

# Ground state energy of quantum dots using the coupled cluster method

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Something about coupled-cluster... Preferably doubles.

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## I. INTRODUCTION

In this project we will study the ground state energy of quantum dots.

## II. THEORY

In this project we will study a system of  $N$  interacting electrons. We will be looking at a Hamiltonian consisting of a one-body and a two-body part. The one-body part is given by

$$h(\mathbf{r}_i) = -\frac{1}{2}\nabla_i^2 + \frac{1}{2}\omega^2\mathbf{r}_i^2, \quad (1)$$

where we use natural units  $\hbar = c = e = 1$  and set the mass to unity. The two-body part is the Coulomb interaction potential.

$$w(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2)$$

We thus get the total Hamiltonian

$$H = h + W = \sum_{i=1}^N h(\mathbf{r}_i) + \sum_{i<j}^N w(\mathbf{r}_i, \mathbf{r}_j), \quad (3)$$

where  $h$  is the full one-body operator and  $W$  the full two-body operator, i.e., over the entire system. Working in a basis of  $L$  single particle functions,  $\{|p\rangle\}_{p=1}^L$ . We define the reference Slater determinant as

$$|\Phi_0\rangle \equiv |1, 2, \dots, N\rangle, \quad (4)$$

i.e., a tensorproduct of the  $N$  first single particle functions,  $|i\rangle$ , of the system. We call these single particle functions *occupied* as they are contained in the Slater determinant. We will denote the occupied indices with  $i, j, k, l, \dots \in \{1, \dots, N\}$ , the *virtual* states with  $a, b, c, d, \dots \in \{N+1, \dots, L\}$  and general indices with  $p, q, r, s, \dots \in \{1, \dots, L\}$ . In terms of sets of basis functions we can write this as

$$\{|p\rangle\}_{p=1}^L = \{|i\rangle\}_{i=1}^N \cup \{|a\rangle\}_{a=N+1}^L, \quad (5)$$

i.e., the general indexed states consists of both occupied and virtual states. Note that the single particle functions are orthonormal, i.e.,

$$\langle p|q\rangle = \delta_{pq}. \quad (6)$$

We can construct other Slater determinants in this basis by exciting or relaxing the reference determinant. A general excitation is labeled  $|\Phi_{ij\dots}^{ab\dots}\rangle$  which means that we have removed the single particle functions with indices  $i, j, \dots$  from the reference and added  $a, b, \dots$ . Note that

$$\langle \Phi_{ij\dots}^{ab\dots} | \Phi_0 \rangle = 0, \quad (7)$$

for any excitation.

### A. Second quantization

Employing the creation operators,  $a_p^\dagger$ , and the destruction operators,  $a_p$ , we can write the Hamiltonian as

$$H = \sum_{pq} h_q^p a_p^\dagger a_q + \frac{1}{4} \sum_{pqrs} \langle pq||rs \rangle a_p^\dagger a_q^\dagger a_s a_r, \quad (8)$$

where the sums are general indices over all  $L$  basis states and the matrix elements are defined as

$$h_q^p \equiv \langle p|h|q\rangle, \quad (9)$$

$$\langle pq||rs \rangle \equiv \langle pq|w|rs \rangle - \langle pq|w|sr \rangle. \quad (10)$$

Note that we use the chemists notation to label the antisymmetric matrix elements.

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\* Project code: <https://github.com/Schoyen/FYS4411>

### B. The coupled cluster approximation

We approximate the true wavefunction,  $|\Psi\rangle$ , of the system by the coupled cluster wavefunction,  $|\Psi_{CC}\rangle$ , defined by

$$|\Psi_{CC}\rangle \equiv e^T |\Phi_0\rangle = \left( \sum_{n=0}^{\infty} \frac{1}{n!} T^n \right) |\Phi_0\rangle, \quad (11)$$

where the *cluster operator*,  $T$ , is given by a sum of  $p$ -excitation operators labeled  $T_p$ . They consist of *cluster amplitudes*,  $t_{i,\dots}^a$ , and creation and annihilation operators.

