Linear Differential Equation :

It is necessary to study the differential equations as it has various applications in engineering field such as – oscillation of mechanical system, electrical system, conduction of heat etc.

Pre-requisite :

1. $D^2 - a^2 = (D-a)(D+a) = 0 \Rightarrow D = \pm a$
2. $D^2 + a^2 = (D-ia)(D+ia) = 0 \Rightarrow D = \pm ia$
3. $(D-a)^2 = D^2 - 2aD + a^2 = 0 \Rightarrow D = a, a$
4. $(D+a)^2 = D^2 + 2aD + a^2 = 0 \Rightarrow D = -a, -a$
5. $D^3 + a^3 = (D+a)(D^2 - aD + a^2)$
6. $D^3 - a^3 = (D-a)(D^2 + aD + a^2)$
7. $D^4 - a^4 = (D^2 - a^2)(D^2 + a^2) = 0$

   $\Rightarrow D = a, -a, ia, -ia$
8. For quadratic equation,

   $ax^2 + bx + c = 0$,

   the roots are :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Special :

(i) $D^4 + a^4 = D^4 + 2a^2D^2 + a^4 - 2a^2D^2$

$= (D^2 + a^2)^2 - (\sqrt{2}aD)^2$

$= (D^2 + a^2 + \sqrt{2}aD)(D^2 + a^2 - \sqrt{2}aD)$

(ii) $D^4 - a^4 = (D^2+a^2)(D^2-a^2)$

$= (D+ia)(D-ia)(D+a)(D-a)$

Synthetic Division Method :

In the process of finding roots of polynomial equation $f(D)=0$ by synthetic division method, the first root is obtained by trial and error method.

e.g. $D^4 - 2D^3 - 3D^2 + 4D + 4 = 0$

1	1	−2	−3	4	4
+	0	1	−1	−4	0
	1	−1	−4	0	$\boxed{4}$ ← non-zero

As the last term we got 4, try for the next

−1	1	−2	−3	4	4
+	0	−1	3	0	−4
	1	−3	0	4	$\boxed{0}$

Hence $D=-1$ is one of the root and it gives

$$D^3 - 3D^2 + 0\cdot D + 4 = 0 \quad \text{——— (1)}$$

Again,

−1	1	−3	0	4
	0	−1	4	−4
	1	−4	4	$\boxed{0}$

$\Rightarrow D^2 - 4D + 4 = 0$

$$\Rightarrow (D-2)^2 = 0$$
$$\Rightarrow D = +2, +2$$

Hence roots of eq^n (1) are —
$$D = -1, -1, +2, +2$$

Linear Differential Equation of n^{th} order with constant coefficients :

A differential equation in which the dependent variable and it's derivative occur only in the first degree and not multiplied together is called Linear Differential Equation.

LDE With Constant Coefficients :

The general form of this equation is,

$$k_0 \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = f(x)$$

where $k_0, k_1, \dots k_n$ are constants ~~and~~.

Differential Operator :

Denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, $\frac{d^3}{dx^3} = D^3$, ... $\frac{d^n}{dx^n} = D^n$ etc.

D satisfies algebraic property :

(1) If c is constant $D(cy) = c(Dy)$

(ii) Distributive property:

$$D(y \pm z) = Dy \pm Dz \; ;$$
$$D^n(y \pm z) = D^n y \pm D^n z$$

(iii) Commutative property:

$$(D-a)(D-b) = (D-b)(D-a)$$

(iv) Factor property:

$$(D-a)(D-b) = D^2 - (a+b)D + ab$$

(v) Index property:

$$D^m(D^n)y = D^{m+n} y$$

Complete solution of the Differential Equation:

In symbolic form, we can write D-E as —

$$(K_0 D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = f(x)$$

The complete (general) solution is, $y =$ Complementary fn + Particular Integral

$$y = y_c + y_p$$

Method to find Complimentary function (C.F):

Given Differential Equation (D.E) is,

$$(K_0 D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = f(x)$$

$$f(D) y = \phi(D) y = f(x)$$

Complementary function is a solution of $\phi(D) y = 0$ or $f(D) y = 0$

For C.F, Auxilliary equation is -

$$K_0 D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n = 0$$

Suppose $m_1, m_2, \cdots m_n$ are the roots of this equation.

Linear Differential Equation $f(D)y = 0$.

Consider Linear D.E

$$f(D)y = 0 \quad \text{——— (1)}$$

Since $f(D)$ is a polynomial in D of degree n, there are n roots of the polynomial equation $f(D) = 0$.

Suppose these roots are $m_1, m_2, \cdots m_n$

We can write,

$$(D-m_1)(D-m_2) \cdots (D-m_n)y = 0$$

The polynomial equation $f(D) = 0$ is known as auxilliary equation

Since, $(D-m_n)y = 0$

$$Dy - m_n y = 0$$

$$\therefore \frac{dy}{dx} = m_n y$$

$$\frac{dy}{y} = m_n dx$$

$$\therefore \log y = m_n x$$

$$\boxed{y = e^{m_n x}}$$

Similarly, there are $y = e^{m_1 x}, y = e^{m_2 x} \cdots y = e^{m_n x}$

are solutions of $(D-m_1)y=0$, $(D-m_2)y=0$ ... ... $(D-m_n)y=0$ respectively.

Hence general solution is given by -

$$C.F = \boxed{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}}$$

The general solution of $f(D)y=0$ depends on the nature of the roots of the auxilliary eq^n $f(D)=0$

CASE-I : If all roots are real and distinct

then
$$\boxed{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}}$$

CASE-II : If all roots are real and some are repeated.

Suppose some roots are repeated

say $m_1 = m_2$

$\therefore (D-m_1)^2 y = 0$

$\therefore (D-m_1)(D-m_1)y = 0$

If $(D-m_1)y = u$. —— (*)

then $(D-m_1)u = 0$

$\therefore$ sol^n is $u = c_1 e^{m_1 x}$

(*) becomes.

$(D-m_1)y = c_1 e^{m_1 x}$

$\therefore \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$

which is LDE $\left(\frac{dy}{dx} + P_1 y = Q\right)$

Hence $P = -m_1$, $Q = C_1 e^{m_1 x}$

and $\underline{\underline{I.F = e^{\int P\,dx}}} = e^{\int -m_1 dx}$
$= e^{-m_1 x}$

Hence solution is -

$$y(I.F) = \int Q(I.F)\,dx + C_2$$

$$\therefore y e^{-m_1 x} = \int C_1 e^{m_1 x} \cdot e^{-m_1 x}\,dx + C_2$$

$$\therefore y e^{-m_1 x} = C_1 x + C_2$$

$$\therefore y = (C_1 x + C_2)\, e^{m_1 x}$$

Similarly, If $\underline{\underline{m_1 = m_2 = m_3}}$

~~then sol^n is~~ $\cancel{y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x}}$

then sol^n is $\boxed{y = (C_1 + C_2 x + C_3 x^2)\, e^{m_1 x}}$

CASE III : If some roots are complex

If $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

Then solution is
$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x}\left[C_1 e^{i\beta x} + C_2 e^{-i\beta x}\right]$$

$$y_c = e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$

$$y_c = e^{\alpha x}[(c_1+c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x]$$

$$y_c = e^{\alpha x}[c_1'\cos\beta x + c_2'\sin\beta x]$$

is the solution.

CASE IV: If some roots are complex and repeated.

If $m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$.

$$y_c = (c_1 + c_2 x)e^{(\alpha+i\beta)x} + (c_3 + c_4 x)e^{(\alpha-i\beta)x}$$

$$y_c = e^{\alpha x}[(c_1+c_2x)(\cos\beta x + i\sin\beta x) + (c_3 + c_4 x)(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(c_1+c_3) + (c_2+c_4)x]\cos\beta x + [(ic_1 - ic_3) + (ic_2 - ic_4)x]\sin\beta x$$

$$y_c = e^{\alpha x}[(c_1' + c_2' x)\cos\beta x + (c_3' + c_4' x)\sin\beta x]$$

is the solution.

Methods of finding Particular Integral (P.I)

Particular method :

The Linear Differential Equation is -

$$\phi(D)y = f(x) \quad \text{---} \quad (1)$$

Since, $\phi(D)\left[\frac{1}{\phi(D)} f(x)\right] = f(x)$,

$\frac{1}{\phi(D)} f(x)$ satisfies eqn (1)

Hence

$$P.I = y_p = \frac{1}{\phi(D)} f(x)$$

General method to find Particular Integral (P.I) :

(i) Consider $Dy = f(x)$

$$\frac{dy}{dx} = f(x)$$

$$\therefore dy = f(x)dx$$

$$\Rightarrow y = \int f(x)dx$$

$$\therefore y_p = \frac{1}{D} f(x) = \int f(x)dx.$$

Similarly,

$$\frac{1}{D^2} f(x) = \iint f(x)dx$$

$$\frac{1}{D^3} f(x) = \iiint f(x)dx \text{ and so on}$$

(ii) Now $(D-a)y = f(x)$

$$\frac{dy}{dx} - ay = f(x)$$

$$\left\{ \text{LDE } \frac{dy}{dx} + Py = Q \right\}$$

$\therefore P = -a, \quad Q = f(x)$

$$\therefore I.F = e^{\int P dx} = e^{\int -a\,dx} = e^{-ax}$$

Hence solution is, –

$$y(I.F) = \int Q\,(I.F)\,dx$$

$$\therefore y \cdot e^{-ax} = \int f(x) \cdot e^{-ax}\,dx$$

$$\therefore y = e^{ax} \int f(x) \cdot e^{-ax}\,dx$$

$$\frac{1}{(D-a)} f(x) = y_p = e^{ax} \int e^{-ax} \cdot f(x)\,dx$$

$$\boxed{y_p = \frac{1}{(D-a)} f(x) = e^{+ax} \int e^{-ax} f(x)\,dx}$$

By using this formula, we can find P.I by two ways (i) Direct method

(ii) by partial fraction.

