Indian Institute of Technology Madras

ED5330:
Control of Automotive Systems

Project 3 – Suspension Control

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Contents

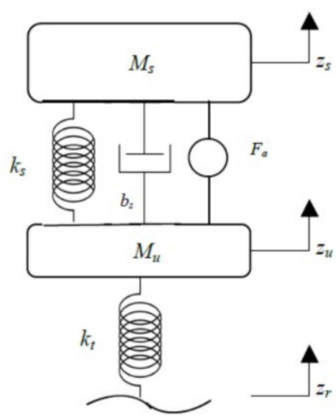
1 Introduction

The objective of the project is to understand suspension control in depth. We analyse the crucial role of active suspension as well. We used Control System Toolbox from MATLAB, Simulink to plot bode plots.

The following sections describe the tasks involved and their results.

2 Part A: Quarter Car Modelling

The given system of Quarter Car Model is,



Equations involved in are:

$$m_s \ddot{z_s} = -b_s(\dot{z}_s - \dot{z}_u) - k_s(z_s - z_u) + f_a$$

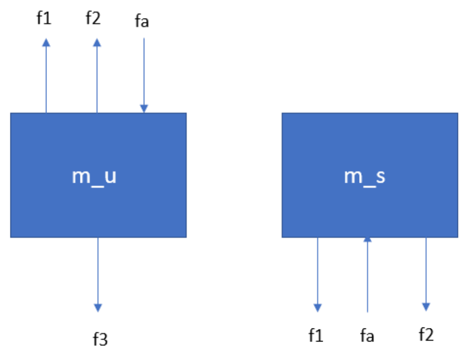
and

$$m_u \ddot{z_u} = b_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) - f_a + b_t(\dot{z}_r - \dot{z}_u) + k_t(z_r - z_u)$$

a)Equations Derivation as follows:

$$f_1 = k_s(z_s - z_u) \tag{1}$$
$$f_2 = b_s(\dot{z}_s - \dot{z}_u) \tag{2}$$
$$f_3 = k_t(z_u - z_r) \tag{3}$$

The FBD for the above equations:



Substituting all the analysis done, we get

$$m_u \ddot{z_u} = f_1 + f_2 - f_a - f_3 \tag{4}$$

$$m_s \ddot{z_s} = f_a - f_1 - f_2 \tag{5}$$

And that leads us to,

$$m_s \ddot{z_s} = -b_s(\dot{z}_s - \dot{z}_u) - k_s(z_s - z_u) + f_a \tag{6}$$

$$m_u \ddot{z_u} = b_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) - f_a + b_t(\dot{z}_r - \dot{z}_u) + k_t(z_r - z_u) \tag{7}$$

b)Natural frequencies of the system, as per the given conditions
Assumption: Natural frequencies, only conform for masses; hence have no relation of the springs and dampers

$$[M]\ddot{X} + [K]X = 0 \tag{8}$$

$$\begin{bmatrix} m_s & 0 \\ \dot{0} & m_u \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} + \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_u \end{bmatrix} \begin{bmatrix} z_s \\ z_u \end{bmatrix} = 0 \tag{9}$$

$$\begin{bmatrix} 300 & 0 \\ 0 & 40 \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} + \begin{bmatrix} 15000 & -15000 \\ -15000 & 165000 \end{bmatrix} \begin{bmatrix} z_s \\ z_u \end{bmatrix} = 0$$

$$\begin{bmatrix} 300 & 0 \\ 0 & 40 \end{bmatrix} \begin{bmatrix} \ddot{z}_s \\ \ddot{z}_u \end{bmatrix} + \begin{bmatrix} 15000 & -15000 \\ -15000 & 165000 \end{bmatrix} \begin{bmatrix} z_s \\ z_u \end{bmatrix} = 0$$

$$det[A] = det[[K] - w^2[M]] \tag{10}$$

$$= det \left[\begin{bmatrix} 15000 & -15000 \\ -15000 & 165000 \end{bmatrix} - w^2 \begin{bmatrix} 300 & 0 \\ 0 & 40 \end{bmatrix} \right]$$

$$= det \begin{bmatrix} 15000 - 300w^2 & -15000 \\ -15000 & 165000 - 40w^2 \end{bmatrix}$$

$$12w^4 - 50100w^2 - 2250000 = 0 \tag{11}$$

$$w_1 = 6.708 rad/s \tag{12}$$

$$w_2 = 64.25 rad/s \tag{13}$$

$$f_1 = \frac{6.708}{\pi} rad/s$$

$$f_2 = \frac{64.25}{\pi} rad/s$$

$$f_1 = 2.136 Hz \tag{14}$$

$$f_2 = 20.46 Hz \tag{15}$$

Another method would be by eigen values

$$[A] = [M]^{-1} K \tag{16}$$

$$Eig A = \lambda_1, \lambda_1$$

$$\lambda_1 = w_1^2 \tag{17}$$

$$\lambda_2 = w_2^2 \tag{18}$$

$$[A] = [M]^{-1}[K] = \begin{bmatrix} 300 & 0 \\ 0 & 40 \end{bmatrix}^{-1} \begin{bmatrix} 15000 & -15000 \\ -15000 & 165000 \end{bmatrix} \tag{19}$$

This brings us to equation (11), and then subsequent steps give us the same results

c)State Space Representation

$$x(t) = \begin{bmatrix} z_s(t) - z_u(t) \\ \dot{z}_s(t) \\ z_u(t) - z_r(t) \\ \dot{z}_u(t) \end{bmatrix}$$

$$x(t) = Ax(t) + F_a(t) + I\dot{z}_r(t) \tag{20}$$

$$\begin{bmatrix} \dot{z}_s(t) - \dot{z}_u(t) \\ \ddot{z}_s(t) \\ \dot{z}_u(t) - \dot{z}_r(t) \\ \ddot{z}_u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{k_s}{m_s} & \frac{-b}{m_s} & 0 & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & \frac{-k_t}{m_u} & \frac{-b_s}{m_u} \end{bmatrix} \begin{bmatrix} z_s(t) - z_u(t) \\ \dot{z}_s(t) \\ z_u(t) - z_r(t) \\ \dot{z}_u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_s} \\ 0 \\ -1/m_u \end{bmatrix} [F_a] + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} [\dot{z}_r(t)] \tag{21}$$

$$\dot{x}(t) = A \cdot x(t) + b \cdot F_a(t) + I \cdot \dot{z}_r(t) \tag{22}$$

3 Part B: Open-loop Suspension Performance Analysis

(i)
From equations (6) and (7) we get,
m$_s s^2 z_s + b_s(sz_s - sz_u) + k_s(z_s - z_u) = 0 m_u s^2 z_u + b_s(sz_u - sz_s) + k_s(z_u - z_s) + k_t(z_u - z_r) = 0 z_s(m_s \cdot s^2 + b_s \cdot s + k_s) = z_u(b_s \cdot s + k_s)$(23)

$$z_u(m_u \cdot s^2 + b_s \cdot s + k_s + k_t) = z_r(k_t) + z_s(b_s \cdot s + k_s) \quad (24)$$

and combining both of them leads us to,

$$z_u = \frac{z_r(k_t) + z_s(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t} \quad (25)$$

$$z_s(m_s \cdot s^2 + b_s \cdot s + k_s) = (\frac{z_r(k_t) + z_s(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t})(b_s \cdot s + k_s)$$

$$z_s(m_s \cdot s^2 + b_s \cdot s + k_s - \frac{(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t}) = (\frac{z_r(k_t) \cdot (b_s \cdot s + k_s)}{m_u s^2 + sb_s + k_s + k_t})$$

$$z_s(m_s \cdot s^2 + b_s \cdot s + k_s - \frac{(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t}) = (\frac{z_r(k_t) \cdot (b_s \cdot s + k_s)}{m_u s^2 + sb_s + k_s + k_t})$$

$$\frac{z_s}{z_r} = \frac{(\frac{(k_t) \cdot (b_s \cdot s + k_s)}{m_u s^2 + sb_s + k_s + k_t})}{(m_s \cdot s^2 + b_s \cdot s + k_s - \frac{(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t})} \quad (26)$$

Now for the Acceleration Transfer Function (TF1)

$$\frac{s^2 \cdot z_s}{s \cdot z_r} = \frac{((k_t) \cdot (b_s \cdot s + k_s))}{A}$$

$$A = (s^4(m_s m_u) + s^3(m_s b_s + b_s m_u) + s^2(m_s k_s + m_s k_t + b_s^2) + s(2b_s k_s + b_s k_t) + k_s^2 + k_s k_t - (sb_s + k_s))$$

