ENRICHED SCHUBERT PROBLEMS IN THE GRASSMANNIAN OF LAGRANGIAN SUBSPACES IN 8-SPACE

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Abstract. We explain how to formulate Schubert problems on the lagrangian Grassmannian, and then describe the three enriched Schubert problems on the Lagrangian Grassmannian $LG(4)$ of isotropic 4-planes in 8-space, and use that to determine their Galois groups. (This is in progress.)

Assume that $\mathbb{K}$ is a field not of characteristic 2. Suppose that $\langle\ ,\ \rangle$ is a nondegenerate alternating form on a $\mathbb{K}$-vector space V. Nondegenerate means that the map $\varphi\colon V \to V^*$ is an isomorphism from V to its dual space, or rather that for all $0 \neq v \in V$, there is some $u \in V$ such that $\langle v, u\rangle \neq 0$. Alternating means that for all $u, v \in V$, we have $\langle v, u\rangle = -\langle u, v\rangle$. The existence of such a form implies that V has even dimension $2n$ for some positive integer n.

For a linear subspace $H \subset V$, its *annihilator* is

$$H^\perp \ :=\ \{u \in V \mid \langle u, h\rangle = 0 \qquad \forall\, h \in H\}\,.$$

A linear subspace $H \subset V$ is *isotropic* if $H \subset H^\perp$, so that $\langle h, h'\rangle = 0$ for all $h, h' \in H$. As $\langle , \rangle$ is a nondegenerate, this implies that $\dim H \leq \dim H^\perp = 2n - \dim H$, so that $\dim H \leq n$.

An isotropic subspace L of dimension n is necessarily maximal (and vice-versa). We call such a maximal isotropic subspace *Lagrangian*. The set of all Lagrangian subspaces forms the *Lagrangian Grassmannian*, $LG(V)$ or $LG(n)$. It has a transitive action by the *symplectic group*, which is the group of all linear transformations of V that preserve the form $\langle\ ,\ \rangle$,

$$\mathrm{Sp}(V) \ =\ \mathrm{Sp}(n) \ :=\ \{g \in \mathrm{GL}(V) \mid \langle gv, gu\rangle = \langle v, u\rangle \qquad \forall u, v \in V\}\,.$$

A complete flag $F_\bullet\colon \{0\} \subset F_1 \subset \cdots \subset F_{2n-1} \subset V$ ($\dim F_j = j$) of subspaces in V is *isotropic* if $F_i^\perp = F_{2n-i}$. Consequently, this is determined by its restriction $F_1 \subset \cdots \subset F_n$ to F_n, which is isotropic. The attitude of a Lagrangian subspace L with respect to an isotropic flag is the sequence $A(L, F_\bullet) := (\dim L \cap F_j \mid j = 1, \ldots, 2n)$.

As L and $F_\bullet$ are isotropic any attitude A is necessarily symmetric in that $j \in A \leftrightarrow 2n - j \notin A$. Consequently, A is determined by its intersection with $[n] := \{1, \ldots, n\}$. For reasons that will become clear in the sequel, we define $\lambda(A) := \{n + 1 - j \mid j \in A \cap [n]\}$. It is an exercise that all subsets λ of $[n]$ may occur, and for each $\lambda \subset [n]$, the set

$$\{L \in LG(n) \mid \lambda = \lambda(A(L, F_\bullet))\}$$

is isomorphic to the affine space $\mathbb{K}^{\binom{n+1}{2} - |\lambda|}$, where $|\lambda| = \sum_{\ell \in \lambda} \ell$.

We have just described the Schubert decomposition of $LG(n)$ into Schubert cells

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1. Formulating Schubert Problems on $LG(n)$

2. Preliminary calculations

Using the Frobenius algorithm, we determined that three of the 44 essential Schubert problems on $LG(4)$ are enriched. For one, with 384 solutions, we are still computing Frobenius elements. We have yet to be able to compute an eliminant for a problem with 768 solutions.

Using strict partitions to represent Schubert conditions, these three problems are

$$\ydiagram{2,1+1}^{\,2}\cdot\ydiagram{3,1+1}\ =\ 4\,,\quad \ydiagram{2,1+1}^{\,2}\cdot\ydiagram{3}\cdot\ydiagram{1}\ =\ 4\,,\quad \text{and}\quad \ydiagram{2,1+1}^{\,3}\cdot\ydiagram{1}\ =\ 8\,.$$

Let $V \simeq \mathbb{C}^8$ be a vector space equipped with a nondegenerate alternating form $\langle\bullet,\bullet\rangle$. We call $(V,\langle\bullet,\bullet\rangle)$ a *symplectic vector space*. The annihilator of a linear space H of V is $H^{\angle} := \{v \in V \mid \langle u,v\rangle = 0\ \forall u \in H\}$. As $\langle\bullet,\bullet\rangle$ is nondegenerate, $\dim H + \dim H^{\angle} = \dim V$. A subspace $H \subset V$ is *isotropic* if $H \subset H^{\angle}$. Then the dimension of an isotropic subspace H is at most $\frac{1}{2}\dim V$, and it is *Lagrangian* (maximal isotropic) if $\dim H = \frac{1}{2}\dim V$. Write $LG(V)$ or $LG(4)$ for the space of Lagrangian subspaces of V. This is a ten-dimensional smooth subvariety of $Gr(4,V)$, the Grassmannian of 4-planes in V. We will assume that the reader is familiar with our terminology, as well as the basics of Schubert calculus on $LG(V)$.

