

# Ground state energy of quantum dots using the coupled cluster method

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Something about coupled-cluster... Preferably doubles.

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## I. INTRODUCTION

In this project we will study the ground state energy of quantum dots.

## II. THEORY

In this project we will study a system of  $N$  interacting electrons. We will be looking at a Hamiltonian consisting of a one-body and a two-body part. The one-body part is given by

$$h(\mathbf{r}_i) = -\frac{1}{2}\nabla_i^2 + \frac{1}{2}\omega^2\mathbf{r}_i^2, \quad (1)$$

where we use natural units  $\hbar = c = e = 1$  and set the mass to unity. The two-body part is the Coulomb interaction potential.

$$w(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2)$$

We thus get the total Hamiltonian

$$H = h + W = \sum_{i=1}^N h(\mathbf{r}_i) + \sum_{i<j}^N w(\mathbf{r}_i, \mathbf{r}_j), \quad (3)$$

where  $h$  is the full one-body operator and  $W$  the full two-body operator, i.e., over the entire system. Working in a

basis of  $L$  single particle functions,  $\{|p\rangle\}_{p=1}^L$ . We define the reference Slater determinant as

$$|\Phi_0\rangle \equiv |1, 2, \dots, N\rangle, \quad (4)$$

i.e., a tensorproduct of the  $N$  first single particle functions,  $|i\rangle$ , of the system. We call these single particle functions *occupied* as they are contained in the Slater determinant. We will denote the occupied indices with  $i, j, k, l, \dots \in \{1, \dots, N\}$ , the *virtual* states with  $a, b, c, d, \dots \in \{N+1, \dots, L\}$  and general indices with  $p, q, r, s, \dots \in \{1, \dots, L\}$ . In terms of sets of basis functions we can write this as

$$\{|p\rangle\}_{p=1}^L = \{|i\rangle\}_{i=1}^N \cup \{|a\rangle\}_{a=N+1}^L, \quad (5)$$

i.e., the general indexed states consists of both occupied and virtual states. Note that the single particle functions are orthonormal, i.e.,

$$\langle p|q\rangle = \delta_{pq}. \quad (6)$$

We can construct other Slater determinants in this basis by exciting or de-exciting the reference determinant. A general excitation is labeled  $|\Phi_{ij\dots}^{ab\dots}\rangle$  which means that we have removed the single particle functions with indices  $i, j, \dots$  from the reference and added  $a, b, \dots$ . Note that

$$\langle \Phi_{ij\dots}^{ab\dots} | \Phi_0 \rangle = 0, \quad (7)$$

for any excitation.

### A. Second quantization

Employing the creation operators,  $a_p^\dagger$ , and the destruction operators,  $a_p$ , we can write the Hamiltonian as

$$H = \sum_{pq} h_q^p a_p^\dagger a_q + \sum_{pqrs} w_{rs}^{pq} a_p^\dagger a_q^\dagger a_s a_r, \quad (8)$$

where the sums are general indices over all  $L$  basis states and the matrix elements are defined as

$$h_q^p \equiv \langle p|h|q\rangle, \quad (9)$$

$$w_{rs}^{pq} \equiv \langle pq|v|rs\rangle. \quad (10)$$

Note that the two-body matrix elements are not antisymmetric yet.

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\* Project code: <https://github.com/Schoyen/FYS4411>

## B. The coupled cluster approximation

We approximate the true wavefunction,  $|\Psi\rangle$ , of the system by the coupled cluster wavefunction,  $|\Psi_{CC}\rangle$ , defined by

$$|\Psi_{CC}\rangle \equiv e^T |\Phi_0\rangle = \left( \sum_{n=0}^{\infty} \frac{1}{n!} T^n \right) |\Phi_0\rangle, \quad (11)$$

where the *cluster operator*,  $T$ , is given by a sum of  $p$ -excitation operators labeled  $T_p$ . They consist of *cluster amplitudes*,  $t_{i\dots}^a$ , and creation and annihilation operators.

$$T = T_1 + T_2 + \dots + T_p \quad (12)$$

$$= \sum_{ia} t_i^a a_a^\dagger a_i + \left( \frac{1}{2!} \right)^2 \sum_{ijab} t_{ij}^{ab} a_a^\dagger a_b^\dagger a_i a_j + \dots \quad (13)$$

In the doubles approximation we limit the cluster operator to

$$T \equiv T_2 = \frac{1}{4} \sum_{ijab} t_{ij}^{ab} a_a^\dagger a_b^\dagger a_j a_i. \quad (14)$$

The first part of the coupled cluster method consists of constructing the cluster amplitudes using the *amplitude equations*. After we have found the amplitudes we can compute the energy.

