Cours 2: Modèle de Malik et Perona

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Plan du cours

- Modèle physique de la diffusion de la chaleur
- Equation de diffusion anisotrope

Modèle physique

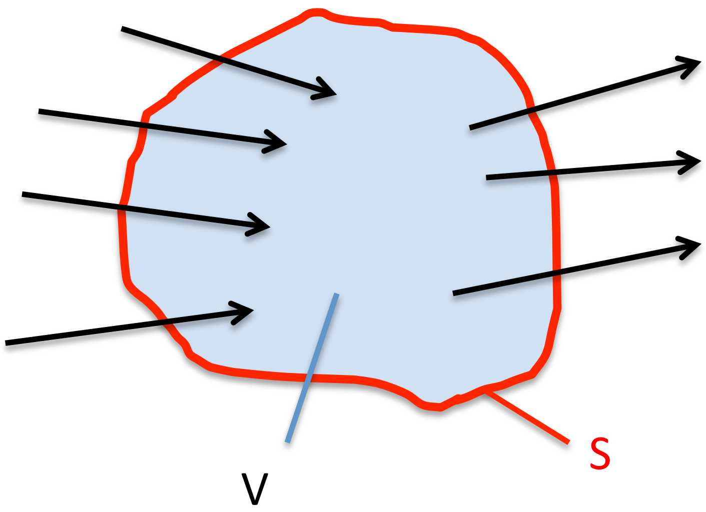
L'équation de la chaleur

Justification physique par l'équation de conservation:

c(x,y,t): température, concentration etc

J: flux

f: apport de chaleur, produit etc



$$\underbrace{\frac{\partial}{\partial t}\int_V c(\mathbf{x},t)\,dv}_{\text{Variation au cours du temps dans le volume V}} = \underbrace{-\int_S \mathbf{J}\cdot\mathbf{ds}}_{\text{Flux sur la surface du volume (entrée/sortie)}} + \underbrace{\int_V f\,dv}_{\text{Apport local}},$$

Modèle physique

- Loi de conservation dans le monde discret
- Intégration par parties revisitée en discret
  - Comment passer d'une intégrale de surface à une intégrale de Volume ?
  - Application « amusante »: calcul du volume à l'intérieur d'un maillage fermé, uniquement à l'aide de la triangulation !

Intégration par parties

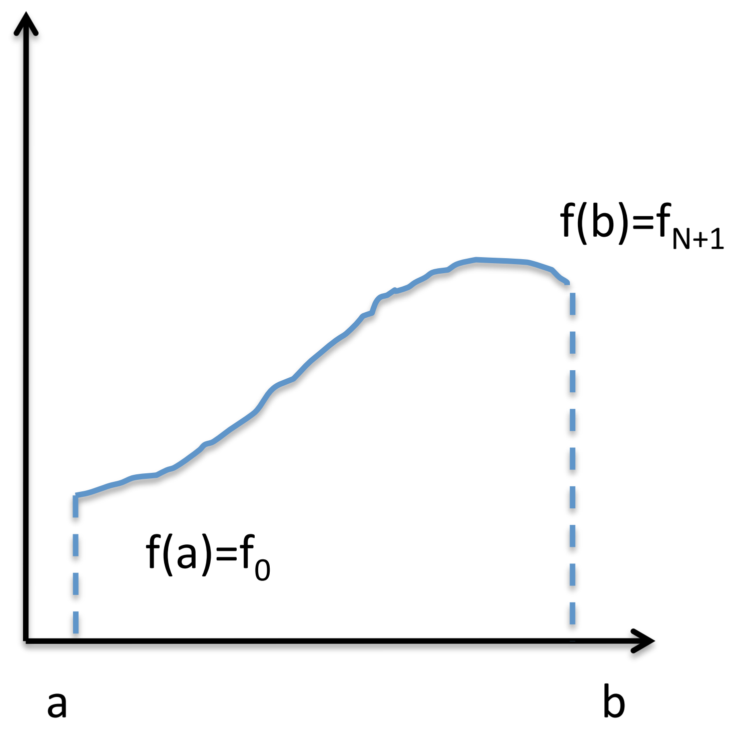
En 1D

$$\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx + \left[f(x)g(x)\right]_a^b$$

