A Comprehensive Guide to Numeral Systems

Binary, Hexadecimal, Octal, and Decimal

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Contents

1 Introduction

This document discusses the basic numeral systems: Binary (base 2), Octal (base 8), Decimal (base 10), and Hexadecimal (base 16). Detailed methods for converting numbers between these systems are provided along with numerous examples and solved exercises. The goal is to offer a deep understanding of these numeral systems and demonstrate practical conversion techniques.

2 General Concepts of Numeral Systems

A numeral system is a method for representing numbers using a set of symbols (digits) and a specific base b. Any number can be represented in base b as:

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0,$$

where $0 \leq a_i < b$. The primary numeral systems discussed in this document are:

- **Binary:** $b = 2$ (digits: 0, 1);
- **Octal:** $b = 8$ (digits: 0, 1, . . . , 7);
- **Decimal:** $b = 10$ (digits: 0, 1, . . . , 9);
- **Hexadecimal:** $b = 16$ (digits: 0, 1, . . . , 9, A, B, C, D, E, F).

3 The Binary System

The binary system is based on two digits: 0 and 1. Each digit represents a power of 2. For example, the number:

$$1011_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 8 + 0 + 2 + 1 = 11_{10}.$$

3.1 Conversion from Binary to Decimal

To convert a binary number to decimal, multiply each digit by the corresponding power of 2 and sum the results:

$$\text{If } N = d_n d_{n-1} \ldots d_0, \text{ then } N_{10} = \sum_{i=0}^{n} d_i \cdot 2^i.$$

Example: Convert 1101_2 to decimal.

$$1101_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 4 + 0 + 1 = 13_{10}.$$

4 The Octal System

The octal system uses digits from 0 to 7. Each digit corresponds to a power of 8:

$$N = a_n 8^n + a_{n-1} 8^{n-1} + \cdots + a_0.$$

4.1 Conversion from Octal to Decimal

Example: Convert 725_8 to decimal.

$$725_8 = 7 \cdot 8^2 + 2 \cdot 8^1 + 5 \cdot 8^0 = 7 \cdot 64 + 2 \cdot 8 + 5 = 448 + 16 + 5 = 469_{10}.$$

5 The Decimal System

The decimal system is the standard numeral system, with base 10. It uses the digits 0 through 9.

5.1 Conversion from Decimal to Other Systems

To convert a decimal number to another numeral system, perform successive division by the target base. The remainders, read in reverse order, form the converted number.

Example: Convert 345_{10} to binary.

1. $345 \div 2 = 172$ with a remainder of 1.
2. $172 \div 2 = 86$ with a remainder of 0.
3. $86 \div 2 = 43$ with a remainder of 0.
4. $43 \div 2 = 21$ with a remainder of 1.
5. $21 \div 2 = 10$ with a remainder of 1.
6. $10 \div 2 = 5$ with a remainder of 0.
7. $5 \div 2 = 2$ with a remainder of 1.
8. $2 \div 2 = 1$ with a remainder of 0.
9. $1 \div 2 = 0$ with a remainder of 1.

Reading the remainders in reverse order gives:

$$345_{10} = 101011001_2.$$

6 The Hexadecimal System

The hexadecimal system is based on 16. In addition to the digits 0–9, the letters A, B, C, D, E, and F are used to represent values 10 through 15.

6.1 Conversion from Hexadecimal to Decimal

Example: Convert $1A3_{16}$ to decimal.

$$1A3_{16} = 1 \cdot 16^2 + 10 \cdot 16^1 + 3 \cdot 16^0 = 256 + 160 + 3 = 419_{10}.$$

7 Conversion Methods Between Numeral Systems

This section describes the main algorithms for converting numbers between different numeral systems.

7.1 Binary ↔ Decimal

- **Binary to Decimal:** Multiply each binary digit by the corresponding power of 2 and sum the results.
- **Decimal to Binary:** Use successive division by 2 and record the remainders.

7.2 Binary ↔ Octal

Conversion between binary and octal is simplified because $8 = 2^3$. To convert:

1. Group the binary number into sets of 3 bits starting from the right.
2. Replace each group with the corresponding octal digit.

Example: Convert 1011101_2 to octal.

$$1011101_2 \quad \Rightarrow \quad \underline{1}\,011\,101.$$

Add leading zeros if necessary (e.g., 001) to form groups: 001, 011, 101.

- $001_2 = 1_8$,
- $011_2 = 3_8$,
- $101_2 = 5_8$.

Thus, $1011101_2 = 135_8$.

7.3 Binary $\leftrightarrow$ Hexadecimal

Since $16 = 2^4$, the process is analogous:

1. Group the binary number into sets of 4 bits starting from the right.
2. Replace each group with the corresponding hexadecimal digit.

Example: Convert 10101100_2 to hexadecimal.

$$10101100_2 \quad \Rightarrow \quad 1010 \quad 1100.$$

- $1010_2 = A_{16}$ (since $10_{10} = A$),
- $1100_2 = C_{16}$ (since $12_{10} = C$).

Thus, $10101100_2 = AC_{16}$.

7.4 Hexadecimal $\leftrightarrow$ Octal

Conversion between hexadecimal and octal is often performed via binary:

1. Convert the hexadecimal number to binary (by replacing each hexadecimal digit with 4 bits).
2. Convert the resulting binary number to octal by grouping the bits into sets of 3.

8 Detailed Examples and Exercises

8.1 Exercise 1: Convert from Binary to Hexadecimal

Problem: Convert 11010110_2 to hexadecimal.