(i) Direct method :

$$P.I : \; y_p = \frac{1}{(D-a)(D-b)} f(x)$$

$$= \frac{1}{(D-a)}\left[e^{bx}\int e^{-bx} f(x)\,dx\right]$$

$$= \frac{1}{(D-a)} f_1(x)$$

where $f_1(x) = e^{bx}\int e^{-bx} f(x)\,dx$

$\therefore y_p = e^{ax}\int e^{-ax} f_1(x)$ and solve further.

(ii) By Partial fraction :

$$P.I : \; y_p = \frac{1}{(D-a)(D-b)} f(x)$$

$$= \frac{1}{a-b}\left[\frac{1}{(D-a)} - \frac{1}{(D-b)}\right] f(x)$$

← By partial fraction

$$= \frac{1}{a-b}\left[\frac{1}{(D-a)} f(x) - \frac{1}{(D-b)} f(x)\right]$$

$$y_p = \frac{1}{a-b}\left[e^{ax}\int e^{-ax} f(x)\,dx - e^{bx}\int e^{-bx} f(x)\,dx\right]$$

& then evaluate it.

Type-I : P.I By General Method.

Ex: 1. Solve: $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{1}{1+e^x}$

<u>Soln</u>: Given D.E is,

$$(D^2+D)y = \frac{1}{1+e^x} \quad \text{--- } (1)$$

The general solution of above DE is,

$$y = y_c + y_p$$
$$y = C.F + P.I \quad \text{--- } (2)$$

<u>Step-I</u> : <u>To find C.F</u> :

Auxilliary Equation is,

$$f(D)=0 \Rightarrow D^2+D=0$$
$$D(D+1)=0$$
$$\Rightarrow D=0, -1$$

Hence,

$$y_c = c_1e^{0x} + c_2e^{-x}$$
$$y_c = c_1 + c_2e^{-x} \quad \text{--- } (3)$$

<u>Step-II</u> :

$$y_p = \frac{1}{D^2+D}\left(\frac{1}{1+e^x}\right)$$
$$= \frac{1}{D(D+1)}\left(\frac{1}{1+e^x}\right)$$
$$= \frac{1}{D}\, e^{-x}\int e^x \cdot \frac{1}{1+e^x}\,dx$$

, $a = -$

$$= \frac{1}{D}\, e^{-x}\log(1+e^x)$$

ILATE : I ≡ Inverse

L ≡ Logarithmic

A ≡ Algebraic

T ≡ Trig.

E ≡ Expone.

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$$y_p = \frac{1}{D} e^{-x} \log(1+e^x)$$

$$y_p = \int \underset{u}{\log(1+e^x)} \cdot \underset{v}{e^{-x}}$$

$$\int uv\,dx = u \int v\,dx - \int \left[\left(\int v\,dx \right) \frac{du}{dx} \right] dx$$

Hence,

$$y_p = \log(1+e^x)\left(\frac{e^{-x}}{-1}\right) - \int \frac{e^x}{1+e^x}\left(\frac{e^{-x}}{-1}\right) dx$$

$$y_p = -e^{-x} \log(1+e^x) + \int \frac{1}{1+e^x}\, dx$$

$$= -e^{-x} \log(1+e^x) + \int \frac{(1+e^x) - e^x}{(1+e^x)}\, dx$$

$$= -e^{-x} \cdot \log(1+e^x) + \int \left(1 - \frac{e^x}{1+e^x} \right) dx$$

$$y_p = -e^{-x} \cdot \log(1+e^x) + x - \log(1+e^x) \quad (4)$$

Step-III : $y = y_c + y_p$

$$= c_1 + C_2 e^{-x} - e^{-x} \cdot \log(1+e^x) + x - \log(1+e^x)$$

H.W

Solve

2. ~~H.W~~ : $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$

3. Solve $(D^2-D-2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

4. Solve $(D^2+3D+2)y = e^{e^x} - \sin(e^x)$

5. Solve $(D^2+3D+2)y = e^{e^x}$

Answers.

1. $y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4} e^{-2x}[4\tan x - 3]$

3. $y = c_1 e^{2x} + c_2 e^{-x} - \log x$

4. $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}\left[e^{e^x} + \sin(e^x)\right]$

5. $y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

x

6. $(D^2 - 9D + 18) = e^{e^{-3x}}$

Soln: The complete general solution of given D.E is -

$$y = y_c + y_p = C.F + P.I$$

Step-I: Auxilliary Equation is -

$$D^2 - 9D + 18 = 0$$
$$(D-6)(D-3) = 0$$
$$D = 3, 6.$$

$$\therefore \quad y_c = c_1 e^{3x} + c_2 e^{6x}$$

Step-II : P.I: $y_p = \frac{1}{D^2-9D+18} e^{e^{-3x}}$

$$y_p = \frac{1}{(D-6)(D-3)} e^{e^{-3x}}$$

$$= \frac{1}{(D-6)} \cdot \frac{1}{(D-3)} e^{e^{-3x}}$$

$$y_p = \frac{1}{(D-6)} e^{3x} \int e^{-3x} \cdot e^{e^{-3x}} \, dx$$

$$\left[\text{As } \frac{1}{D-a} f(x) = e^{ax} \int e^{-ax} f(x) \, dx \right.$$

put $e^{-3x} = t$

$$\Rightarrow e^{-3x}(-3)\,dx = dt$$

$$e^{-3x}\,dx = \frac{dt}{-3}$$

$$y_p = \frac{1}{D-6} e^{3x} \int e^{t} \left(\frac{dt}{-3}\right)$$

$$= \frac{1}{(D-6)} e^{3x} \left(-\frac{1}{3}\right) \int e^{t}\,dt$$

$$= -\frac{1}{3} \frac{1}{(D-6)} e^{3x} e^{t} = -\frac{1}{3} \frac{1}{(D-6)} e^{3x} e^{e^{-3x}}$$

$$= -\frac{1}{3} \left[e^{6x} \int e^{-6x} \cdot e^{3x} \cdot e^{e^{-3x}} \, dx \right.$$

$$y_p = -\frac{1}{3} e^{6x} \int e^{-3x} \cdot e^{e^{-3x}} dx$$

Put $e^{-3x} = t$

$\therefore e^{-3x}(-3)\,dx = dt$

$\rightarrow e^{-3x}\,dx = \frac{dt}{-3}$

Hence,

$$y_p = -\frac{1}{3} e^{6x} \int e^t \left(\frac{dt}{-3}\right)$$

$$= +\frac{1}{9} e^{6x} \cdot e^t$$

$$y_p = \frac{1}{9} e^{6x} \cdot e^{e^{-3x}}$$

Hence,

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{6x} + c_2 e^{3x} + \frac{1}{9} e^{6x} \cdot e^{e^{-3x}}}$$

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Ex: 7. Solve $(D^2+9)y = \sec 3x$

Soln: The general solution of given D.E is —

$$y = y_c + y_p = C.F + P.I$$

Step-I: A.E (Auxiliary Equation) is,

$$D^2+9=0$$

$$\Rightarrow D^2=-9 \Rightarrow D=\pm 3i$$

$$\therefore y_c = C_1\cos 3x + C_2 \sin 3x$$

Step-II:
$$y_p = \frac{1}{D^2+9}\sec 3x$$

$$= \frac{1}{(D+3i)(D-3i)}\sec 3x$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right]\sec 3x$$

$$\left[\text{Since, } \frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right]\right.$$

Here $a=3i,\ b=-3i$

$a-b = 6i$

$$y_p = \frac{1}{6i}\left[\frac{1}{D-3i}\sec 3x - \frac{1}{D+3i}\sec 3x\right]$$

$$= \frac{1}{6i}\left\{e^{3ix}\int e^{-3ix}\sec 3x\,dx - e^{-3ix}\int e^{3ix}\sec 3x\,dx\right.$$

$$y_p = \frac{1}{6i}\left[e^{3ix}\int\left(\frac{\cos 3x - i\sin 3x}{\cos 3x}\right)dx - e^{-3ix}\int\left(\frac{\cos 3x + i\sin 3x}{\cos 3x}\right)dx\right]$$

$$y_p = \frac{1}{6i}\left[e^{3ix}\int(1 - i\tan 3x)\,dx - e^{-3ix}\int(1 + i\tan 3x)\,dx\right]$$

$$= \frac{1}{6i}\left[e^{3ix}\left(x - i\frac{\log(\sec 3x)}{3}\right) - e^{-3ix}\left(x + i\frac{\log(\sec 3x)}{3}\right)\right]$$

$$= \frac{1}{6i}\left[x\left(e^{3ix} - e^{-3ix}\right) - i\frac{\log(\sec 3x)}{3}\left(e^{3ix} + e^{-3ix}\right)\right]$$

$$y_p = \frac{1}{6i}\left[x(2i\sin 3x) - i\frac{\log(\sec 3x)}{3}(2\cos 3x)\right]$$

$$\left[\text{As } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right]$$

Here. $\theta = 3x$

$$y_p = \frac{2i}{6i}\left[x\sin 3x - \frac{\cos 3x}{3}\log(\sec 3x)\right]$$

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$$y_p = \frac{1}{3}\left[x\sin 3x - \frac{1}{3}\cos 3x \cdot \log(\sec 3x)\right]$$