Now for Rattle Space Transfer Function (TF2)

$$\frac{z_s - z_u}{sz_r} = \frac{-m_s s}{b_s s + k_s}(\frac{z_s}{z_r})$$

$$\frac{z_s - z_u}{sz_r} = \frac{-m_s s}{b_s s + k_s}(\frac{(\frac{(k_t) \cdot (b_s \cdot s + k_s)}{m_u s^2 + sb_s + k_s + k_t})}{(m_s \cdot s^2 + b_s \cdot s + k_s - \frac{(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t})}) \quad (27)$$

Now for Tyre Deflection Transfer Function (TF3)

$$\frac{z_u - z_r}{sz_r} = \frac{z_u}{sz_r} - \frac{1}{s}$$

$$\frac{z_u - z_r}{sz_r} = \frac{\frac{z_s(m_s \cdot s^2 + b_s \cdot s + k_s)}{(b_s \cdot s + k_s)}}{sz_r} - \frac{1}{s}$$

$$\frac{z_u - z_r}{sz_r} = \frac{z_s(m_s \cdot s^2 + b_s \cdot s + k_s)}{(b_s \cdot s + k_s) \cdot sz_r} - \frac{1}{s}$$

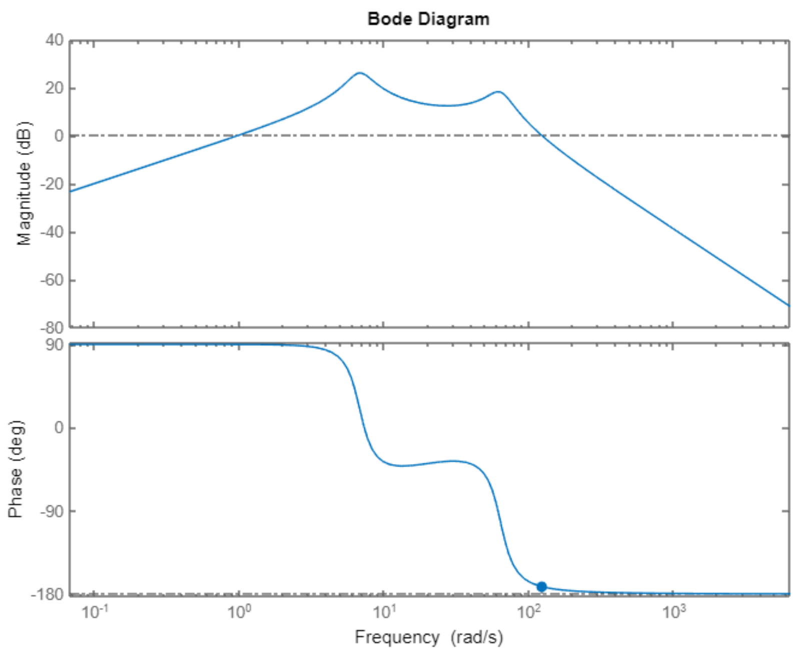
$$\frac{z_u - z_r}{sz_r} = \frac{z_s(m_s \cdot s^2 + b_s \cdot s + k_s)}{(b_s \cdot s + k_s) \cdot sz_r} - \frac{1}{s}$$

$$\frac{z_u - z_r}{sz_r} = \frac{(s^2(m_s - b_s z_r) + s(b_s - k_s z_r) + k_s)}{(b_s \cdot s + k_s) \cdot s^2} \cdot (\frac{z_s}{z_r})$$

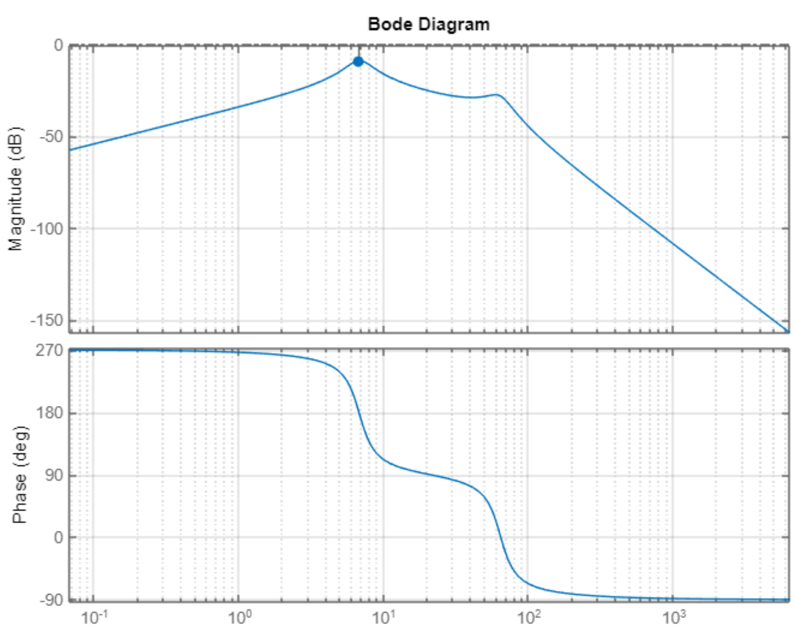
$$\frac{z_u - z_r}{sz_r} = \frac{(s^2(m_s - b_s z_r) + s(b_s - k_s z_r) + k_s)}{(b_s \cdot s + k_s) \cdot s^2} \cdot (\frac{(\frac{(k_t) \cdot (b_s \cdot s + k_s)}{m_u s^2 + sb_s + k_s + k_t})}{(m_s \cdot s^2 + b_s \cdot s + k_s - \frac{(sb_s + k_s)}{m_u s^2 + sb_s + k_s + k_t})}) \tag{28}$$

(ii) We have natural frequencies 6.708 and 64.25 rad/s
for 2 decade below and 2 decade above frequency range,
w$_m in = 6.708/100$
$w_m ax = 64.25 * 100$

