SET THEORY

SETS: A set is a collection of distinct objects.

OR

A set is a collection of well defined objects.

Ex: 1. The collection of red cars
2. The collection of positive numbers
3. The collection of people born before 1990
4. The collection of greatest football players.

Ex. 1, 2, 3 are well defined objects
But Ex. 4 is not well defined.
& hence Ex 1, 2, 3 are sets but 4 is not.

Notation: Sets are usually denoted by capital letters A, B, C, ...
And elements are usually denoted by small letters a, b, c, ...

Note: If 'a' is an element of A, then we write $a \in A$

otherwise $a \notin A$ ('a' is not an element of A)

Standard Notations :

1. $\mathbb{N} \equiv$ set of natural numbers $= \{1, 2, 3 \cdots\}$
2. $\mathbb{W} \equiv$ set of whole numbers $= \{0, 1, 2, 3, \cdots\}$
3. $\mathbb{Z} \equiv$ set of integers $= \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$
4. $\mathbb{Z}^+ / \mathbb{Z}^- \equiv$ A set of +ve/−ve integers
5. $\mathbb{Q} \equiv$ set of all rational numbers.
6. $\mathbb{Q}^+ / \mathbb{Q}^- \equiv$ set of all +ve/−ve rational numbers.
7. $\mathbb{R} \equiv$ A set of real numbers
8. $\mathbb{R}^+ / \mathbb{R}^- \equiv$ A set of all +ve/−ve real numbers
9. $\mathbb{C} \equiv$ A set of all complex numbers.

Methods of describing a set:

(i) Roster / Listing method / Tabular form.

In this method, a set is described by listing elements separated by commas, within braces.

e.g. $A = \{a, e, i, o, u\}$

(ii) Set builder / Rule method:

In this method, we write a property or rule which gives us all the elements of the set by that rule.

eg. $A = \{x \,/\, x$ is a vowel of English alphabet$\}$

$B = \{x \,/\, x \in \cancel{\mathbb{Z}},\ x \text{ is even}\}$

$C = \{x \,/\, x \in \mathbb{Z},\ x \text{ is odd}\}$

$D = \{x \,/\, x \in \mathbb{Z},\ x \text{ is even and divisible by } 4\}$

$= \{4, 8, 12, 16, 20, \ldots\}$

Types of sets:

1. Finite set: A set containing finite number of elements or no elements.

2. Cardinal number of a finite set:
The number of elements in a given finite set is called cardinal number of a finite set denoted by $n(A)$ or $|A|$

3. Infinite set : A set containing infinite number of elements.

4. Empty / Null / Void set : A set containing no elements ~~it~~ is denoted by ϕ or $\{ \}$.

5. ~~A~~ Singleton set : A set containing a single element is called singleton set.

6. Equal sets : Two sets A and B are said to be equal, if every element of A is a member of B and every element of B is a member of A

We write $A = B$.

7. Equivalent sets : Two sets A and B are said to be equivalent sets if they have same number of elements.

Note : It is not necessary that they have same elements or they are a subset of each other.

If A & B are equivalent then

$n(A) = n(B)$

8. Subset ; Let A and B be ~~any~~ two sets. If every element of A is an element of B, then A is called subset of B And B is called superset of A

$A \subseteq B$ or $B \supseteq A$

9. Proper subset : If A is a subset of B and $A \neq B$, then A is called proper subset of B.

Notation : $A \subset B$

10. Universal set (U) A set consisting of all possible elements which occur under consideration is called a universal set.

11. Comparable sets : Two sets A and B are comparable, if $A \subseteq B$ or $B \subseteq A$

12. Non-comparable sets : For two sets A and B, if neither $A \subseteq B$ nor $B \subseteq A$, then A and B are called non-comparable sets.

13. Power set (P) The set of all subsets of A is called power set of A

Notation : $P(A)$

Ex: 1. $A = \{a, b, c\}$

$$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$$

Note : If $|A| = n$ then $|P(A)| = 2^n$

In above Ex: $|A| = 3$, $|P(A)| = 2^3 = 8$

2. If $|A| = 4$, then $|P(A)| = 2^4 = 16$

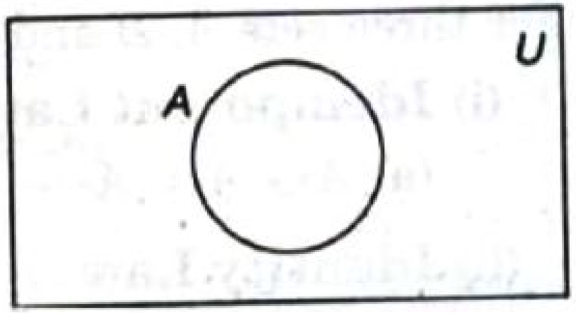
3. If $A = \{\}$ or ϕ $P(A) = \phi$

$|A| = 0$, $|P(A)| = |\{\phi\}| = 1$

4. If $A = \{3, 9, 11\}$ find $P(A)$

Venn Diagram

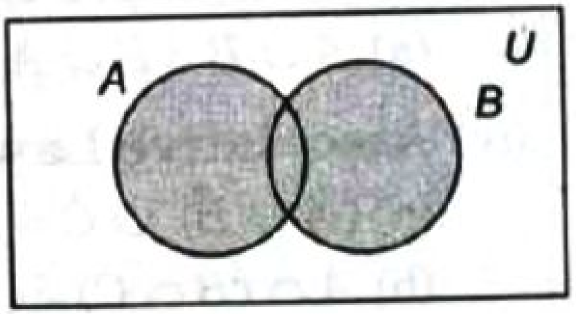
In a Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set.



Operations on Sets

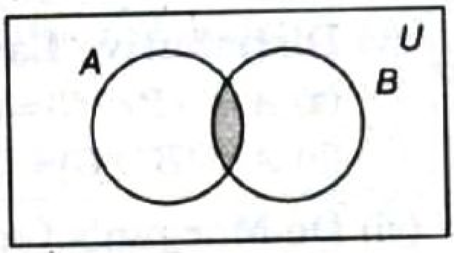
1. Union of Sets

The union of two sets A and B, denoted by A ∪ B is the set of all those elements, each one of which is either in A or in B or both in A and B.



2. Intersection of Sets

The intersection of two sets A and B, denoted by A ∩ B, is the set of all those elements which are common to both A and B.

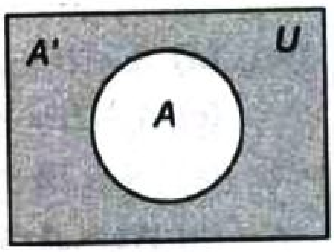


If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their intersection is denoted b

$$\bigcap_{i=1}^{n} A_i \text{ or } A_1 \cap A_2 \cap \ldots \cap A_n$$

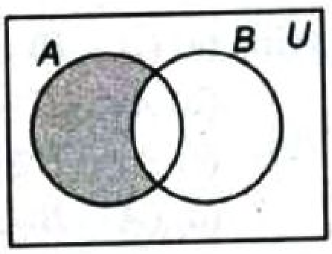
3. Complement of a Set

If A is a set with U as universal set, then complement of a set, denoted by A' or A^c is the set U – A .



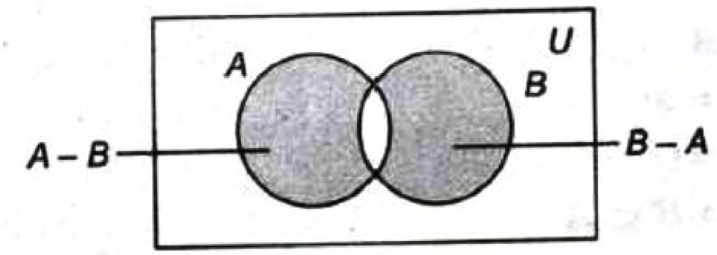
4. Difference of Sets

For two sets A and B, the difference A – B is the set of all those elements of A which do not belong to B.



5. Symmetric Difference

For two sets A and B, symmetric difference is the set (A – B) ∪ (B – A) denoted by A Δ B.



Laws of Algebra of Sets

For three sets A, B and C

(i) Commutative Laws

$A \cap B = B \cap A$
$A \cup B = B \cup A$

(ii) Associative Laws

$(A \cap B) \cap C = A \cap (B \cap C)$
$(A \cup B) \cup C = A \cup (B \cup C)$

(iii) Distributive Laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) Idempotent Laws

$A \cap A = A$
$A \cup A = A$

(v) Identity Laws

$A \cup \Phi = A$
$A \cap \mathbf{U} = A$

(vi) De Morgan's Laws

(a) $(A \cap B)' = A' \cup B'$
(b) $(A \cup B)' = A' \cap B'$
(c) $A - (B \cap C) = (A - B) \cap (A - C)$
(d) $A - (B \cup C) = (A - B) \cup (A - C)$

(vii) (a) $A - B = A \cap B'$
(b) $B - A = B \cap A'$
(c) $A - B = A \Leftrightarrow A \cap B = (\Phi)$

(d) $(A - B) \cup B = A \cup B$
(e) $(A - B) \cap B = (\Phi)$
(f) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
(g) $A \cup (A \cap B) = A$
(h) $A \cap (A \cup B) = A$

(viii) (a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
(b) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(c) $A \cap (B \Delta C) = (A \cap B)\ A\ (A \cap C)$
(d) $(A \cap B) \cup (A - B) = A$
(e) $A \cup (B - A) = (A \cup B)$

(ix) (a) $U' = (\Phi)$
(b) $\Phi' = U$
(c) $(A')' = A$
(d) $A \cap A' = (\Phi)$
(e) $A \cup A' = U$
(f) $A \subseteq B \Leftrightarrow B' \subseteq A'$

Important Points to be Remembered

- Every set is a subset of itself i.e., $A \subseteq A$, for any set A.
- Empty set Φ is a subset of every set i.e., $\Phi \subset A$, for any set A.
- For any set A and its universal set U, $A \subseteq U$
- If $A = \Phi$, then power set has only one element i.e., $n(P(A)) = 1$
- Power set of any set is always a non-empty set.
  Suppose $A = \{1, 2\}$, then$P(A) = \{\{1\}, \{2\}, \{1, 2\}, \Phi\}$.(a) $A \notin P(A)$
  (b) $\{A\} \in P(A)$
- (vii) If a set A has n elements, then P(A) or subset of A has 2^n elements.
- (viii) Equal sets are always equivalent but equivalent sets may not be equal.

The set $\{\Phi\}$ is not a null set. It is a set containing one element Φ.

Results on Number of Elements in Sets

- $n(A \cup B) = n(A) + (B) - n(A \cap B)$
- $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint.
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- n (number of elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
- n (number of elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
- $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
- $n(B - A) = n(B) - n(A \cap B)$

Ordered Pair

An ordered pair consists of two objects or elements in a given fixed order.

Equality of Ordered Pairs Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$.

Cartesian Product of Sets

For two sets A and B (non-empty sets), the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called Cartesian product of the sets A and' B, denoted by A x B.