$$T = T_1 + T_2 + \dots + T_p \quad (12)$$

$$= \sum_{ia} t_i^a a_a^\dagger a_i + \left( \frac{1}{2!} \right)^2 \sum_{ijab} t_{ij}^{ab} a_a^\dagger a_b^\dagger a_i a_j + \dots \quad (13)$$

In the doubles approximation we limit the cluster operator to

$$T \equiv T_2 = \frac{1}{4} \sum_{ijab} t_{ij}^{ab} a_a^\dagger a_b^\dagger a_j a_i. \quad (14)$$

The first part of the coupled cluster method consists of constructing the cluster amplitudes using the *amplitude equations*. After we have found the amplitudes we can compute the energy.

### C. Energy of the coupled cluster approximation

When we're going to compute the energy of a system using the coupled cluster approximation we would ideally want to find the expectation value of the energy using the coupled cluster wavefunction.

$$E_{CC} = \langle \Psi_{CC} | H | \Psi_{CC} \rangle. \quad (15)$$

As it turns out, this is an uncomfortable way of finding the energy as  $T \neq T^\dagger$ . Instead we will define what we call the *similarity transformed Hamiltonian*. We plug the coupled cluster wavefunction into the Schrödinger equation.

$$H |\Psi_{CC}\rangle = E_{CC} |\Psi_{CC}\rangle. \quad (16)$$

Next, we left multiply with the inverse of the cluster expansion, i.e.,

$$e^{-T} H |\Psi_{CC}\rangle = e^{-T} E_{CC} |\Psi_{CC}\rangle = E_{CC} |\Phi_0\rangle. \quad (17)$$

Projecting this equation on the reference state we get

$$E_{CC} = \langle \Phi_0 | e^{-T} H | \Psi_{CC} \rangle = \langle \Phi_0 | e^{-T} H e^T | \Phi_0 \rangle, \quad (18)$$

where in the latter inner-product we have located the similarity transformed Hamiltonian defined by

$$\bar{H} \equiv e^{-T} H e^T. \quad (19)$$

To simplify the energy equation and the amplitude equations we use the normal ordered Hamiltonian.

$$H = H_N + \langle \Phi_0 | H | \Phi_0 \rangle. \quad (20)$$

The energy equation thus becomes

$$E_{CC} = \langle \Phi_0 | \bar{H} | \Phi_0 \rangle = E_0 + \langle \Phi_0 | e^{-T} H_N e^T | \Phi_0 \rangle, \quad (21)$$

where the reference energy is given by

$$E_0 = \langle \Phi_0 | H | \Phi_0 \rangle. \quad (22)$$

We now define the normal ordered similarity transformed Hamiltonian as

$$\bar{H}_N \equiv e^{-T} H_N e^T. \quad (23)$$

By expanding the exponentials of this Hamiltonian and recognizing the commutators we get the Baker-Campbell-Hausdorff expansion.

$$\bar{H}_N = H_N + [H_N, T] + \frac{1}{2!} [[H_N, T], T] + \dots \quad (24)$$

From the connected cluster theorem we know that the only nonzero terms in the Baker-Campbell-Hausdorff expansion will be the terms where the normal ordered Hamiltonian has at least one contraction<sup>1</sup> with every cluster operator on its right. This lets us write the expansion as

$$\bar{H}_N = H_N + (H_N T)_c + \frac{1}{2!} (H_N T^2)_c + \dots, \quad (25)$$

where the subscript  $c$  signifies that only contributions where at least one contraction between  $H_N$  and  $T$  has been performed will be included.

#### 1. Coupled cluster doubles energy equation

Using the doubles approximation with the cluster operator  $T_2$  defined in Equation 14 the energy equation becomes

$$E_{CCD} = E_0 + \langle \Phi_0 | e^{-T_2} H_N e^{T_2} | \Phi_0 \rangle. \quad (26)$$

As the doubles cluster operator doubly excites the reference and using the expansion in Equation 25 we see that we can write the energy equation as

$$E_{CCD} = E_0 + \langle \Phi_0 | H_N | \Phi_0 \rangle + \langle \Phi_0 | (H_N T_2)_c | \Phi_0 \rangle, \quad (27)$$

as the Hamiltonian is only able to relax one pair of single particle functions. By construction we have that

$$\langle \Phi_0 | H_N | \Phi_0 \rangle = 0. \quad (28)$$

<sup>1</sup> In the Wick's theorem sense.