Step-III : Hence, complete solution is,

$$y = y_c + y_p$$

$$y = C_1\cos 3x + C_2\sin 3x + \frac{1}{3}\left[x\sin 3x - \frac{1}{3}\cos 3x \cdot \cancel{\log}\ \log(\sec 3x)\right]$$

SHORT METHODS to find P.I (Particular Integral)

Rule-1: $f(x) = e^{ax}$

If $f(x) = e^{ax}$ then $P.I = \frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}$

(provided $\phi(a) \neq 0$)

We have, $\boxed{\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax} \quad (\text{provided } \phi(a) \neq 0)}$

$\boxed{\text{If } \phi(a) = 0 \text{ then } \frac{1}{\phi(D)} e^{ax} = \frac{x}{\phi'(a)} e^{ax} \quad \text{provided } \phi'(a) \neq 0}$

$\boxed{\text{If } \phi'(a) = 0, \text{ then } \frac{1}{\phi(D)} e^{ax} = \frac{x^2}{\phi''(a)} e^{ax} \quad \text{provided } \phi''(a) \neq 0}$

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Explanation:

Since $D(e^{ax}) = ae^{ax}$

$$D^2(e^{ax}) = a^2e^{ax}$$

$$D^3(e^{ax}) = a^3e^{ax}$$ and so on.

$$\boxed{\phi(D)\, e^{ax} = \phi(a)\, e^{ax}} \quad \text{——— } (*)$$

operating $\frac{1}{\phi(D)}$ on both sides.

$$\frac{1}{\phi(D)}\phi(D)e^{ax} = \frac{1}{\phi(D)}\phi(a)\, e^{ax}$$

$$e^{ax} = \frac{1}{\phi(D)}\phi(a)\, e^{ax}$$

$$\Rightarrow \frac{\cancel{e^{ax}}}{\cancel{\phi(a)}}$$

$$\frac{1}{\phi(a)} e^{ax} = \frac{1}{\phi(D)} e^{ax}$$

Hence,

$$\boxed{\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}, \text{ provided } \phi(a) \neq 0}$$

If $\phi(a) = 0$ then

$$\phi(D) = (D-a)\, g(D)$$

// As $\phi(D) = D^4 - 2D^3 - 3D^2 + 4D + 4 = 0$

Now, $\phi(-1) = 0$

$\Rightarrow$ $\phi(D) = \underline{(D+1)} (D^3 - 3D^2 + 4)$ //

$[(D-(-1))]$

$\therefore \dfrac{1}{\phi(D)} e^{ax} = \dfrac{1}{(D-a)g(a)} \cdot e^{ax}$

$= \dfrac{1}{g(a)} \dfrac{1}{(D-a)} e^{ax}$

$= \dfrac{1}{g(a)} e^{ax} \int e^{-ax} e^{ax} dx$

$= \dfrac{1}{g(a)} e^{ax} \int 1 \, dx$

$= \dfrac{1}{g(a)} e^{ax} \cdot x$

$\boxed{\dfrac{1}{\phi(D)} e^{ax} = \dfrac{x}{g(a)} e^{ax}}$ ——— (1)

Now, $\phi(D) = (D-a) g(D)$

$\phi'(D) = (D-a) g'(D) + g(D) \times 1$

$\underline{\phi'(a) = g(a)}$

put in (1)

$$\therefore \boxed{\frac{1}{\phi(D)} e^{ax} = \frac{x}{\phi'(a)} e^{ax}, \text{ provided } \phi'(a) \neq 0}$$

and so on.

<u>Note that</u>: (i) $\frac{1}{(D-a)} e^{ax} = \frac{x^r}{r!} e^{ax}$.

(ii) $\frac{1}{(D-a)^r g(D)} e^{ax} = \frac{x^r}{g(a) \cdot r!} e^{ax}$

provided $g(a) \neq 0$

<u>Note</u>: In the following case, we can use Rule-1.

(i) We can write $b^x = e^{\log b^x} = e^{x \log b} = e^{ax}$

where $a = \log b$

(ii) Also we can write $b = be^{0x}$, where $a = 0$

(iii) $\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

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Examples on Short Methods

Ex: 1 Solve $(D^5 - D)y = 4e^x + 2^{-x}$

Soln: The general solution of the given D.E is –

$$y = y_c + y_p = C.F + P.I$$

Step-I: A.E (Auxilliary Eqⁿ) is –

$$\phi(D) = 0 \Rightarrow D^5 - D = 0$$

$$D(D^4 - 1) = 0$$

$$\Rightarrow D(D^2-1)(D^2+1) = 0$$

$$\Rightarrow D = 0,\ D = \pm 1,\ D = \pm i$$

$$\therefore \boxed{y_c = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-x} + e^{0x}(c_4 \cos x + c_5 \sin x)}$$

Step-II: $P.I \doteq y_p = \frac{1}{D^5 - D}(4e^x + 2^{-x})$

$$\therefore y_p = \frac{1}{D^5 - D} 4e^x + \frac{1}{D^5 - D} 2^{-x}$$

$(a = 1)$ $(a = -\log 2)$

Since $2^{-x} = e^{\log 2^{-x}} = e^{-x \log 2}$

$$= e^{(-\log 2)x}$$

$$a = -\log 2$$

$$y_p = \frac{x}{5D^4 - 1} 4e^x + \frac{1}{(-\log 2)^5 - (-\log 2)} 2^{-x}$$

put $D=1$

$$y_p = \frac{x}{5-1} 4e^x - \frac{1}{(\log 2)^5 - (\log 2)} 2^{-x}$$

$$y_p = \frac{4xe^x}{4} - \frac{1}{(\log 2)^5 - (\log 2)} 2^{-x}$$

$$y_p = xe^x - \frac{1}{(\log 2)^5 - (\log 2)} 2^{-x}$$

Step-III : Hence the complete solution is

$$y = y_c + y_p$$

$$\therefore y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$$
$$+ xe^x - \frac{2^{-x}}{(\log 2)^5 - (\log 2)}$$

Ex: 2. Solve $(D^3 - 5D^2 + 8D - 4)y = 2e^x + e^{2x}$

Soln: The general solution of above D.E is

$$y = y_c + y_p$$

Step-I: A.E (Auxilliary Eqn) is

$$D^3 - 5D^2 + 8D - 4 = 0$$

	1	-5	8	-4
1	0	1	-4	4
	1	-4	4	$\boxed{0}$

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Hence

$$D^3 - 5D^2 + 8D - 4 = 0$$

$$\Rightarrow (D-1)(D^2-4D+4) = 0$$

$$\Rightarrow (D-1)(D-2)^2 = 0$$

$$\therefore D = 1, 2, 2$$

$$\therefore y_C = C_1 e^{x} + (C_2 + C_3 x) e^{2x}$$

<u>Step-II</u>: $y_p = \dfrac{1}{D^3 - 5D^2 + 8D - 4}(2e^x + e^{2x})$

$$= \frac{1}{(D^3 - 5D^2 + 8D - 4)} 2e^x \quad [D=1]$$

$$+ \frac{1}{D^3 - 5D^2 + 8D - 4} e^{2x} \quad [D=2]$$

$$= \frac{x}{3D^2 - 10D + 8} 2e^x + \frac{x}{3D^2 - 10D + 8} e^{2x}$$

(since $\phi(1) = 0$, $\phi(2) = 0$ in prev. step)

$$= \frac{x}{3(1) - 10(1) + 8} 2e^x + \frac{x^2}{6D - 10} e^{2x}$$

since $\phi(2) = 0$ in prev. step

$$= \frac{x}{1} \cdot 2e^x + \frac{x^2}{6(2) - 10} e^{2x}$$

$$y_p = 2xe^x + \frac{x^2}{2} \times e^{2x}$$

Hence, complete solution is -

$$y = y_c + y_p$$

$$y = c_1 e^{x} + (c_2 + c_3 x) e^{2x} + 2x e^{x} + \frac{x^2}{2} e^{2x}$$

Ex: 3. Solve $(D^2 + 6D + 9) y = 5^x - \log 2$

Soln: The general solution of the given D.E is,

$$y = y_c + y_p = C.F + P.I$$

Step-I: A.E (Auxilliary eqn) is -

$$\phi(D) = 0$$

$$\Rightarrow D^2 + 6D + 9 = 0$$

$$(D+3)^2 = 0$$

$$\Rightarrow D = -3, -3$$

$$\therefore \quad y_c = (C_1 + C_2 x) e^{-3x}$$

Step-II: $y_p = \frac{1}{D^2 + 6D + 9} (5^x - \log 2)$

$$= \frac{1}{(D^2 + 6D + 9)} \left(e^{(\log 5)x} - (\log 2) e^{0x} \right)$$

$D = \log 5 \qquad D = 0$

$$= \frac{1}{\underset{(D = \log 5)}{(D+3)^2}} e^{(\log 5)x} - \frac{1}{\underset{(D=0)}{(D+3)^2}} (\log 2) e^{0x}$$

$$= \frac{5^x}{(\log 5 + 3)^2} - \frac{1}{(0+3)^2} (\log 2)$$

$$\therefore y_p = \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Step-III : Hence $y = y_c + y_p$

$$\boxed{\therefore y = (C_1 + C_2 x)e^{-3x} + \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}}$$