$$\sum_{i=1}^{N} f_i\left(g_{i+1} - g_i\right) = \sum_{i=1}^{N} f_i g_{i+1} - \sum_{i=1}^{N} f_i g_i$$

$$= \sum_{i=2}^{N+1} f_{i-1} g_i - \sum_{i=1}^{N} f_i g_i$$

$$= \sum_{i=2}^{N} \left(f_{i-1} - f_i\right) g_i + f_N g_{N+1} - f_0 g_1$$



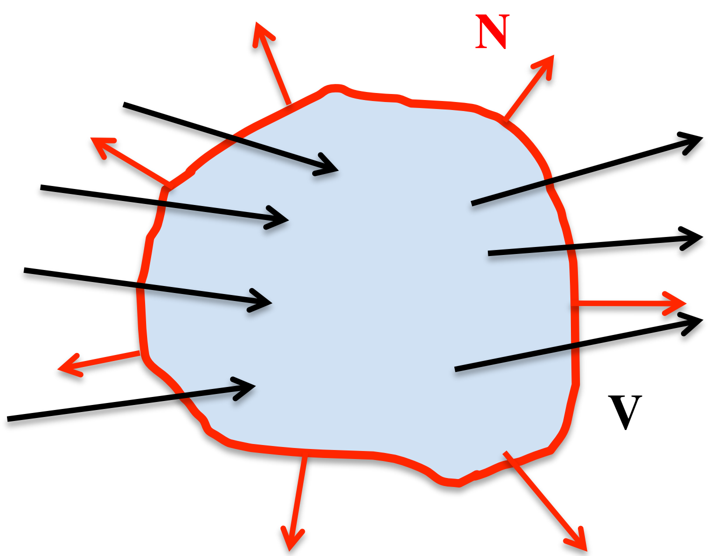
Intégration par parties

En 2D, 3D

Appelé aussi théorème flux/divergence, de Stokes, formule de Green…

$$\iint \mathbf{V}(x,y)\cdot\nabla f(x,y) = -\iint f(x,y)\,\mathrm{div}(\mathbf{V}(x,y)) + \oint f(x,y)\mathbf{V}(x,y)\cdot\mathbf{N}$$

$$\sum_{i=1}^{N} V_{i,j}^1\left(f_{i+1,1} - f_{i,1}\right) + V_{i,j}^2\left(f_{i,j+1} - f_{i,j}\right) = \ldots$$



En utilisant la relation en 1D, on en déduit ce qu’est la **divergence**

$$\left(V_{i+1,j}^1 - V_{i,j}^1\right) + \left(V_{i,j+1}^2 - V_{i,j}^2\right) \simeq \frac{\partial V^1}{\partial x} + \frac{\partial V^2}{\partial y}$$

Intégration par parties

Digression: calcul du volume à l'intérieur d'un maillage

On applique la formule de Green avec f= 1 et un champ de vecteur V tel que div(V)=1. Est ce que cela existe ?

Oui et c'est même très facile d'en trouver un !

Pour résumer, le calcul se réduit à la somme de tous les produits scalaires de V avec la normale en chaque face du maillage.

Equation de la chaleur

L'équation de la chaleur

Après quelques calculs (formule de Green) et simplifications dans le cas où il n'y a pas d'apport de température, concentration etc:

$$\frac{\partial c}{\partial t} + \mathrm{div}\left(\mathrm{J}\right) = 0$$

Opérateur divergence

$$\mathbf{V} = \left(V_1(x,y), V_2(x,y)\right)$$

$$\mathrm{div}\mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}$$

Loi de Fick (1855):

$$\mathrm{J} = -D\nabla c$$

Equation de la chaleur

L'équation de la chaleur

Si la diffusivité D est constante, on aboutit à l'équation de la chaleur:

$$\frac{\partial c}{\partial t} = D\Delta c \longrightarrow$$

Laplacien !

L'équation de la chaleur traduit l'évolution dans un domaine au cours du temps, d'une quantité comme la température.

Sans apport de chaleur extérieur, la température s'uniformise: c'est la propriété recherchée pour le traitement d'images

Equation de diffusion anisotrope

- Diffusivité constante dans l'équation de la chaleur
- Cette diffusivité peut être rendue non constante de la façon suivante

$$\frac{\partial u}{\partial t} = \mathrm{div}\Big(f(x,y)\nabla u\Big)$$

Où f est une fonction qui dépend de l'espace

- Interprétation :

- Là où f est petit, la divergence est faible, peu de variation temporelle.

