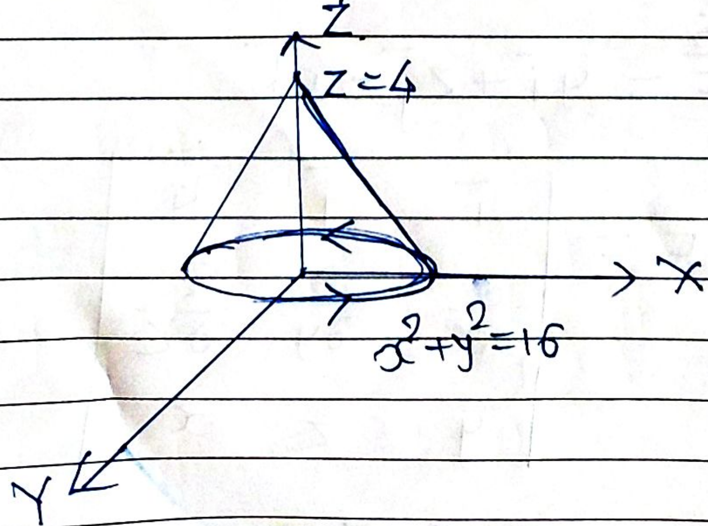
Ex: 4. Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$

where $\bar{F} = (x-y)\,\bar{i} + (x^2+yz)\,\bar{j} - 3xy^2\,\bar{k}$

and S is surface of cone

$z = 4 - \sqrt{x^2+y^2}$ above XOY plane.

Soln :



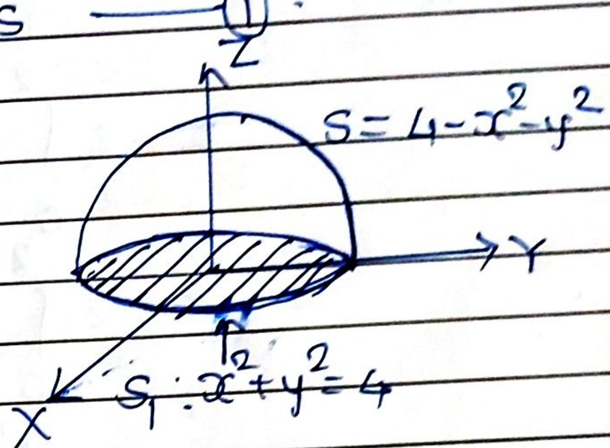
Ans : $I = 16\pi$

Ex: 5 . $\iint (\nabla \times \bar{F}) \cdot \hat{n}\, ds$ for the surface of the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$.

$\bar{F} = y^2 \bar{i} + z j + xy \bar{k}$

Soln : Here $\bar{F} = y^2 \bar{i} + z\bar{j} + xy\bar{k}$

Let $I = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$ —— (1)



consider,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z & xy \end{vmatrix}$$

$$= \bar{i}(x-1) - j(y-0) + k(0-2y)$$
$$= (x-1)\bar{i} - y\bar{j} - 2y\bar{k}$$

Since the paraboloid $z = 4 - x^2 - y^2$ and plane surface of circle say S_1 has common boundry as circle $x^2 - y^2 = 4$

We have,

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_{S_1} (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

Now, for plane surface of circle S_1, $\hat{n} = \bar{k}$

$$(\nabla \times \bar{F}) \cdot \hat{n} = [(x-1)\bar{i} - y\bar{j} - 2y\bar{k}] \cdot \bar{k}$$
$$= -2y$$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_S -2y\, dx = -2\iint_S y\, dx\, dy \quad \text{—— (2)}$$

R is region bounded by circle $x^2+y^2=4$

$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$r: 0 \longrightarrow 2,\quad \theta: 0 \longrightarrow 2\pi.$

From eqⁿ (2)

$$\iint_S (\nabla \times \bar{F})\cdot\hat{n}\,ds = -2\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(r\sin\theta)(r\,dr\,d\theta)$$