Ex: 4. Solve $(D^4 - 4D^3 + 6D^2 - 4D + 1)y = e^x + 2^x + \frac{1}{3}$

Soln: The general solution of above DE is -

$$y = y_c + y_p = C.F + P.I$$

Step-I : A.E (Auxilliary Eqn) is -

$$D^4 - 4D^3 + 6D^2 - 4D + 1 = 0$$

$$\Rightarrow (D-1)^4 = 0$$

$$D = 1, 1, 1, 1.$$

OR

1	1	-4	6	-4	1
	0	1	-3	3	-1
1	1	-3	3	-1	$\boxed{0}$
	0	1	-2	1	
1	1	-2	1	$\boxed{0}$	
	0	1	-1		
1	1	-1	$\boxed{0}$		
	0	1			
	1	$\boxed{0}$			

$$\therefore D = 1, 1, 1, 1$$

Hence, $y_c = (c_1 + c_2 x + c_3 x^2 + c_4 x^4) e^x$

Step-II : To find y_p :

$$y_p = \frac{1}{(D-1)^4}\left(e^x + 2^x + \frac{1}{3}\right)$$

$$= \frac{1}{(D-1)^4} e^x + \frac{1}{(D-1)^4} e^{(\log 2)x} + \frac{1}{(D-1)^4}\frac{1}{3}e^{0x}$$

$$(D=1) \qquad (D=\log 2) \qquad (D=0)$$

$$y_p = \frac{x^4}{4!}e^x + \frac{1}{(\log 2-1)^4}2^x + \frac{1}{3}$$

$$\boxed{\text{As } \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}}$$

Step-III : $y = y_c + y_p$

$$\boxed{\therefore y = (c_1 + c_2 x + c_3 x^2 + c_4 x^4)e^x + \frac{x^4}{4!}e^x + \frac{2^x}{(\log 2-1)^4} + \frac{1}{3}}$$

5. Solve $\dfrac{d^3y}{dx^3} - y = (1+e^x)^2$

OR

$$(D^3-1)y = (1+e^x)^2$$

Ans: $y = c_1 e^{x} + e^{-1/2x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$

$$-1 + \frac{2x}{3}e^x + \frac{e^{2x}}{7}$$

Soln: Given D.E is $(D^3-1)y = (1+e^x)^2$

The general solution of the given D.E is,

$$y = y_c + y_p \quad \text{——— (1)}$$

Step-1: A.E (Auxilliary Eqn) is, $(D^3-1) = 0$

$$\therefore (D-1)(D^2+D+1) = 0$$

// As $a^3-b^3 = (a-b)(a^2+ab+b^2)$

$\Rightarrow D = 1,\ D^2+D+1 = 0$

$$\Rightarrow D = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$= \alpha \pm i\beta$$

Hence,

$$y_c = c_1 e^{x} + e^{-1/2x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] \quad \text{——— (2)}$$

Step-2: $y_p = \dfrac{1}{D^3-1}(1+e^x)^2 = \dfrac{1}{(D^3-1)}(1+2e^x+e^{2x})$

$$= \frac{1}{(D^3-1)}(1) + \frac{1}{(D^3-1)} 2e^{x} + \frac{1}{(D^3-1)} e^{2x}$$

$$= \frac{1}{D^3-1} e^{0x} + \frac{1}{D^3-1} 2e^{x} + \frac{1}{(D^3-1)} e^{2x}$$

[use $D=0$ $\quad D=1 \quad D=2$]

$$= \frac{1}{(0-1)} 1 + \frac{x}{3D^2} 2e^{x} + \frac{1}{8-1} e^{2x}$$

$D=1$

$$= (-1) + \frac{x}{3(1)^2} 2e^{x} + \frac{1}{7} e^{2x}$$

$$y_p = -1 + \frac{2x}{3} e^{x} + \frac{e^{2x}}{7}$$

Hence,

Step-III: $y = y_c + y_p$

$$y = c_1 e^{x} + e^{-1/2 x}\left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right] - 1 + \frac{2x}{3} e^{x} + \frac{e^{2x}}{7}$$

Ex:6 Solve $(D^3 - 3D^2 + 3D - 1)y = e^{2x} \cosh x$

Soln: The general solution of given D.E is

$$y = y_c + y_p = C.F + P.I$$

Step-1: A.E (Auxilliary Equation) is,

$$D^3 - 3D^2 + 3D - 1 = 0$$
$$(D-1)^3 = 0$$
$$D = 1, 1, 1.$$

$$\therefore \quad y_c = (c_1 + c_2 x + c_3 x^2) e^x$$

Step-II : $y_p = \dfrac{1}{(D-1)^3} e^{2x} \cosh x)$

$$= \frac{1}{(D-1)^3} e^{2x} \left(\frac{e^x + e^{-x}}{2} \right)$$

$$= \frac{1}{2} \frac{1}{(D-1)^3} (e^{3x} + e^x)$$

$$= \frac{1}{2} \left[\frac{1}{(D-1)^3} e^{3x} + \frac{1}{(D-1)^3} e^x \right]$$

Use $D=3$, $D=1$

$$= \frac{1}{2} \left[\frac{1}{(3-1)^3} e^{3x} + \frac{x^3}{3!} e^x \right]$$

$$y_p = \frac{1}{2} \left[\frac{e^{3x}}{8} + \frac{x^3}{6} e^x \right]$$

$$y_p = \frac{e^{3x}}{16} + \frac{x^3}{12} e^x$$

Step-III : Hence, complete solution is,

$$y = (c_1 + c_2 x + c_3 x^2) e^x + \frac{e^{3x}}{16} + \frac{x^3}{12} e^x$$

Rule-2 : $f(x) = \sin(ax+b)$ OR $\cos(ax+b)$

If $f(x) = \sin(ax+b)$ then $P.I = \frac{1}{\phi(D^2)}\sin(ax+b)$

$= \frac{1}{\phi(-a^2)}\sin(ax+b)$

provided $\phi(-a^2) \neq 0$

If $\phi(-a^2) = 0$, then,

$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{x}{\phi'(-a^2)}\sin(ax+b)$

provided $\phi'(-a^2) \neq 0$

and so on.

Explanation :

We have, $D\sin(ax+b) = a\cos(ax+b)$

$D^2\sin(ax+b) = -a^2\sin(ax+b)$

$D^3\sin(ax+b) = -a^3\cos(ax+b)$

$D^4\sin(ax+b) = a^4\sin(ax+b)$

$\Rightarrow (D^2)^2\sin(ax+b) = (-a^2)^2\sin(ax+b)$

Hence, in general,

$(D^2)^r\sin(ax+b) = (-a^2)^r\sin(ax+b)$

~~Hence~~, $\Rightarrow \phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$

operating $\frac{1}{\phi(D^2)}$ on both sides.

$$\frac{1}{\phi(D^2)}\phi(D^2)\sin(ax+b) = \frac{1}{\phi(D^2)}\phi(-a^2)\sin(ax+b)$$

$$\therefore \sin(ax+b) = \frac{\phi(-a^2)}{\phi(D^2)}\sin(ax+b)$$

Dividing by $\phi(-a^2)$ provided $\phi(-a^2) \neq 0$

$$\therefore \frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\Rightarrow \boxed{\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)}$$

provided $\phi(-a^2) \neq 0$

Since,

$$\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$$

$$\therefore \frac{1}{\phi(D^2)}\sin(ax+b) = \text{Im.Part of } \frac{1}{\phi(D^2)}e^{i(ax+b)}$$

I.P ≡ Imaginary Part (since $\phi(-a^2)=0$

$$= \text{I.P of } \frac{x}{\phi'(D^2)}e^{i(ax+b)}$$

By Rule-1

where $D^2 = -a^2$

$$\boxed{\therefore \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{x}{\phi'(-a^2)}\sin(ax+b)}$$

provided $\phi(-a^2) \neq 0$

& so on

Similarly,

$$\boxed{\frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{\phi(-a^2)}\cos(ax+b), \quad \text{provided } \phi(-a^2)\neq 0}$$

$$\boxed{\begin{aligned}&\text{If } \phi(-a^2)=0. \\ &\frac{1}{\phi(D^2)}\cos(ax+b) = \frac{x}{\phi'(-a^2)}\cos(ax+b) \quad \text{provided } \phi'(-a^2)\neq 0\end{aligned}}$$

& so on.

