Numerical Solution of ordinary differential Equations:

In numerical analysis, a numerical method is a mathematical tool designed to solve numerical problems.

Note: 1. Many differential equations cannot be solved exactly.
But for practical purposes, in engineering a numeric approximation to the solution is often sufficient.

2. Numerical methods for solving first order IVPs often fall into one of two large categories.
— Linear multisteps methods
— Runga Kutta methods

IVP ≡ Intial Value Problems

We shall consider the solution of ordinary differential equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{with intial condition } y(x_0) = y_0$$

Note: (i) $x \in (x_0, x_{n+1})$

(ii) $y(x_0) = y_0$ is called intial condition

(iii) Problem of solution of a differential equation with given intial condition is called

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an <u>intial value problem</u>:

Given intial condition $y(x_0) = y_0$, We develop method finding $y(x_1) = y(x_0 + h)$

$$y(x_2) = y(x_1 + h)$$

$$\vdots$$

$$y(x_n) = y(x_{n-1} + h)$$

where h is a suitably chosen step length.

This method can be extended to solve system of first order differential equations with set of intial conditions.

Such system of equations is usually expressed in the form.

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \cdots y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \cdots y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \cdots y_n)$$

with intial conditions $y_1(x_0) = a_1$, $y_2(x_0) = a_2$, $\cdots$ $y_n(x_0) = a_n$

Numerical Methods to solve ODE:

1. Euler's method:

Euler's method give the solution in the form of tabulated values.

Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \qquad \text{——— (1)}$$

which is to be solved, subject to intial condition

$$x = x_0, \quad y = y_0$$

Suppose we want to find the value of y at

$x_1 = x_0 + h$

$x_2 = x_1 + h, \quad \ldots\ldots \quad x_{n+1} = x_n + h$

Given: $\frac{dy}{dx} = f(x,y)$

$$dy = f(x,y)\,dx$$

integrating.

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

We assume that in the interval (x_0, x_1), $f(x,y)$ remains stationary (constant) as $f(x_0,y_0)$

Hence.

$$y_1 - y_0 = f(x_0,y_0)\,[x]_{x_0}^{x_1}$$

$$y_1 - y_0 = (x_1 - x_0)\, f(x_0, y_0)$$
$$y_1 - y_0 = h \cdot f(x_0, y_0)$$
$$\therefore\ y_1 = y_0 + h f(x_0, y_0)$$
$$\Rightarrow \boxed{y(x_1) = y_1 = y_0 + h\, f(x_0, y_0)}$$

Similarly,
$$y_2 = y_1 + h f(x_1, y_1)$$
$$y_3 = y_2 + h f(x_2, y_2)$$
$$\vdots$$
$$y_{n+1} = y_n + h f(x_n, y_n)$$

which gives the Euler's formula to find y at $x = x_1, x_2, \ldots x_n$

Ex: 1. Use Euler's method to solve the equation
$$\frac{dy}{dx} = 1 + xy$$

subject to the conditions at $x=0$, $y=1$ and tabulate y for $x = 0, 0.1, 0.5$

Soln. Here $f(x,y) = 1 + xy$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$

Step-1 : To find $y_1 = y(x_1)$.

$$f(x_0, y_0) = 1 + x_0 y_0$$
$$= 1 + 0 \times 1 = 1$$

Now, $y_1 = y_0 + h f(x_0, y_0)$
$$= 1 + 0.1 \times 1 = 1 + 0.1 = 1.1$$

$y_1 = 1.1$, $x_1 = 0 + \cancel{0.1} = 0.1$

Step-2 : $f(x_1, y_1) = 1 + x_1 y_1$

$= 1 + 0.1 \times 1.1$

$= 1 + 0.11$

$= 1.11$

Now, $y_2 = y_1 + h f(x_1, y_1)$

$= 1.1 + 0.1 \times 1.11$

$= 1.1 + 0.111$

$= 1.211$

Step-3 : $y_2 = 1.211$, $x_2 = 0.1 + 0.1 = 0.2$

$\therefore f(x_2, y_2) = 1 + x_2 \cdot y_2$

$= 1 + 0.2 \times 1.211$

$= 1.2422$

$\therefore y_3 = y_2 + h f(x_2, y_2)$

$= 1.211 + 0.1 \times 1.2422$

$= 1.33522$

Step-4 : $y_3 = 1.33522$, $x_3 = 0.3$

$f(x_3, y_3) = 1 + x_3 y_3$

$= 1 + 0.3 \times 1.33522$

$= 1.40056$

~~Step-5~~

$y_4 = y_3 + h f(x_3, y_3)$

$= 1.33522 + 0.1 \times 1.40056$

$= 1.4753$

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Step-5: $y_4 = 1.4753$, $x_4 = 0.4$