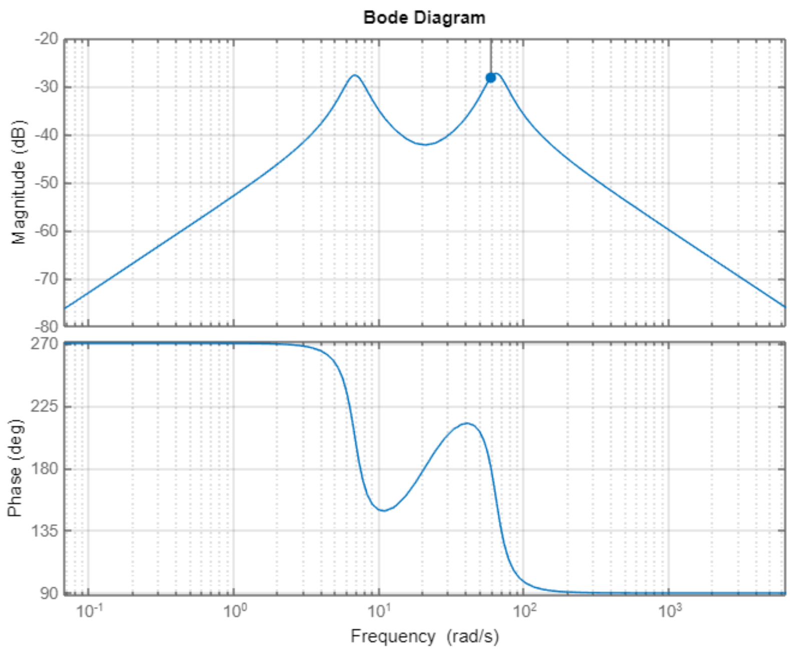
phase margin = 7.81 deg
for TF1



gain margin = 8.6 db
for TF2



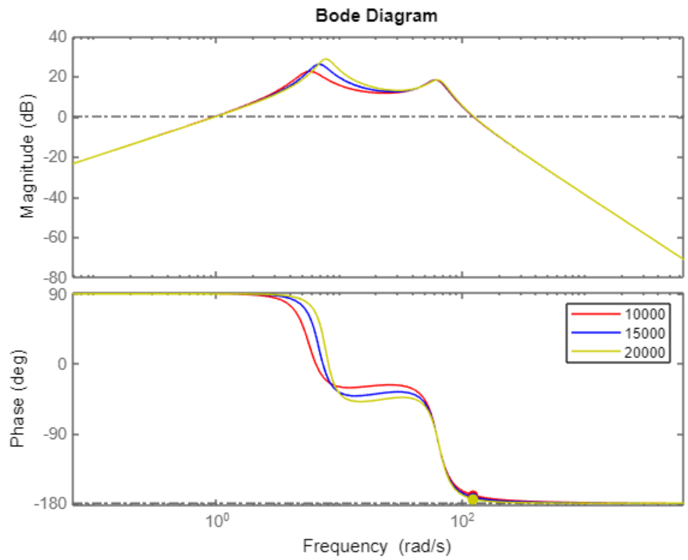
Gain margin = 28.1 db
for TF3



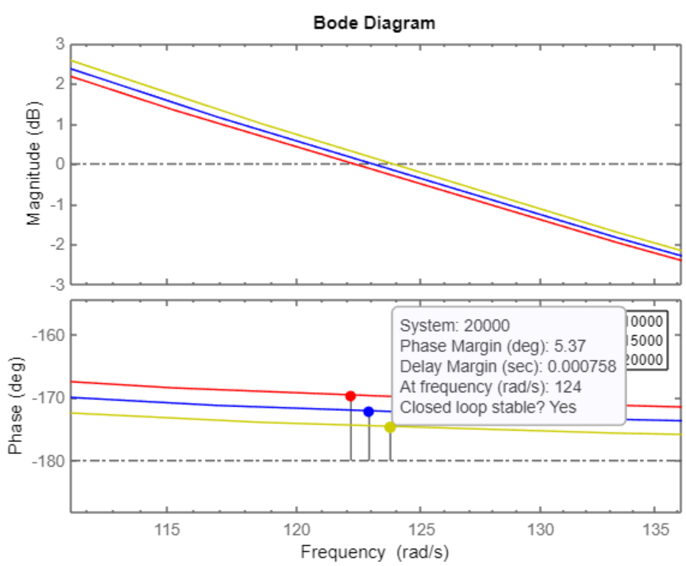
Observations: The approach we use to observe the Bode plots here, is based on the fundamental understanding of Bode plots.

- The system will be unstable for frequency domain magnitude as zero and Phase value at -180 degrees.
- We try to check that how close do our magnitude and phase values get to these instability values and those closest values are called magnitude margin and phase margin respectively.
- We try to check the magnitude and phase margin values in our selected frequency range.
- Negatiev PM and GM values suggest unstability.
- In the bode plots for three transfer functions, the margin values are pretty close to the unstability values and hence there is a scope of improvement in stability with help of closed loop control.

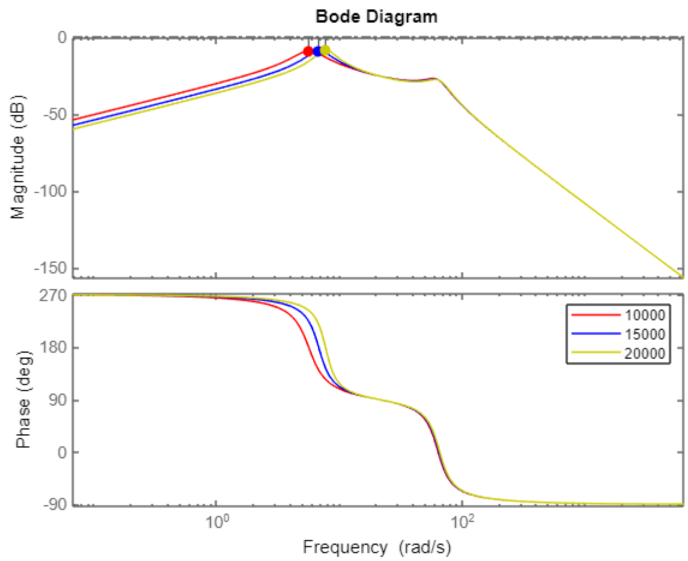
(iii) For different values of k; A,C matrices change their value, hence even the TFs vary. for Ks=10000, 15000, 20000 N/m and TF1



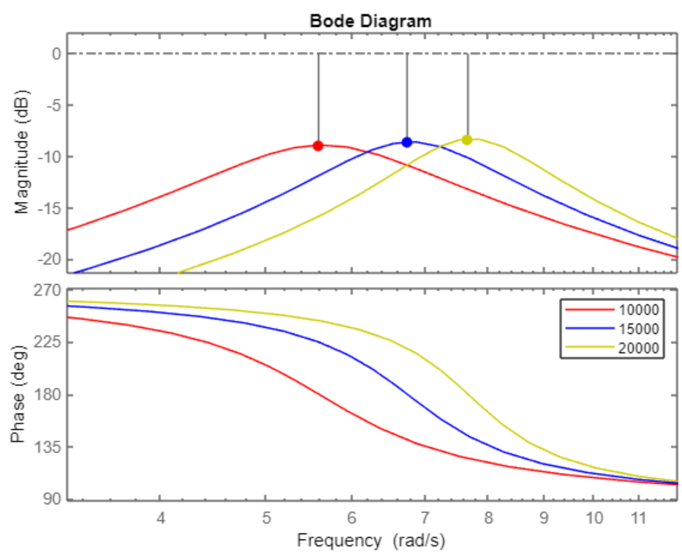
zoomed version



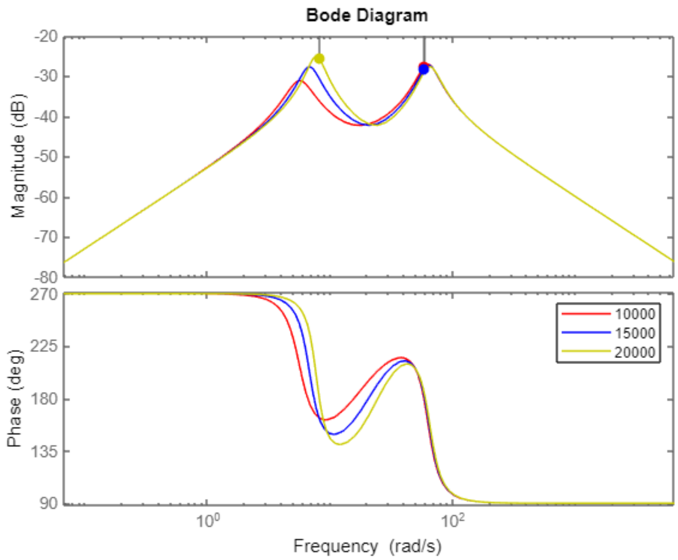
for k=10000,15000,20000 N/m and TF2



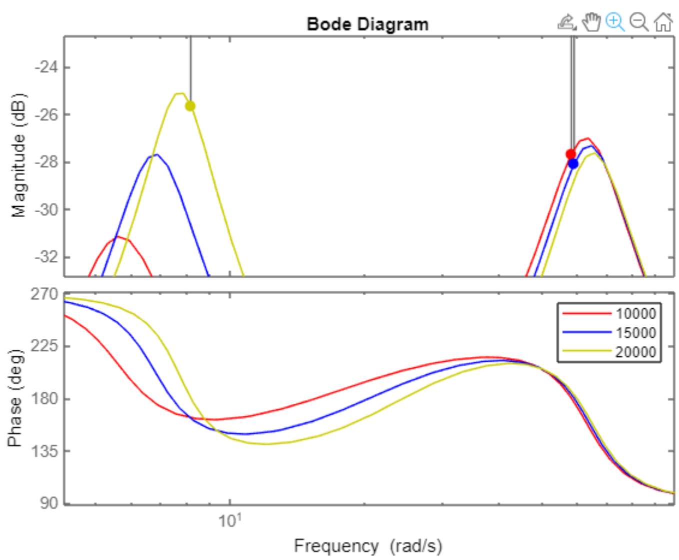
zoomed version



for k=10000,15000,20000 N/m and TF3



zoomed version



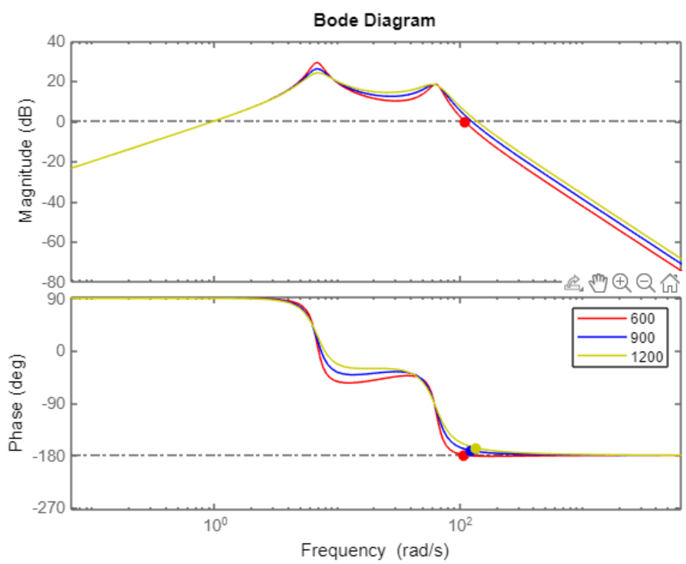
Observations:

- For acceleration transfer function, with an increase in Ks, we observe a decreasing phase margin and hence lesser stability.
- For rattle scale transfer function, with an increase in Ks, we observe

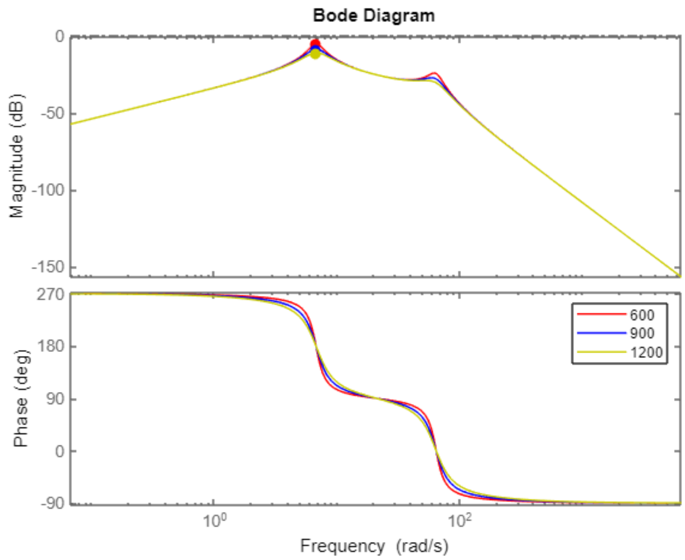
that phase margin values are almost similar for three Ks values and hence almost equally stable.