$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

If there are three sets A, B, C and $a \in A$, be B and $c \in C$, then we form, an ordered triplet (a, b, c). The set of all ordered triplets (a, b, c) is called the cartesian product of these sets A, B and C.

i.e., $A \times B \times C = \{(a,b,c) : a \in A, b \in B, c \in C\}$

Properties of Cartesian Product

For three sets A, B and C

- $n(A \times B) = n(A)\, n(B)$
- $A \times B = \Phi$, if either A or B is an empty set.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B — C) = (A \times B) — (A \times C)$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
- If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subset (B \times D)$
- If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
- $A \times B = B \times A \Leftrightarrow A = B$
- If either A or B is an infinite set, then A x B is an infinite set.
- $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
- $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$
- If A and B be any two non-empty sets having n elements in common, then A x B and B x A have n^2 elements in common.
- If $\neq B$, then $A \times B \neq B \times A$
- If $A = B$, then $A \times B = B \times A$
- If $A \subseteq B$, then $A \times C = B \times C$ for any set C.

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Example Set - 1.

Ex: 1. Let $A = \{a, b, \{a\}, \{a,b\}, \{a,b,c\}\}$

Identify each of the following statements as true or false.
Justify your answers.

① $a \in A$ ② $\{b\} \in A$
③ $\{b, c\} \in A$ ④ $b \in A$
⑤ $\{\{a,b\}\} \subseteq A$

⑥ $\{a,b\} \in A$

⑦ $\{a, \underline{\{b\}}\} \subseteq A$.

Ex: 2. Determine whether each of the following statements are true for arbitrary sets A, B, C
Justify your answer.

① If $A \in B$ & $B \subseteq C$ then $A \in C$

② If $A \in B$ and $B \subseteq C$ then $A \subseteq C$

③ If $A \subseteq B$ & $B \in C$ then $A \in C$

False $A = \{a\}$, $B = \{a, b\}$, $C = \{\{a,b\}\}$

$A \subseteq B$, $B \in C$

But $A = \{a\} \notin C$.

④ If $A \subseteq B$ and $B \in C$ then $A \subseteq C$

False.

Ex: 3 Consider, $U = \mathbb{N}$.

$$A = \{x \mid x \in \mathbb{N},\ 2 \leq x \leq 10\}$$

$$B = \{x \mid x \in \mathbb{N},\ x < 15\}$$

Find $A \cup B$, $A \cap B$, A^c, B^c,

$A - B$, $B - A$, $A \Delta B$

Note that : $\overline{\phi} = U$ & $\overline{U} = \phi$.

Properties :

(i) $\overline{A} = U - A$

(ii) $A - A = \phi$

(iii) $A - \overline{A} = A$ (As $A \cap \overline{A} = \phi$)

(iv) $\overline{A} - A = \overline{A}$

(v) $A - \phi = A$

(vi) $A - B = B - A$ if and only if $A = B$

(vii) $A - B = A$ if and only if $A \cap B = \phi$

(viii) $A - B = \phi$ if and only if $A \subseteq B$

Properties of symmetric difference: (Δ or $\oplus$)

(i) $A \Delta A = \phi$ (OR $A \oplus A = \phi$)

(ii) $A \Delta \phi = A$

(iii) $A \Delta U = \bar{A}$

(iv) $A \Delta \bar{A} = U$

(v) $A \Delta B = (A \cup B) - (A \cap B)$.

pfs: (i) $A \Delta A = (A - A) \cup (A - A)$
$= \phi \cup \phi$
$= \phi$.

(ii) $A \Delta \phi = (A - \phi) \cup (\phi - A)$
$= A \cup A$
$= A$

(iii) $A \Delta U = (A - U) \cup (U - A)$
$= \phi \cup \bar{A}$
$= \bar{A}$

(iv) $A \Delta \bar{A} = (A - \bar{A}) \cup (\bar{A} - A)$.
$= A \cup \bar{A}$
$= U$

(v) $A \Delta B = A \cup B - A \cap B$.

Let $x \in A \cup B - A \cap B$

Then $x \in A \cup B$ but $x \notin A \cap B$

$\Rightarrow x \in A, x \notin B$

&

or $x \in B, x \notin A$

$\Rightarrow x \in (A-B) \cup (B-A)$.

$\Rightarrow x \in A \Delta B$

Conversely, if $x \in A \Delta B$

$\Rightarrow x \in (A-B) \cup (B-A)$

$\Rightarrow x \in A, x \notin B$ or $x \in B, x \notin A$

$\Rightarrow x \in A \cup B$ but $x \notin A \cap B$

$\Rightarrow x \in (A \cup B) - (A \cap B)$

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Algebra of set operations :

1. Commutativity :

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

2. Associativity :

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributivity :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. Idempotent laws :

(i) $A \cup A = A$

(ii) $A \cap A = A$

5. Absorption laws :

(i) $A \cup (A \cap B) = A$

(ii) $A \cap (A \cup B) = A$

6. De' Morgan's laws :

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

7. **Double Complement :**

$$\overline{\overline{A}} = A$$

Q.1 State & Prove distributive law:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

pf: (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

then $x \in A$ or $x \in B \cap C$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ — ①

Similarly, we can prove,

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ — ②

$\Rightarrow$ from ① & ②

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q.2 State and prove D' Morgans laws.

pf: (i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$

(i) $(A \cup B)^c = A^c \cap B^c$

Now, $(A \cup B)^c = \{x / x \notin A \cup B\}$

$= \{x / x \notin A \text{ and } x \notin B\}$

$= \{x / x \in \bar{A} \text{ and } x \in \bar{B}\}$

$= \{x / x \in \bar{A} \cap \bar{B}\}$

$= \bar{A} \cap \bar{B}$

(ii) H.W.

Q.3. If $A = \{a, b, \{a, c\}, \phi\}$, determine the following sets.

(i) $A - \{a, c\}$

(ii) $\{\{a, c\}\} - A$

(iii) $A - \{\{a, b\}\}$

(iv) $\{a, c\} - A$

Soln. (i) $A - \{a, c\} = \{b, \{a, c\}, \phi\}$

(ii) $\{\{a, c\}\} - A = \phi$

(iii) $A - \{\{a,b\}\} = A$

(iv) $\{a,c\} - A = \{c\}$

Ex. 4. If $U = \{n / n \in N, n \leq 15\}$

$A = \{n / n \in N, 4 < n < 12\}$

$B = \{n / n \in N, 8 < n < 15\}$

$C = \{n / n \in N, 5 < n < 10\}$

Find $\bar{A} - \bar{B}$ and $\bar{C} - \bar{A}$

Soln : $\bar{A} - \bar{B} = \{12, 13, 14\}$

$\bar{C} - \bar{A} = \{5, 10, 11\}$

Ex: 5 Let A, B, C be subsets of the universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$.
Is it necessary that $B = C$?
Justify your answer.

Soln : Yes. It is necessary that $B = C$

We can write B as -

$$B = B \cap U = B \cap (A \cup \bar{A})$$

$$= (B \cap A) \cup (B \cap \bar{A})$$

— Distributive law

$= (A \cap B) \cup (\bar{A} \cap B)$ — commutative law

$= (A \cap C) \cup (\bar{A} \cap C)$ — Given condⁿ

$= (A \cup \bar{A}) \cap C$ — Distri. law

$= U \cap C$

$= C.$

Ex: 6. (i) Given that $A \cup B = A \cup C$,
Is it necessary that $B = C$?

(ii) Given that $A \cap B = A \cap C$
Is it necessary that $B = C$?

<u>Solⁿ</u>: (i) No, Let $A = \{a, b, c\}$
$B = \{b\}$
$C = \{c\}$

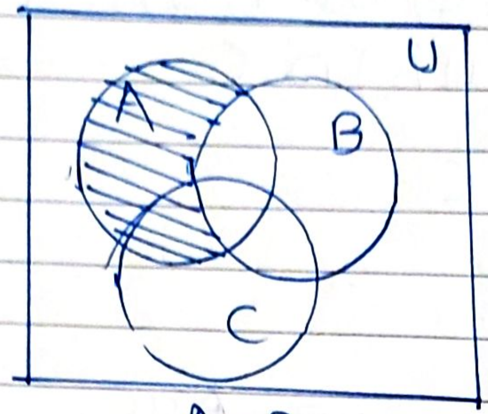
$A \cup B = \{a, b, c\} = A \cup C$
But $B \neq C$.

(ii) No, Let ~~$A = \{a, b, c, d\}$~~
~~$B = \{a, b, c\}$~~
~~$C = \{b, c\}$~~

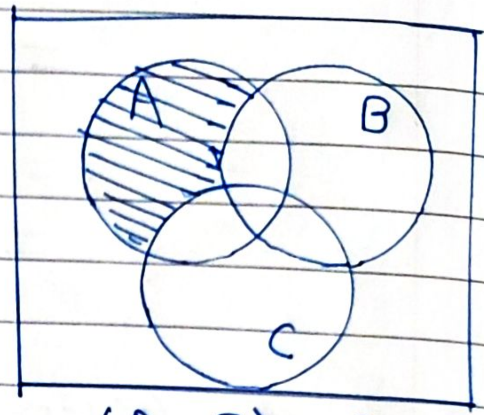
~~$A \cap B = \{b, c\} = A \cap C$~~
~~But $B \neq C$.~~

Ex: 7. Show that $(A-B)-C = A-(B\cup C)$ using venn diagram.

Soln :

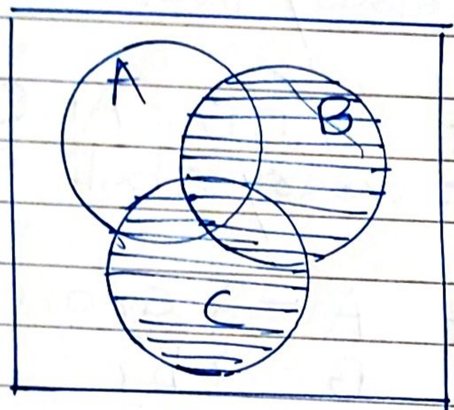


$A-B$

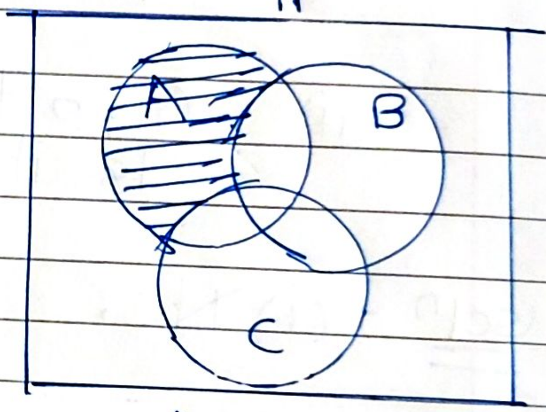


$(A-B)-C$

$\|$



$B\cup C$



$A-(B\cup C)$

Ex: 8. Using Venn diagram,
Prove that:

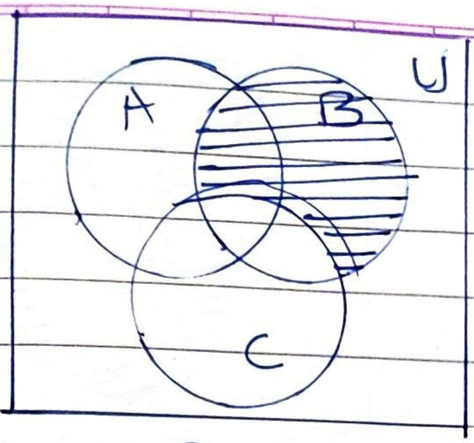
$$A-(B-C) = (A-B)\cup(A\cap B\cap C)$$

Ex: 9 Using Venn diagram, prove or disprove.