In the second term only the normal ordered two-body operator can contribute as the cluster operator gives a total excitation of +2. As we are projecting onto the reference we have to relax to zero again. The normal ordered Fock operator is at most able to excite and relax by 1 and does therefore not contribute to the overall expression.

$$\langle \Phi_0 | (W_N T_2)_c | \Phi_0 \rangle = \frac{1}{4} \sum_{ijab} \langle ij || ab \rangle t_{ij}^{ab}. \quad (29)$$

In total the energy equation reduces to

$$E_{\text{CCD}} = \sum_i h_i^i + \frac{1}{2} \sum_{ij} \langle ij || ij \rangle + \frac{1}{4} \sum_{ijab} \langle ij || ab \rangle t_{ij}^{ab}, \quad (30)$$

where the first two terms come from the reference energy as shown in Equation A10.

#### D. Coupled cluster amplitude equations

In order for us to solve the energy equation using the coupled cluster approximation we need to figure out what the cluster amplitudes,  $t_{ij\dots}^{ab\dots}$ , are. This is done by projecting Equation 17 onto an excited Slater determinant, i.e.,

$$\langle \Phi_{ij\dots}^{ab\dots} | e^{-T} H e^T | \Phi_0 \rangle = 0. \quad (31)$$

Note that in the amplitude equations we can use both the regular and the normal ordered Hamiltonian. They are equal as the reference energy term disappears due to Equation 7. The order of the excitation in the projection determines the order of the amplitudes you will find. In our case we are only interested in the second order amplitudes found in the doubles approximation, hence we will solve the equation

$$\langle \Phi_{ij}^{ab} | e^{-T} H_N e^T | \Phi_0 \rangle = 0, \quad (32)$$

to find an expression that can be used to solve for  $t_{ij}^{ab}$ .

#### E. Constructing the matrix elements

Having found the equations needed in order to find an estimate to the ground state energy using the coupled cluster doubles approximation is a well and dandy. But, we need basis functions to create the matrix elements needed to feed into the coupled cluster code. Often these basis functions are not known and we have to use an approximation or utilize Hartree-Fock to create more optimized basis functions.

##### 1. Harmonic oscillator basis

We will be looking at a system of two-dimensional quantum dots with a Coulomb repulsion. If we assume,

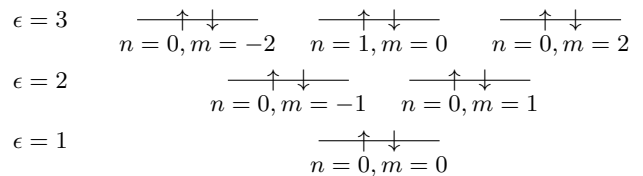


FIG. 1: In this plot we can see the energy degeneracy of the lowest three energy levels in the two-dimensional quantum dot. Each arrow represents a spin up or a spin down state with the quantum numbers  $n$  and  $m$  as listed below.

or make it so, that the repulsive two-body part is small we can use eigenfunctions of the one-body part our basis. In this case we have two-dimensional harmonic oscillator functions as eigenfunctions. We can then compute the matrix elements,  $h_q^p$  and  $u_{rs}^{pq}$ , before feeding these into the coupled cluster code.