- For tyre deflection transfer function, with an increase in Ks, we observe a decreasing phase margin and hence lesser stability.

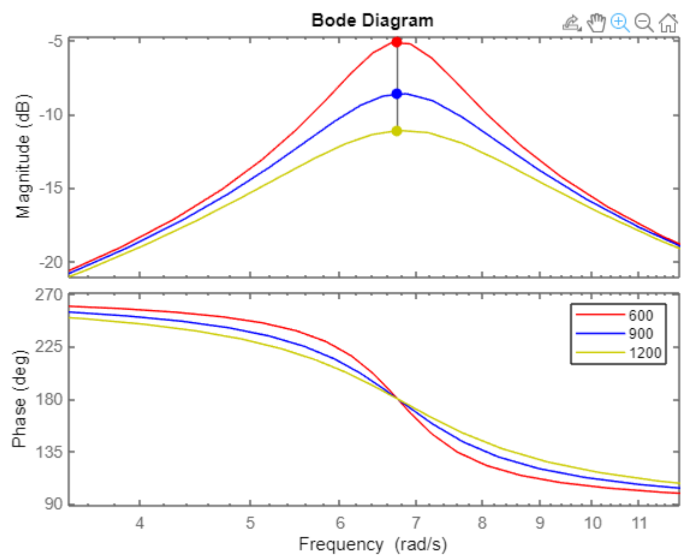
(iv) For variations in damping coefficient as bs = 600, 900, 1200 and TF1



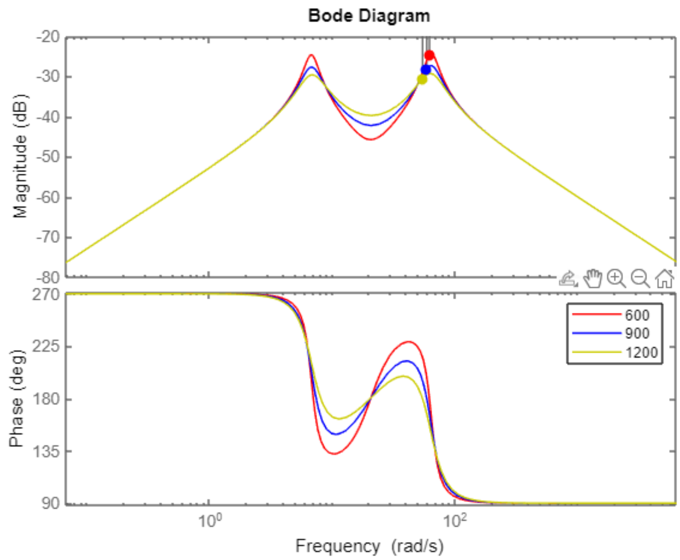
For variations in damping coefficient as bs = 600, 900, 1200 and TF2



zoomed version



For variations in damping coefficient as bs = 600, 900, 1200 and TF3

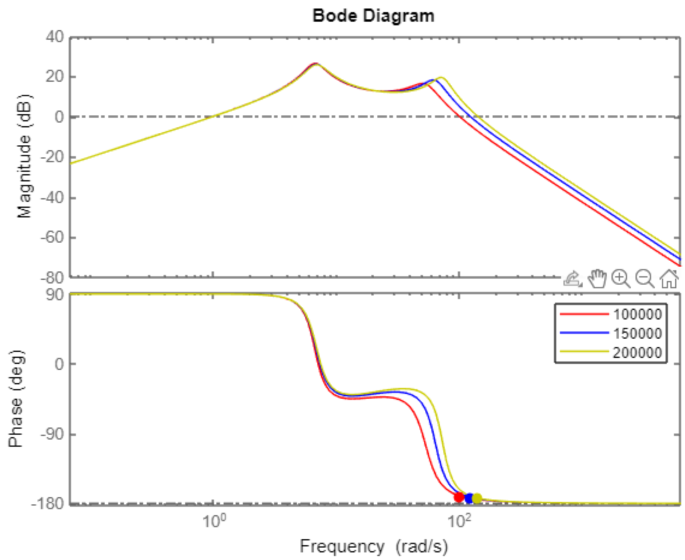


Observations:

- For acceleration transfer function, the phase margin and gain margin are both very close to unstability threshold for 600 Ns/m and high risk of unstability. For other two increasing damping coeff values, system is slightly more stable with increase in bs.
- For rattle scale transfer function, with an increase in bs, we see an increase in gain margin and better stability characteristics.
- For tyre deflection transfer function, system with higher bs value are more stable relatively.

(v)

$$\text{For variations in tyre spring coefficient as } k_t = 100000, 150000, 200000 and TF1$$



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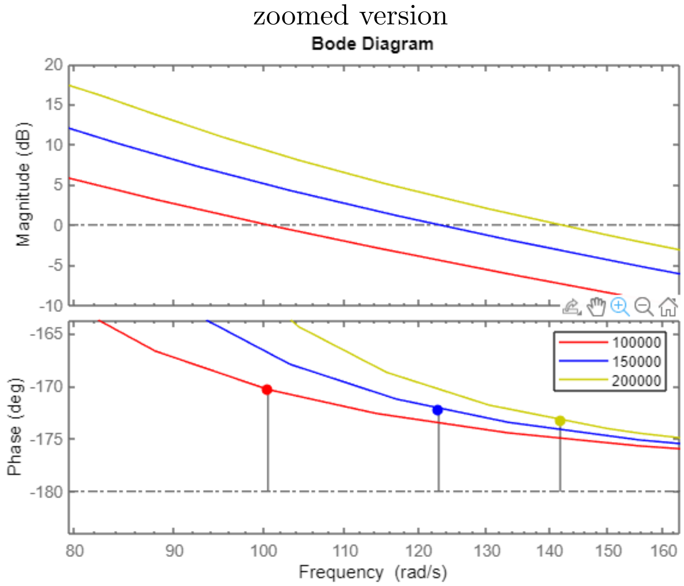
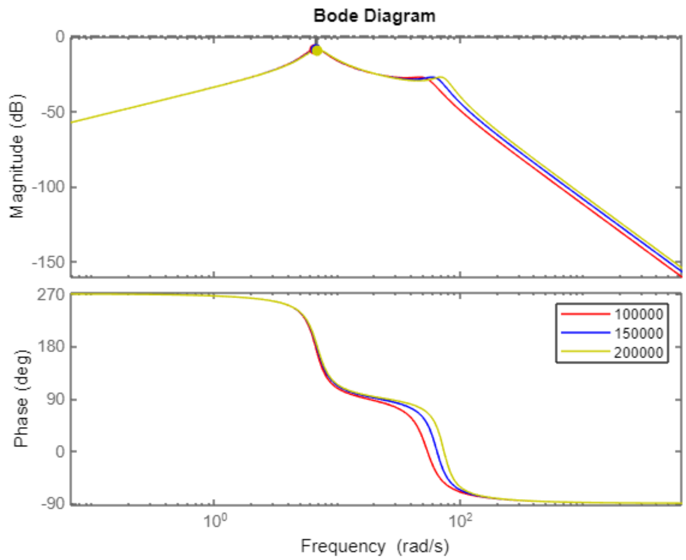
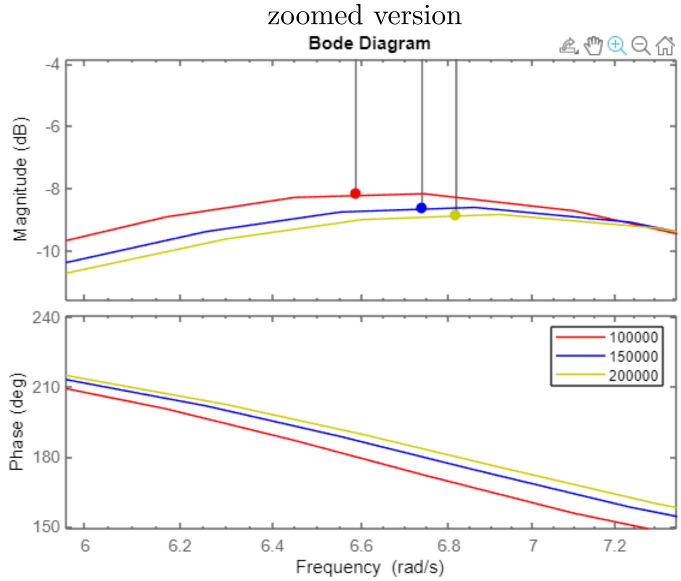