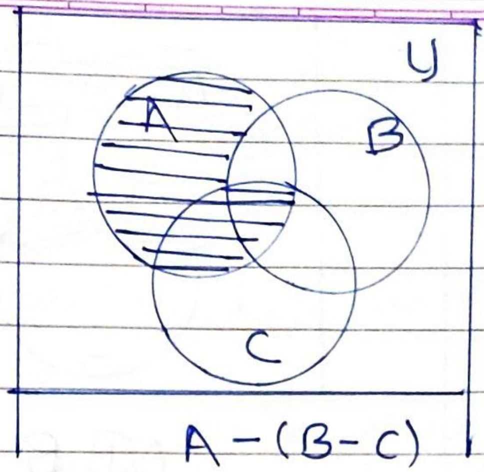
(i) $A\oplus(B\oplus C) = (A\oplus B)\oplus C$

(ii) $A\cap B\cap C = A-[(A-B)\cup(A-C)]$

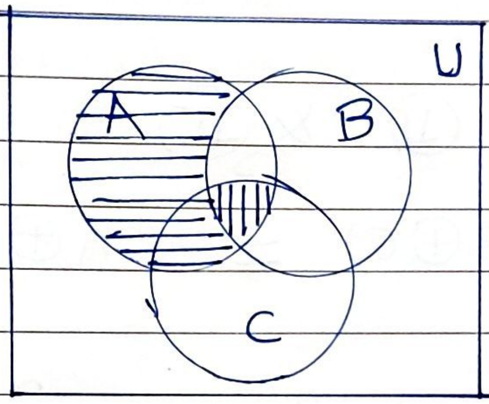
Ex: 8



B − C —— I

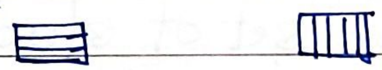


A − (B − C) —— II



—— III

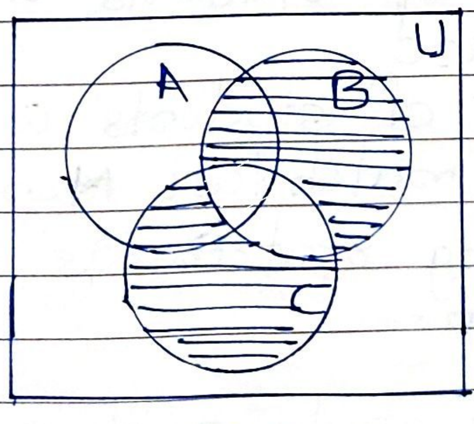
(A − B) ∪ (A ∩ B ∩ C)



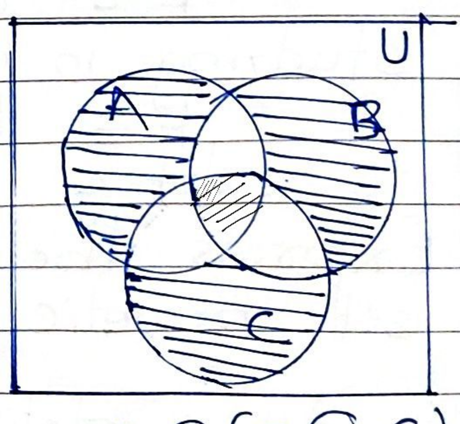
From II & III

$A - (B - C) = (A - B) \cup (A \cap B \cap C)$

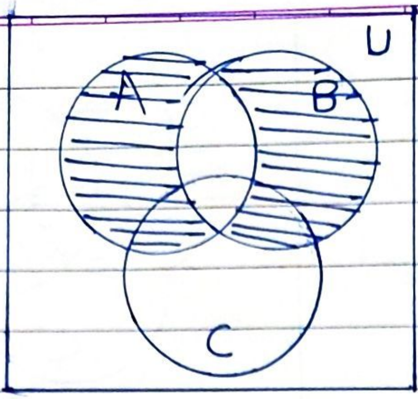
Ex: 9



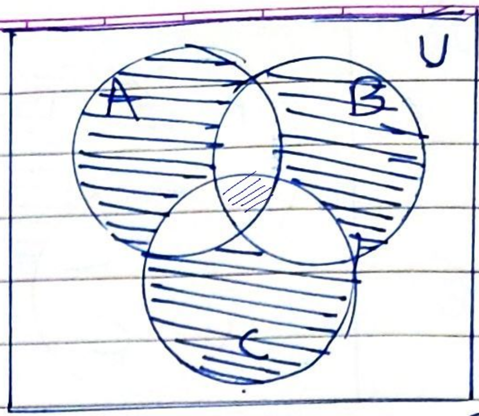
B ⊕ C —— I



A ⊕ (B ⊕ C) —— II



$A \oplus B$ — III



$(A \oplus B) \oplus C$ — IV

From II & IV

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

Ex: 10. Let A denote the set of students who study Data structures (DS)

$B \equiv$ the set of students who study Discrete Mathematics (DM)

$C \equiv$ the set of students who study Assembly Language programming.

$D \equiv$ the set of students who study computer science.

$E \equiv$ the set of students who are studying in hostel and

$F \equiv$ the set of students who went to watch a cricket match last Monday.

Express the following statements in set theoretic notation.

(i) All hostelites who study neither DS nor DM ~~wh~~ went to watch cricket match last Monday.

Ex: 10. Let A denote the set of students who study Data structures (DS)

$B \equiv$ the set of students who study Discrete Mathematics (DM)

$C \equiv$ the set of students who study Assembly Language programming.

$D \equiv$ the set of students who study computer science.

$E \equiv$ the set of students who are studying in hostel and

$F \equiv$ the set of students who went to watch a cricket match last Monday.

Express the following statements in set theoretic notation.

(i) All hostelites who study neither DS nor DM ~~who~~ went to watch cricket match last Monday.

(ii) The students who went to see cricket match are only those who study Assembly Language programming or DS

(iii) No student who is studying DS went to see cricket match.

(iv) Those and only those students who are studying Theory of CS and DM went for a cricket match.

(v) All went to see cricket match.

Soln: (i) $E \cap \overline{A} \cap \overline{B} \subseteq F$

(ii) $F \subseteq C \cup A$

(iii) $A \cap F = \phi$

(iv) $F \subseteq B \cap D$

(v) If $U = A \cup B \cup C \cup D \cup E$

then $A \cup B \cup C \cup D \cup E = F$

Otherwise $A \cup B \cup C \cup D \cup E \subseteq F$

Ex: 11. Let A denote the set of all automobiles that are manufactured domistically. Let B denote the set of all imported automobiles. Let C denote the set of all automobiles manufactured before 1977. Let D denote the set of all automobiles with a current value less than 2000\$. Let E denote the set of all automobiles owned by students at the university.

Express the following in theoretic notation

(i) The automobiles owned by the students at the university are either domestically manufactured or imported.

(ii) All domestic automobiles manufactured before 1977 have a market value of less than 2000 \$

(iii) All imported automobiles manufactured after 1977 have a market value more than 2000\$

soln: (i) $E \subseteq A \cup B$

(ii) $A \cap C \subseteq D$

(iii) $\overline{C} \cap B \subseteq \overline{D}$

Here $U = A \cup B$

$\equiv$ set of all automobiles manufactured domestically or imported automobiles.

Ex: 12 Tony, Mike and John belong to the Alpine club. Every club member is either a skier or mountain climber or both. No mountain climber likes rain and all skiers like snow. Mike dislikes whatever Tony likes and likes whatever Tony dislikes. Tony likes rain and snow. Is there a member of the Alpine club who is mountain climber but not skier?

Soln : Let $A \equiv$ the set of all members of the Alpine club.

Let $S \equiv$ the set of all skiers

$M \equiv$ the set of all mountain climbers

Then $A \subset S \cup M$

If $x \in M$, then x dislike rain

& if $y \in S$, then y likes snow.

Since Tony likes rain and snow.

$\Rightarrow$ Tony is skier $\Rightarrow$ Tony $\in S$.

& By given information,

Mike dislike rain and snow.

$\Rightarrow$ Mike is mountain climber but not skier.

$\Rightarrow$ Mike $\in M$

Ex. 13 Consider the following assumptions.

S_1 : Poets are happy people

S_2 : Every doctor is wealthy

S_3 : No one who is happy is also wealthy.

Determine the validity of the following arguments, using venn diagram.

(i) No poet is wealthy

(ii) Doctors are happy people

(iii) No one can be both a poet and a doctor.

Soln : Let H ~~be~~ the set of happy people

P be the set of poets.

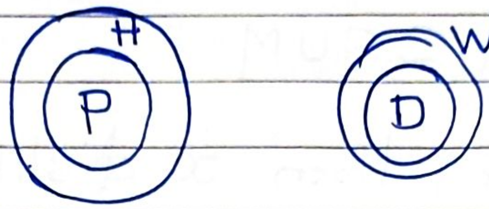
W be the set of wealthy people

D be the set of doctors.

By S_1, $P \subseteq H$

S_2, $D \subseteq W$

S_3, $H \cap W = \phi$



From Venn Dia; $\underline{P \cap W = \phi}$

i.e no poet is wealthy — I

$\therefore$ Argument (i) is valid

~~Also~~

Now, $D \cap H = \phi$

i.e Doctors are not happy people

$\therefore$ Argument (ii) is invalid

As $P \cap D = \phi$

Hence no one can be both a poet and a doctor.

Hence argument (iii) is valid

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Cardinality of finite set :

A very important problem in DM is that of determining the number of objects in a finite set.

1. In computer algorithms, one is often required to count the number of operations executed by various algorithms.
This is necessary to estimate the cost effectiveness of a particular algorithm

2. In the study of data structures of files, determining average and maximum lengths of searches for items stored in a data structure also involve counting.

Defn // Let A be a finite set. The cardinality of A, denoted by $|A|$ is the number of elements in the set.

If $A = \phi$ then $|A| = 0$

If $A \subseteq B$, where B is finite set then $|A| \leq |B|$

Thm : $\boxed{\text{The Addition Principle}}$:

Let A and B be finite sets which are disjoint. Let $|A| = n$, $|B| = m$

Then $|A \cup B| = |A| + |B|$
$= n + m$

proof : If A ~~and~~ or B is the empty set, then proof is trivial.