- Là où f est grand, la divergence est grande, diffusion importante.

Equation de diffusion anisotrope

- **Principe :**

  On veut construire un processus de lissage qui préserve les contours et lisse dans les régions homogènes de l'image.

- **Approche de Malik et Perona (1990) :**

  La fonction f va dépendre de la norme du gradient de l'image (indicateur de contours).

  Plus précisément on veut que f soit faible pour les forts gradients et f plus élevé pour les faibles gradients.

Equation de diffusion anisotrope

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Scale-Space and Edge Detection Using Anisotropic Diffusion

PIETRO PERONA AND JITENDRA MALIK

***Abstract*—The scale-space technique introduced by Witkin involves generating coarser resolution images by convolving the original image with a Gaussian kernel. This approach has a major drawback: it is difficult to obtain accurately the locations of the "semantically meaningful" edges at coarse scales. In this paper we suggest a new definition of scale-space, and introduce a class of algorithms that realize it using a diffusion process. The diffusion coefficient is chosen to vary spatially in such a way as to encourage intraregion smoothing in preference to interregion smoothing. It is shown that the "no new maxima should be generated at coarse scales" property of conventional scale space is preserved. As the region boundaries in our approach remain sharp, we obtain a high quality edge detector which successfully exploits global information. Experimental results are shown on a number of images. The algorithm involves elementary, local operations replicated over the image making parallel hardware implementations feasible.**

***Index Terms*—Adaptive filtering, analog VLSI, edge detection, edge enhancement, nonlinear diffusion, nonlinear filtering, parallel algorithm, scale-space.**

I. Introduction

THE importance of multiscale descriptions of images has been recognized from the early days of computer vision, e.g., Rosenfeld and Thurston [20]. A clean formalism for this problem is the idea of scale-space filtering introduced by Witkin [21] and further developed in Koenderink [11], Babaud, Duda, and Witkin [1], Yuille and Poggio [22], and Hummel [7], [8].

The essential idea of this approach is quite simple: embed the original image in a family of derived images $I(x, y, t)$ obtained by convolving the original image $I_0(x, y)$ with a Gaussian kernel $G(x, y; t)$ of variance t:

$$I(x, y, t) = I_0(x, y) * G(x, y; t). \tag{1}$$

Larger values of t, the scale-space parameter, correspond to images at coarser resolutions. See Fig. 1.

As pointed out by Koenderink [11] and Hummel [7], this one parameter family of derived images may equivalently be viewed as the solution of the heat conduction, or diffusion, equation

$$I_t = \Delta I = (I_{xx} + I_{yy}) \tag{2}$$

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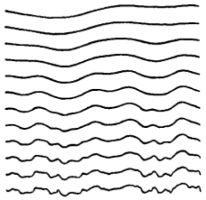

Fig. 1. A family of 1-D signals $I(x, t)$ obtained by convolving the original one (bottom) with Gaussian kernels whose variance increases from bottom to top (adapted from Witkin [21]).

with the initial condition $I(x, y, 0) = I_0(x, y)$, the original image.

Koenderink motivates the diffusion equation formulation by stating two criteria.

1) Causality: Any feature at a coarse level of resolution is required to possess a (not necessarily unique) "cause" at a finer level of resolution although the reverse need not be true. In other words, no spurious detail should be generated when the resolution is diminished.

2) Homogeneity and Isotropy: The blurring is required to be space invariant.

These criteria lead naturally to the diffusion equation formulation. It may be noted that the second criterion is only stated for the sake of simplicity. We will have more to say on this later. In fact the major theme of this paper is to replace this criterion by something more useful.

It should also be noted that the causality criterion does not force uniquely the choice of a Gaussian to do the blurring, though it is perhaps the simplest. Hummel [7] has made the important observation that a version of the maximum principle from the theory of parabolic differential equations is equivalent to causality. We will discuss this further in Section IV-A.