$$\therefore f(x_4, y_4) = 1 + x_4 y_4$$
$$= 1 + 0.4 \times 1.4753$$
$$= 1.59012$$

Hence,
$$y_5 = y_4 + h f(x_4, y_4)$$
$$= 1.4753 + 0.1 \times 1.59012$$
$$= 1.6343$$

The tabulated solution is –

x	0	0.1	0.2	0.3	0.4	0.5
y	1	1.1	1.211	1.3352	1.4753	1.6343

Ex: 2 (a) Use Euler's method to solve the ODE
$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$
From $x = 0$ to $x = 4$ with step size of 0.5
The intial condition is $y(0) = 1$
(b) Giving proper example illustrate the effect of step size on stability of Euler's method.

Soln: (a) Here, $f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$,
$h = 0.5$, $x_0 = 0$, $y_0 = 1$

Step-1: To find y_1: $[x_0 = 0, y_0 = 1]$
$$f(x_0, y_0) = -2(0) + 12(0) - 20(0) + 8.5$$
$$= 8.5$$

Hence $y_1 = y_0 + h f(x_0, y_0)$

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$$y_1 = 1 + 0.5 \times 8.5$$
$$= 5.25$$

Step-2 : $y_2 = y_1 + hf(x_1, y_1)$ $[x_1 = 0.5,\ y_1 = 5.25$

$$y_2 = 5.25 + 0.5\left[-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5\right]$$
$$= 5.875$$

Step-3 : $y_3 = y_2 + hf(x_2, y_2)$ $[x_2 = 1,\ y_2 = 5.875]$
$$= 5.875 + 0.5\left[-2(1)^3 + 12(1)^2 - 20(1) + 8.5\right]$$
$$= 5.125$$

Step-4 : $y_4 = y_3 + hf(x_3, y_3)$ $[x_3 = 1.5,\ y_3 = 5.125]$
$$= 5.125 + 0.5\left[-2(1.5)^3 + 12(1.5)^2 - 20(1.5) + 8.5\right]$$
$$= 4.5$$

Step-5 : $y_5 = y_4 + hf(x_4, y_4)$ $[x_4 = 2,\ y_4 = 4.5]$
$$= 4.5 + 0.5\left[-2(2)^3 + 12(2)^2 - 20(2) + 8.5\right]$$
$$= 4.75$$

Step-6 : $y_6 = y_5 + hf(x_5, y_5)$ $[x_5 = 2.5,\ y_5 = 4.75]$
$$= 4.75 + 0.5\left[-2(2.5)^3 + 12(2.5)^2 - 20(2.5) + 8.5\right]$$
$$= 5.875$$

Step-7 : $y_7 = y_6 + hf(x_6, y_6)$ $[x_6 = 3,\ y_6 = 5.875$
$$= 5.875 + 0.5\left[-2(3)^3 + 12(3)^2 - 20(3) + 8.5\right]$$
$$= 7.125$$

Step-8 : $y_8 = y_7 + h f(x_7, y_7)$ $[x_7 = 3.5,\ y_7 = 7.125]$

$$= 7.125 + 0.5\left[-2(3.5)^3 + 12(3.5)^2 - 20(3.5) + 8.5\right]$$

$$= 7$$

The tabulated solution ~~is~~ by Euler's method is –

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1	5.25	5.875	5.125	4.5	4.75	5.875	7.125	7

(b) Exact solution by direct integration satisfying the intial condition is –

$$y = -\frac{x^4}{2} + 4x^3 - 10x^2 + 8.5x + 1.$$

Exact solution:

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	1	3.219	3	2.219	2	2.719	4	4.719	3

Note: Comparing two solutions shows that approximate solution by Euler's method deviates too much from the exact solution. This is because step size $h = 0.5$ is too large

H.W.