If say $A = \phi$ then we have $A \cup B = B$
$\Rightarrow |A| = 0$

$|A \cup B| = |B| = m = 0 + m.$

Similarly, if $B = \phi \Rightarrow |A \cup B| = |A| = n$
$= n + 0$

Hence, let us assume that $A \neq \phi$, $B \neq \phi$

Since A and B are finite disjoint sets,

Let $A = \{a_1, a_2, \dots a_n\}$, $|A| = n$

& $B = \{b_1, b_2, \dots b_m\}$, $|B| = m$.

where $a_i \neq b_j$, $1 \leq i \leq n$, $1 \leq j \leq m$

Then $A \cup B = \{a_1, a_2, \dots a_n, b_1, b_2, \dots b_m\}$

$\Rightarrow |A \cup B| = n + m = |A| + |B|$

Thus thm is proved.

<u>Note</u>: The above thm can be extended to a finite collection of mutually disjoint sets say $A_1, A_2, \dots A_n$.

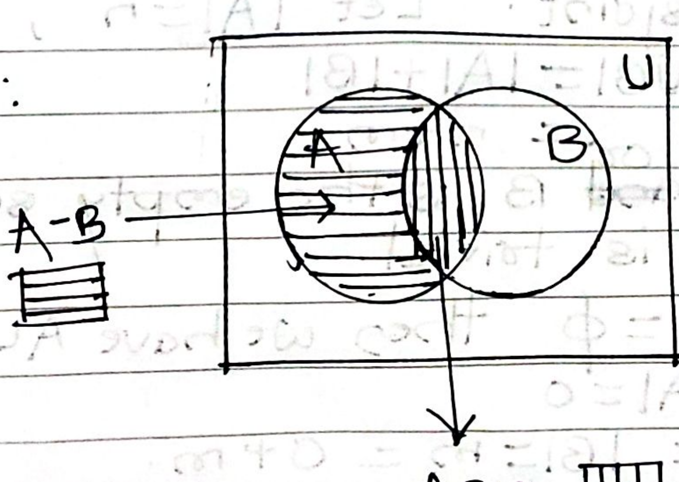
Then

$$|A_1 \cup A_2 \cup \dots A_n| = |A_1| + |A_2| + \dots + |A_n|$$

<u>Thm-2</u>: Let A be a finite set and let B be any set (not necessarily finite)

Then $|A - B| = |A| - |A \cap B|$

<u>proof</u>:



From Venn diagram, it is clear that

$$A = (A-B) \cup (A \cap B)$$

[$(A-B)$ & $(A \cap B)$ are mutually disjoint sets]

Hence by addition principle,

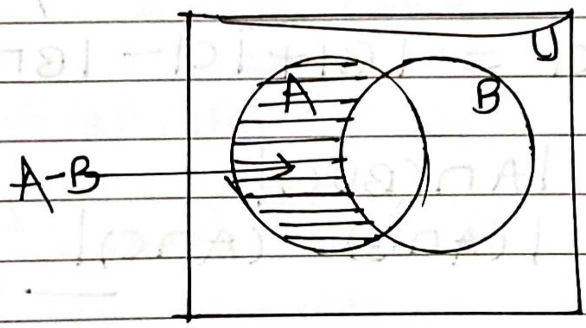
$$|A| = |A-B| + |A \cap B|$$

$$\therefore \quad |A-B| = |A| - |A \cap B|$$

— Hence proof

<u>Thm3:</u> Let A and B be finite sets.
Then $|A \cup B| = |A| + |B| - |A \cap B|$
[<u>Principle of inclusion - exclusion</u>]

<u>proof</u>:



We can write,

$$A \cup B = (A-B) \cup B$$

As $(A-B)$ and B are disjoint sets, by addition principle,

$$|A \cup B| = |A-B| + |B|$$

$$= |A| - |A \cap B| + |B|$$

— by thm 2

Hence $|A \cup B| = |A| + |B| - |A \cap B|$

Principle of inclusion-Exclusion for three sets:

Let A, B, C be finite sets then,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

proof: Let $B \cup C = D$

then $A \cup B \cup C = A \cup D$

Now,

$$|A \cup D| = |A| + |D| - |A \cap D| \quad \text{--- (1)}$$

— by thm 3

where,

$$|D| = |B \cup C| = |B| + |C| - |B \cap C|$$

$$\& \ |A \cap D| = |A \cap (B \cup C)|$$

$$= |(A \cap B) \cup (A \cap C)|$$

— by distr. law

$$= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$$

$$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

Hence eq^n (1) becomes -

$$|A \cup D| = |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |A \cap B \cap C|\}$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

— Hence proof

Hence.

$$|U-(A\cup B)| = |U|-|A\cup B|$$
$$= 2500-1600$$
$$= 900$$

Hence 900 people read neither.

Ex: 2. Among the integers 1 to 300. find how many are divisible by 3, nor by 5. Find also, how many are divisble by 3, but not by 7.

Soln: Let $A \equiv$ set of integers 1 to 300 which are divisible by 3

$B \equiv$ set of integers 1 to 300 which are divisible by 5

$C \equiv$ set of integers 1 to 300 which are divisible by 7.

To find: (i) $|\bar{A}\cap\bar{B}|$ (ii) $A-C$

By De Morgan's law;

$$(A\cup B)^c = A^c \cap B^c$$

Now,

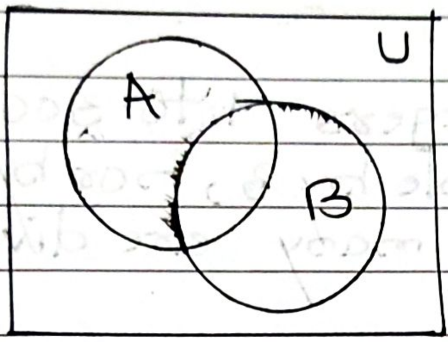
$$|\overline{A\cup B}| = |U| - |A\cup B|$$

As $|A\cup B| = |A|+|B|-|A\cap B|$

$$|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100$$

Ex: 1. In a survey, 2500 people were asked whether they read India Today or Business Times. It was found that 1200 read India Today, 1000 read Business Times and 600 read both.
Find how many read atleast one magazine and how many read neither?

soln:



Let A ≡ the set of people who read India Today
& B ≡ the set of people who read Business Times

Now,

$|U| = 2500$; U ≡ total number of People in a survey

$|A| = 1200$

$|B| = 1000$

$|A \cap B| = 600$

[At least ≡ union ; Both ≡ Intersection]

By the inclusion - exclusion principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 1200 + 1000 - 600$$
$$= 2200 - 600$$
$$= 1600$$

Hence, 1600 people read atleast one magazine.

[Neither ≡ $|U| - |A \cup B|$
≡ $|U|$ − Atleast]

Hence.

$$|U-(A\cup B)| = |U|-|A\cup B|$$
$$= 2500-1600$$
$$= 900$$

Hence 900 people read neither.

Ex: 2. Among the integers 1 to 300. find how many are divisible by 3, nor by 5. Find also, how many are divisible by 3, but not by 7.

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$C \equiv$ set of integers 1 to 300 which are divisible by 7.

To find: (i) $|\bar{A}\cap\bar{B}|$ (ii) $A-C$

By De Morgan's law;

$$(A\cup B)^c = A^c \cap B^c$$

Now,

$$|\overline{A\cup B}| = |U| - |A\cup B|$$

As $|A\cup B| = |A|+|B|-|A\cap B|$

$$|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100$$

$|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60$

$|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20$

Hence,

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 100 + 60 - 20$$
$$= 160 - 20$$
$$= 140$$

Hence, $|\overline{A \cup B}| = |U| - |A \cup B| = 300 - 140$
$= 160$

Hence, there are 160 integers between 1 to 300 which are not divisible by 3 nor by 5.

Now, $|A - C| = |A| - |A \cap C|$

$|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = \lfloor 14.28 \rfloor = 14$

Hence, $|A - C| = 100 - 14 = 86$

Hence, there are 86 integers which are divisible by 3, but not by 7.

Greatest integer not greater than x

$\Longrightarrow$ $\lfloor x \rfloor$ ______ is a function that gives the greatest integer less than or equal to a given number.

We will round off the given number to the nearest integer that is less than or equal to number itself.

Ex: $\lfloor 5.1 \rfloor = 5$, as $5 \leq 5.1 \leq 6$

$\lfloor -5.1 \rfloor = -6$, $-6 \leq -5.1 \leq -5$

$\lfloor 2.76 \rfloor = 2$

$\lfloor -0.5 \rfloor = -1$

$\lfloor -7.5 \rfloor = -8$

$\lfloor 1.84 \rfloor = 1$

Ex: 1. In a computer laboratory out of 6 computers :

(i) 2 have floating point arithmatic unit

(ii) 5 have magnetic disc memory.

(iii) 3 have graphics display.

(iv) 2 have both floating point arithmatic unit and magnetic disc memory.

(v) 3 have both magnetic disc memory and graphics display.

(vi) 1 has both floating point arithmatic unit and graphics display.

(vii) 1 has floating point arithmatic, magnetic disk memory and graphics display.

How many have atleast one specification ?

Soln: Let A = be the set of computers having floating point arithmatic

B = having magnetic disc memory

C = be the set of computers having graphics display.

Then, $|A|=2, |B|=5, |C|=3$

$|A\cap B|=2, |B\cap C|=3, |A\cap C|=1$

$|A\cap B\cap C|=1$

We have to determine $|A\cup B\cup C|$

So

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C| - |A\cap C|+|A\cap B\cap C|$$

$$= 2+5+3-2-3-1+1=5$$

Ex: 2. Among the integers 1 to 1000
(i) How many of them are not divisible by 3, nor by 5, nor by 7 ?
(ii) How many are not divisible by 5 and 7 but divisible by 3 ?

Soln: Let A, B, C denote respectively the set of integers from 1 to 1000 divisible by 3, by 5 and by 7 respectively.

Then $\overline{A} \cap \overline{B} \cap \overline{C}$ denote the set of integers not divisible by 3, nor by 5 nor by 7.

By De Morgan's law,

$$\overline{A} \cap \overline{B} \cap \overline{C} = \overline{(A \cup B \cup C)}$$

Hence,

$$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$$

Now,

$$|A| = \left\lfloor \frac{1000}{3} \right\rfloor = 333, \quad |B| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$|C| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$$

$$|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

$$|B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor = 28, \quad |A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor = 47$$

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{105} \right\rfloor = 9$$

Hence

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

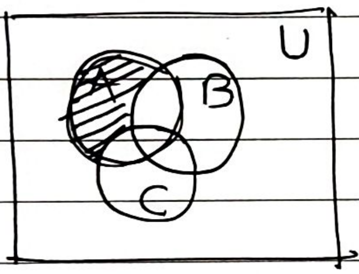
$= 333 + 200 + 142 - 66 - 28 - 47 + 9$

$= 543$

Hence,

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 1000 - 543 = 457$$

(ii)



Shaded portion denotes numbers which are not divisible by 5 and 7 but divisible by 3.