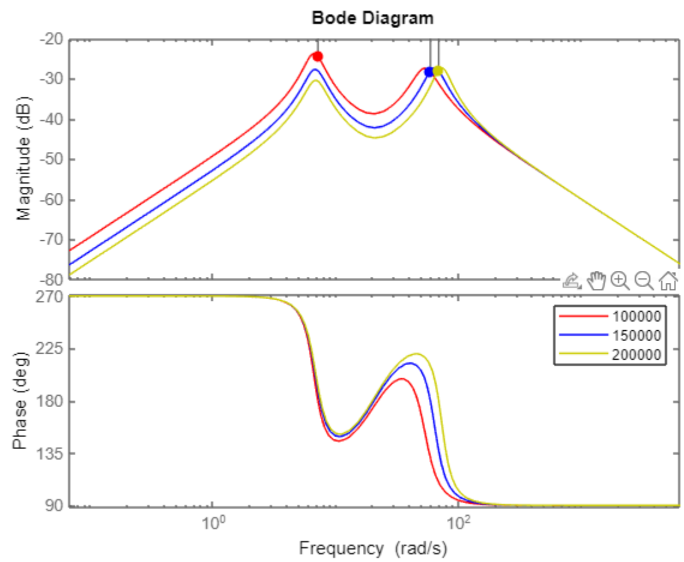
For variations in tyre spring coefficient as
$k_t = 100000, 150000, 200000 and TF2$



zoomed version



For variations in tyre spring coefficient as
$k_t = 100000, 150000, 200000 and TF3$



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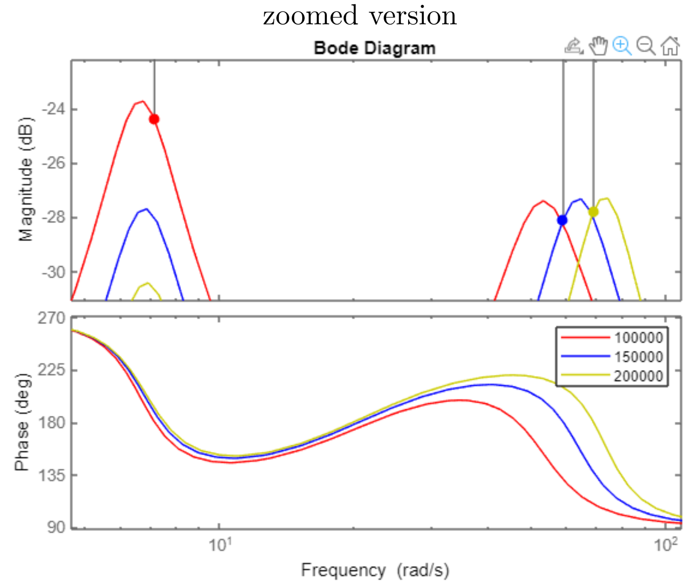


Observations:

* For acceleration transfer function, with an increase in Kt, we observe a decreasing phase margin and hence lesser stability.
* For rattle scale transfer function, with an increase in Ks, we observe that gain margin values are almost similar for three Ks values and hence almost equally stable and still very close

to 0 magnitude so close to unstability.

* For tyre deflection transfer function, with an increase in Ks, we observe a slightly increasing gain margin and hence relatively higher stability.

4 Part C: Closed-loop Suspension Performance Analysis

(i) Given,

$$J = [\int_0^\infty (\ddot{z}_s^2 + \rho_1(z_s - z_u)^2 + \rho_2\dot{z}_s^2 + \rho_3(z_u - z_r)^2 + \rho_4\dot{z}_u^2)dt]$$

That is,

$$\int_0^\infty [(\frac{-k_s}{m_s}(z_s - z_u) - \frac{b_s}{m_s}\dot{z}_s + \frac{b_s}{m_s}\dot{z}_u + \frac{1}{m_s}F_a)^2 + \rho_1(z_s - z_u)^2 + \rho_2\dot{z}_s^2 + \rho_3(z_u - z_r)^2 + \rho_4(\dot{z}_u^2)]dt$$

$$\int_0^\infty [(\frac{k_s^2}{m_s^2}(z_s - z_u)^2 + \frac{b_s^2}{m_s^2}\dot{z}_s^2 + \frac{b_s^2}{m_s^2}\dot{z}_u^2 + \frac{1}{m_s^2}F_a^2 - 2\frac{b_s^2}{m_s^2}\dot{z}_u\dot{z}_s$$

$$+2\frac{b_s}{m_s^2}(\dot{z}_u - \dot{z}_s)F_a - 2\frac{k_s}{m_s}\frac{b_s}{m_s}(z_s - z_u)(\dot{z}_u - \dot{z}_s)$$

$$-2\cdot\frac{k_s}{m_s}(z_s - z_u)\frac{F_a}{m_s}) + \rho_1(z_s - z_u)^2 + \rho_2\dot{z}_s^2 + \rho_3(z_u - z_r)^2 + \rho_4(\dot{z}_u^2)]dt$$

$$\int_0^\infty [(z_s - z_u)^2(\rho_1 + \frac{k_s^2}{m_s^2}) + \dot{z}_s^2(\rho_2 + \frac{b_s^2}{m_s^2}) + \dot{z}_u^2(\rho_4 + \frac{b_s^2}{m_s^2})$$

$$+\rho_3(z_u - z_r)^2 - 2\frac{k_s}{m_s}\frac{b_s}{m_s}(z_s - z_u)\dot{z}_u + 2\frac{k_s}{m_s}\frac{b_s}{m_s}(z_s - z_u)\dot{z}_s$$

$$-2\frac{b_s^2}{m_s^2}\dot{z}_s\dot{z}_u + 2\frac{b_s}{m_s^2}\dot{z}_u F_a - 2\frac{b_s}{m_s^2}\dot{z}_s F_a - 2\frac{k_s}{m_s^2}(z_s - z_u)F_a + \frac{1}{m_s^2}F_a^2]dt$$

Now we need an equation of the form,

$$XQX + 2XNF + FRF$$

$$[X][Q][X] = \begin{bmatrix} z_s(t) - z_u(t) & \dot{z}_s(t) & z_u(t) - z_r(t) & \dot{z}_u(t) \end{bmatrix} Q \begin{bmatrix} z_s(t) - z_u(t) \\ \dot{z}_s(t) \\ z_u(t) - z_r(t) \\ \dot{z}_u(t) \end{bmatrix}$$

$$[Q] = \begin{bmatrix} \rho_1 + \frac{k_s^2}{m_s^2} & \frac{k_s b_s}{m_s^2} & 0 & -\frac{k_s b_s}{m_s^2} \\ \frac{k_s b_s}{m_s^2} & \rho_2 + \frac{k_s^2}{m_s^2} & 0 & -\frac{b_s^2}{m_s^2} \\ 0 & 0 & \rho_3 & 0 \\ -\frac{k_s^2}{m_s^2} & -\frac{b_s^2}{m_s^2} & 0 & \rho_4 + \frac{b_s^2}{m_s^2} \end{bmatrix}$$

$$2[X][N][F] = 2 \begin{bmatrix} z_s(t) - z_u(t) & \dot{z}_s(t) & z_u(t) - z_r(t) & \dot{z}_u(t) \end{bmatrix} \begin{bmatrix} \frac{-k_s}{m_s^2} \\ \frac{-b_s}{m_s^2} \\ 0 \\ \frac{b_s}{m_s^2} \end{bmatrix} [F_a]$$

$$[F][R][F] = [F_a][\frac{1}{m_s^2}[F_a]$$

Legend: With Controller Values are depicted in bode diagram with blue lines, while without with orange lines

(ii)Set1

$$\rho = \begin{bmatrix} 0.3 & 0.1 & 0.3 & 0.1 \end{bmatrix}$$

By using the matlab command, [K, S, P] = lqr(A, B, Q, R, N)
The optimum regulator gain matrix is:

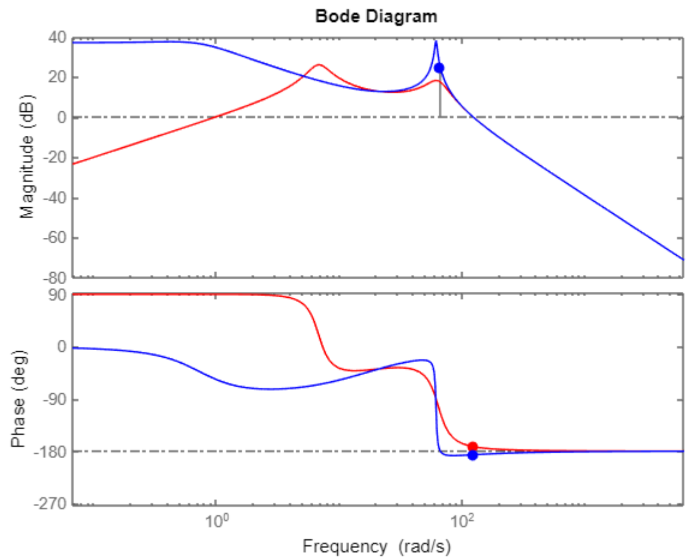
$$K = 10^4 \begin{bmatrix} -1.4836 & -0.0572 & 0.0060 & 0.0805 \end{bmatrix}$$

Now we use K matrix for obtaining closed loop state matrix

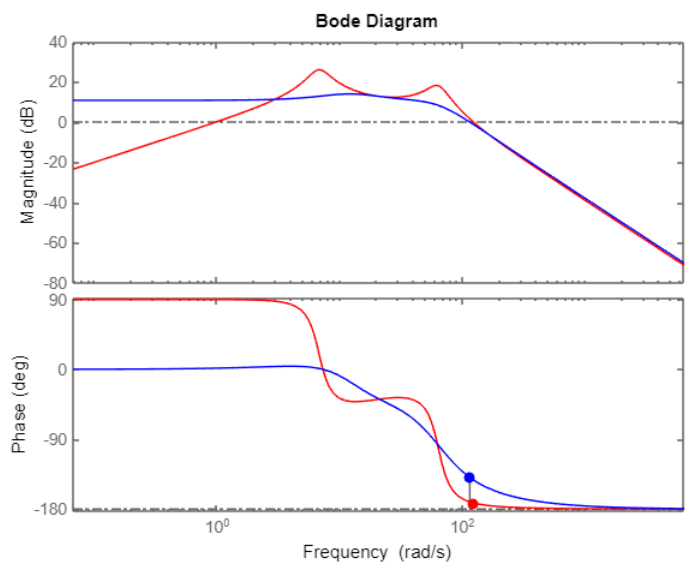
$$A_{cl} = A - (B * K)$$

We now use this Acl with B, C and D matrices to find the system closed loop transfer function. Here, orange curve represents without control and blue represents controlled system bode plot curve.

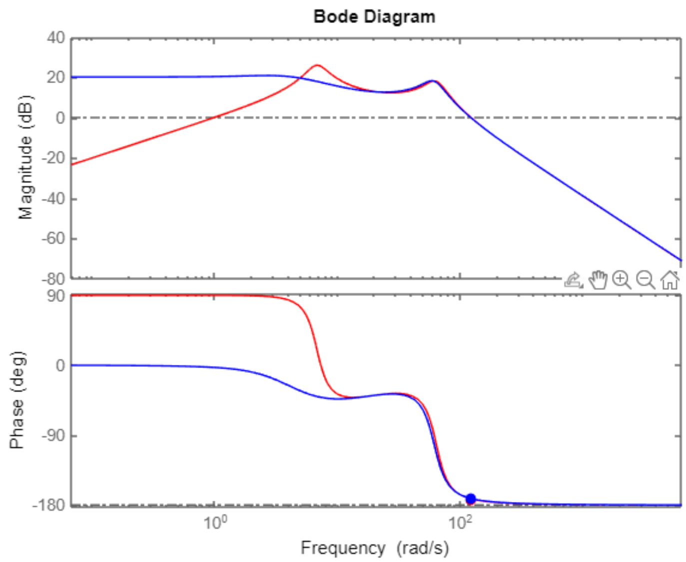
$$TF1$$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

$TF2$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

$TF3$



Set2

$$\rho = \begin{bmatrix} 30000 & 100 & 30000 & 100 \end{bmatrix}$$

By using the matlab command, [K, S, P] = lqr(A, B, Q, R, N)
The optimum regulator gain matrix is:

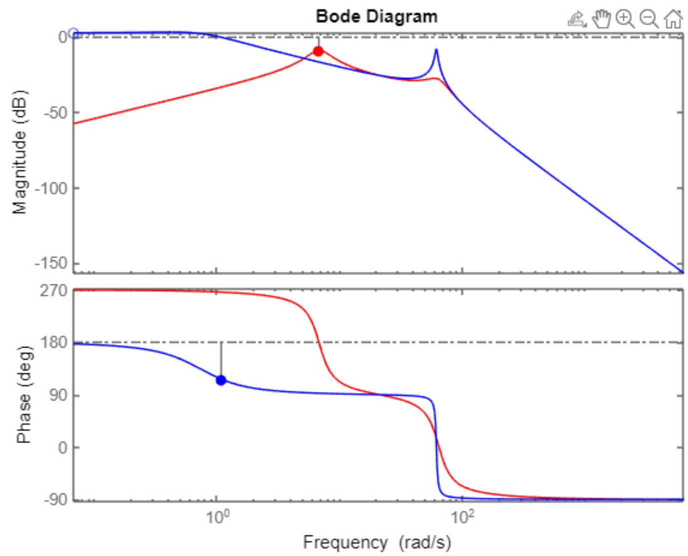
$$K = 10^4 \begin{bmatrix} 3.6962 & 0.5719 & -1.7513 & -0.2125 \end{bmatrix}$$

Now we use K matrix for obtaining closed loop state matrix

$$A_{cl} = A - (B * K)$$

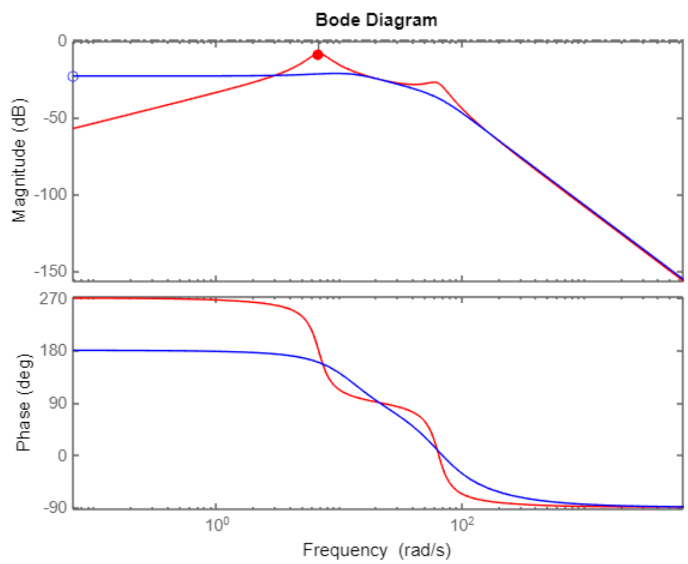
We now use this Acl with B, C and D matrices to find the system closed loop transfer function. Here, orange curve represents without control and blue represents controlled system bode plot curve.

$$TF1$$



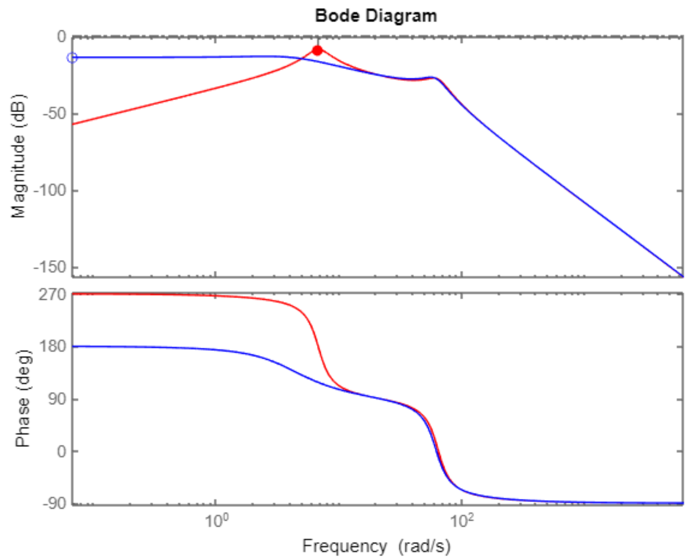
Bode Diagram
Magnitude (dB)
0
-50
-100
-150
Phase (deg)
270
180
90
0
-90
10^0
10^2
Frequency (rad/s)