Let $L, M \in LG(V)$ be two general Lagrangian subspaces. In particular $L \cap M = \{0\}$ so that the map $L \oplus M \to V$ defined by $u \oplus v \mapsto u + v$ is an isomorphism. For $0 \neq v \in M$ consider the linear function $\Lambda_v \colon L \to \mathbb{C}$ defined by $\Lambda_v(u) = \langle u, v\rangle$. As L is Lagrangian and $L \cap M = \{0\}$, this linear form is nonzero on L. In particular, $v \mapsto \Lambda_v$ identifies M with the linear dual $L^* := \mathrm{Hom}(L,\mathbb{C})$ of L.

Suppose that $N \in LG(V)$ is a third Lagrangian subspace in general position with respect to both L and M. Then the projections π_L and π_M of N to the summands in $L \oplus M \simeq V$ are isomorphisms. This identifies N as the graph of a linear isomorphism

$$\varphi_N \ :=\ \pi_M \circ \pi_L^{-1} \ :\ L \ \xrightarrow{\ \sim\ }\ M\,.$$

This linear isomorphism $\varphi_N \colon L \to M \simeq L^*$ induces a nondegenerate bilinear form $(\bullet,\bullet)_N$ on L which is defined for $u, u \in L$ by $(u,u')_N := \langle u, \varphi_N(u')\rangle$.

The bilinear form $(\bullet,\bullet)_N$ is symmetric. Indeed, as N is isotropic, we have that for $u, u' \in L$, $u + \varphi_N(u)$ and $u' + \varphi_N(u')$ lie in N so that

$$\begin{aligned} 0 \ &=\ \langle u + \varphi_N(u)\,,\, u' + \varphi_N(u')\rangle \\ &=\ \langle u\,,\, u'\rangle \ +\ \langle u\,,\, \varphi_N(u')\rangle \ +\ \langle \varphi_N(u)\,,\, u'\rangle \ +\ \langle \varphi_N(u)\,,\, \varphi_N(u')\rangle\,. \end{aligned}$$

As $u, u' \in L$ and $\varphi_N(u), \varphi_N(u') \in M$, we see that $0 = \langle u, \varphi_N(u')\rangle + \langle \varphi_N(u), u'\rangle$, so that $\langle u, \varphi_N(u')\rangle = \langle u', \varphi_N(u)\rangle$, as $\langle\bullet,\bullet\rangle$ is alternating. Thus $(u,u')_N = (u',u)_N$ is symmetric.

Define $\ydiagram{2,1+1}(L) := \{H \in LG(V) \mid \dim H \cap L \geq 2\}$, which is a Schubert subvariety of codimension three in $LG(V)$. It is the intersection of $LG(V)$ with the Schubert subvariety $\Omega_{\ydiagram{2,2}}(L)$ of the Grassmannian $Gr(4,V)$. Set $X(L,M) := \ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M)$, a Richardson variety. If $H \in X(L,M)$, then $H \cap L \in Gr(2,L)$ and $H \cap M \in Gr(2,M)$. If we set $h := H \cap L$ and $h' := H \cap M$, then $H = h \oplus h'$. As H is isotropic, $\langle h, h'\rangle \equiv 0$, which implies that h' is the annihilator $h^{\perp}$ of h in $M = L^*$.

Following work on the Pieri formula in isotropic Schubert calculus [2] (see also [1]), it is useful to define the union, $Z(L,M)$, of the linear spaces in $X(L,M)$,

$$Z(L,M) \;:=\; \bigcup\{H \;\mid\; H \in X(L,M)\}\,,$$

which we consider to be a subvariety of the projective space $\mathbb{P}(V)$. More formally and working projectively, let

$$C(1,4;V) \;:=\; \{(\ell,H) \mid H \in LG(V) \text{ and } \ell \in \mathbb{P}(H)\}$$

be the symplectic flag variety of isotropic lines lying on Lagrangian subspaces in V. This has projections to projective space $\mathbb{P}(V)$ and to the Lagrangian Grsassmannian.