Solution:

1. Group the binary number into sets of 4 bits starting from the right:

$$11010110_2 \quad \Rightarrow \quad 1101 \quad 0110.$$

2. Convert each group:
   - $1101_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 4 + 0 + 1 = 13 \Rightarrow D_{16}$,
   - $0110_2 = 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 0 + 4 + 2 + 0 = 6 \Rightarrow 6_{16}$.
3. Therefore, $11010110_2 = D6_{16}$.

8.2 Exercise 2: Convert from Hexadecimal to Binary

Problem: Convert $2F4_{16}$ to binary.

Solution:

1. Convert each hexadecimal digit to a 4-bit binary number:
   - $2_{16} = 0010_2$,
   - $F_{16} = 15_{10} = 1111_2$,
   - $4_{16} = 0100_2$.
2. Combine the groups:
$$2F4_{16} = 0010\,1111\,0100_2.$$
3. The final binary number is 001011110100_2 (leading zeros may be omitted).

8.3 Exercise 3: Convert from Octal to Decimal

Problem: Convert 657_8 to decimal.

Solution:

$$657_8 = 6 \cdot 8^2 + 5 \cdot 8^1 + 7 \cdot 8^0 = 6 \cdot 64 + 5 \cdot 8 + 7 = 384 + 40 + 7 = 431_{10}.$$

8.4 Exercise 4: Convert from Decimal to Binary

Problem: Convert 237_{10} to binary.

Solution: Use successive division by 2:

1. $237 \div 2 = 118$ with remainder 1.
2. $118 \div 2 = 59$ with remainder 0.
3. $59 \div 2 = 29$ with remainder 1.
4. $29 \div 2 = 14$ with remainder 1.
5. $14 \div 2 = 7$ with remainder 0.
6. $7 \div 2 = 3$ with remainder 1.
7. $3 \div 2 = 1$ with remainder 1.
8. $1 \div 2 = 0$ with remainder 1.

Reading the remainders in reverse order: $1\,1\,1\,0\,1\,1\,0\,1$. Thus,

$$237_{10} = 11101101_2.$$

8.5 Exercise 5: Convert from Binary to Octal

Problem: Convert 101110111_2 to octal.

Solution:

1. Group the binary number into sets of 3 bits starting from the right:

$$101110111_2 \quad \Rightarrow \quad 101\,110\,111.$$

2. Convert each group:
   - $101_2 = 5_8$,
   - $110_2 = 6_8$,
   - $111_2 = 7_8$.
3. Hence, $101110111_2 = 567_8$.

8.6 Exercise 6: Convert from Hexadecimal to Octal

Problem: Convert $3B_{16}$ to octal.

Solution:

1. First, convert the hexadecimal number to binary by replacing each hexadecimal digit with 4 bits:
   - $3_{16} = 0011_2$,
   - $B_{16} = 11_{10} = 1011_2$ (pad with zeros if necessary).
2. Combine the binary digits: $3B_{16} = 0011\,1011_2$.
3. Group the binary number into sets of 3 bits starting from the right:

$$0011\,1011_2 \quad \Rightarrow \quad 00\,111\,011.$$

   Add leading zeros to the first group if needed.
4. Convert each group:
   - $000_2 = 0_8$,
   - $111_2 = 7_8$,
   - $011_2 = 3_8$.
5. Therefore, $3B_{16} = 073_8$ (leading zero may be omitted, yielding 73_8).

9 Notes and Tips

- When converting from binary to octal or hexadecimal, be sure to group the bits correctly (3 bits for octal, 4 bits for hexadecimal). If the total number of bits is not divisible by 3 or 4, add leading zeros.
- To verify the conversion, try converting back to the original numeral system.
- In numeral systems with a base greater than 10, remember the correspondence between digits and letters (e.g., $A = 10$, $B = 11$, $\ldots, F = 15$).

10 Conclusion

This document has explored the fundamentals of various numeral systems and provided methods for converting numbers between binary, octal, decimal, and hexadecimal systems. Detailed examples and exercises help reinforce the theoretical knowledge with practical application. Mastery of these conversion techniques is essential for students in computer science and programming, as these numeral systems are widely used in computing and algorithms.

References

1. Kulikov, A.V. "Fundamentals of Discrete Mathematics."
2. Levin, A. "Computer Science: Theory and Practice."
3. Methodological materials from the Department of Informatics.

Appendix: Additional Examples

Example A. Convert 1001110_2 to hexadecimal.

1. Group into 4-bit groups: $0100\ 1110_2$.
2. $0100_2 = 4_{16}$, $1110_2 = E_{16}$.
3. Answer: $1001110_2 = 4E_{16}$.

Example B. Convert 572_{10} to octal.

1. Divide by 8:
   - $572 \div 8 = 71$ with remainder 4.
   - $71 \div 8 = 8$ with remainder 7.
   - $8 \div 8 = 1$ with remainder 0.

- $1 \div 8 = 0$ with remainder 1.

2. Remainders read in reverse order: 1 0 7 4.
3. Answer: $572_{10} = 1074_8$.

Example C. Convert $3C9_{16}$ to decimal.

$$3C9_{16} = 3 \cdot 16^2 + 12 \cdot 16^1 + 9 \cdot 16^0 = 3 \cdot 256 + 192 + 9 = 768 + 192 + 9 = 969_{10}.$$

Note: This material is intended as a comprehensive study resource for students learning the fundamentals of numeral systems.