$= A \cap \bar{B} \cap C$

$= A - (B \cup C)$

Hence, $|A - (B \cup C)| = |A| - |(A \cap B) \cup (A \cap C)|$

$= |A| - \{|A \cap B| + |A \cap C| - |A \cap B \cap C|\}$

$= 333 - \{66 + 47 - 9\}$

$= 333 - 104$

$= 229$

Ex. 3 How many integers between 1–1000 are divisible by 2, 3, 5 or 7?

H.W

Ans: 772

Cartesian Product of sets :

Sometimes, we want a collection of ordered pairs.

Let's say we interested in finding all possible combinations of X and Y co-ordinates where X and Y should not exceed 3.

Say ~~A =~~ $X \times Y = \{(1,1), (1,2), (1,3),$
$(2,1), (2,2), (2,3)$
$(3,1), (3,2) (3,3)\}$

$(a,b) \in X \times Y$
↳ order pair.

Note: (a,b) and (b,a) are not equal. Hence in order pair (a,b), order matter.

Defⁿ: Let A and B are two sets. The cartesian product of A and B denoted by $A \times B$, is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Ex: 1 $A = \{1,2\}$, $B = \{a,b,c\}$

$$A \times B = \{(1,a), (1,b), (1,c)$$
$$(2,a), (2,b), (2,c)\}$$

Multiset : [Mset]

Multiset is generalization of a set.

We know, set is a collection of distinct objects. However, an object can occur more than once.

Ex: A ≡ the collection of books in a library can contain multiple copies of the same book.

B ≡ names of persons

C ≡ birth months of individuals.

Note: 1. Multiset is also called 'bag', 'heap', 'bunch', 'weighted set'

2. Brackets used to denote mset are square brackets.

Ex: mset $A = [a, b, a, b, c]$

underlying set for A is $A = \{a, b, c\}$

3. A multiset containing no elements is denoted by [ ] corresponding to the empty set ϕ.

Multiplicity of an element:

The multiplicity of an element in a mset is defined as the number of times the element appears in the mset.

Ex: $A = [1, 1, 1, 2, 2, 3]$

Multiplicity of 1 is 3
— " — of 2 is 2
— " — of 3 is 1
— " — of 4 is 0

Multiplicity function: The multiplicity function μ is defined as

$$\mu : A \longrightarrow \{1, 2, 3, \dots\}$$

where A is mset,

$$\mu(a) = k, \quad a \in A$$

& k is number of times the element occurs in the mset.

Ex: $A = [a, b, c, c, a, c]$

$\mu(a) = 2, \quad \mu(b) = 1, \quad \mu(c) = 3$

Equality of msets: If the number of occurrences of each element is the same in both the msets, then the msets are equal.

Ex: $[a, b, a, a] = [a, a, a, b]$

However, $[a, b, a] \neq [a, b]$

Multisubset (msubset): A mset 'A' is said to be msubset of mset B if multiplicity of each element in A is less than or equal to it's multiplicity in B.

Ex: $[1, 2, 2, 3] \subseteq [1, 1, 1, 2, 2, 3, 3]$

Union of msets:
If A and B are two msets, then $A \cup B$ is the mset such that for each element $x \in A \cup B$

$$\mu(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Ex: ① $A = [a, b, b, c]$, $B = [b, c, c, d]$
Then
$A \cup B = [a, b, b, c, c, d]$

② $A = [1, 2, 2, 4, 4, 5]$, $B = [1, 1, 2, 2, 2, 4, 5, 5]$

$A \cup B = [1, 1, 2, 2, 2, 4, 4, 5, 5]$

Intersection of msets:
If A and B are two msets then $A \cap B$ is defined as the mset such that for each element $x \in A \cap B$

$$\mu(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Ex: ① $A = [a, a, b, c, c]$, $B = [a, b, c]$

$A \cap B = [a, b, c]$

② $A = [1, 1, 2, 3, 3, 3]$, $B = [1, 2, 2, 3, 3]$

$A \cap B = [1, 2, 3, 3]$

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Difference of msets :

For multisets A and B, the difference A − B is an mset such that for each $x \in A - B$,

$\boxed{\mu(x) = \mu_A(x) - \mu_B(x)}$, if difference is greater than zero.

$\boxed{\mu(x) = 0}$, if the difference is zero or negative.

Note : 1. $A - A = [\ ]$

Ex: $A = [a, b, c, c, c]$, $B = [b, c, d, d]$

Then $A - B = [a, c, c]$

As, $\mu(a) = \mu_A(a) - \mu_B(a)$
$= 1 - 0 = 1$

$\mu(b) = \mu_A(b) - \mu_B(b)$
$= 1 - 1 = 0$

$\mu(c) = \mu_A(c) - \mu_B(c)$
$= 3 - 1 = 2$

Hence, $A - B = [a, c, c]$

Sum of msets :

This concept is not defined for ordinary sets.

However, for multisets A and B, we define A+B as follows.

for each element $x \in A+B$

$$\mu(x) = \mu_A(x) + \mu_B(x)$$

Ex: $A = [1, 1, 1, 2, 3]$, $B = [2, 3, 3, 3]$

$$A+B = [1, 1, 1, 2, 2, 3, 3, 3, 3]$$

$$A+A = [1, 1, 1, 1, 1, 1, 2, 2, 3, 3]$$

Note: $A+A \neq A$

[Idempotent law is not true for sum of msets]

H.W: Verify:

(i) $(A+B) \cup C = (A \cup C) + (B \cup C)$

(ii) $(A+B) \cap C = (A \cap C) + (B \cap C)$

(iii) $A \cup (B+C) = (A \cup B) + (A \cup C)$

(iv) $A \cap (B+C) = (A \cap B) + (A \cap C)$

Note: 1. The laws of associativity, commutativity, distributivity, absorption and idempotent are satisfied for unions and intersections.

2. The concept of symmetric difference, however, cannot be carried over to that of multisets.

Recall;

$$A \oplus B = A \Delta B = (A-B) \cup (B-A)$$
$$= (A \cup B) - (A \cap B)$$

Symmetric difference satisfies the associative law of sets.

For msets, this law is not valid.

Ex: $A = [2,2,3,3]$, $B = [1,1,2]$, $C = [3,3,2]$

Then if

$$A \oplus B = A \cup B - A \cap B$$
$$= [2,2,1,1,3,3] - [2]$$
$$= [1,1,2,3,3]$$

$$(A \oplus B) \oplus C = ([1,1,2,3,3] \cup [3,3,2]) - ([1,1,2,3,3] \cap [3,3,2])$$
$$= \cancel{[1,1,2,2,3,3,3,3}$$
$$= [1,1,2,3,3] - [2,3,3]$$
$$= [1,1] \quad \text{(1)}$$

On the other hand,

$$B \oplus C = (B \cup C) - (B \cap C)$$
$$= [1,1,2,3,3] - [2]$$
$$= [1,1,3,3]$$

$$A \oplus (B \oplus C) = ([2,2,3,3] \cup [1,1,3,3]) - ([2,2,3,3] \cap [1,1,3,3])$$

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$$= [2, 2, 1, 1, 3, 3] - [3, 3]$$

$$= [2, 2, 1, 1]$$ ——— (2)

From (1) & (2)

$$(A \oplus B) \oplus C \neq A \oplus (B \oplus C)$$

Countable set :

A set is said to be countable if it is either finite or countably infinite

Ex: $A = \{1, 4, 6\}$

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ etc.

Uncountable sets :

A set is said to be uncountable if it is not countable.

i.e neither finite nor countably finite

Ex: $P(\mathbb{N})$, $\mathbb{R}$, $\mathbb{Q}^c$ etc.

Note that : $[a, b]$ is uncountable so it's super set $\mathbb{R}$ is also uncountable.

Introduction to Fuzzy Set:

Is an apple RED?

CRISP SET	FUZZY SET
1. It defines either value is 0 or 1	1. It defines value between 0 and 1 including 0 & 1
2. It is also called classical set	2. It specifies the degree to which something is true.
3. It shows full membership Yes or NO True or False 1 or 0	3. It shows partial membership Yes ------> NO True ----> False 1 ----> 0
4. Ex: ① She is 18 years old ② Rahul is 1.6m tall	4. Ex: ① She is about 18 years old ② Rahul is about 1.6m tall.
5. An apple is red.	5. An apple is perfectly red or slightly red, no red etc.

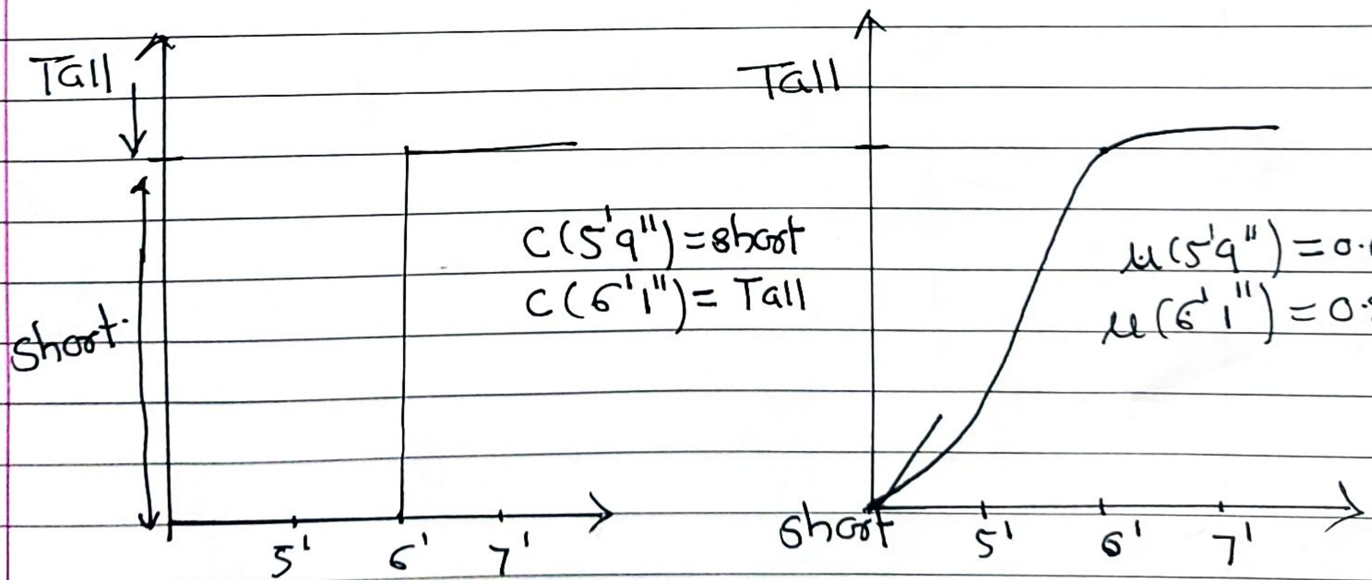
Defⁿ//. The membership function $\mu_A(x)$ of a fuzzy set A is a function $\mu_A : X \longrightarrow [0,1]$

So every element $x \in X$ has membership degree : $\mu_A(x) \in [0,1]$

A is completely determined by the set of tuples : $A = \{ (x, \mu_A(x)) \mid x \in X \}$

// $\mu_A(\text{apple is red}) = 0.01 \approx$ False (no red)

$\mu_A(\text{apple is red}) = 0.99 \approx$ True (perfectly red)



Example: Suppose someone wants to describe the class of cars having the property of being expensive by considering BMW, Rolls Royce, Mercedes, Ferrari, Fiat, Honda and Renault. Some cars like Ferrari and Rolls Royce are definitely expensive and some like Fiat and Renault are not expensive in comparison and do not belong to the set. Using a fuzzy set, the fuzzy set of expensive cars can be described as:

{(Ferrari, 1), (Rolls Royce, 1), (Mercedes, 0.8), (BMW, 0.7), (Honda,0.4)}. Obviously, Ferrari and Rolls Royce have membership value of 1 whereas BMW, which is less expensive, has a Membership value of 0.7 and Honda 0.4.