$TF2$



Bode Diagram
Magnitude (dB)
0
-50
-100
-150
Phase (deg)
270
180
90
0
-90
10^0
10^2
Frequency (rad/s)

$TF3$



Set3

$$\rho = \begin{bmatrix} 300 & 10 & 300 & 10 \end{bmatrix}$$

By using the matlab command, [K, S, P] = lqr(A, B, Q, R, N)
The optimum regulator gain matrix is:

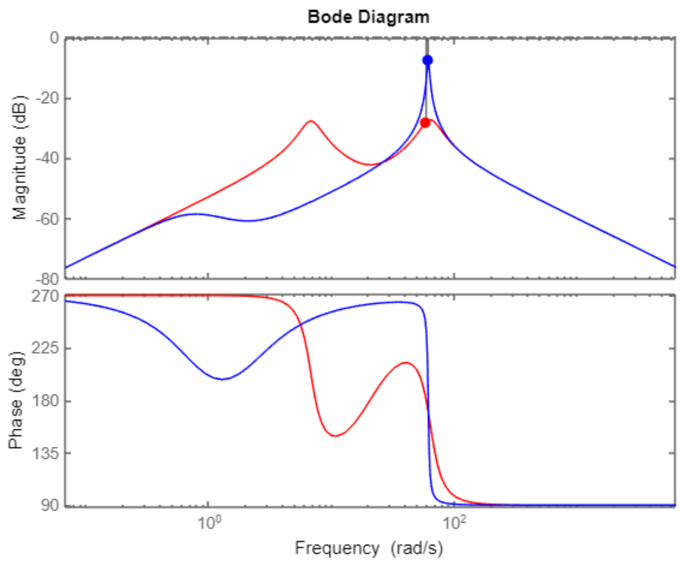
$$K = 10^3 \begin{bmatrix} -9.8038 & 1.1107 & -1.2227 & -0.0505 \end{bmatrix}$$

Now we use K matrix for obtaining closed loop state matrix

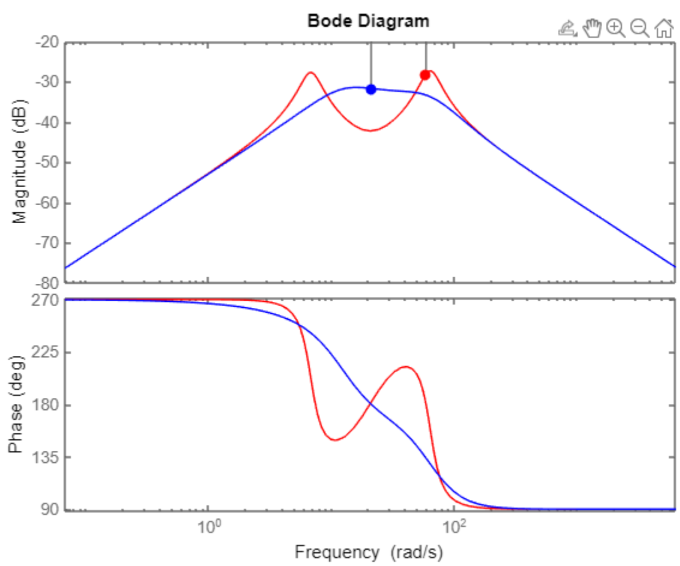
$$A_{cl} = A - (B * K)$$

We now use this Acl with B, C and D matrices to find the system closed loop transfer function. Here, orange curve represents without control and blue represents controlled system bode plot curve.

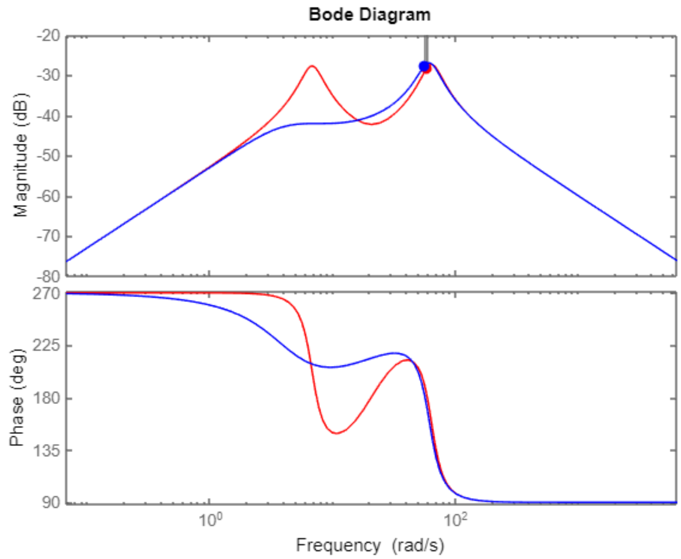
$$TF1$$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

$TF2$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

$TF3$

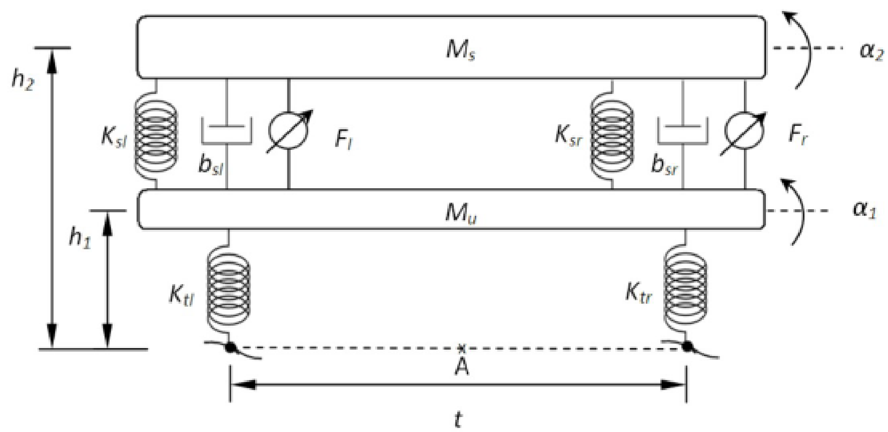


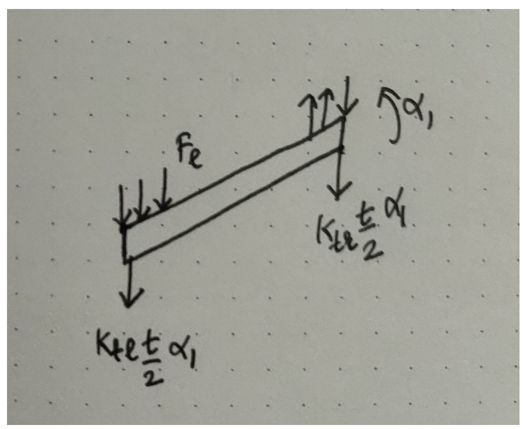
Observations:

- For acceleration transfer function TF1, set 1 values of rho give a system that has negative gain margin and hence unstability is there. For set 3, controller doesn't change the stability situation and in set 3 closed loop control gives a higher phase margin values and better stability.
- For rattle scale transfer function TF2, the first set of rho values gives a negative PM and hence leads to unstability. Rest of the sets, set 2 and set 3 values give no PM and GM in the observed frequency range and are stable.
- For tyre deflection transfer function TF3, we observe that for first set of rho values stability decreases after control application as GM becomes less. For set 2, stability in controlled system is slighly better than uncontrolled. For set 3, no significant change in stability is observed.