In polar coordinates we can write the harmonic oscillator wavefunction for a single particle in two dimensions as<sup>2</sup>.

$$\phi_{nm}(r, \theta) = N_{nm} (ar)^{|m|} L_n^{|m|}(a^2 r^2) e^{-a^2 r^2/2} e^{im\theta}, \quad (33)$$

where  $a = \sqrt{m\omega/\hbar}$  is the Bohr radius,  $L_n^{|m|}$  is the associated Laguerre polynomials,  $n$  and  $m$  are the principal and azimuthal quantum numbers respectively and  $N_{nm}$  is a normalization constant given by

$$N_{nm} = a \sqrt{\frac{n!}{\pi(n+|m|)!}}. \quad (34)$$

Included is also the spin,  $\sigma$ , of the wavefunction, which can be either up or down. This means each level,  $(n, m)$ , is doubly occupied. We also have that the wavefunctions are orthonormal

$$\langle n_1 m_1 \sigma_1 | n_2 m_2 \sigma_2 \rangle = \delta_{n_1 n_2} \delta_{m_1 m_2} \delta_{\sigma_1 \sigma_2}. \quad (35)$$

The eigenenergy of a single harmonic oscillator is given by

$$\epsilon_{nm} = \hbar\omega(2n + |m| + 1). \quad (36)$$

Our next job is now to create a mapping from the three quantum numbers  $n$ ,  $m$  and  $\sigma$  to a single quantum number  $\alpha$  as the matrices  $h$  and  $u$  use single indices for each wavefunction. In Figure 1 we can see the energy levels that needs to be mapped.

Starting from the bottom and working our way upwards from left to right we can label each line from 0 to

<sup>2</sup> Note that this is without spin. As we are looking at fermions this means that each mode of the harmonic oscillator functions will be repeated twice.

$n$  in increasing order. This enumeration will serve as our common quantum number  $\alpha^3$ .

The normalization condition now reads

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta}, \quad (37)$$

when  $|\alpha\rangle$  and  $|\beta\rangle$  share the same spin direction. The one-body matrix will now be a diagonal matrix with the eigenenergies of single particle harmonic oscillator functions as elements.

$$\langle\alpha|h|\beta\rangle = \epsilon_\beta\delta_{\alpha\beta}. \quad (38)$$

The two-body matrix elements are a little harder to work out as the harmonic oscillator wavefunctions are not eigenfunctions to the correlation operator. Luckily, from E. Anisimovas and A. Matulis[1] we can get an analytical expression for the two-body matrix elements.

### Appendix A: The normal ordered Hamiltonian

When constructing the normal ordered Hamiltonian we use Wick's theorem to write the one-body,  $h$ , and the two-body,  $W$ , operators onto a normal ordered form. Specifically we define the normal ordered form in terms of the *Fermi vacuum*<sup>4</sup>. That is, an operator on normal ordered form destroys the reference Slater determinant.

We start by writing the one-body operator,  $h$ , to its normal-ordered form.

$$h = \sum_{pq} h_q^p a_p^\dagger a_q = \sum_{pq} h_q^p \left( \{a_p^\dagger a_q\} + \overline{\{a_p^\dagger a_q\}} \right) \quad (A1)$$

$$= \sum_{pq} h_q^p \{a_p^\dagger a_q\} + \sum_{pq} h_q^p \delta_{p \in i} \delta_{pq} \quad (A2)$$

$$= h_N + \sum_i h_i^i, \quad (A3)$$

where we have used  $\delta_{p \in i}$  to mean that  $p$  must be an occupied index. Doing the same for the two-body operator is a slightly more tedious endeavor. For brevity we will only write out the operator strings and only keep the non-zero contributions.

$$\begin{aligned} a_p^\dagger a_q^\dagger a_s a_r &= \{a_p^\dagger a_q^\dagger a_s a_r\} + \overline{\{a_p^\dagger a_q^\dagger a_s a_r\}} + \overline{\{a_p^\dagger a_q^\dagger a_s a_r\}} \\ &\quad + \{a_p^\dagger a_q^\dagger a_s a_r\} + \overline{\{a_p^\dagger a_q^\dagger a_s a_r\}} \\ &\quad + \overline{\{a_p^\dagger a_q^\dagger a_s a_r\}} + \overline{\{a_p^\dagger a_q^\dagger a_s a_r\}} \end{aligned} \quad (A4)$$

$$\begin{aligned} &= \{a_p^\dagger a_q^\dagger a_s a_r\} - \delta_{p \in i} \delta_{ps} \{a_q^\dagger a_r\} + \delta_{p \in i} \delta_{pr} \{a_q^\dagger a_s\} \\ &\quad + \delta_{q \in i} \delta_{qs} \{a_p^\dagger a_r\} - \delta_{q \in i} \delta_{qr} \{a_p^\dagger a_s\} \\ &\quad - \delta_{p \in i} \delta_{ps} \delta_{q \in j} \delta_{qr} + \delta_{p \in i} \delta_{pr} \delta_{q \in j} \delta_{qs}. \end{aligned} \quad (A5)$$