Note: // 1. Membership function is a function that specifies the degree to which a given input belongs to a set.

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Degree of membership: A value between 0 and 1 (including both) represents the degree of membership, also called membership value of element x in set A. It is the output of membership function.

* Membership function can be defined as a technique to solve practical problem by experience rather by knowledge (degree of truth).

Note: Membership function are used in the fuzzification and defuzzification of a fuzzy Logic system.

Properties of Fuzzy sets :

Fuzzy sets follow the same properties as crisp sets. Since membership values of crisp sets are a subset of the interval [0, 1]; classical sets can be thought of as generalization of fuzzy sets.

1. Commutativity : $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$
   $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$

2. Associativity : $\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$
   $\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

3. Distributivity : $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$

4. Idempotency : $\tilde{A} \cup \tilde{A} = \tilde{A}$
$\tilde{A} \cap \tilde{A} = \tilde{A}$

5. Identity : $\tilde{A} \cup \phi = \tilde{A}$, $\tilde{A} \cap X = \tilde{A}$
$\tilde{A} \cap \phi = \phi$, $\tilde{A} \cup X = X$

6. Transitivity : $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$. then $\tilde{A} \subseteq \tilde{C}$

7. Involution : $\bar{\bar{\tilde{A}}} = \tilde{A}$

Operations on Fuzzy sets :

1. Equal fuzzy sets : Two sets $A(x)$ and $B(x)$ are said to be equal if $\boxed{\mu_A(x) = \mu_B(x)}$, $\forall x \in X$

It is expressed as follows :
$A(x) = B(x)$, if $\mu_A(x) = \mu_B(x)$

Note : Two fuzzy sets $A(x)$ and $B(x)$ are said to be unequal if $\mu_A(x) \neq \mu_B(x)$, for atleast one $x \in X$

Ex: 1 $A(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

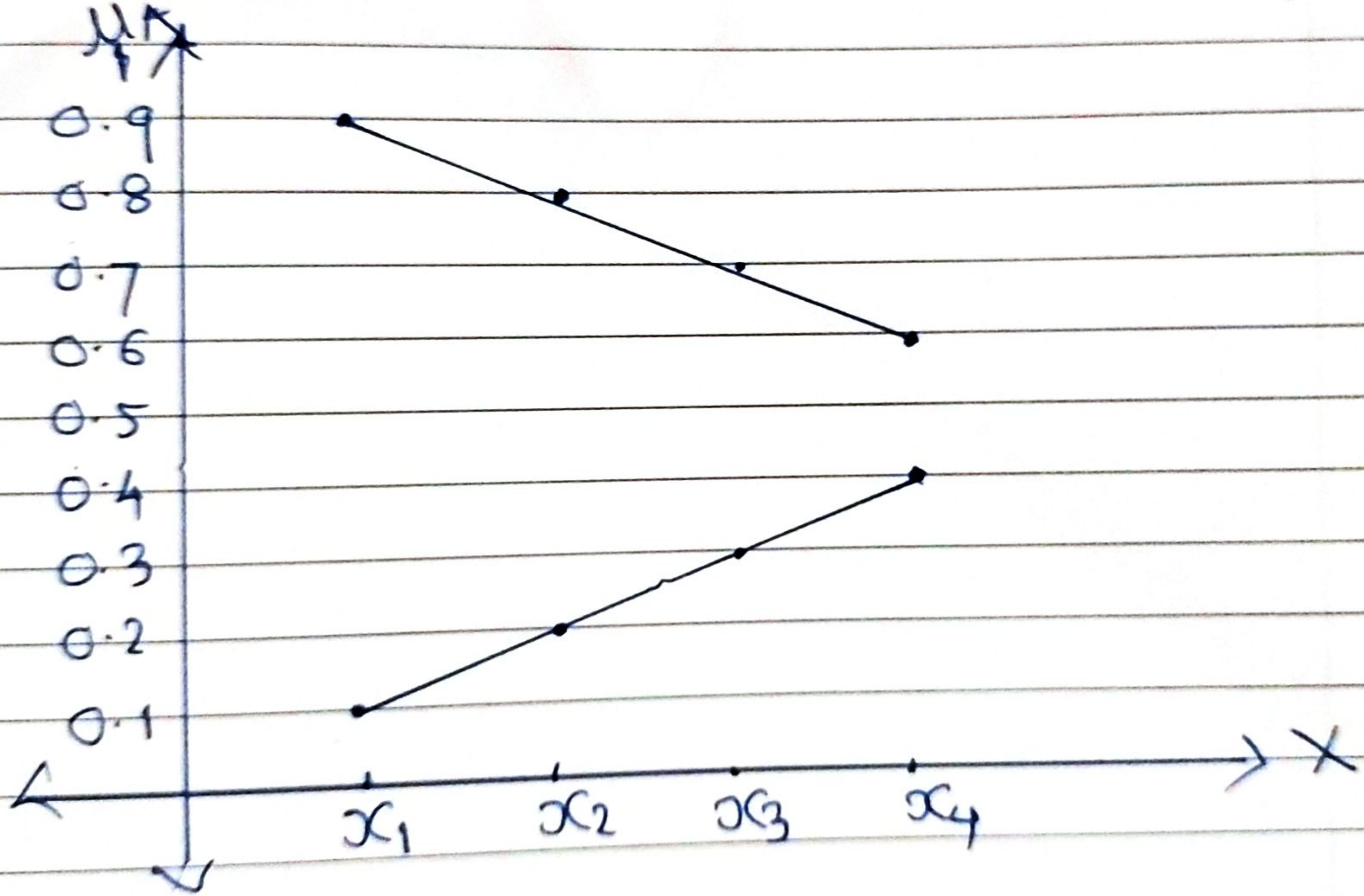
$B(x) = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.3), (x_4, 0.6)\}$

As $\mu_A(x) \neq \mu_B(x)$, for different $x \in X$, Hence $A(x) \neq B(x)$.

2. Complement of fuzzy set $A(x)$:

The complement is the opposite of the set. The complement of a fuzzy set is denoted by $\bar{A}(x)$ and is defined ~~as~~ with respect to the universal set X as follows:

$$\bar{A}(x) = 1 - A(x), \quad \forall\, x \in X$$



- Find complement of fuzzy set A ?

$A = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

$\bar{A} = \{(x_1, 0.9), (x_2, 0.8), (x_3, 0.7), (x_4, 0.6)\}$

3. Intersection of fuzzy sets:

Intersection of a fuzzy sets define how much of the ~~complement~~ element belongs to both sets.

Let A(x) and B(x) are two fuzzy, the intersection is denoted by $(A \cap B)(x)$ and the membership function value is given as follows.

$$\mu_{(A \cap B)}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Note: Intersection is analogous to logical AND operation.

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Ex: $A(x) = \{ (x_1, 0.7), (x_2, 0.3), (x_3, 0.9), (x_4, 0.1) \}$

$B(x) = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 0.4) \}$

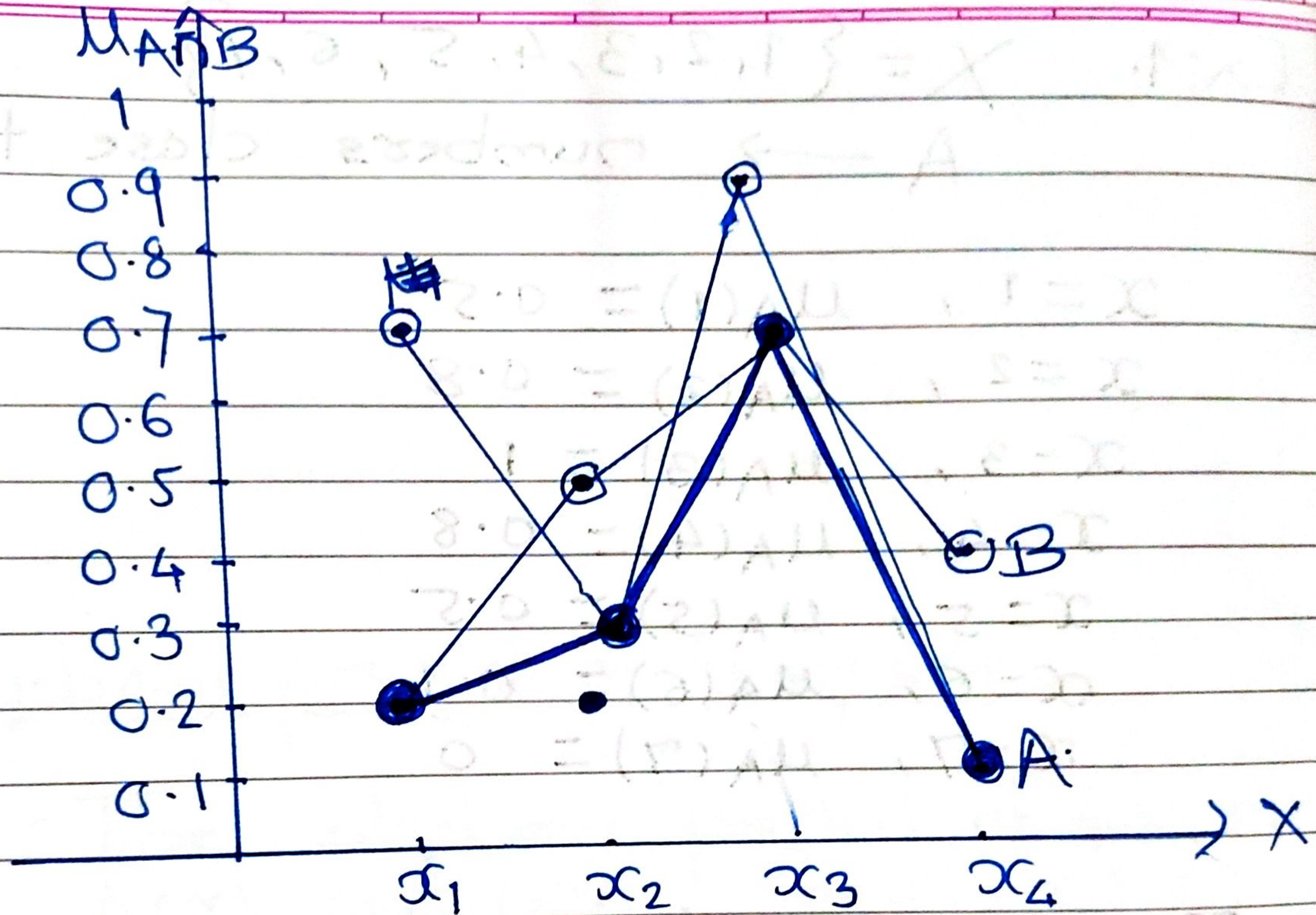
$$\mu_{(A \cap B)}(x_1) = \min\{\mu_A(x_1), \mu_B(x_1)\}$$
$$= \min\{0.7, 0.2\}$$
$$= 0.2$$

$$\mu_{(A \cap B)}(x_2) = \min\{\mu_A(x_2), \mu_B(x_2)\}$$
$$= \min\{0.3, 0.5\}$$
$$= 0.3$$

$$\mu_{(A \cap B)}(x_3) = \min\{\mu_A(x_3), \mu_B(x_3)\}$$
$$= \min\{0.9, 0.7\}$$
$$= 0.7$$

$$\mu_{(A \cap B)}(x_4) = \min\{\mu_A(x_4), \mu_B(x_4)\}$$
$$= \min\{0.1, 0.4\}$$
$$= 0.1$$

The graphical representation of the intersection operator is given below.