This paper is organized as follows: Section II critiques the standard scale space paradigm and presents an additional set of criteria for obtaining "semantically meaningful" multiple scale descriptions. In Section III we show that by allowing the diffusion coefficient to vary, one can satisfy these criteria. In Section IV-A the maximum principle is reviewed and used to show how the causality criterion is still satisfied by our scheme. In Section V some

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Pietro Perona was born in Padua, Italy, on September 3, 1961. He received the Doctor degree in electrical engineering cum laude from the University of Padua in 1985 with a thesis on dynamical systems theory.

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Dr. Malik received a Presidential Young Investigator Award in 1989.

Malik and Perona, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on pattern Analysis and machine intelligence, 1990

Article cité plus de 13000 fois ! (Google scholar)

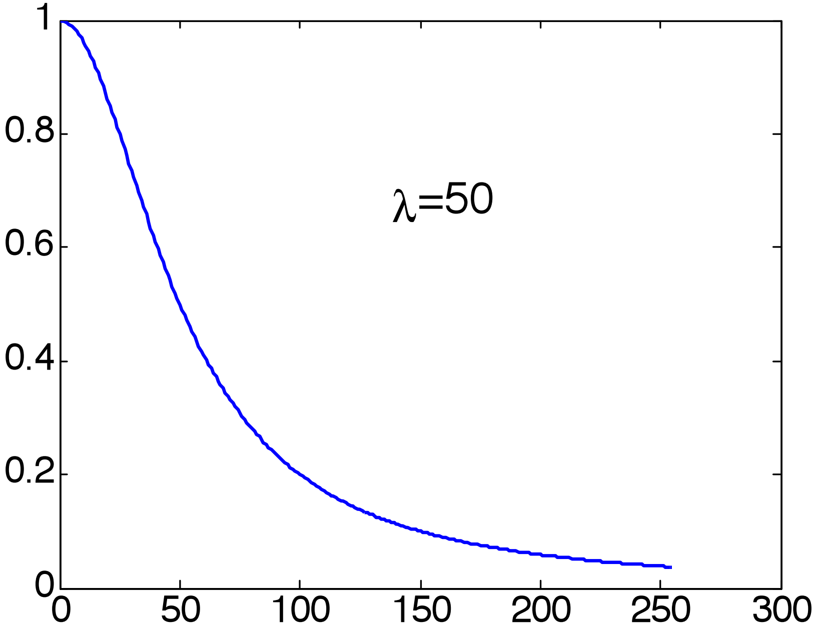
Equation de diffusion anisotrope

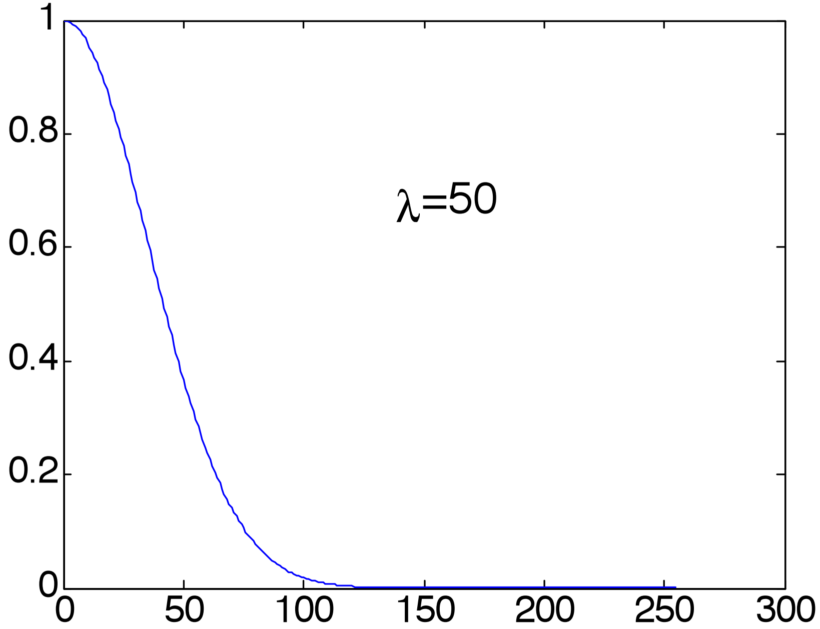
- **Quelle fonction de diffusivité ?**

  On va choisir une fonction décroissante. Malik et Perona proposent :