$$= -2\int_0^{2\pi}\int_0^2 r^2\sin\theta\,dr\,d\theta$$

$$= -2\int_0^{2\pi}\sin\theta\left[\int_0^2 r^2\,dr\right]d\theta$$

$$= -2\int_0^{2\pi}\sin\theta\left(r^3/3\right)_0^2\,d\theta$$

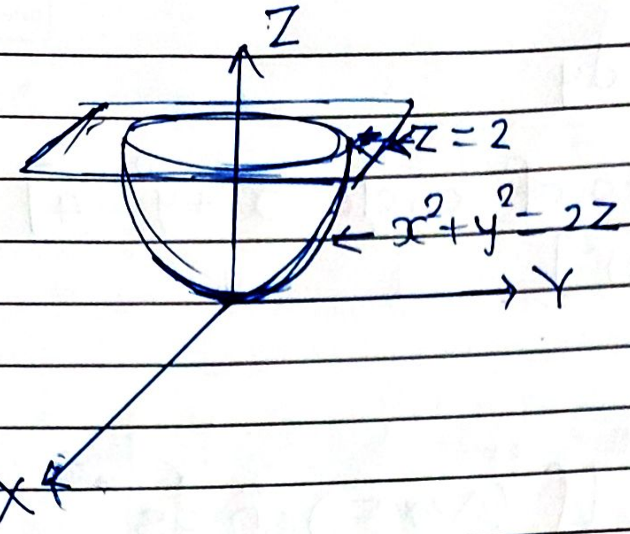
$$= -2\left(2^3/3 - 0\right)\int_0^{2\pi}\sin\theta\,d\theta$$

$$= (-2)\left(8/3\right)0$$

$$= 0 \qquad \left(\because \int_0^{2\pi}\sin^n\theta\,d\theta = 0,\ \text{if } n \text{ is odd int.}\right)$$

Ex: 6. Evaluate $\iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$ where S is the curved surface of the paraboloid $x^2+y^2=2z$ bounded by the plane $z=2$ where

$$\bar{F} = 3(x-y)\bar{i} + 2xz\bar{j} + xy\bar{k}$$



Soln: Here $\bar{F} = 3(x-y)\,\bar{i} + 2xz\,\bar{j} + xy\,\bar{k}$

Let $I = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$ —— (1)

Consider,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x-y) & 2xz & xy \end{vmatrix}$$

$$= \bar{i}(x-2x) - \bar{j}(y-0) + \bar{k}(2z-(-3))$$

$$= -x\bar{i} - y\bar{j} + (2z+3)\bar{k}$$

For given surface $\hat{n} = \bar{k}$

$$\therefore (\nabla \times \bar{F}) \cdot \hat{n} = (-x\bar{i} - y\bar{j} + (2z+3)\bar{k}) \cdot \bar{k}$$

$$= (2z+3)$$

Also $ds = dx\,dy$

$$\therefore I = \iint_S (2z+3)\, dx\,dy$$

Here $z=2$ & S is circle $x^2+y^2=4$

$$\therefore I = \iint 7\,dx\,dy$$

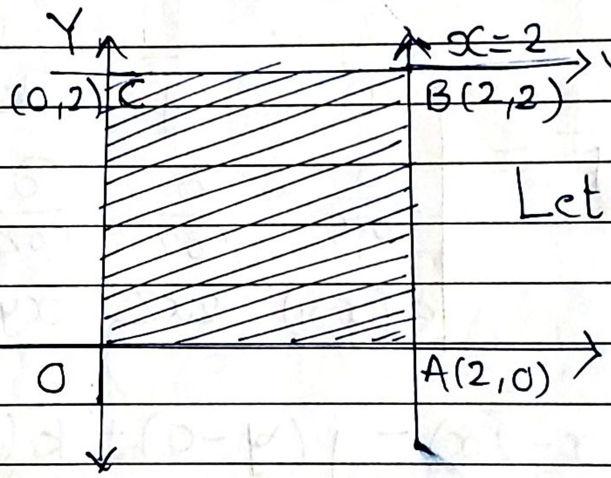
$$= 7\,[\text{Area of circle } x^2+y^2=4]$$

$$= 7\,[\pi(2)^2]$$

$$= 28\pi$$

Ex. 7. Evaluate $\iint_S (\nabla \times \bar{F})\cdot \hat{n}\, ds$ for $\bar{F} = x^2\,\bar{i} + y^2 z\,\bar{j} + xy\,\bar{k}$ for the plane surface S bounded by $x=0,\ y=0,\ x=2,\ y=2,\ z=0$.

Soln:



Given surface S is the plane lamina on XY plane.