## C. Energy of the coupled cluster approximation

When we're going to compute the energy of a system using the coupled cluster approximation we would ideally want to find the expectation value of the energy using the coupled cluster wavefunction.

$$E_{CC} = \langle \Psi_{CC} | H | \Psi_{CC} \rangle. \quad (15)$$

As it turns out, this is an uncomfortable way of finding the energy as  $T \neq T^\dagger$ . Instead we will define what we call the *similarity transformed Hamiltonian*. We plug the coupled cluster wavefunction into the Schrödinger equation.

$$H |\Psi_{CC}\rangle = E_{CC} |\Psi_{CC}\rangle. \quad (16)$$

Next, we left multiply with the inverse of the cluster expansion, i.e.,

$$e^{-T} H |\Psi_{CC}\rangle = e^{-T} E_{CC} |\Psi_{CC}\rangle = E_{CC} |\Phi_0\rangle. \quad (17)$$

Projecting this equation on the reference state we get

$$E_{CC} = \langle \Phi_0 | e^{-T} H | \Psi_{CC} \rangle = \langle \Phi_0 | e^{-T} H e^T | \Phi_0 \rangle, \quad (18)$$

where in the latter inner-product we have located the similarity transformed Hamiltonian defined by

$$\bar{H} \equiv e^{-T} H e^T. \quad (19)$$

To simplify the energy equation and the amplitude equations we use the normal ordered Hamiltonian.

$$H = H_N + \langle \Phi_0 | H | \Phi_0 \rangle. \quad (20)$$

The energy equation thus becomes

$$E_{CC} = \langle \Phi_0 | \bar{H} | \Phi_0 \rangle = E_0 + \langle \Phi_0 | e^{-T} H_N e^T | \Phi_0 \rangle, \quad (21)$$

where the reference energy is given by

$$E_0 = \langle \Phi_0 | H | \Phi_0 \rangle. \quad (22)$$

We now define the normal ordered similarity transformed Hamiltonian as

$$\bar{H}_N \equiv e^{-T} H_N e^T. \quad (23)$$

By expanding the exponentials of this Hamiltonian and recognizing the commutators we get the Baker-Campbell-Hausdorff expansion.

$$\bar{H}_N = H_N + [H_N, T] + \frac{1}{2!} [[H_N, T], T] + \dots \quad (24)$$

From the connected cluster theorem we know that the only nonzero terms in the Baker-Campbell-Hausdorff expansion will be the terms where the normal ordered Hamiltonian has at least one contraction<sup>1</sup> with every cluster operator on its right. This lets us write the expansion as

$$\bar{H}_N = H_N + (H_N T)_c + \frac{1}{2!} (H_N T^2)_c + \dots, \quad (25)$$

where the subscript  $c$  signifies that only contributions where at least one contraction between  $H_N$  and  $T$  has been performed will be included.

### 1. Coupled cluster doubles energy equation

Using the doubles approximation with the cluster operator  $T_2$  defined in Equation 14 the energy equation becomes

$$E_{CCD} = E_0 + \langle \Phi_0 | e^{-T_2} H_N e^{T_2} | \Phi_0 \rangle. \quad (26)$$

As the doubles cluster operator doubly excites the reference and using the expansion in Equation 25 we see that we can write the energy equation as

$$E_{CCD} = E_0 + \langle \Phi_0 | H_N (1 + T_2) | \Phi_0 \rangle, \quad (27)$$

as the Hamiltonian is only able to de-excite one pair of single particle functions.

<sup>1</sup> In the Wick's theorem sense.

## D. Coupled cluster amplitude equations

In order for us to solve the energy equation using the coupled cluster approximation we need to figure out what the cluster amplitudes,  $t_{ij\dots}^{ab\dots}$ , are. This is done by projecting Equation 17 onto an excited Slater determinant, i.e.,

$$\langle \Phi_{ij\dots}^{ab\dots} | e^{-T} H e^T | \Phi_0 \rangle = 0. \quad (28)$$

Note that in the amplitude equations we can use both the regular and the normal ordered Hamiltonian. They are equal as the reference energy term disappears due to Equation 7. The order of the excitation in the projection determines the order of the amplitudes you will find. In our case we are only interested in the second order amplitudes found in the doubles approximation, hence we will solve the equation

$$\langle \Phi_{ij}^{ab} | e^{-T} H_N e^T | \Phi_0 \rangle = 0, \quad (29)$$

to find an expression that can be used to solve for  $t_{ij}^{ab}$ .

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- [1] T. D. Crawford and H. F. Schaefer, Reviews in Computational Chemistry, Volume 14 , 33 (2007).

## Appendix A: The normal ordered Hamiltonian

When constructing the normal ordered Hamiltonian we use Wick's theorem to write the one-body,  $h$ , and the two-body,  $W$ , operators onto a normal ordered form. Specifically we define the normal ordered form in terms of the *Fermi vacuum*<sup>2</sup>. That is, an operator on normal ordered form destroys the reference Slater determinant.

We start by writing the one-body operator,  $h$ , to its normal-ordered form.

$$h = \sum_{pq} h_q^p a_p^\dagger a_q = \sum_{pq} h_q^p \left( \{a_p^\dagger a_q\} + \{a_p^\dagger \overline{a_q}\} \right) \quad (A1)$$

$$= \sum_{pq} h_q^p \{a_p^\dagger a_q\} + \sum_{pq} h_q^p \delta_{p \in i} \delta_{pq} \quad (A2)$$

$$= h_N + \sum_i h_i^i, \quad (A3)$$

where we have used  $\delta_{p \in i}$  to mean that  $p$  must be an occupied index. Doing the same for the two-body operator is a slightly more tedious endeavor. For brevity we will only write out the operator strings and only keep the non-zero contributions.

$$a_p^\dagger a_q^\dagger a_s a_r \quad (A4)$$

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<sup>2</sup> Fermi vacuum defines the reference state, i.e.,  $|\Phi_0\rangle$ , as the vacuum.