$$f(x) = \frac{1}{1 + \frac{x^2}{\lambda^2}}$$

$$f(x) = \exp\left(-\frac{x^2}{\lambda^2}\right)$$





Equation de diffusion anisotrope

- **Résultats**

Paramètre λ= 20

10 itérations

20 itérations

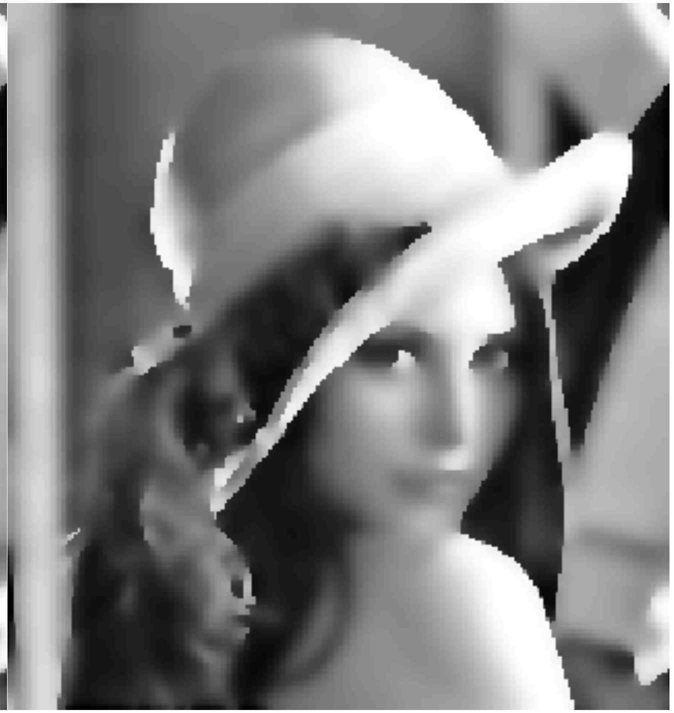
50 itérations

Equation de diffusion anisotrope

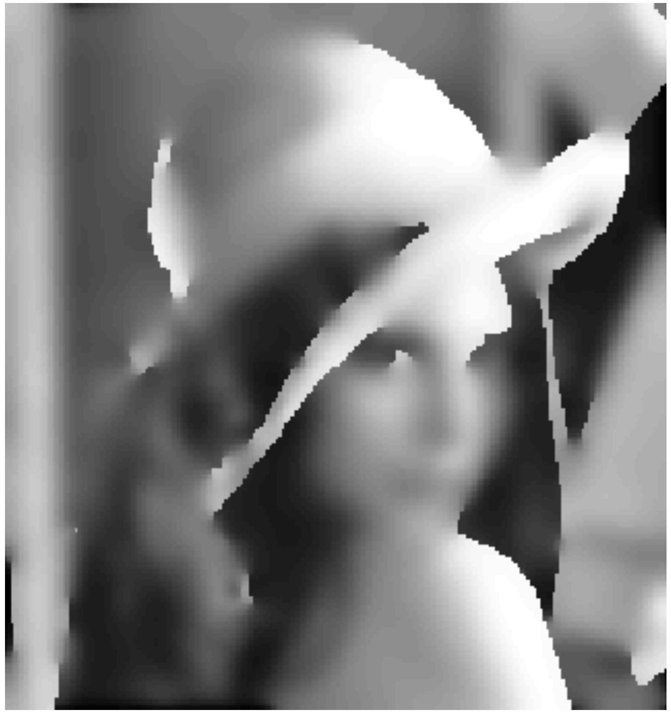
- **Résultats**

Paramètre λ= 50

10 itérations



20 itérations



50 itérations

Equation de diffusion anisotrope

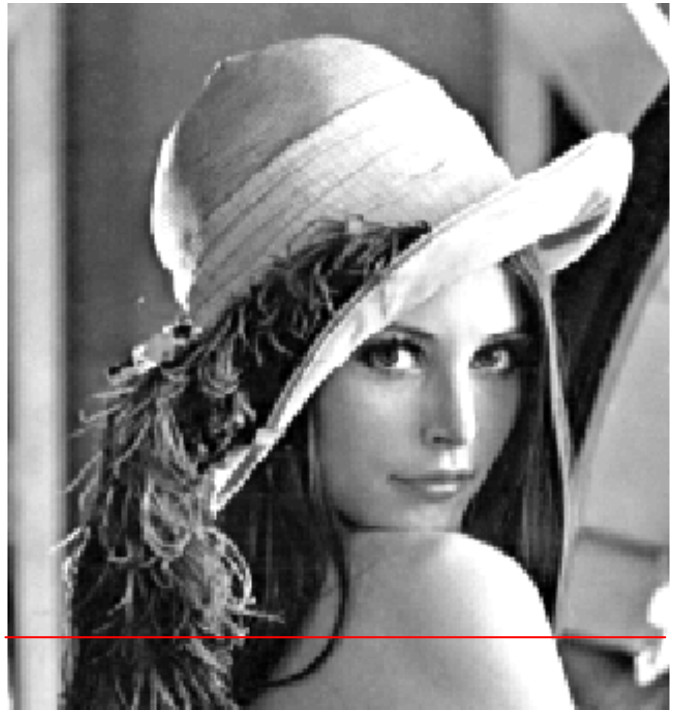
- **Bruit**

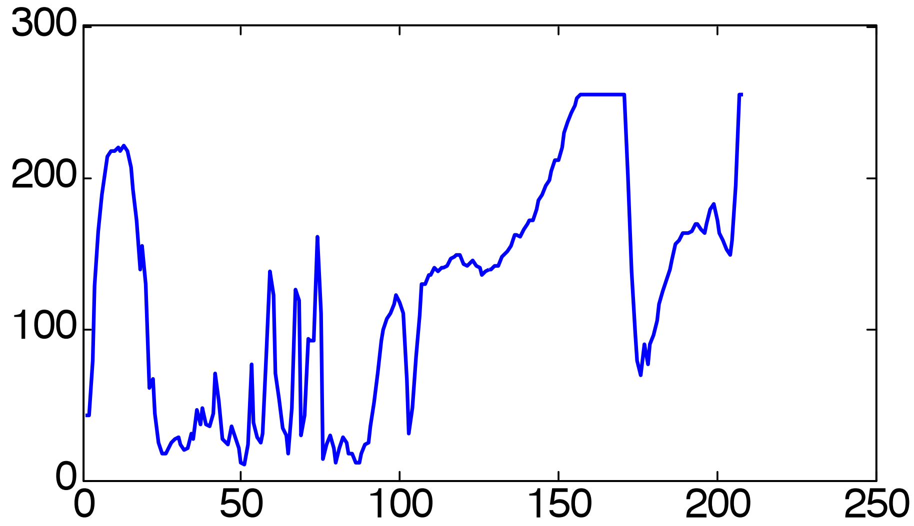