Let $I = \iint_S (\nabla \times \bar{F})\cdot \hat{n}\, ds$ —— (1)

Consider,

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 z & xy \end{vmatrix}$$

$$= \bar{i}\,(x - y^2) - \bar{j}\,(y - 0) + \bar{k}\,(0-0)$$

$$= (x - y^2)\,\bar{i} - y\,\bar{j}$$

For surface S, $\hat{n} = \bar{k}$

$$(\nabla \times \bar{F})\cdot \hat{n} = [(x-y^2)\,\bar{i} - \bar{j}]\,\bar{k}$$

$$= 0$$

Hence ~~say~~

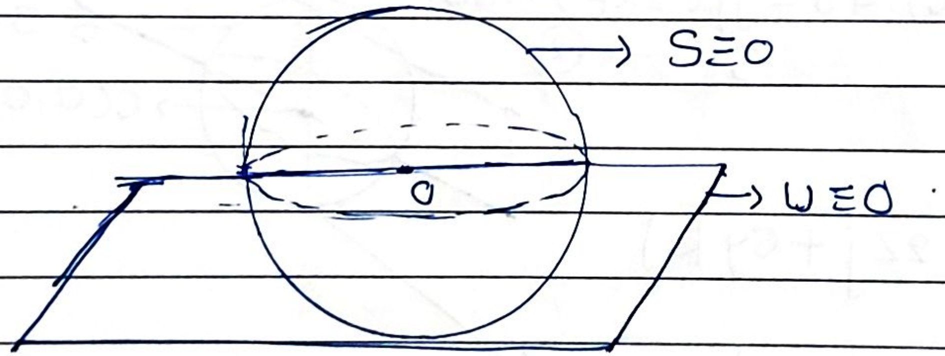
$$I = \iint_S (\nabla \times \bar{F}) \cdot \hat{n} \, ds$$

$$= 0$$

Ex:8. Verify Stoke's thm for $\bar{F} = -y^3 \bar{i} + x^3 \bar{j}$ and the closed curve C is the boundry of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Evaluation of Line Integral by using Stoke's thm

Note: 1. Section of sphere by plane is circle.



2. Section of sphere by plane passing through centre of sphere is called great circle.

3. Centre of great circle and sphere coincides and hence their radii are equal.

4. General equation of sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

centre of sphere $= (-u, -v, -w)$

Radius $= \sqrt{u^2+v^2+w^2-d}$

5. General equation of plane is

$$ax+by+cz+d=0.$$

Equation of plane in intercept form is,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

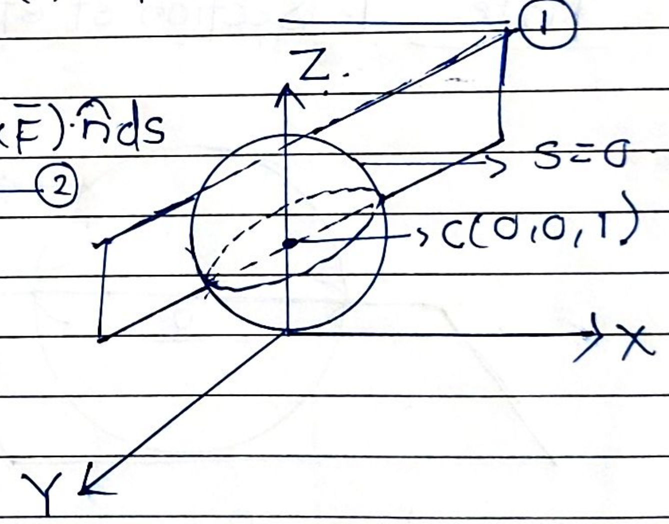
Ex: 1. Use stoke's thm to evaluate

$\oint_C (4y\bar{i}+2z\bar{j}+6y\bar{k})\cdot d\bar{r}$ where C is curve of intersection of $x^2+y^2+z^2=2z$ & $x=z-1$.

<u>Soln</u>: By stoke's thm,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$$

$$\oint_C (4y\bar{i}+2z\bar{j}+6y\bar{k})\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds \quad \text{--- (2)}$$



Here $\bar{F}=(4y\bar{i}+2z\bar{j}+6y\bar{k})$

Given equation of sphere is $x^2+y^2+z^2-2z=0$

centre $(0,0,1)$

Radius $=\sqrt{0+0+1-0}=1$

Equation of plane is $x-z=-1$ which can be written as,

$$\frac{x}{-1}+\frac{z}{1}=1$$

We know that section of sphere by given plane is circle C and circular lamina is bounded by circle C and surface S

$\therefore S \equiv x - z + 1$.