1. Solve $\frac{dy}{dx} = x+y$. with boundry conditions $y=1$ at $x=0$.
Find approximate value of y ~~at~~ for $x=0.1$.

$$h = \frac{b-a}{n} = \frac{0.1-0}{5} = 0.02.$$

0.06

0 x_1 x_2 x_3 x_4 $0.1 = x_5$

x_0 0.02 0.04 0.08

$y_1 = 1.02$, $y_2 = 1.0408$, $y_3 = 1.0624$

$y_4 = 1.0848$, $y_5 = 1.1081$

Euler's method to solve D.E of higher order:

Consider the D.E

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

with condition at $x = x_0$, $y = y_0$, $y'(x_0) = y_0'$

To solve such equations,

consider, $\frac{dy}{dx} = z$ so that

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = f(x, y, z)$$

Given equation of higher order is equivalent to system of simultaneous equations of first order such ~~the~~ as -

$$\frac{dz}{dx} = f(x, y, z)$$

$$; \quad \frac{dy}{dx} = z = \phi(x, y, z) \text{ (say)}$$

with conditions at $x = x_0$, $y = y_0$, $z = z_0 = y_0'$

If h is step length in x,

then values of z_1, y_1 at $x_1 = x_0 + h$ are given by Euler's ~~modified~~ method as -

$$z_1 = z_0 + h f(x_0, y_0, z_0)$$

$$y_1 = y_0 + h\,\phi(x_0, y_0, z_0)$$

In general;

$$z_{n+1} = z_n + h f(x_n, y_n, z_n)$$

$$y_{n+1} = y_n + h\,\phi(x_n, y_n, z_n)$$

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Ex: 1. Solve the equation :

$$2\frac{d^2y}{dx^2} = 3x\frac{dy}{dx} - 9y + 9$$

Subject to the conditions $y(0)=1$, $y'(0)=-2$ using Euler's method and compute y for $\underset{\uparrow \atop x_1}{x = 0.1}\ \underset{\uparrow \atop h}{(0.1)}\ (0.4)$.

Soln : Given equation can be written as -

$$\frac{d^2y}{dx^2} = \frac{3}{2}x\frac{dy}{dx} - \frac{9}{2}y + \frac{9}{2} \qquad (1)$$

Let $\frac{dy}{dx} = z$ so that $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Hence eq^n (1) can be written as -

$$\frac{dz}{dx} = 1.5x.z - 4.5y + 4.5 \qquad (2)$$

$$\phi(x,y,z) = \frac{dy}{dx} = z \qquad (3)$$

with $x_0 = 0$, $y_0 = 1$, $z_0 = y'(0) = -2$

Take $h = 0.1$, we have,

step-1 : $z_1 = z_0 + hf(x_0, y_0, z_0)$ [$x_0 = 0, y_0 = 1, z_0 = -2$]

$= -2 + (0.1)[1.5(0)(-2) - 4.4(1) + 4.5]$

$= -2$

$y_1 = y_0 + h\phi(x_0, y_0, z_0)$

$= y_0 + hz_0$

$= 1 + (0.1)(-2)$

$= 0.8$

$[x_1 = 0.1,\ y_1 = 0.8,\ z_1 = -2]$

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step-2 :
$$Z_2 = Z_1 + h f(x_1, y_1, z_1)$$
$$= -2 + (0.1)[1.5(0.1)(-2) - 4.5(0.8) + 4.5]$$
$$= -1.94$$

$$y_2 = y_1 + h Z_1$$
$$= 0.8 + (0.1)(-2)$$
$$= 0.6$$

step-3 :
$$Z_3 = Z_2 + h f(x_2, y_2, Z_2) \quad [x_2 = 0.2,\ y_2 = 0.6,\ Z_2 = -1.94]$$
$$= -1.94 + (0.1)[1.5(0.2)(-1.94) - 4.5(0.6) + 4.5]$$
$$= -1.8182$$