Bruit blanc (amplitude 100)

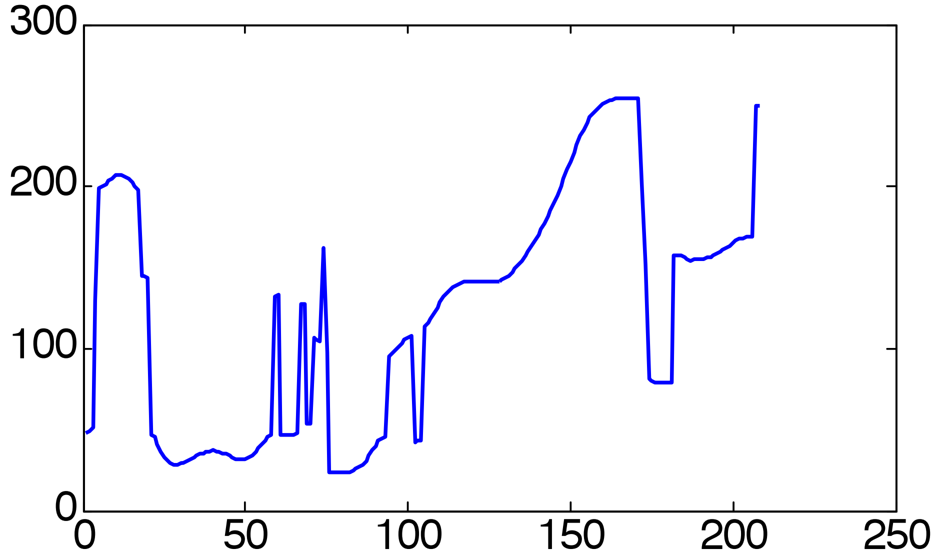
50 itérations

Equation de diffusion anisotrope

- **Contours**







Equation de diffusion anisotrope

- **Propriétés**

  - Lissage **non-uniforme**, ralenti sur les contours
  - Meilleure préservation des angles que médian ou Nagao
  - Tendance à créer une image **constante par morceaux**
  - Zones d'intensité croissante se retrouvent en escalier
  - Vaste choix de fonctions f
  - Choix du paramètre λ en regardant l'histogramme de la norme du gradient