Note:

(i) $$\frac{1}{D^2+a^2}\sin(ax+b) = -\frac{x}{2a}\cos(ax+b)$$

(ii) $$\frac{1}{D^2+a^2}\cos(ax+b) = -\frac{x}{2a}\sin(ax+b)$$

(iii) $$\frac{1}{(D^2+a^2)^n}\sin(ax+b) = \left(-\frac{x}{2a}\right)^n\frac{1}{n!}\sin\left(ax+b+\frac{n\pi}{2}\right)$$

(iv) $$\frac{1}{(D^2+a^2)^n}\cos(ax+b) = \left(-\frac{x}{2a}\right)^n\frac{1}{n!}\cos\left(ax+b+\frac{n\pi}{2}\right)$$

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NOTE :

1. $\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

2. $\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$

3. $\sin x \cdot \sin y = -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$

4. $\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$

5. $\cos^2 x = \frac{1+\cos 2x}{2}$

6. $\sin^2 x = \frac{1-\cos 2x}{2}$

7. $\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$

8. $\cos^3 x = \frac{1}{4}\left[\cos 3x + 3\cos x\right]$

Ex: 1 Solve : $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$

Soln : The given D.E is

$$(D^3 + D^2 - D - 1)y = \cos 2x$$

The general solution of D.E is :

$$y = y_c + y_p$$

Step-I : A.E (Auxilliary Eqn) is -

$$D^3 + D^2 - D - 1 = 0$$

$$\Rightarrow D^2(D+1) - 1(D+1) = 0$$

$$(D+1)(D^2-1) = 0$$

$$\Rightarrow (D+1)(D-1)(D+1) = 0$$

$$\Rightarrow D = 1, -1, -1$$

$$\therefore y_c = c_1 e^{x} + (c_2 + c_3 x)e^{-x}$$

Step-II : $y_p = \frac{1}{(D+1)(D^2-1)} \cos 2x$.

Use $D^2 = -4$, $\cos 2x = \cos ax$

$// D^2 = -a^2 //$ $a = 2$

$$= \frac{1}{(D+1)(-4-1)} \cos 2x$$

$$= -\frac{1}{5} \cdot \frac{1}{(D+1)} \cos 2x$$

$a = -1$

$$= -\frac{1}{5}\left[e^{-x} \int e^{x} \cos 2x \, dx \right]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right] + C$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right] + C$$

$$y_p = -\frac{1}{5}\left[e^{-x}\int e^{x} \cos 2x \, dx\right]$$

$a = 1, b = 2$

$$= -\frac{1}{5}\left[e^{-x} \frac{e^{x}}{1+4}\left[1 \cos 2x + 2 \sin 2x\right]\right]$$

$$= -\frac{1}{5 \times 5}\left[\cos 2x + 2 \sin 2x\right]$$

$$y_p = -\frac{1}{25}\left(\cos 2x + 2 \sin 2x\right)$$

Hence,
complete solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{x} + (c_2 + c_3 x)e^{-x} + \left(-\frac{1}{25}\right)\left(\cos 2x + 2 \sin 2x\right)}$$

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OR $y_p = \frac{1}{(D+1)(D^2-1)} \cos 2x$ $\cos 2x = \cos ax$

use $D^2 = -4$; $a = 2$

$= \frac{1}{(D+1)(-4-1)} \cos 2x$

$= -\frac{1}{5} \frac{1}{(D+1)} \times \frac{D-1}{D-1} \cos 2x$

$= \frac{-(D-1)}{5(D^2-1)} \cos 2x$

use $D^2 = -4$

$= \frac{-(D-1)}{5(-4-1)} \cos 2x$

$= +\frac{1}{25} (D-1) \cos 2x$

$= \frac{1}{25} (-2\sin 2x - \cos 2x)$

$y_p = -\frac{1}{25} (2\sin 2x + \cos 2x)$

Ex: 2. Solve: $(D^3+4D)y = \sin 2x + 2^x$

Soln: The general soln of given D.E is -

$y = y_c + y_p$

Step-I: A.E (Auxilliary Eqn) is -

$D^3 + 4D = 0$

$D(D^2+4) = 0$

$D = 0$, $D^2 + 4 = 0$

$D^2 = -4 \Rightarrow D = \pm 2i$

$$D = 0, \pm 2i$$

$$\alpha = 0, \beta = 2$$

Hence $C.F = y_c = c_1 e^{0x} + e^{0x}[c_2 \cos 2x + c_3 \sin 2x]$

$$y_c = c_1 + c_2 \cos 2x + c_3 \sin 3x$$

Step-II : $y_p = \frac{1}{D^3+4D}(\sin 2x + 2^x)$

$$y_p = \frac{1}{D^3+4D} \sin 2x + \frac{1}{D^3+4D} 2^x$$

$\underline{(a=2)}$

Use $D^2 = -4$ Use $D = \log 2$

As $\phi(D^2) = \phi(-4) = 0$

$$y_p = \frac{x}{3D^2+4} \sin 2x + \frac{1}{(\log 2)^3 + 4(\log 2)} 2^x$$

Use $D^2 = -4$

$$= \frac{x}{3(-4)+4} \sin 2x + \frac{1}{(\log 2)^3 + 4(\log 2)} 2^x$$

$$= \frac{x}{-8} \sin 2x + \frac{1}{(\log 2)^3 + 4(\log 2)} 2^x$$

Hence,

Complete soln is $y = y_c + y_p$

$$\therefore y = c_1 + c_2 \cos 2x + c_3 \sin 3x - \frac{x}{8} \sin 2x + \frac{2^x}{(\log 2)^3 + 4(\log 2)}$$

HW.

3. Solve $(D^4-16)y = 2\cos^2 x$

Ans: $y = C_1 e^{2x} + C_2 e^{-2x} + C_3\cos 2x + C_4 \sin 2x - \frac{1}{16}\left[1 + \frac{x\sin 2x}{2}\right]$

4. Solve: $(D^3+4D)y = \sin 5x \cdot \cos 3x$.

Ans: $y = C_1 + C_2\cos 2x + C_3\sin 2x + \frac{1}{16}\left[\frac{\cos 8x}{60} - x\sin 2x\right]$

$\boxed{\text{Rule-3 : If } f(x) = \sinh ax \text{ OR } \cosh ax}$

If $f(x) = \sinh(ax+b)$ then

$$P.I = y_p = \frac{1}{\phi(D^2)}\sinh(ax+b) = \frac{1}{\phi(a^2)}\sinh(ax+b)$$

provided $\phi(a^2) \neq 0$

If $\phi(a^2) = 0$

$$\frac{1}{\phi(D^2)}\sinh(ax+b) = \frac{x}{\phi'(a^2)}\sinh(ax+b)$$

provided $\phi'(a^2) \neq 0$

& so on.

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Ex: 1. Solve: $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = \sinh 2x$

Soln: Given D.E is -

$$(D^3 - 4D)y = \sinh 2x$$

The general solution of the given D.E is -

$$y = y_c + y_p$$

Step-I: A.E (Auxilliar Eqn) is —

$$\phi(D) = 0$$

$$\Rightarrow D^3 - 4D = 0$$

$$\therefore D(D^2 - 4) = 0$$

$$\therefore D = 0, \quad D^2 - 4 = 0$$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

Hence $D = 0, 2, -2$

Hence $y_c = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$

$$= c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Step-II: To find y_p:

$$y_p = \frac{1}{\phi(D)} f(x) = \frac{1}{D^3 - 4D} \sinh 2x$$

[use $D^2 = a^2 = 4$]

$$= \frac{x}{3D^2 - 4} \sinh 2x$$

[use $D^2 = 4$]

$$\therefore y_p = \frac{x}{3 \times 4 - 4} \sinh 2x$$

$$= \frac{x}{8} \sinh 2x$$

Step-III : Hence complete solution is

$$y = y_c + y_p$$

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{x}{8} \sinh 2x$$

Ex: 2 : Solve: $(D^3 - 4D)y = 2\cosh^2 2x$

Soln. The general solution of the given D.E is -

$$y = y_c + y_p$$

Step-I : A.E (Auxilliary Eqn) is -

$$D^3 - 4D = 0$$
$$D(D^2 - 4) = 0$$
$$D = 0, 2, -2$$

Hence,

$$y_c = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$$
$$y_c = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Step-II : To find y_p :

$$y_p = \frac{1}{D^3 - 4D} 2\cosh^2 2x$$

$$= \frac{1}{D^3 - 4D} (1 + \cosh 4x)$$

$$= \frac{1}{D^3 - 4D} 1 e^{0x} + \frac{1}{D^3 - 4D} \cosh 4x \qquad a = 4$$

Use $D = 0$ $\qquad$ Use $D^2 = 16 = a^2$

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$$\therefore y_p = \frac{x}{3D^2-4} e^{0x} + \frac{1}{D(16)-4D} \cosh 4x$$

use $D=0$

$$\therefore y_p = -\frac{x}{4} + \frac{1}{12D} \cosh 4x$$

$$= -\frac{x}{4} + \frac{1}{12} \int \cosh 4x \, dx$$

$$= \frac{-x}{4} + \frac{1}{12} \frac{\sinh 4x}{4}$$

$$= \frac{-x}{4} + \frac{1}{48} \sinh 4x$$

Step-III :