5 Part D: Half-Car (Roll) System Modeling

(i)





$$J = [\int_0^\infty (\rho_1\alpha_1^2 + \rho_2\dot{\alpha}_1^2 + \rho_3\dot{\alpha}_2^2 + \rho_4\dot{\alpha}_2^2 + \rho_4F_r^2 + \rho_6F_l^2 + \rho_6F_l^2)dt]$$

Now since the centre of gravity will be shifted, we'll take the renewed I's around point A

$$I_{au} = I_u + M_u \cdot h_1^2$$

$$I_{as} = I_s + M_s \cdot h_2^2$$

This leads us to the following equations, for unsprung mass

$$I_{au}\ddot{\alpha}_1 = m_u a_y h_1 + (k_{sr} + k_{sl})(\tfrac{t}{2})^2(\alpha_2 - \alpha_1) +$$
$$(\mathrm{b}_{sr} + b_{sl})(\tfrac{t}{2})^2(\dot{\alpha}_2 - \dot{\alpha}_1) + (F_l - F_r)\tfrac{t}{2} - \alpha_1(\tfrac{t}{2})^2(k_{tr} + k_{tl})$$

(29)

Similarly, for sprung mass,

$$I_{as}\ddot{\alpha}_2 = m_s a_y h_2 + (k_{sr}+k_{sl})(\frac{t}{2})^2(-\alpha_2+\alpha_1) + (b_{sr}+b_{sl})(\frac{t}{2})^2(-\dot{\alpha}_2+\dot{\alpha}_1) + (F_r - F_l)\frac{t}{2} \tag{30}$$

(ii)
Writing the above set of governing equations in SS form,

$$\dot{x}(t) = Ax(t) + Bf(t) + da_t(t)$$

$$x(t) = \begin{bmatrix} \alpha_1(t) \\ \dot{\alpha}_1(t) \\ \alpha_2(t) \\ \dot{\alpha}_2(t) \end{bmatrix}$$

f(t)= $\begin{bmatrix} F_r(t) \\ F_l(t) \end{bmatrix}$

$$\dot{x}(t) = \begin{bmatrix} \dot{\alpha}_1(t) \\ \ddot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \ddot{\alpha}_2(t) \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} \dot{\alpha}_1(t) \\ \ddot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \ddot{\alpha}_2(t) \end{bmatrix} = [A] \begin{bmatrix} \alpha_1(t) \\ \dot{\alpha}_1(t) \\ \alpha_2(t) \\ \dot{\alpha}_2(t) \end{bmatrix} + [B][F(t)] + [d][a_t(t)]$$

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{sr}+k_{sl}+k_{tr}+k_{tl}}{I_{au}}(\frac{t}{2})^2 & -\frac{b_{sr}+b_{sl}}{I_{au}}(\frac{t}{2})^2 & \frac{k_{sr}+k_{sl}}{I_{au}}(\frac{t}{2})^2 & \frac{b_{sr}+b_{sl}}{I_{au}}(\frac{t}{2})^2 \\ 0 & 0 & 0 & 1 \\ \frac{k_{sr}+k_{sl}}{I_{as}}(\frac{t}{2})^2 & \frac{b_{sr}+b_{sl}}{I_{as}}(\frac{t}{2})^2 & -\frac{k_{sr}+k_{sl}}{I_{as}}(\frac{t}{2})^2 & -\frac{b_{sr}+b_{sl}}{I_{as}}(\frac{t}{2})^2 \end{bmatrix}$$

$$[d] = \begin{bmatrix} 0 \\ \frac{m_u h_l}{I_a u} \\ 0 \\ \frac{m_s h_r}{I_a r} \end{bmatrix}$$

B=$\begin{bmatrix} 0 & 0 \\ -\frac{t}{2I_{au}} & \frac{t}{2I_{au}} \\ 0 & 0 \\ \frac{t}{2I_{as}} & -\frac{t}{2I_{as}} \end{bmatrix}$

(iii) Given,

$$J = [\int_0^\infty (\rho_1 \alpha_1^2 + \rho_2 \dot{\alpha}_1^2 + \rho_3 \alpha_2^2 + \rho_4 \dot{\alpha}_2^2 + \rho_5 F_r^2 + \rho_6 F_l^2) dt]$$

Now we need an expression (inside integral) of the form,

$$XQX + 2XNF + FRF$$

$$[X][Q][X] = \begin{bmatrix} \alpha_1 & \dot{\alpha}_1 & \alpha_2 & \dot{\alpha}_2 \end{bmatrix} Q \begin{bmatrix} \alpha_1 \\ \dot{\alpha}_1 \\ \alpha_2 \\ \dot{\alpha}_2 \end{bmatrix}$$

$$[Q] = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 0 & 0 & 0 \\ 0 & 0 & \rho_3 & 0 \\ 0 & 0 & 0 & \rho_4 \end{bmatrix}$$

$$2[X][N][F] = 2 \begin{bmatrix} \alpha_1 & \dot{\alpha}_1 & \alpha_2 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_r \\ F_l \end{bmatrix}$$

$$[F][R][F] = \begin{bmatrix} F_r \\ F_l \end{bmatrix} \begin{bmatrix} \rho_5 & 0 \\ 0 & \rho_6 \end{bmatrix} \begin{bmatrix} F_r \\ F_l \end{bmatrix}$$

$$[N] = [0]$$

$$[R] = \begin{bmatrix} \rho_5 & 0 \\ 0 & \rho_6 \end{bmatrix}$$

We have the following system specifications for Half car model:

M$_s$ = 1250Kg
ksl = ksr = 50000 N/m
bsl = bsr = 3000 Ns/m
Mu = 125 Kg
ktl = ktr = 350000 N/m
t = 2.5 m
h1 = 1 m
h2 = 2.2 m
Is = 2000 Kg.m^2
Iu = 200 Kg.m^2

$$I_{sc} = I_s + M_s * h_2^2$$

$$I_{uc} = I_u + M_u * h_1^2$$

By using the matlab command, [K, S, P] = lqr(A, B, Q, R, N)
The LQR gains are:

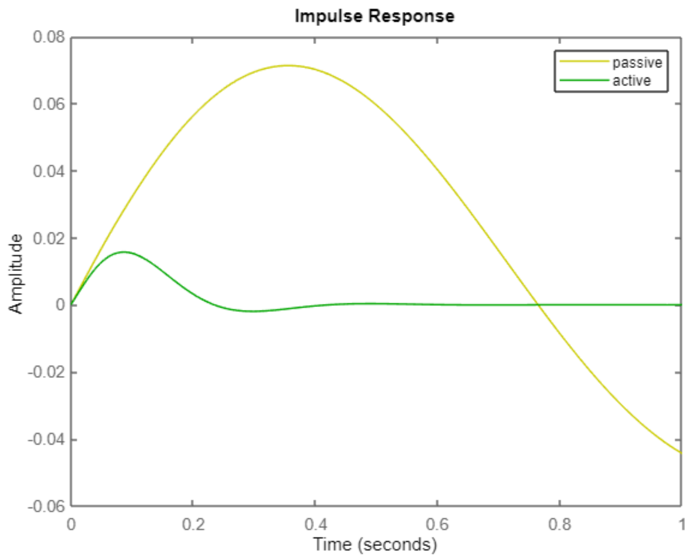
$$K = 10^6 \cdot \begin{bmatrix} -2.8039 & -0.0709 & 4.410 & 0.4707 \\ 2.8039 & 0.0709 & -4.4101 & -0.4707 \end{bmatrix}$$

$$S = 10^3 \cdot \begin{bmatrix} 0.5725 & -0.0024 & -0.6675 & -0.0784 \\ -0.0024 & 0.0000 & 0.0076 & 0.0006 \\ -0.6675 & 0.0076 & 4.2433 & 0.2171 \\ -0.0784 & 0.0006 & 0.2171 & 0.0181 \end{bmatrix}$$

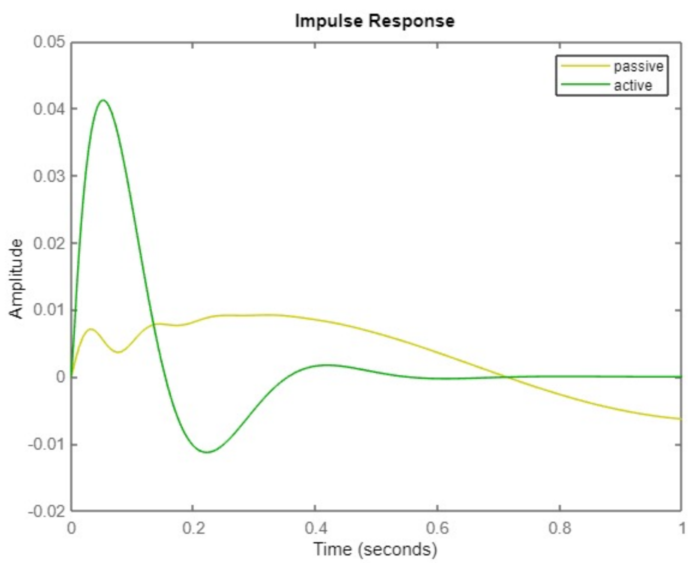
$$P = 10^2 \cdot \begin{bmatrix} -0.0935 + 0.1625i \\ -0.0935 - 0.1625i \\ -0.1946 + 0.0000i \\ -6.8358 + 0.0000i \end{bmatrix}$$

(iv)
Here in this part we look at the system response as roll angle to the input of lateral acceleration. We see at roll angle response for impulse lateral acceleration input.

Impulse Response (Sprung)



Impulse Response (Unsprung)



We observe that in case of Sprung mass (Vehicle body), active suspension is very effective at reducing the impact/response of given impulse lateral acceleration input and improves the ride comfort.

The impulse response for unsprung mass is more complex and shows that the active suspension shows a little more turbulent roll response before getting back to 0 as compared to passive suspension.