Equation de diffusion anisotrope

- **Aspects numériques**

  - Pas de solutions analytiques comme dans l'équation de la chaleur.

  - Nécessité d'une discrétisation de l'équation.

  - Schéma de discrétisation simple :

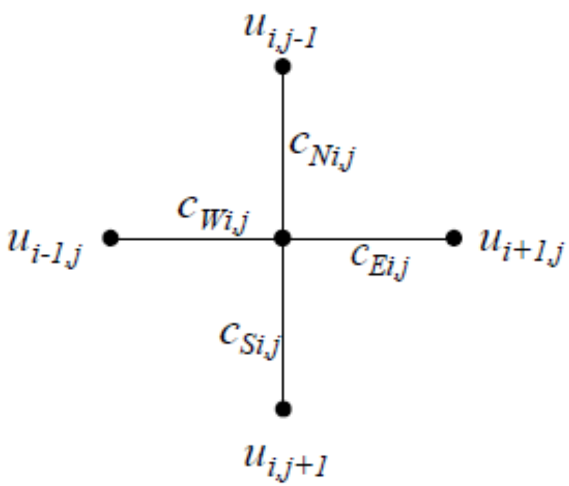
$$u_{i,j}^{t+1} = u_{i,j}^{t} + \Delta t \Big[c_N \nabla_N u + c_S \nabla_S u + c_W \nabla_W u + c_E \nabla_E u \Big]_{i,j}^{t}$$

Equation de diffusion anisotrope

- **Aspects numériques**

$$u_{i,j}^{t+1} = u_{i,j}^{t} + \Delta t \Big[c_N \nabla_N u + c_S \nabla_S u + c_W \nabla_W u + c_E \nabla_E u \Big]_{i,j}^{t}$$

$$\begin{aligned}
\nabla_N u &= u_{i,j-1} - u_{i,j} \\
\nabla_S u &= u_{i,j+1} - u_{i,j} \\
\nabla_W u &= u_{i-1,j} - u_{i,j} \\
\nabla_E u &= u_{i+1,j} - u_{i,j}
\end{aligned}$$