Hence complete solution is -

$$y = y_c + y_p$$
$$= C_1 + C_2 e^{2x} + C_3 e^{-2x} + \frac{1}{4}\left[\frac{1}{12}\sinh 4x - x\right]$$

Ex: 3. Solve: $(D^3+3D)y = \cosh 2x \sinh 3x$

$= \sinh 3x \cdot \cosh 2x$

<u>Solⁿ</u>: The general solution of given D.E is –

$y = y_c + y_p = C.F + P.I$

<u>Step-I</u>: A.E (Auxilliary Eqⁿ) is –

$D^3+3D = 0$

$D(D^2+3) = 0$

$$\Rightarrow D=0\ ;\ D^2+3=0 \Rightarrow D^2=-3 \Rightarrow D=\pm\sqrt{3}\,i$$

$$\alpha=0\ ;\ \beta=\sqrt{3}$$

$$\therefore y_c = c_1 e^{0x} + e^{0x}\left[c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right]$$

$$y_c = c_1 + c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x$$

<u>Step-II</u> : $P.I. = y_p = \dfrac{1}{D^3+3D}\sinh 3x\cdot\cosh 2x$

$$\left[\text{As}\quad \sinh 3x\cdot\cosh 2x = \frac{1}{2}\left[\sinh 5x + \sinh x\right]\right]$$

Now, $y_p = \dfrac{1}{D^3+3D}\,\dfrac{1}{2}\left[\sinh 5x+\sinh x\right]$

$$= \frac{1}{2}\,\frac{1}{D^3+3D}\underset{a=5}{\sinh 5x} + \frac{1}{2}\,\frac{1}{D^3+3D}\underset{a=1}{\sinh x}$$

$$\left[\text{Use: } D^2 = 25 = 5^2\right] \qquad \left[\text{Use } D^2 = 1^2 = 1\right]$$

$$\therefore y_p = \frac{1}{2}\,\frac{1}{25D+3D}\sinh 5x + \frac{1}{2}\,\frac{1}{D+3D}\sinh x$$

$$= \frac{1}{2}\,\frac{1}{28D}\sinh 5x + \frac{1}{2}\,\frac{1}{4D}\sinh x$$

$$= \frac{1}{56}\int \sinh 5x\,dx + \frac{1}{8}\int \sinh x\,dx$$

$$= \frac{1}{56}\,\frac{\cosh 5x}{5} + \frac{1}{8}\cdot\frac{\cosh x}{1}$$

$$y_p = \frac{\cosh 5x}{280} + \frac{\cosh x}{8}$$

Hence,

$y = y_c + y_p$

$y = c_1 + c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x + \frac{1}{280}\cosh 5x + \frac{1}{8}\cosh x$

$\boxed{\text{Rule-4 : If } f(x) = x^m}$

If $f(x) = x^m$, then $P.I = \frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$, $m>0$

Expand $[\phi(D)]^{-1}$ in ascending powers of D up to the highest power of D is m and then operate on x^m

Note: 1. $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$

2. $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$

In general,

$f(x) =$ polynomial in x of the form

$p(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_n$

$PI = y_p = \frac{1}{\phi(D)} p(x)$

Take lowest degree term from denominator and write $\phi(D)$ in the form

$\phi(D) = 1 \pm g(D)$

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$$\therefore \; y_p = \frac{1}{1 \pm g(D)} p(x)$$

$$y_p = [1 \pm g(D)]^{-1} p(x)$$

Expand $[1 \pm g(D)]^{-1}$ in ascending powers of x and operate each terms of D on $p(x)$

Steps to find P.I:

Step-I: Write $\phi(D)$ in ascending powers of D

e.g. $\frac{1}{D^2+D+4}$ as $\frac{1}{4+D+D^2}$

Step-II: Common out lowest degree term of $\phi(D)$ and write $\frac{1}{\phi(D)}$ in the form of $\frac{1}{1+Z}$

e.g. $\frac{1}{4+D+D^2}$ as $\frac{1}{4\left[1+\left(\frac{D+D^2}{4}\right)\right]}$

Step-III: Write $\frac{1}{1+Z}$ as $(1+Z)^{-1}$ and

use $(1+Z)^{-1} = 1 - Z + Z^2 - Z^3 + Z^4 \ldots$ upto highest powers of x

e.g. $\frac{1}{4\left[1+\left(\frac{D+D^2}{4}\right)\right]}$ as $\frac{1}{4}\left[1+\left(\frac{D+D^2}{4}\right)\right]^{-1}$

$$= \frac{1}{4}\left[1-\left(\frac{D+D^2}{4}\right)+\left(\frac{D+D^2}{4}\right)^2-\left(\frac{D+D^2}{4}\right)^3\cdots\right]$$

Step-IV : Apply D, D^2, D^3, etc on $f(x)$

$$\therefore D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}$$

Ex: 1. Solve : $(D^2+2D+2)y = x^3-4x$

Soln : The general soln of given D.E is

$$y = y_c + y_p$$

Step-I : A.E (Auxilliary Eqn) is -

$$D^2+2D+2=0$$

$$\therefore D = \frac{-2 \pm \sqrt{4-4(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\alpha = -1, \beta = 1$$

$$\therefore y_c = e^{-x}\left[c_1 \cos x + c_2 \sin x\right]$$

Step-II : $$y_p = \frac{1}{D^2+2D+2}(x^3-4x)$$

$$= \frac{1}{2+2D+D^2}(x^3-4x)$$

$$= \frac{1}{2\left[1+\left(\frac{2D+D^2}{2}\right)\right]}(x^3-4x)$$

$$\therefore y_p = \frac{1}{2}\left[1+\left(\frac{2D+D^2}{2}\right)\right]^{-1}(x^3-4x)$$

$$= \frac{1}{2}\left[1-\left(\frac{2D+D^2}{2}\right)+\left(\frac{2D+D^2}{2}\right)^2 - \left(\frac{2D+D^2}{2}\right)^3 \cdots\right](x^3-4x)$$

$$= \frac{1}{2}\left[1-D-\frac{D^2}{2}+\frac{1}{4}(4D^2+4D^3+D^4) - \frac{8D^3}{8}\right](x^3-4x)$$

$$y_p = \frac{1}{2}\left[1-D+\frac{D^2}{2}\right](x^3-4x)$$

$$= \frac{1}{2}\left[(x^3-4x)-(3x^2-4)+\frac{1}{2}(6x)\right]$$

$$= \frac{1}{2}\left[x^3-3x^2-x+4\right]$$

Hence, complete solution is —

$$y = y_c + y_p$$

$$y = e^{-x}\left[c_1\cos x + c_2\sin x\right] + \frac{1}{2}\left(x^3-3x^2-x+4\right]$$

Ex: Solve: $(D^3+3D^2-4)y = x^2+x+1$

Rule-4 : If $f(x) = x^m$

If $f(x) = x^m$, then $P.I = \frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$,

$m > 0$

Expand $[\phi(D)]^{-1}$ in ascending powers of D up to the highest power of D is m and then operate on x^m

Note: 1. $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$

2. $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$

In general,

$f(x) =$ polynomial in x of the form

$p(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_n$

$$PI = y_p = \frac{1}{\phi(D)} p(x)$$

Take lowest degree term from denominator and write $\phi(D)$ in the form

$$\phi(D) = 1 \pm g(D)$$

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$$\therefore y_p = \frac{1}{1 \pm g(D)} p(x)$$

$$y_p = [1 \pm g(D)]^{-1} p(x)$$

Expand $[1 \pm g(D)]^{-1}$ in ascending powers of x and operate each terms of D on $p(x)$

Steps to find P.I:

Step-I : Write $\phi(D)$ in ascending powers of D

e.g. $\frac{1}{D^2+D+4}$ as $\frac{1}{4+D+D^2}$

Step-II : Common out lowest degree term of $\phi(D)$ and write $\frac{1}{\phi(D)}$ in the form of $\frac{1}{1+Z}$

e.g. $\frac{1}{4+D+D^2}$ as $\frac{1}{4\left[1+\left(\frac{D+D^2}{4}\right)\right]}$

Step-III : Write $\frac{1}{1+Z}$ as $(1+Z)^{-1}$ and

use $(1+Z)^{-1} = 1 - Z + Z^2 - Z^3 + Z^4 \ldots$ upto highest powers of x.

e.g. $\frac{1}{4\left[1+\left(\frac{D+D^2}{4}\right)\right]}$ as $\frac{1}{4}\left[1+\left(\frac{D+D^2}{4}\right)\right]^{-1}$

$$= \frac{1}{4}\left[1 - \left(\frac{D+D^2}{4}\right) + \left(\frac{D+D^2}{4}\right)^2 - \left(\frac{D+D^2}{4}\right)^3 \cdots \right]$$