Inserted into the full two-body operator and sorting out the sums we get

$$\begin{aligned} W &= \frac{1}{4} \sum_{pqrs} \langle pq||rs \rangle \{a_p^\dagger a_q^\dagger a_s a_r\} - \frac{1}{4} \sum_{iqr} \langle iq||ri \rangle \{a_q^\dagger a_r\} \\ &\quad + \frac{1}{4} \sum_{iqs} \langle iq||is \rangle \{a_q^\dagger a_s\} + \frac{1}{4} \sum_{pir} \langle pi||ri \rangle \{a_p^\dagger a_r\} \\ &\quad - \frac{1}{4} \sum_{pis} \langle pi||is \rangle \{a_p^\dagger a_s\} - \frac{1}{4} \sum_{ij} \langle ij||ji \rangle \\ &\quad + \frac{1}{4} \sum_{ij} \langle ij||ij \rangle. \end{aligned} \quad (A6)$$

Using the antisymmetric properties of the two-body matrix elements,

$$\langle pq||rs \rangle = -\langle pq||sr \rangle = -\langle qp||rs \rangle = \langle qp||sr \rangle, \quad (A7)$$

and relabeling of the indices we can rearrange and collect some terms.

$$W = W_N + \sum_{pir} \langle pi||ri \rangle \{a_p^\dagger a_r\} + \frac{1}{2} \sum_{ij} \langle ij||ij \rangle, \quad (A8)$$

where the normal ordered two-body operator is

$$W_N = \frac{1}{4} \sum_{pqrs} \langle pq||rs \rangle \{a_p^\dagger a_q^\dagger a_s a_r\}. \quad (A9)$$

When we now construct the full Hamiltonian we can collect some terms. The constants in both the one-body and the two-body operator in total constitutes the reference energy.

$$E_0 \equiv \langle \Phi_0 | H | \Phi_0 \rangle = \sum_i h_i^i + \frac{1}{2} \sum_{ij} \langle ij||ij \rangle. \quad (A10)$$

Combining the normal ordered one-body operator and the second term in the two-body operator, i.e., the term with a single creation and annihilation operator pair, we get the normal ordered Fock-operator.

$$F_N = \sum_{pq} h_q^p \{a_p^\dagger a_q\} + \sum_{pqi} \langle pi||qi \rangle \{a_p^\dagger a_q\} \quad (A11)$$

$$= \sum_{pq} f_q^p \{a_p^\dagger a_q\}, \quad (A12)$$

where we have defined the Fock matrix elements as

$$f_q^p = h_q^p + \sum_i \langle pi||qi \rangle. \quad (A13)$$

In total we get the full Hamiltonian

$$H = F_N + W_N + \langle \Phi_0 | H | \Phi_0 \rangle \quad (A14)$$

$$= H_N + \langle \Phi_0 | H | \Phi_0 \rangle, \quad (A15)$$

which is what we wanted to show.[2]

<sup>3</sup> Note that we use greek letters  $\alpha, \beta, \dots$  for the harmonic oscillator wavefunctions as opposed to latin letters for the general indices.

<sup>4</sup> Fermi vacuum defines the reference state, i.e.,  $|\Phi_0\rangle$ , as the vacuum.

## Appendix B: Coupled cluster doubles diagrams

In order to get an expression for the energy equation and the amplitude equations we use a diagrammatic approach.

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- [1] E. Anisimovas and A. Matulis, Journal of Physics: Condensed Matter **10**, 601 (1998).
  - [2] T. D. Crawford and H. F. Schaefer, Reviews in Computational Chemistry, Volume 14 , 33 (2007).