$$y_3 = y_2 + h Z_2$$
$$= 0.6 + (0.1)(-1.94)$$
$$= 0.406$$

step-4 :
$$Z_4 = Z_3 + h f(x_3, y_3, Z_3) \quad [x_3 = 0.3,\ y_3 = 0.406,\ Z_3 = -1.8182]$$
$$= -1.8182 + (0.1)[1.5(0.3)(-1.8182) - 4.5(0.406) + 4.5]$$
$$= -1.6327$$

$$y_4 = y_3 + h Z_3$$
$$= 0.41818 + (0.1)(-1.8182)$$
$$= 0.23636$$

Values when tabulated are :

x	0	0.1	0.2	0.3	0.4
y	1	0.8	0.6	0.406	0.23636
$\frac{dy}{dx} = Z$	-2	-2	-1.94	-1.8182	-1.6327

Modified Euler's method :

In order to get better approximate value of y, we first obtain y_1 by Euler's method i.e initial approximation $y_1^{(0)}$ by using formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

Then we ~~modi~~ modify it by using modified Euler's method using formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

Put $n = 0, 1, 2, \cdots$

$n=0$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$n=1$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

& so on.

Note: 1. This procedure is terminated depending upon the accuracy required.
If two consecutive values are equal say $y_1^{(k)} = y_1^{(k+1)}$ then

$$y_1 = y_1^{(k)}$$

2. Proceeding in the same way $y_2, y_3, \cdots$ are calculated.

Ex: 1. Given $\frac{dy}{dx} = x^2+y$., with $y(0)=1$

Find $y(0.02)$ & $y(0.04)$

by Euler's modified method.

Soln: Consider, $f(x,y) = \frac{dy}{dx} = x^2+y$

$x_0 = 0$, $y_0 = 1$.

$x_1 = 0.02$, $y_1 = ?$

$x_2 = 0.04$ $y_2 = ?$

$h = x_1 - x_0 = 0.02$

Step-1: $x_0 = 0$, $y_0 = 1$, $f(x_0, y_0) = 1$

Now, $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.02 \times (0+1)$

$= 1 + 0.02$

$= 1.02$

Now,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.02}{2}\left[(0^2+1) + \left[(0.02)^2 + 1.02\right]\right]$$

$$= 1 + 0.01\left[1 + 1.0204\right]$$

$$= 1.020204$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.02}{2}\left[1 + \left[(0.02)^2 + 1.020204\right]\right]$$

$$= 1 + 0.01\left[1 + \left(0.0004 + 1.020204\right)\right]$$

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$$y_1^{(2)} = 1.0202064$$

Hence, As $y_1^{(1)} \approx y_1^{(2)} = 1.02020 = y_1$ upto 5 digits

Step-2 : $x_1 = 0.02, y_1 = 1.02020$, $f(x_1, y_1) = 1.0206$

Now,

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1)$$
$$= 1.02020 + 0.02 \times ((0.02)^2 + 1.02020)$$
$$= 1.040612$$

n = 1

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.02020 + \frac{0.02}{2}\left[1.0206 + ((0.04)^2 + 1.040612)\right]$$

$$= 1.040828$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.02020 + \frac{0.02}{2}\left[1.0206 + ((0.04)^2 + 1.040828)\right]$$

$$= 1.040830$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 1.02020 + \frac{0.02}{2}\left[1.0206 + ((0.04)^2 + 1.040830)\right]$$
$$= 1.0408303$$

As $y_2^{(2)} = y_2^{(3)} = 1.040830$ upto 6 digits -

Hence $y_2 = 1.04083$

Ex: 2 Solve the equation

$$\frac{dy}{dx} = 1 + xy, \quad x_0 = 0, \; y_0 = 1$$

to find y at $x = 0.1$ and $x = 0.2$ using Euler's modified method taking $h = 0.1$