Step-IV : Apply D, D^2, D^3, etc on $f(x)$

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}$$

Ex: 1. Solve : $(D^2+2D+2)\,y = x^3 - 4x$

Soln : The general soln of given D.E is

$$y = y_c + y_p$$

Step-I : A.E (Auxilliary Eqn) is -

$$D^2+2D+2 = 0$$

$$\therefore D = \frac{-2 \pm \sqrt{4-4(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\alpha = -1, \beta = 1$

$$\therefore y_c = e^{-x}\left[C_1 \cos x + C_2 \sin x\right]$$

Step-II :

$$y_p = \frac{1}{D^2+2D+2}(x^3-4x)$$

$$= \frac{1}{2+2D+D^2}(x^3-4x)$$

$$= \frac{1}{2\left[1+\left(\frac{2D+D^2}{2}\right)\right]}(x^3-4x)$$

$$\therefore y_p = \frac{1}{2}\left[1+\left(\frac{2D+D^2}{2}\right)\right]^{-1}(x^3-4x)$$

$$= \frac{1}{2}\left[1-\left(\frac{2D+D^2}{2}\right)+\left(\frac{2D+D^2}{2}\right)^2 - \left(\frac{2D+D^2}{2}\right)^3 \ldots\right](x^3-4x)$$

$$= \frac{1}{2}\left[1-D-\frac{D^2}{2}+\frac{1}{4}(4D^2+4D^3+D^4) - \frac{8D^3}{8}\right](x^3-4x)$$

$$y_p = \frac{1}{2}\left[1-D+\frac{D^2}{2}\right](x^3-4x)$$

$$= \frac{1}{2}\left[(x^3-4x)-(3x^2-4)+\frac{1}{2}(6x)\right]$$

$$= \frac{1}{2}\left[x^3-3x^2-x+4\right]$$

Hence, complete solution is –

$$y = y_c + y_p$$

$$y = e^{-x}\left[c_1\cos x + c_2\sin x\right) + \frac{1}{2}(x^3-3x^2-x+4\right]$$

Ex:2. Solve: $(D^3+3D^2-4)y = x^2+x+1$ —— (1)

soln: The complete general solution of given D.E (1) is –

$$y = y_c + y_p$$

step-1: A.E (Auxilliary Equation) is –

$$D^3+3D^2-4=0$$

$$\begin{array}{c|cccc} 1 & 1 & 3 & 0 & -4 \\ & 0 & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$\therefore D = 1, \quad D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

$D = -2, -2.$

$\therefore D = 1, -2, -2$

Hence, $y_c = c_1 e^{x} + (c_2 + c_3 x) e^{-2x}$

Step-II: $y_p = \dfrac{1}{D^3 + 3D^2 - 4} (x^2 + x + 1)$

$$y_p = \frac{1}{-4 + 3D^2 + D^3} (x^2 + x + 1)$$

$$= \frac{1}{-4\left[1 + \left(\frac{3D^2 + D^3}{-4}\right)\right]} (x^2 + x + 1)$$

$$= -\frac{1}{4}\Bigg[1 + \underbrace{\left(\frac{3D^2 + D^3}{-4}\right)}_{z}\Bigg]^{-1} (x^2 + x + 1)$$

$$= -\frac{1}{4}\left[1 - \left(\frac{3D^2 + D^3}{-4}\right) + \left(\frac{3D^2 + D^3}{-4}\right)^2\right] (x^2 + x + 1)$$

$$= -\frac{1}{4}\left[1 + \frac{1}{4}(3D^2 + D^3)\right] (x^2 + x + 1)$$

$$= -\frac{1}{4}\left[1 + \frac{3D^2}{4}\right] (x^2 + x + 1)$$

[Neglecting more than two powers of D]

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$$y_p = -\frac{1}{4}\left[(x^2+x+1)+\frac{3}{4}(2)\right]$$

$$y_p = -\frac{1}{4}\left(x^2+x+\frac{5}{2}\right)$$

$f(x) = x^2+x+1$
$D = 2x+1$
$D^2 = 2$

Hence complete solution is —

$$y = c_1e^{x}+(c_2+c_3x)e^{-2x}-\frac{1}{4}\left(x^2+x+\frac{5}{2}\right)$$

Ex: 3. $(D^4+D^2+1)y = 36x^2-17$

Soln: The complete general soln of given D.E is —

$$y = y_c + y_p$$

Step-I: To find y_c:

Auxilliary Equation (A.E) is -

$$D^4+D^2+1 = 0.$$

$$D^4+2D^2+1-D^2 = 0$$

$$(D^2+1)^2 - D^2 = 0$$

$$(D^2+1+D)(D^2+1-D) = 0.$$

$$\therefore (D^2+D+1)(D^2-D+1) = 0$$

$$D = \frac{-1\pm\sqrt{1-4(1)}}{2}, \quad D = \frac{1\pm\sqrt{1-4(1)}}{2}$$

$$D = \frac{-1\pm\sqrt{3}\,i}{2}, \quad D = \frac{1\pm\sqrt{3}\,i}{2}$$

$$\therefore D = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \quad D = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore y_c = e^{-1/2 x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$
$$+ e^{1/2 x}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right]$$

Step-II : To find y_p:

$$y_p = \frac{1}{D^4 + D^2 + 1}(36x^2 - 17)$$

$$= \frac{1}{1 + D^2 + D^4}(36x^2 - 17)$$

$$= \left[1 + (D^2 + D^4)\right]^{-1}(36x^2 - 17)$$

$$= \left[1 - (D^2 + D^4) + (D^2 + D^4)^2\right](36x^2 - 17)$$

$$= \left[1 - (D^2 + D^4)\right](36x^2 - 17)$$

$$= \left[1 - D^2\right]\left[36x^2 - 17\right]$$

$$= 36x^2 - 17 - D^2(36x^2 - 17)$$

$$= 36x^2 - 17 - 72$$

$$y_p = 36x^2 - 89$$

$f(x) = 36x^2 - 17$
$Df(x) = 72x$
$D^2 f(x) = 72$

Hence, $y = y_c + y_p$

$$y = e^{-1/2 x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$
$$+ e^{-1/2 x}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right] + 36x^2 - 89$$

HW: Solve: $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=x^2+7x+9$

Ans: $y=c_1e^{-x}+c_2e^{-4x}+\frac{1}{4}\left[x^2+\frac{9}{2}x+\frac{23}{8}\right]$

Rule 5: If $f(x)=e^{ax}V$
where V is function of x

If $f(x)=e^{ax}V$ then $P.I=\frac{1}{\phi(D)}e^{ax}V=e^{ax}\frac{1}{\phi(D+a)}V$

Explanation: We have,

$$D(e^{ax}u)=e^{ax}Du+ae^{ax}u$$
$$=e^{ax}(D+a)u$$

$$D^2(e^{ax}u)=e^{ax}(D+a)^2u$$

In general,
$$D^n(e^{ax}u)=e^{ax}(D+a)^nu$$
$$\therefore \phi(D)(e^{ax}u)=e^{ax}\phi(D+a)u$$

Operating $\frac{1}{\phi(D)}$ on both sides,

$$\frac{1}{\phi(D)}\cdot\phi(D)(e^{ax}u)=\frac{1}{\phi(D)}\left[e^{ax}\phi(D+a)u\right]$$

$$e^{ax}u=\frac{1}{\phi(D)}\left[e^{ax}\phi(D+a)u\right]$$

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Now, put $\phi(D+a)u = V$

i.e $u = \frac{1}{\phi(D+a)}V$

Hence above eq^n (4) becomes.

$$e^{ax}\frac{1}{\phi(D+a)}V = \frac{1}{\phi(D)}(e^{ax}V)$$

$$\text{i.e } \frac{1}{\phi(D)}(e^{ax}V) = e^{ax}\frac{1}{\phi(D+a)}V$$

Ex: 1. Solve: $(D^2-4D+4)y = e^{2x}\sin 3x$

Soln: The complete general soln of above D.E is

$$y = y_c + y_p$$

Step-I: To find y_c:

Auxilliary Eq^n is –

$$\phi(D) = 0$$

$$D^2-4D+4 = 0$$

$$(D-2)^2 = 0$$

$$D = 2, 2$$

Hence, $y_c = (C_1 + C_2 x)e^{2x}$

Step-II: To find y_p

$$y_p = \frac{1}{\phi(D)}e^{2x}\sin 3x$$

$$y_p = \frac{1}{D^2-4D+4} e^{2x} \sin 3x$$

$$= \frac{1}{(D-2)^2} e^{2x} \sin 3x.$$

$a=2$ $\qquad D = D+2$

$$= e^{2x} \frac{1}{((D+2)-2)^2} \sin 3x$$

$$= e^{2x} \frac{1}{D^2} \sin 3x \qquad a=3 \qquad \underline{\text{Rule-2}} : D^2 = -a^2 = -9$$

$$= e^{2x} \frac{1}{-9} \sin 3x$$

$$y_p = -\frac{e^{2x} \sin 3x}{9}$$

Hence, complete solution is –

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x) e^{2x} - \frac{e^{2x} \cdot \sin 3x}{9}$$