Intersection of A & B

4. **Union of fuzzy sets:**

Union of fuzzy sets consists of every element that falls into either set. The value of the membership value is the largest membership value of the element in either set.

Let $A(x)$ and $B(x)$ are two fuzzy sets for all $x \in X$. Union of fuzzy sets is denoted by $(A \cup B)(x)$ and membership ~~value~~ function value is determined as follows.

$$\mu_{(A \cup B)}(x) = \max \{ \mu_A(x), \mu_B(x) \}$$

Ex: $A(x) = \{ (x_1, 0.7), (x_2, 0.3), (x_3, 0.9), (x_4, 0.1) \}$

$B(x) = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 0.4) \}$

$$\mu_{(A\cup B)}(x_1) = \max\{\mu_A(x_1), \mu_B(x_1)\}$$
$$= \max\{0.7, 0.2\}$$
$$= 0.7$$

$$\mu_{(A\cup B)}(x_2) = \max\{\mu_A(x_2), \mu_B(x_2)\}$$
$$= \max\{0.3, 0.5\}$$
$$= 0.5$$

$$\mu_{(A\cup B)}(x_3) = \max\{\mu_A(x_3), \mu_B(x_3)\}$$
$$= \max\{0.9, 0.7\}$$
$$= 0.9$$

$$\mu_{(A\cup B)}(x_4) = \max\{\mu_A(x_4), \mu_B(x_4)\}$$
$$= \max\{0.1, 0.4\}$$
$$= 0.4.$$

Note : Union is analogous to logical 'OR' operation.

5. <u>Algebraic product of fuzzy sets:</u>

The algebraic product of two fuzzy sets $A(x)$ and $B(x)$, for all $x \in X$, is denoted by $A(x) \cdot B(x)$ and defined as follows -

$$A(x) \cdot B(x) = \{ (x, \mu_A(x) \cdot \mu_B(x)), x \in X \}$$

Ex: 1.

$$A(x) = \{(x_1, 0.7), (x_2, 0.3), (x_3, 0.9), (x_4, 0.1)\}$$

$$B(x) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 0.4)\}$$

$$\mu_{(A(x) \cdot B(x))}(x_1) = \mu_A(x_1) \cdot \mu_B(x_1)$$
$$= 0.7 \times 0.2 = 0.14$$

$$\mu_{A(x) \cdot B(x)}(x_2) = 0.3 \times 0.5 = 0.15$$

$$\mu_{A(x) \cdot B(x)}(x_3) = 0.9 \times 0.7 = 0.63$$

$$\mu_{A(x) \cdot B(x)}(x_4) = 0.1 \times 0.4 = 0.04$$

6. <u>Multiplication of fuzzy sets by a crisp number</u>

The product of fuzzy set $A(x)$ and a crisp number 'd' is expressed as follows

$$A(x) \cdot B(x) = \{ (x, d \cdot \mu_A(x)) / x \in X \}$$

Ex: Let us consider a fuzzy set $A(x)$ such that

$$A(x) = \{ (x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4) \}$$

$$d = 0.2$$

then

$$d \cdot A(x) = \{ (x_1, 0.02), (x_2, 0.04), (x_3, 0.06), (x_4, 0.08) \}$$

7. Power of a fuzzy set:

The p^{th} power of a fuzzy set $A(x)$ yields another fuzzy set $A^P(x)$, whose membership value can be determined as follows.

$$\mu_{A^P(x)} = \{ (\mu_A(x))^P, x \in X \}$$

If $P \geq 1 \longrightarrow A^P(x)$ is called <u>concentration</u>

$P < 1 \longrightarrow A^P(x)$ is called <u>dilation</u>

Ex: Let us consider a fuzzy set $A(x)$

$$A(x) = \{ (x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4) \}$$

$P = 2$

Then $A^2(x) = \{ (x_1, 0.01), (x_2, 0.04), (x_3, 0.09), (x_4, 0.16) \}$

8. Algebraic sum of two fuzzy sets:

The algebraic sum of two fuzzy sets $A(x)$ and $B(x)$ for all $x \in X$, is denoted by $A(x) + B(x)$ and defined as follows:

$$A(x) + B(x) = \{ (x, \mu_{A+B}(x) / x \in X \}$$

where,

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Ex: $A(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

$B(x) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.8), (x_4, 0.9)\}$

Now,

$$A(x) + B(x) = \{(x_1, 0.6 - 0.05), (x_2, 0.9 - 0.14), (x_3, 1.1 - 0.24) (x_4, 1.3 - 0.36)\}$$

$$= \{(x_1, 0.55), (x_2, 0.76), (x_3, 0.86), (x_4, 0.94)\}$$

9. Bounded sum of two fuzzy sets:

The bounded sum of two fuzzy sets $A(x)$ and $B(x)$ for all $x \in X$ is denoted by $A(x) \oplus B(x)$ and defined as follows.

$$A(x) \oplus B(x) = \{(x, \mu_{A \oplus B}(x)) / x \in X\}$$

where,

$$\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Ex: $A(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

$B(x) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.8), (x_4, 0.9)\}$

$$A(x) \oplus B(x) = \{(x_1, 0.6), (x_2, 0.9), (x_3, 1.0), (x_4, 1.0)\}$$

$$\begin{aligned} \mu_{A \oplus B}(x_1) &= \min\{1, \mu_A(x_1) + \mu_B(x_1)\} \\ &= \min\{1, 0.1 + 0.5\} \\ &= \min\{1, 0.6\} \\ &= 0.6 \end{aligned}$$

||'ly find other values.

10. **Algebraic difference of two fuzzy sets:**

The algebraic difference of two fuzzy sets $A(x)$ and $B(x)$, for all $x \in X$, is denoted by $A(x) + B(x)$ and defined as follows:

$$A(x) - B(x) = \{(x, \mu_{A-B}(x) / x \in X\}$$

where $\mu_{A-B}(x) = \mu_{A \cap \bar{B}}(x)$.

$$A(x) - B(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.2), (x_4, 0.1)\}$$

11. Bounded sum of two fuzzy sets:

The bounded difference of two fuzzy sets $A(x)$ and $B(x)$, $\forall\ x \in X$, is denoted by $A(x) - B(x)$ and defined as follows:

$$A(x) \ominus B(x) = \{(x, \mu_{(A \ominus B)}(x) / x \in X\}$$

where,

$$\mu_{(A \ominus B)}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Ex: $A(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

$B(x) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.8), (x_4, 0.9)\}$

$$A(x) \ominus B(x) = \{(x_1, 0), (x_2, 0), (x_3, 0.1), (x_4, 0.3)\}$$

12. Cartesian product of two fuzzy sets:

Let us consider two fuzzy sets $A(x)$ and $B(y)$ defined on the universal sets X and Y respectively.
The cartesian product of fuzzy sets $A(x)$ and $B(y)$, is denoted by $A(x) \times B(y)$ such that $x \in X$, $y \in Y$.

It is determined so that the following conditions satisfy

Date:

$$\mu_{(A\times B)}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$$

Ex: $A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$

$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$\min\{\mu_A(x_1), \mu_B(y_1)\} = \min\{0.2, 0.8\} = 0.2$

$\min\{\mu_A(x_1), \mu_B(y_2)\} = \min\{0.2, 0.6\} = 0.2$

$\min\{\mu_A(x_1), \mu_B(y_3)\} = \min\{0.2, 0.3\} = 0.2$

$\min\{\mu_A(x_2), \mu_B(y_1)\} = \min\{0.3, 0.8\} = 0.3$

$\min\{\mu_A(x_2), \mu_B(y_2)\} = \min\{0.3, 0.6\} = 0.3$

$\min\{\mu_A(x_2), \mu_B(y_3)\} = \min\{0.3, 0.3\} = 0.3$

$\min\{\mu_A(x_3), \mu_B(y_1)\} = \min\{0.5, 0.8\} = 0.5$

$\min\{\mu_A(x_3), \mu_B(y_2)\} = \min\{0.5, 0.6\} = 0.5$

$\min\{\mu_A(x_3), \mu_B(y_3)\} = \min\{0.5, 0.3\} = 0.3$

$\min\{\mu_A(x_4), \mu_B(y_1)\} = \min\{0.6, 0.8\} = 0.6$

$\min\{\mu_A(x_4), \mu_B(y_2)\} = \min\{0.6, 0.6\} = 0.6$

$\min\{\mu_A(x_4), \mu_B(y_3)\} = \min\{0.6, 0.3\} = 0.3$

$$A \times B = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix}$$

Relation :

Let A and B be two nonempty sets. A relation from A to B is any subset of $A \times B$.

It is denoted by $R: A \longrightarrow B$

e.g. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ then

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$$

$$R_1 = \{(a,1), (b,2), (b,3), (c,3)\}$$

$$R_2 = \{(a,3), (c,3)\}$$

$$R_3 = \{\phi\}.$$

are relations from A to B.

$R_4 = \{(1,b), (2,b)\}$ is not relation from A to B. whereas R_4 is the relation from B to A.