Solution Step-1: Here $x_0 = 0$, $y_0 = 1$; $x_1 = 0.1$

$$y_1^{(0)} = y_0 + h f(x_0, y_0) \quad ; \quad f(x_0, y_0) = 1 + 0 \times 1 = 1$$

$$= 1 + 0.1\,(1 + 0 \times 1)$$

$$= 1 + 0.1\,(1)$$

$$= 1 + 0.1$$

$$= 1.1$$

To find $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$= 1 + \frac{0.1}{2}\left[1 + (1 + 0.1 \times 1.1)\right]$$

$$= 1 + 0.05\left[1 + 1 + 0.11\right]$$

$$= 1.1055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[1 + (1 + (0.1)(1.1055))\right]$$

$$= 1.1055275$$

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If the accuracy is required upto fourth decimal place, then.

$$y_1 = y_1^{(1)} \approx y_1^{(2)} = 1.1055$$

~~Step-2~~:

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + 0.05\left[1 + (1 + 0.1 \times 1.1055275)\right]$$

$$= 1.105527638$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + 0.05\left[1 + (1 + 0.1 \times 1.105527638)\right]$$

$$= 1.105527638$$

Now,

$y_1 = y_1^{(3)} \approx y_1^{(4)} = 1.1055276$ correct upto seven decimal places.

Step-2 : To find $y_2 = y(0.2)$

$x_1 = 0.1, \ y_1 = 1.1055276, \ x_2 = 0.2.$

$y_2^{(0)} = y_1 + hf(x_1, y_1)$; $f(x_1, y_1) = 1 + x_1 y_1$

$= 1 + 0.1 \times 1.1055276$

$= 1.11055276$

$y_2^{(0)} = 1.1055276 + 0.1 \times 1.11055276$

$= 1.216582876$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.1055276 + \frac{0.1}{2}\left[1.11055276 + \left[1 + (0.2 \times 1.216582876)\right]\right]$$

$$y_2^{(1)} = 1.22322$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.1055276 + \frac{0.1}{2}\left[1.11055276 + 1 + 0.2 \times 1.22322\right]$$

$$= 1.22329$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 1.1055276 + \frac{0.1}{2}\left[1.11055276 + 1 + 0.2 \times 1.22329\right]$$

$$y_2^{(3)} = 1.22329$$

Observe that $y_2^{(2)} \simeq y_2^{(3)} = 1.22329$ correct upto 5 decimal places.

Ex: 3. Determine using modified Euler's method the value of y when $x = 0.1$, given that

$$\frac{dy}{dx} = x^2 + y, \quad y(0) = 1$$

Soln.: $f(x,y) = \dfrac{dy}{dx} = x^2 + y, \quad x_0 = 0, \; y_0 = 1,$

take $h = 0.05$

Step-1

$$y_1^{(0)} = 1.05$$

$$y_1^{(1)} = 1.0513$$

$$y_1^{(2)} = 1.0513$$

$$\Rightarrow \quad y_1 = 1.0513$$

Step-2

$$y_2^{(0)} = 1.1040$$

$$y_2^{(1)} = 1.1055$$

$$y_2^{(2)} = 1.1055$$

$$\Rightarrow \quad y_2 = 1.1055$$

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Runge-Kutta Methods:

The Runge-Kutta methods are designed to give more greater accuracy with the advantage of requiring function values only at some selected points on the subinterval.

Consider the DE

$$\frac{dy}{dx} = f(x, y); \quad f(x_0) = y_0$$

(A) Second order Runge-Kutta method:

If we substitute $y_1 = y_0 + h f(x_0, y_0)$ in the RHS of modified Euler's method ~~we obtain~~;

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

we obtain;

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_0 + h f(x_0, y_0))\right]$$

$$= y_0 + \frac{1}{2}\left[h f(x_0, y_0) + h f(x_0 + h, y_0 + h f(x_0, y_0))\right]$$

$\cancel{= y_0 + \frac{1}{2}[}$ Set $k_1 = h f(x_0, y_0)$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

$$\& \; k = \frac{1}{2}(k_1 + k_2)$$

$$\therefore y_1 = y_0 + k$$

$$y_1 = y_{x_0+h} = y_0 + k$$

which is the second order RK method

Note: The error in this formula can be shown to be of order h^3

Ex: Solve: $\dfrac{dy}{dx} = y - x = f(x,y)$

by second order RK method subject to condition $y(0)=2$ & calculate y at $x=0.2$ taking $h=0.1$