Ex: 2. <u>Solve</u>: $(D^2+4D+4)y = \dfrac{e^{-2x}}{x^2}$

<u>Soln</u>: The complete general soln of D.E is

$$y = y_c + y_p$$

<u>Step-I</u>: Auxilliary Eqn is –

$$\phi(D) = 0$$

$$\Rightarrow D^2 + 4D + 4 = 0$$

$$\therefore (D+2)^2 = 0$$

$$D = -2, -2$$

Hence,

$$y_c = (c_1 + c_2 x)e^{-2x}$$

Step-II: To find y_p:

$$y_p = \frac{1}{\phi(D)} \frac{e^{-2x}}{x^2}$$

$$= \frac{1}{D^2+4D+4} \frac{e^{-2x}}{x^2}$$

$$= \frac{1}{(D+2)^2} \frac{e^{-2x}}{x^2}$$

$a = -2$

$D = D + a$
$= D - 2$

$$= e^{-2x} \frac{1}{((D-2)+2)^2} \frac{1}{x^2}$$

$$= e^{-2x} \frac{1}{D^2} \frac{1}{x^2}$$

$$= e^{-2x} \frac{1}{D}\left(\frac{1}{D} \frac{1}{x^2}\right)$$

$$= e^{-2x} \frac{1}{D}\left(\int \frac{1}{x^2} dx\right)$$

$$= e^{-2x} \frac{1}{D}\left(\frac{x^{-2+1}}{-2+1}\right)$$

$$= -e^{-2x}\left(\int 1/x \, dx\right)$$

$$= -e^{-2x} \log x$$

Hence,

$$y = (c_1 + c_2 x)e^{-2x} - e^{-2x} \log x$$

H.W

Ex: 3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1+x)$

Ans:

$$y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{e^{2x}}{12}\left(x + \frac{17}{12}\right)$$

Ex: 4. $(D^2+6D+9)y = e^{-3x}(x^3+\sin 3x)$

Ans:

$$y = (c_1+c_2x)e^{-3x} + e^{-3x}\left[\frac{x^5}{20} - \frac{\sin 3x}{9}\right]$$

Ex: 5 $(D^2+2D+1)y = xe^{-x}\cos x$

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Rule 6: If $f(x) = xV$ ~~where V is f~~

where V is function of x

If $f(x) = xV$ then $P.I = \frac{1}{\phi(D)} xV$

$$= \left\{ x - \frac{1}{\phi(D)} \phi'(D) \right\} \frac{1}{\phi(D)} V$$

Note: $e^{iax} = \cos ax + i \sin ax$.

$$\Rightarrow Re(e^{iax}) = \cos ax$$

$$Im(e^{iax}) = \sin ax.$$

Ex: 1. Solve $(D^2+9)y = x \sin 2x$

Soln: The general (complete) solution of DE is -

$$y = y_c + y_p$$

Step-I: To find C.F.

Consider Auxilliary. Eqn

$$\phi(D) = 0$$

$$D^2+9 = 0 \Rightarrow D^2 = -9 \Rightarrow D = \pm 3i$$

$$y_c = C_1 \cos 3x + C_2 \sin 3x$$

~~Hence,~~

$$\cancel{y_p = \frac{1}{D^2+9} x \sin 2x}$$

Step-II: To find y_p:

$$y_p = \left\{ x - \frac{1}{\phi(D)} \phi'(D) \right\} \frac{1}{\phi(D)} V$$

$$y_p = \left\{ x - \frac{1}{D^2+9} 2D \right\} \frac{1}{D^2+9} \sin 2x$$

$$= x \frac{1}{D^2+9} \sin 2x - \frac{2D}{(D^2+9)^2} \sin 2x$$

Use : $D^2 = -4$

$$= \frac{x \cdot \sin 2x}{-4+9} - \frac{2D}{(-4+9)^2} \sin 2x$$

$$= \frac{x \cdot \sin 2x}{5} - \frac{2D}{25} \sin 2x$$

$$= \frac{x \cdot \sin 2x}{5} - \frac{2}{25} \cos 2x \cdot 2$$

$$y_p = \frac{x \cdot \sin 2x}{5} - \frac{4}{25} \cos 2x$$

Step-III : $y = y_c + y_p$

$$= c_1 \cos 3x + c_2 \sin 3x + \frac{x \sin 2x}{5} - \frac{4}{25} \cos 2x.$$

Ex: 2. Solve $(D^2+D+1)y = x \sin x$

Soln: The general solution of D.E is -

$$y = y_c + y_p$$

Step-I : To find y_c :

Consider Auxilliary Eqn is -

$\phi(D) = 0$

$\therefore D^2 + D + 1 = 0$

$\therefore D = \dfrac{-1 \pm \sqrt{1-4(1)}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\alpha = -\frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$

$\therefore y_c = e^{-1/2 x}\left[\cos \frac{\sqrt{3}}{2} x + i \sin \frac{\sqrt{3}}{2} x\right]$

Step-III: To find y_p:

$y_p = \dfrac{1}{D^2+D+1} x \sin x$

$= \left[x - \dfrac{1}{\phi(D)} \phi'(D)\right] \dfrac{1}{\phi(D)} \sin x$

$= \left[x - \dfrac{1}{D^2+D+1}(2D+1)\right] \dfrac{1}{(D^2+D+1)} \sin x$

$= x \cdot \dfrac{1}{D^2+D+1} \sin x - \dfrac{2D+1}{(D^2+D+1)^2} \sin x$

Use: $D^2 = -1$

$= x \cdot \dfrac{\sin x}{\cancel{-1} + D + \cancel{1}} - \dfrac{2D+1}{(\cancel{-1} + D + \cancel{1})^2} \sin x$

$= x \cdot \int \sin x \, dx - \dfrac{2D+1}{(-1)} \sin x$

$= x(-\cos x) + 2\cos x + \sin x$

$\therefore y = e^{-1/2 x}\left[\cos \frac{\sqrt{3}}{2} x + i \sin \frac{\sqrt{3}}{2} x\right] - x\cos x + 2\cos x + \sin x$

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Ex: 3 : Solve: $(D^2-5D+6)y = x\cos 2x$

Sol^n: The general solution of D.E is –

$$y = y_c + y_p$$

Step-I : To find y_c:

Consider Auxilliary Eq^n is –

$$\phi(D) = 0$$

$$D^2-5D+6 = 0$$

$$(D-3)(D-2) = 0$$

$$D = 2, 3$$

Hence,

$$y_c = C_1 e^{2x} + C_2 e^{3x}$$

Step-II : To find y_p:

$$y_p = \frac{1}{\phi(D)} f(x) = \frac{1}{D^2-5D+6} x\cos 2x$$

$$y_p = \left[x - \frac{1}{\phi(D)}\phi'(D)\right]\frac{1}{\phi(D)}\cos 2x$$

$$= x \cdot \frac{1}{D^2-5D+6}\cos 2x - \frac{1}{(D^2-5D+6)^2}(2D-5)\cos 2x$$

Use: $D^2 = -4$

$$= x \cdot \frac{1}{-4-5D+6}\cos 2x - \frac{1}{(-4-5D+6)^2}(2D-5)\cos 2x$$

$$= x \cdot \frac{1}{2-5D}\cos 2x - \frac{(2D-5)}{(2-5D)^2}\cos 2x$$

$$= x \cdot \frac{1}{2-5D} \times \frac{(2+5D)}{(2+5D)}\cos 2x - \frac{(2D-5)}{(4-20D+25D^2)}\cos 2x$$

$$\therefore y_p = x \cdot \frac{2+5D}{4-25D^2}\cos 2x - \frac{2D-5}{4-20D+25D^2}\cos 2x$$

Use $D^2 = -4$

$$= x \cdot \frac{(2+5D)}{4-25(-4)}\cos 2x - \frac{(2D-5)}{4-20D+25(-4)}\cos 2x$$

$$= x \cdot \frac{(2+5D)}{104}\cos 2x - \frac{(2D-5)}{(-96-20D)}\cos 2x$$

$$= x \cdot \frac{(2\cos 2x + 5\cdot(-\sin 2x)\cdot 2)}{104} + \frac{(2D\mp 5)}{4(5D+24)}\cos 2x$$

$$= x \cdot \frac{(\cos 2x - 5\sin 2x)}{52} + \frac{(2D\mp 5)(5D-24)}{4(25D^2-576)}\cos 2x$$

Use : $D^2 = -4$

$$= \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{(10D^2 - 48D \mp 25D \pm 120)}{4(25\times(-4)-576)}\cos 2x$$

$$y_p = \frac{x}{52}(\cos 2x - 5\sin 2x) + \frac{10(-4\cos 2x) - 73(-2\sin 2x) + 120\cos 2x}{4(-676)}$$

$f(x) = \cos 2x$

$Df(x) = -\sin 2x \cdot 2$

$D^2 f(x) = -4\cos 2x$

$$y_p = \frac{x}{52}(\cos 2x - 5\sin 2x) - \frac{1}{1352}\left[40\cos 2x + 73\sin 2x\right]$$

Hence,

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{x}{52}(\cos 2x - 5\sin 2x) - \frac{1}{1352}\left[40 \cdot \cos 2x + 73\sin 2x\right]$$