$$\begin{tikzcd} & C(1,4;V) \arrow[dl, "pr"'] \arrow[dr, "\pi"] & \\ \mathbb{P}(V) & & LG(V) \end{tikzcd}$$

Each realizes $C(1,4;V)$ as a fibre bundle, with $\pi^{-1}(H) = \mathbb{P}(H) \simeq \mathbb{P}^3$ and $pr^{-1}(\ell) = LG(3,\ell^{\angle}/\ell)$. Then $Z(L,M) := pr\circ\pi^{-1}(X(L,M))$. Define

$$Y(L,M) \;:=\; \pi^{-1}(X(L,M)) \subset C(1,4;V)\,.$$

For $0 \neq u \in L$, let $u^\perp \subset M$ be its annihilator, which is 3-dimensional. Similarly, for $0 \neq v \in M$, let $v^\perp \subset L$ be its annihilator.

Lemma 2.1. *In the coordinates $\{(u,v) \mid u \in L$ and $v \in M\}$ for $\mathbb{P}(V)$, the variety $Z(L,M)$ is the quadratic hypersurface with equation $\langle u,v\rangle = 0$. The map $pr\colon Y(L,M) \to Z(L,M)$ has fibre over a point $(u,v) \in Z(L,M)$ identified with $\mathbb{P}(v^\perp/u)$. When u and v are nonzero, this is isomorphic to $\mathbb{P}^1$; otherwise it is isomorphic to $\mathbb{P}^2$.*

Let $(u,v) \in Z(L,M)$ with u and v both nonzero. If we restrict the maps π, pr to $Y(L,M)$, then the set

$$\bigcup\{H \in X(L,M) \mid (u,v) \in H\} \;=\; pr\circ\pi^{-1}\circ\pi\circ pr^{-1}(u,v)\,, \tag{1}$$

is the quadric hypersurface $Z(L,M,u,v) := Z(L,M) \cap \mathbb{P}(v^\perp \oplus u^\perp)$ in $\mathbb{P}(v^\perp \oplus u^\perp) \simeq \mathbb{P}^5$, and the map between $\pi\circ pr^{-1}(u,v) \subset LG(V)$ and $Z(L,M,u,v)$ is birational away from the exceptional divisor $\mathbb{P}(u+v^\perp) \cup \mathbb{P}(u^\perp + v)$.

Proof. A Lagrangian subspace $H \in X(L,M)$ has the form $h \oplus h^\perp$ for $h \in Gr(2,L)$. Thus if $(u,v) \in H$, then $u \in h$ and $v \in h^\perp$, so that $\langle u,v\rangle = 0$, and we have that $u \in h \subset v^\perp$. A point $(u,v) \in V$ with $\langle u,v\rangle = 0$ has $u \in v^\perp \subset L$. Given any $h \in Gr(2,L)$ with $u \in H \subset v^\perp$, we have $v \in h^\perp$ so that $(u,v) \in h \oplus h^\perp \in X(L,M)$. This shows that $Z(L,M)$ equals the quadratic hypersurface and that the fibre $pr^{-1}(u,v) = \mathbb{P}(v^\perp/u)$. Since at most one of u or v may be zero, this is ispmorphic to $\mathbb{P}^2$ if one is zero and $\mathbb{P}^1$ if neither is zero.

For the last statement, note that the set (1) is contained in $Z(L,M) \cap \mathbb{P}(v^\perp \oplus u^\perp)$. Indeed, suppose that $(u,v) \in H$ and $H \in X(L,M)$. Then $H = h \oplus h^\perp$ and $u \in h \subset v^\perp$ and $v \in h^\perp \subset u^\perp$. If $(a,b) \in H$, then $a \in v^\perp$ and $b \in u^\perp$, and $\langle a,b\rangle = 0$.

Let $(a,b) \in Z(L,M) \cap \mathbb{P}(v^\perp \oplus u^\perp)$. Suppose that a is linearly independent of u and b is linearly indepedent of v. As $u,a \in L$, $v,b \in M$, $a \in v^\perp$ and $b \in u^\perp$, $\operatorname{span}\{a,u\}$ annihilates span$\{b, v\}$, so that span$\{a, b, u, v\}$ is the unique Lagrangian subspace containing these four points. $\square$

The quadric $Z(L, M, u, v)$ is singular; it is the cone over a quadric isomorphic to $\mathbb{P}(v^\perp/u)\times\mathbb{P}(u^\perp/v)$ in a $\mathbb{P}^3$ with vertex $\mathbb{P}(u + v)$, a $\mathbb{P}^1$.