Soln: We have second order RK formula,

$$y_1 = y_0 + k$$

where $k = \frac{1}{2}(k_1 + k_2)$

$$k_1 = hf(x_0, y_0), \quad k_2 = hf(x_0+h, y_0+k_1)$$

Step-1: $h = 0.1, \quad y_0 = 2, x_0 = 0$

$$\begin{aligned} k_1 &= hf(x_0, y_0) \\ &= 0.1\,(y_0 - x_0) \\ &= 0.1\,(2-0) \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0+h, y_0+k_1) \\ &= 0.1\, f(0+0.1, 2+0.2) \\ &= 0.1\, f(0.1, 2.2) \\ &= 0.1 \times (2.2 - 0.1) \\ &= 0.1 \times (2.1) \\ &= 0.21 \end{aligned}$$

Hence,

$$k = \frac{1}{2}(k_1 + k_2)$$

$$= \frac{1}{2}(0.2 + 0.21) = 0.205$$

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Hence,

$$y_1 = y_0 + k$$
$$= 2 + 0.205$$

$$y_{x_1} = y_{0.1} = y_1 = 2.205$$

Step-2 : Now, to find y at $x = 0.2$,

~~i.e y_2 at $x_2 = 0.2$~~

Consider, $x_0 = 0.1$, $y_0 = 2.205$, $h = 0.1$

$$\therefore k_1 = h f(x_0, y_0) = 0.1 [2.205 - 0.1]$$
$$= 0.2105$$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$
$$= 0.1 f(0.1 + 0.1, 2.205 + 0.2105)$$
$$= 0.1 f(0.2, 2.4155)$$
$$= 0.1 (2.4155 - 0.2)$$
$$= 0.22155$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.2105 + 0.22155)$$

$$k = 0.216025$$

$$y_{0.2} = y_1 = y_0 + k$$
$$= 2.205 + 0.216025$$
$$y_1 = 2.421025$$

Fourth order Runge-Kutta method:

Fourth order Runge-Kutta method is of great practical importance because of more accuracy.

RK4 formulae are —

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

And

$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

The required approximate value is given by

$$y_{1\,x_0+h} = y_0 + k$$

Note: The error in this formula can be shown to be order h^5.

Ex: Using 4^{th} order Runge-Kutta method, solve the equation $\frac{dy}{dx} = \sqrt{x+y}$ subject to the conditions $x=0$, $y=1$ and find y at $x = 0.2$ taking $h = 0.2$

Soln: To determine y at $x = 0.2$,
we have, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$
Then we obtain;

$$k_1 = hf(x_0, y_0) = hf(0,1) = 0.2 \times \sqrt{0+1}$$
$$= 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$
$$= hf(0 + 0.1, 1 + 0.1)$$
$$= 0.2\, f(0.1, 1.1)$$
$$= 0.2 \sqrt{0.1 + 1.1}$$
$$= 0.2 \sqrt{1.2}$$
$$= 0.2191$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf\left(0 + \frac{0.2}{2}, 1 + \frac{0.2191}{2}\right)$$

$$k_3 = 0.2\, f(0.1, 1 + 0.10955)$$
$$= 0.2120.$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(0 + 0.2, 1 + 0.2120)$$
$$= 0.2 \sqrt{0.2 + 1.2120}$$
$$= 0.2377$$

Hence,
$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k = \frac{1}{6}\left[0.2 + 2 \times 0.2191 + 2 \times 0.2120 + 0.2377\right]$$

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$k = 0.2167$

Hence,

$$y|_{x=0.2} = y_0 + k = 1 + 0.2167$$

$$y|_{x=0.2} = 1.2167$$

Ex 2. Using fourth order RK method, solve the equation $\frac{dy}{dx} = \sqrt{x+y}$ subject to the conditions $x=0, y=1$ to find y at $x=0.2$ taking $h=0.1$.