Note : 1. If $(a,b) \in R$ then it is denoted as aRb

2. If R is a relation from A to B then $R \subseteq A \times B$

3. If $R \subseteq A \times A$, then R is relation from A to A and R is called a Binary relation on A.

4. If R is a relation from A to B, then the set of all first elements of the ordered pairs $(a,b) \in R$ is called the domain of R

$$D(R) = \{ a \,/\, (a,b) \in R \}$$

& $Range(R) = R(R) = \{ b \,/\, (a,b) \in R \}$

5. The null set is the subset of $A \times B$
$\therefore$ ϕ is a relation called null relation or empty relation.

Ex: Let $A = \{1,2,3\}$, $B = \{x,y\}$

$R_1 = \{(1,x), (1,y), (3,y)\}$

$\therefore D(R_1) = \{1,3\}$, $R(R_1) = \{x,y\}$

Def^n// Let $\{A_1, A_2, \cdots A_n\}$ be a finite collection of sets.

A subset R of $A_1 \times A_2 \times \cdots \times A_n$ is called n-ary relation on $A_1, A_2, \cdots A_n$

If $R = \phi$, then R is called void or empty relation

If $R = A_1 \times A_2 \times \cdots A_n$, then R is called the universal relation.

If $n = 1, 2, 3$, then R is called a unary, binary, or ternary relation respecti

Ex: 1 Let $\mathbb{Z}$ be the set of all integers. Then

$$R = \{ \cancel{x}\ x \in \mathbb{Z} / x \text{ is even} \}$$

2. Let $A = \{1, 2, 3\}$ and let R be the relation characterised by the property "$x+y$ is less than z"
Then $R = \{(1,1,3)\}$, which is ternary.

$$A^3 = \{(1,1,1), (2,2,2), (3,3,3), (1,1,2), (1,2,1), (2,1,1), \underline{(1,1,3)} \ldots \}$$

$$|A^3| = 3^3 = 27$$

3. Let $A = \{2, 3, 4\}$ and let R be the relation characterized by the property "$x+y$ is divisible by z"
Then
$$R = \{\underline{(2,2,2)}, (2,2,4), (2,4,2), (2,4,3), (3,3,2), \underline{(3,3,3)}, (4,2,2), (4,2,3), (4,4,2), \underline{(4,4,4)}\}$$
which is ternary.

Complement of a relation ($\bar{R}$)

A relation as a set has it's complement, which is defined as -

$$\bar{R} = \{(a,b) / (a,b) \notin R\}$$

i.e $a\bar{R}b \iff a \not{R} b$

Ex: 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$
Let $R = \{(1,a), (1,b), (2,c), (3,a), (4,b)\}$

$S = \{(1,b), (1,c), (2,a), (3,b), (4,b)\}$

Find (i) $\overline{R}$ and $\overline{S}$

(ii) Verify De Morgan's law for R and S.

Soln :(i)

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (4,a), (4,b), (4,c)\}$

$\overline{R} = \{(1,c), (2,a), (2,b), (3,b), (3,c), (4,a), (4,c)\}$

$\overline{S} = \{(1,a), (2,b), (2,c), (3,a), (3,c), (4,a), (4,c)\}$

(ii) De Morgan's law states that:

ⓐ $\overline{R \cup S} = \overline{R} \cap \overline{S}$ ⓑ $\overline{(R \cap S)} = \overline{R} \cup \overline{S}$

ⓐ $R \cup S = \{(1,a), (1,b), (1,c), (2,a), (2,c), (3,a), (3,b), (4,b)\}$

$\overline{R \cup S} = \{(2,b), (3,c), (4,a), (4,c)\}$ — ①

$\overline{R} \cap \overline{S} = \{(2,b), (3,c), (4,a), (4,c)\}$ — ②

From ① & ②

$\overline{R \cup S} = \overline{R} \cap \overline{S}$

ⓑ $R \cap S = \{(1,b), (4,b)\}$

$\overline{R \cap S} = \{(1,a), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (4,a), (4,c)\}$ — ③

$\overline{R} \cup \overline{S} = \{(1,a),(1,c),(2,a),(2,b),(2,c),$
$(3,a),(3,b),(3,c),(4,a),(4,c)\}$ — ④

Hence $\overline{R \cap S} = \overline{R} \cup \overline{S}$

Converse of a relation:

Given a relation from A to B, we can find a converse relation from B to A.

Let R be a relation from A to B. Then converse of R, denoted by R^C. is the relation from B to A, defined as

$$R^C = \{(b,a) / (a,b) \in R\}$$

clearly, $R^C \subseteq B \times A$

Ex: Let $A = \mathbb{N}$ = set of all natural numbers.

If relation $R = <$ (less than relation)
then $R^C = >$ (greater than relation)

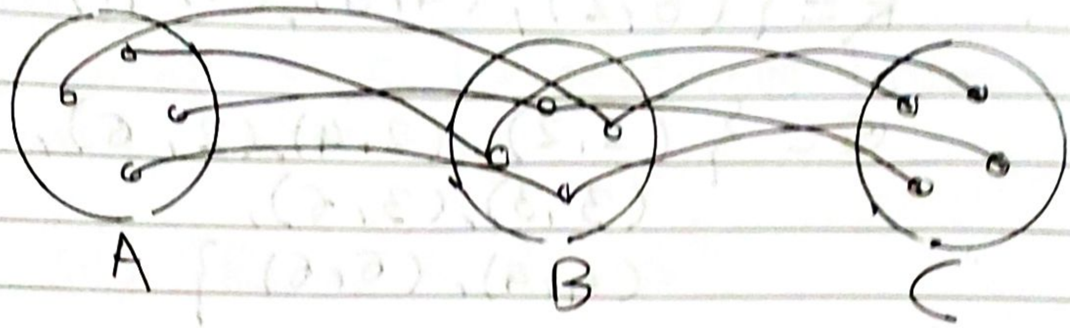
Note: The converse relation is also called as the inverse relation and is denoted by R^{-1}

Composition of Binary Relations :

The concept of composite relations plays an important role in execution of programs, where a sequence of data conversions takes place, from decimal to binary and from binary to floating point.

Def^n // Let R_1 be a relation from A to B and R_2 be a relation from B to C.
The composite relation from A to C, is denoted by $R_1 \circ R_2$ (or $R_1 R_2$) and defined as,

$$R_1 \circ R_2 = \{ (a, c) \mid a \in A \text{ and } c \in C \text{ and } \exists\, b \in B \text{ such that } (a, b) \in R_1 \text{ and } (b, c) \in R_2 \}$$



Note that: ① If R_1 is relation from A to B and R_2 is relation from C to D

Then $R_1 \circ R_2$ is not defined unless $B = C$

② In general if $\{A_1, A_2, \cdots A_{n+1}\}$ is finite collection of sets

~~wh~~ And $R_1 : A_1 \longrightarrow A_2$

$R_2 : A_2 \longrightarrow A_3$

$\cdots\cdots R_n : A_n \longrightarrow A_{n+1}$.

Then $R_1 \circ R_2 \circ R_3 \cdots \circ R_n$ is a relation from A_1 to A_{n+1}

③ In particular, if $A_1 = A_2 = \cdots = A_{n+1} = A$

and $R_1 = R_2 = \cdots = R_n = R$.

then we denote $R_1 \circ R_2 \circ \cdots \circ R_n$ by $\underline{R^n}$ which is relation on A.

Hence, given R, one can compute, R^2, R^3, and so on.

④ clearly, $R_1 R_2 \neq R_2 R_1$

5.

Theorem: Let R_1, R_2 and R_3 be relations from A to B, B to C and C to D

Then $(R_1R_2)R_3 = R_1(R_2R_3)$

Proof: To prove: $(R_1R_2)R_3 \subseteq R_1(R_2R_3)$
& $(R_1R_2)R_3 \supseteq R_1(R_2R_3)$

Let $(a,d) \in (R_1R_2)R_3$, where $a \in A$ & $d \in D$ —— (1)

Note that R_1R_2 is a relation from A to C
$\Rightarrow$ there exist ($\exists$) an element $c \in C$
such that ($\ni$) $(a,c) \in R_1R_2$
and $(c,d) \in R_3$

Now, $(a,c) \in R_1R_2 \Rightarrow \exists$ an element $b \in B$
such that $(a,b) \in R_1$ and $(b,c) \in R_2$

Since $(b,c) \in R_2$ and $(c,d) \in R_3$
$\Rightarrow (b,d) \in R_2R_3$

Now $(a,b) \in R_1$ and $(b,d) \in R_2R_3$

$\Rightarrow (a,d) \in R_1(R_2R_3)$ —— (2)

Hence, from (1) & (2)

$$(R_1R_2)R_3 \subseteq R_1(R_2R_3)$$

Similarly, we can prove
$(R_1R_2)R_3 \supseteq R_1(R_2R_3)$ Hence proof

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Thm-2: Let R_1 be a relation from A to B and R_2 from B to C.

Then $(R_1 R_2)^c = R_2^c R_1^c$

proof: Given: $R_1 : A \longrightarrow B$

& $R_2 : B \longrightarrow C$

Hence,

$R_1^c : B \longrightarrow A$ & $R_2^c : C \longrightarrow B$

To prove: $(R_1 R_2)^c \subseteq R_2^c R_1^c$

& $(R_1 R_2)^c \supseteq R_2^c R_1^c$

Let ~~$R_1 R_2$~~ $(c, a) \in (R_1 R_2)^c$ ——— ①

$\Rightarrow (a, c) \in (R_1 R_2)$

Hence $\exists$ $b \in B$ such that

$(a, b) \in R_1$ and $(b, c) \in R_2$

$\Rightarrow (b, a) \in R_1^c$ and $(c, b) \in R_2^c$

$\Rightarrow (c, b) \in R_2^c$ and $(b, a) \in R_1^c$

$\Rightarrow$ $(c, a) \in R_2^c R_1^c$ ——— ②

$\Rightarrow$ From ① & ②

$(R_1 R_2)^c \subseteq R_2^c R_1^c$

Similarly, we can prove,

$(R_1 R_2)^c \supseteq R_2^c R_1^c$

Hence, $(R_1 R_2)^c = R_2^c R_1^c$

Ex: 1. Let $A = \{a, b, c, d\}$ where

$$R_1 = \{(a,a), (a,b), (b,d)\} \text{ and}$$

$$R_2 = \{(a,d), (b,c), (b,d), (c,b)\}$$

Find R_1R_2, R_2R_1, R_2^2, R_2^3

Sol^n:

$$R_1R_2 = \{(a,d), (a,c), (a,d)\}$$

$$R_2R_1 = \{(c,d)\}$$

$$R_1^2 = R_1R_1 = \{(a,a), (a,b), (a,d)\}$$

$$R_2^2 = R_2R_2 = \{\cancel{(b,d)}, (b,b), (c,c), (c,d)\}$$

$$R_2^3 = R_2R_2^2\cancel{R_2} = \{(b,c), (b,d), (c,b)\}$$

Ex: 2. Let $A = \{2, 3, 4, 5, 6\}$ and let R_1, R_2 be relations on A such that

$$R_1 = \{(a,b) / a - b = 2\} \text{ and}$$

$$R_2 = \{(a,b) / a + 1 = b \text{ or } a = 2b\}$$

Find the composite relations.

(i) R_1R_2 (ii) R_2R_1 (iii) $R_1R_2R_1$ (iv) R_1^2

(v) $R_1R_2^2$