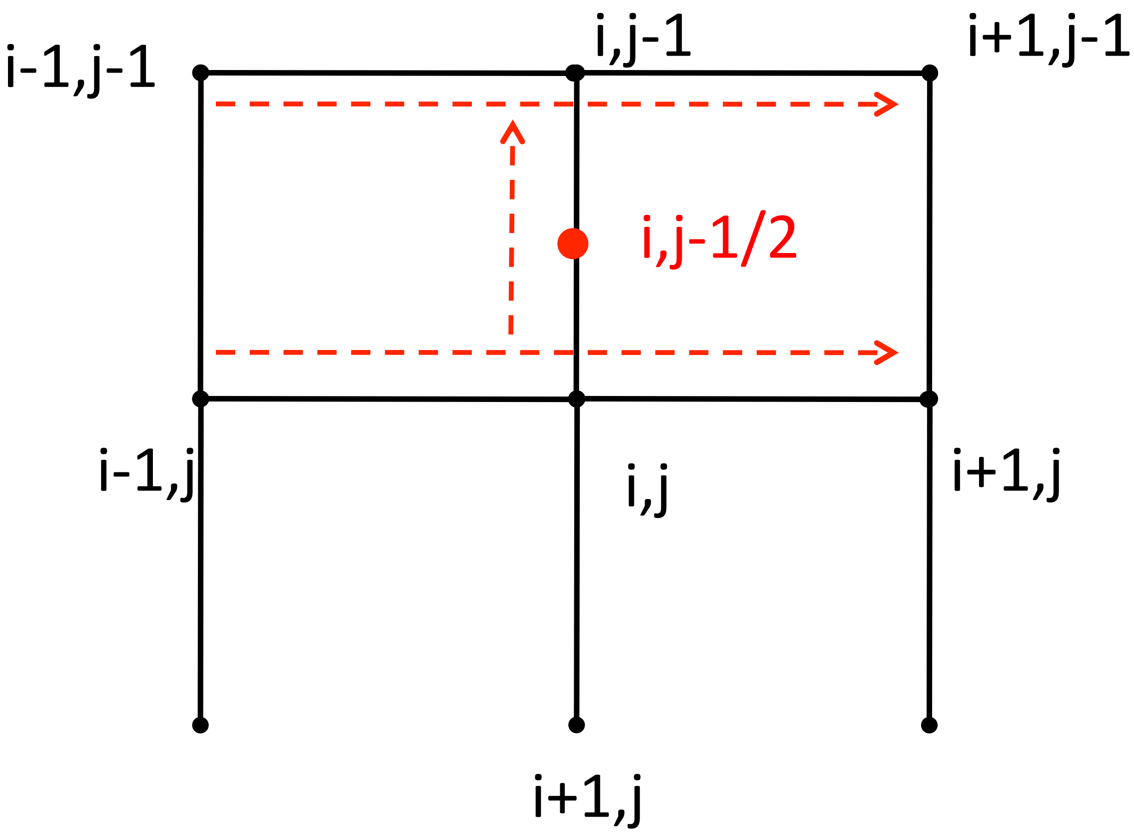


$$\begin{aligned}
(c_N)_{i,j}^t &= f(\|\nabla u_{i,j-1/2}^t\|) \\
(c_S)_{i,j}^t &= f(\|\nabla u_{i,j+1/2}^t\|) \\
(c_W)_{i,j}^t &= f(\|\nabla u_{i-1/2,j}^t\|) \\
(c_E)_{i,j}^t &= f(\|\nabla u_{i+1/2,j}^t\|)
\end{aligned}$$

Equation de diffusion anisotrope

- **Aspects numériques**

  Calcul de $f(\|\nabla u^t_{i,j-1/2}\|)$ par interpolation



$$\nabla u^t_{i,j-1/2} = \Big[\frac{1}{4}(u_{i+1,j} - u_{i-1,j}) + \frac{1}{4}(u_{i+1,j-1} - u_{i-1,j-1}),\; u^t_{i,j} - u^t_{i,j-1}\Big]$$

Equation de diffusion anisotrope

- **Aspects numériques**

  - En pratique on préfère les approximations suivantes, plus simples et qui ont de bonnes propriétés :

$$
\begin{aligned}
(c_N)_{i,j}^t &= f(\nabla_N u_{i,j}) \\
(c_S)_{i,j}^t &= f(\nabla_S u_{i,j}) \\
(c_W)_{i,j}^t &= f(\nabla_W u_{i,j}) \\
(c_E)_{i,j}^t &= f(\nabla_E u_{i,j})
\end{aligned}
$$

  - Le schéma numérique vérifie le principe du maximum discret

$$
\min_{i,j} I_{i,j} \leq I_{i,j}^n \leq \max_{i,j} I_{i,j}
$$

Equation de diffusion anisotrope

- **Implémentation**

Entrées: Image I, dt, n

Sortie: Image O

Fonctions auxiliaires: gradientD, f

```
On déclare 2 images, O, TMP
Pour i=1 à n
    TMP=copie(O); % important
    Pour p pixel
        Pour k=1 à 4
            grad=gradientD(TMP,p,k);
            O[p]=O[p]+dt*f(grad)*grad;
```

Equation de diffusion anisotrope

- **Convergence**

  - Comme pour l'équation de la chaleur, convergence vers la moyenne des niveaux de gris de l'image

  - Pas de critère universel pour déterminer le nombre d'itérations

  - Condition CFL sur le pas de temps : $\Delta t <= 0.25$

  - Pour le problème du bruit, on lisse en général l'image avant de calculer son gradient.

Equation de diffusion anisotrope

- **Modèles plus généraux**

  - Modèle général avec *D* matrice 2x2

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(D\nabla u\right)$$

  - Tenseur de structure (Tschumperlé)