Ans: step-1: $y_0 = 1, x_0 = 0, h = 0.1$

$$y|_{x=0.1} = y_0 + k = 1 + 0.1049$$

$$= 1.1049$$

step-2: $x_0 = 0.1, y_0 = 1.1049; h = 0.1$

$$y|_{x=0.2} = y_0 + k = 1.1049 + 0.1145$$

$$= 1.2194$$

Predictor - Corrector Methods :

Note : 1. We discussed the methods to solve a DE over an interval, say from $x = x_n$ to $x = x_{n+1}$ which require function value at the beginning of the interval i.e at $x = x_n$.

2. Predictor - corrector methods are multistep methods for the intial solution of IVP which require intial values at $x_n, x_{n-1}, x_{n-2}, \ldots$ for the computation of the function value at x_{n+1}.

3. We use explicit predictor formula for predicting a value y_{n+1} and then use implicit corrector formula to improve the value of y_{n+1}, iteratively until the convergence is obtained.

Milne's method :

Consider an intial value problem

$$\frac{dy}{dx} = f(x, y), \quad x = x_0, \quad y = y_0$$

To find an approximate value of y_{n+1} using Newton- Gregory forward difference interpolation formula, Milne's Predictor formula can be obtained as —

$$y_4^P = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

and corrector formula is given by —

$$y_4^C = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

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Here $f_1 = f(x_1, y_1)$, $y_2 = f(x_2, y_2)$ etc.

General forms of Milne's formulae are –

Predictor formula:

$$y^{p}_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right]$$

Corrector Formula:

$$y^{c}_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right]$$

P ≡ (predictor formula) Predicted values
C ≡ (Corrector formula) Corrected values.

Note: Single step methods are self starting (just intial conditions are required) whereas multistep methods require four starting values y_0, y_1, y_2, y_3

Ex: 1: Solution of the equation $5x\frac{dy}{dx} + y^2 - 2 = 0$ is tabulated as

	x_0	x_1	x_2	x_3
x	4	4.1	4.2	4.3
y	1.0	1.0049	1.0097	1.0143

Use Milne's predictor - corrector method to find y at $x = 4.4$ (x_4) and 4.5 (x_5)

soln: Given D.E is –

$$\frac{dy}{dx} = f(x, y) = \frac{2 - y^2}{5x}$$

We first determine starting values of Milne's method from the table as —

$x_0 = 4$ $\quad y_0 = 1.0$ $\quad \therefore f_0 = f(x_0, y_0) = \dfrac{2-y_0^2}{5x_0}$

$= 0.05$

$x_1 = 4.1$, $\quad y_1 = 1.0049$, $\quad \therefore f_1 = f(x_1, y_1) = \dfrac{2-y_1^2}{5x_1}$

$= 0.0483$

$x_2 = 4.2$, $\quad y_2 = 1.0097$ $\quad \therefore f_2 = f(x_2, y_2) = \dfrac{2-y_2^2}{5x_2}$

$= 0.0467$

$x_3 = 4.3$, $\quad y_3 = 1.0143$, $\quad \therefore f_3 = f(x_3, y_3) = \dfrac{2-y_3^2}{5x_3}$

$= 0.0452$

Using Predictor ~~value~~ formula,

$$y_4^p = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4 \times 0.1}{3}\left[2(0.0483) - 0.0467 + 2(0.0452)\right]$$

$$= 1.01871$$

This predicted value is used as to calculate f_4 as —

$x_4 = 4.4$, $\quad y_4^p = 1.01871 = y_4$

$\therefore f_4 = (x_4, y_4)$

$$= \frac{2 - y_4^2}{5x_4}$$

$$= 0.04374$$

Using corrector formula;

$$y_4^c = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4\times 0.0452 + 0.04374\right]$$

$$= 1.01871$$

Since the predicted value and corrected value is same, we terminate the procedure and accept

$$y_4 = 1.01871$$

To find y_5, we use predictor formula,

$$y_5^p = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 1.0049 + \frac{4(0.1)}{3}\left[2(0.0467) - 0.0452 + 2(0.04374)\right)$$

$$= 1.023$$

This predicted value is used to calculate f_5 as -

$x_5 = 4.5$, $y_5^p = 1.023 = y_5$

$$\therefore f_5 = f(x_5, y_5) = \frac{2 - y_5^2}{5x_5}$$

$$= 0.04238$$

By corrector formula;

$$y_5^c = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + f_5\right]$$

$$y_5^c = 1.0143 + \frac{0.1}{3}\left[0.0452 + 4(0.04374) + 0.04238\right]$$

$$= \underline{1.02305}$$

If we want to modify this value.