2.1. The Galois group of $\young(\ \ ,:\ )^2 \cdot \young(\ \ \ ) \cdot \young(\ ) = 4$ is D_4.

Let L, M, and N be general Lagrangian subspaces in V as before, and let m be an isotropic 2-plane, also in general position. Observe that

$$\young(\ \ ,:\ )(L) \cap \young(\ \ ,:\ )(M) \cap \young(\ \ \ )(m) \;=\; \pi\big(pr^{-1}(m \cap Z(L, M))\big)\,. \tag{2}$$

By Lemma 2.1, $Z(L, M)$ is a quadric. Thus it meets m in two points (u, v) and (u', v'), showing that the intersection (2) has two components.

Let W be the component of (2) coming from (u, v). By Lemma 2.1 again, if we restrict π to $Y(L, M)$, then $pr(\pi^{-1}(W)) = Z(L, M) \cap \mathbb{P}(u^\perp \oplus v^\perp) = Z(L, M, u, v)$ is a quadric hypersurface in the $\mathbb{P}^5 \simeq \mathbb{P}(u^\perp \oplus v^\perp)$. Each of the two points of intersection of N with $Z(L, M, u, v)$ gives a solution to the Schubert problem

$$\young(\ \ ,:\ )(L) \cap \young(\ \ ,:\ )(M) \cap \young(\ \ \ )(m) \cap \young(\ )(N)\,. \tag{3}$$

With the other point (u', v') of $m \cap Z(L, M)$, this gives four solutions to the Schubert problem (3). Note that N is spanned by its intersections with $Z(L, M, u, v)$ and $Z(L, M, u', v')$ As its Galois group must preserve the partition coming from the two points (u, v) and (u', v'), it is a subgroup of D_4. We have computed Frobenius elements which show that the Galois group is D_4.

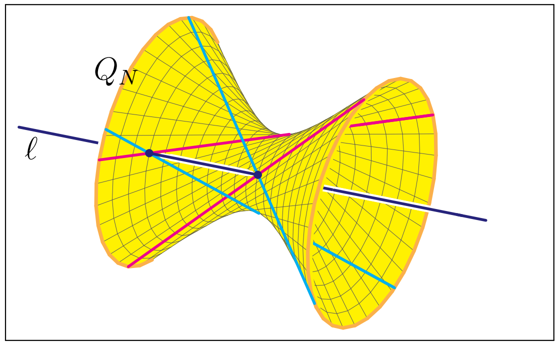
For an alternative proof, note that it is possible to find a the monodromy loop that fixes L, M, m (and hence the points (u, v) and (u', v')), as well as the two points $N \cap Z(L, M, u, v)$, but interchanges the other two points $N \cap Z(L, M, u', v')$. Indeed, let $\{x, y\} = N \cap Z(L, M, u, v)$. Then the set of Lagrangian planes containing $h :=$ spanx, y is identified with $LG(h^\angle/h)$, and any two points in $\mathbb{P}(x^\angle) \cap \mathbb{P}(y^\angle)$ that are independent. Fix this. It is important to make these kinds of arguments.

2.2. The Galois group of $\young(\ \ ,:\ )^2 \cdot \young(\ \ \ ,:\ ) = 4$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let L, M, N be as before, and consider a Lagrangian subspace $H \in \young(\ \ ,:\ )(L) \cap \young(\ \ ,:\ )(M) \cap \young(\ \ ,:\ )(N)$. As $H \in \young(\ \ ,:\ )(L) \cap \young(\ \ ,:\ )(M)$, it has the form $h \oplus h^\perp$ for $h \in Gr(2, L)$, and it is not hard to see that $h^\perp = \varphi_N(h)$. These together imply that $(h, h)_N \equiv 0$, so that h is an isotropic 2-plane in the linear space $L \simeq \mathbb{C}^4$ equipped with the nondegenerate symmetric form $(\bullet, \bullet)_N$. Let us work in $\mathbb{P}(L)$. Then h lies in one of the two families of lines that rule the quadric surface $Q_N := \{u \in \mathbb{P}(L) \mid (u, u)_N = 0\}$ in $\mathbb{P}(L)$. Now let $\ell \subset L$ be an isotropic 2-plane in L, which is a line in $\mathbb{P}(L)$. This will meet Q in two points, and through each point there will be two lines—one in each ruling. These four solutions h give the four solutions $h \oplus h^\perp$ to the Schubert problem.

The partition of the four solutions by the corresponding points of intersection $\ell \cap Q$ show that the Galois group is a subgroup of D_4. To analyze this further, let p and q be the two points in $\ell \cap Q_N$, and let the four lines on Q_N meeting these points be h_p^1, h_p^2, h_q^1, and h_q^2, with the upper index representing the ruling of Q_N the line lies in and the lower indicating the point of $\ell \cap Q_N$ it meets. However, there are two solution lines h in each ruling and the

Galois group must preserve their intersections. Consequently, the Galois group is the Klein 4-group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.



2.3. **The Galois group of $\yng(2,1)^3 \cdot \square = 8$ is not yet determined.**

References

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