(Since, co-ordinates of centre (0,0,1) of sphere satisfies equation of plane.
$\therefore$ Given circle is great circle!
Radius of sphere and radius of circle is equal)

$\hat{n}$ is unit outward normal to surface $S \equiv x - z + 1$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Now,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x - z + 1)$$

$$= \bar{i} - \bar{k}$$

since $\phi \equiv x - z + 1$.

$$\therefore \hat{n} = \frac{\bar{i} - \bar{k}}{\sqrt{2}}$$

$$(\nabla \times F) \cdot \hat{n} = (4\bar{i} - 4\bar{k}) \cdot \frac{(\bar{i} - \bar{k})}{\sqrt{2}} = \frac{8}{\sqrt{2}}$$

$$\iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint_R \frac{8}{\sqrt{2}}\, ds = \frac{8}{\sqrt{2}} \iint_R ds$$

$$= \frac{8}{\sqrt{2}} \text{(Area of circle}$$

$$= \frac{8}{\sqrt{2}} \pi (1)^2 = 4\sqrt{2}\pi$$

$(r = 1)$

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$$\oint_C (4y\bar{i} + 2z\bar{j} + 6y\bar{k}) \cdot d\bar{r} = 4\sqrt{2}\pi$$

Ex: 2. Apply Stoke's thm, to prove that

$$\oint_C (y\bar{i} + z\bar{j} + x\bar{k}) \cdot d\bar{r} = -2\sqrt{2}\pi a^2,$$

where C is curve given by $x^2+y^2+z^2-2ax-2ay=0$; $x+y=2a$.

Soln: By Stoke's thm;

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

$$\oint_C (y\bar{i} + z\bar{j} + x\bar{k}) \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds \quad \text{--- (1)}$$

$\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and $S \equiv x+y-2a=0$

Given curve C is circle of intersection of sphere $x^2+y^2+z^2-2ax-2ay=0$ by the plane $x+y-2a=0$ ($\because$ centre of sphere $=(a,a,0)$)

Radius of sphere $= \sqrt{a^2+a^2+0-0} = \sqrt{2a^2} = a\sqrt{2}$

Since $(a, a, 0)$ satisfies eq^n of plane $x+y=2a$,
$\therefore$ given circle C is great circle.
$(\because a+a=2a)$

Hence Radius of circle $= a\sqrt{2}$

Consider,
$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\bar{i}-\bar{j}-\bar{k}$$

$\hat{n}$ is unit outward normal surface S.
$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{where } \phi = x+y-2a$$

Consider,
$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x+y-2a) = \bar{i}+\bar{j}$$
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{j}}{\sqrt{2}} \quad (\because \phi = x+y-2a)$$

$$(\nabla\times\bar{F})\cdot\hat{n} = (-\bar{i}-\bar{j}-\bar{k})\cdot\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right) = -\sqrt{2} \quad \text{--- (3)}$$

From eq^n(s) (1) & (3) we get,
$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S -\sqrt{2}\,ds = -\sqrt{2}\,(\text{Area bounded by surfaces})$$
$$= -\sqrt{2} \times (\text{Area of circle})$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = -\sqrt{2}\,[\pi(2a^2)]$$
$$= -2\sqrt{2}\pi a^2$$

Ex: 3. Apply Stoke's thm, to prove that
$\oint_C 4y\,dx + 2z\,dy + 6y\,dz = 36\pi\sqrt{2}$, where curves C is intersection of
$x^2+y^2+z^2=6z$ and $z=x+3$

Soln: By Stoke's thm;

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds \quad \text{--- (1)}$$

Here $\bar{F} = 4y\bar{i} + 2z\bar{j} + 6y\bar{k}$

$$\oint_C 4y\,dx + 2z\,dy + 6y\,dz = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$$

Given curve C is a circle of intersection of sphere $x^2+y^2+z^2-6z=0$ by the plane $z-x-3=0$. Centre of sphere $(0,0,3)$ which satisfies eqn of plane. $\therefore$ given circle is a great circle.
$\therefore$ Radius of circle $= 3$

Consider,

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix} = 4\bar{i} - 4\bar{k}$$

$\hat{n}$ is unit outward normal to surface S.

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}, \text{ where } \phi = z-x-3$$

Date:

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(z - x - 3)$$

$$= -\bar{i} + \bar{k}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-\bar{i}+\bar{k}}{\sqrt{2}} \quad (\because \phi \equiv z - x - 3)$$

$$(\nabla\times\bar{F})\cdot\hat{n} = (4\bar{i} - 4\bar{k})\cdot\left(\frac{-\bar{i}+\bar{k}}{\sqrt{2}}\right) = -8/\sqrt{2} \quad \text{(3)}$$

From eqⁿ (1) & (3), we get,

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S -8/\sqrt{2}\, ds = -8/\sqrt{2}\,(\text{Area bounded by surfaces})$$

$$= -8/\sqrt{2}\,(\text{Area of circle})$$

$$= -8/\sqrt{2} \times \pi(3)^2$$

$$= -36\pi\sqrt{2}$$