$$f_5 = f(x_5, y_5^c) = \frac{2-(1.02305)^2}{5(4.5)} = 0.04238$$

Hence $\underline{y_5 = 1.02305}$ as f_5 is same as before,

Hence,

x	4	4.1	4.2	4.3	4.4	4.5
y	1.0	1.0049	1.0097	1.0143	1.01871	1.02305

Ex: 2. Numerical solution of the DE $\frac{dy}{dx} = 2 + \sqrt{xy}$ is tabulated as

x	1.0	1.2	1.4	1.6
y	1.0	1.6	2.2771	3.0342

Find y at $x = 1.8$ by Milne's predictor-corrector method taking $h = 0.2$

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Soln: Given DE is -

$$\frac{dy}{dx} = 2 + \sqrt{xy}$$

We first determine starting values of Milne's method from the table as -

$x_0 = 1.0$	$y_0 = 1.0$	$\therefore f_0 = f(x_0, y_0) =$	3.0
$x_1 = 1.2$	$y_1 = 1.6$	$\therefore f_1 = f(x_1, y_1) =$	3.3856
$x_2 = 1.4$	$y_2 = 2.2771$	$\therefore f_2 = f(x_2, y_2) =$	3.7855
$x_3 = 1.6$	$y_3 = 3.0342$	$\therefore f_3 = f(x_3, y_3) =$	4.2033

Using predictor formula,

$$y_4^p = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.2)}{3}\left[2(3.3856) - 3.7855 + 2(4.2033)\right]$$

$$y_4^p = 4.0379$$

This predicted value is used to calculate f_4 as :

$$x_4 = 1.8, \quad y_4^p = 4.0379 = y_4$$

$$\therefore f_4 = f(x_4, y_4) = 2 + \sqrt{x_4 y_4}$$

$$= 2 + \sqrt{(1.8)(4.0379)}$$

$$= 4.696$$

Using corrector formula,

$$y_4^c = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

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$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4(4.2033) + 4.696]$$

$$y_4^c = 3.9634$$

Since predicted and corrected values do not match, we recalculate f_4 as —.

$$x_4 = 1.8, \quad y_4^c = 3.9634 = y_4$$

$$\therefore \quad f_4 = f(x_4, y_4) = 2 + \sqrt{x_4 y_4}$$

$$= 2 + \sqrt{(1.8)(3.9634)}$$

$$= 4.671$$

Again using corrector formula,

$$y_4^c = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4(4.2033) + 4.671]$$

$$= 3.9617$$

Repeating the procedure,

$$x_4 = 1.8, \quad y_4^c = 3.9617 = y_4$$

$$f_4 = f(x_4, y_4) = 2 + \sqrt{(1.8)(3.9617)}$$

$$= 4.67$$

Hence,

$$y_4^c = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= 2.2771 + \frac{0.2}{3}[3.7855 + 4(4.2033) + 4.67]$$

$$= 3.9617$$

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Since two successive values match,
we accept $\underline{\underline{y_4 = 3.9617}}$

Ex: 1. Use Runge-Kutta method of second order to solve

$$\frac{dy}{dx} = \frac{1}{x+y}, \quad x_0 = 0, \; y_0 = 1$$

to find y at $x = 0.4$ taking $h = 0.2$

Soln → Ans: $y|_{x=0.2} = y_0 + k = 1 + 0.17143$
$= 1.17143$

$y|_{x=0.4} = y_0 + k = 1.17143 + 0.131145$
$= 1.302575$

Ex: 2 Use Runge-Kutta method of fourth order to solve :

$$\frac{dy}{dx} = \frac{1}{x+y}, \quad x_0 = 0, \; y_0 = 1$$

to find y at $x = 0.4$, taking $h = 0.2$

⇒ Ans: $y|_{x=0.2} = y_0 + k = 1 + 0.1697$
$= 1.1697$

$y|_{x=0.4} = y_0 + k = 1.1697 + 0.13057$
$= 1.30027$