ENRICHED SCHUBERT PROBLEMS IN THE GRASSMANNIAN OF LAGRANGIAN SUBSPACES IN 8-SPACE

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Abstract. We describe the three enriched Schubert problems on the Lagrangian Grassmannian $LG(4)$ of isotropic 4-planes in 8-space, and use that to determine their Galois groups.

1. Preliminary calculations

Using the Frobenius algorithm, we determined that three of the 44 essential Schubert problems on $LG(4)$ are enriched. For one, with 384 solutions, we are still computing Frobenius elements. We have yet to be able to compute an eliminant for a problem with 768 solutions.

Using strict partitions to represent Schubert conditions, these three problems are

$$\ydiagram{2,1+1}^{\,2}\cdot\ydiagram{3,1+1} \;=\; 4\,,\qquad \ydiagram{2,1+1}^{\,2}\cdot\ydiagram{3}\cdot\ydiagram{1} \;=\; 4\,,\qquad\text{and}\qquad \ydiagram{2,1+1}^{\,3}\cdot\ydiagram{1} \;=\; 8\,.$$

Let $V \simeq \mathbb{C}^8$ be a vector space equipped with a nondegenerate alternating form $\langle \bullet,\bullet\rangle$. We will call $(V,\langle\bullet,\bullet\rangle)$ a *symplectic vector space*. Write $LG(V)$ or $LG(4)$ for the space of *Lagrangian* (maximal isotropic) subspaces of V. This is a ten-dimensional smooth subvariety of $Gr(4,V)$, the Grassmannian of 4-planes in V. We will assume the reader is familiar with our terminology, as well as the basics of Schubert calculus on $LG(V)$.

Let $L,M\in LG(V)$ be two general Lagrangian subspaces. In particular $L\cap M=\{0\}$ so that the map $L\oplus M\to V$ defined by $u\oplus v\mapsto u+v$ is an isomorphism. For $0\neq v\in M$ consider the linear function $\Lambda_v\colon L\to\mathbb{C}$ defined by $\Lambda_v(u)=\langle u,v\rangle$. As L is Lagrangian (maximal isotropic) and $L\cap M=\{0\}$, this linear form is nondegenerate on L. In particular, $v\mapsto\Lambda_v$ identifies M with the linear dual $L^*:=\mathrm{Hom}(L,\mathbb{C})$ of L.

Suppose that $N\in LG(V)$ is a third Lagrangian subspace in general position with respect to L and M. Then projection of N to each summand in $L\oplus M\simeq V$ is an isomorphism, which identifies N as the graph of a linear isomorphism $\varphi_N\colon L\to M$. As N is isotropic, we have that for $u,u'\in L$, $u+\varphi_N(u)$ and $u'+\varphi_N(u')$ lie in N so that

$$\begin{aligned} 0 \;&=\; \langle u+\varphi_N(u)\,,\, u'+\varphi_N(u')\rangle \\ &\qquad=\; \langle u\,,\,u'\rangle \;+\; \langle u\,,\,\varphi_N(u')\rangle \;+\; \langle\varphi_N(u)\,,\,u'\rangle \;+\; \langle\varphi_N(u)\,,\,\varphi_N(u')\rangle\,. \end{aligned}$$

As $u,u'\in L$ and $\varphi_N(u),\varphi_N(u')\in M$, we see that $0=\langle u,\varphi_N(u')\rangle+\langle\varphi_N(u),u'\rangle$, so that $\langle u,\varphi_N(u')\rangle=\langle u',\varphi_N(u)\rangle$. Thus the bilinear form $(\bullet,\bullet)_N$ induced on L by the linear isomorphism $\varphi_N\colon L\to M\simeq L^*$ is symmetric (and nondegenerate).

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Define $\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}(L) := \{H \in LG(V) \mid \dim H \cap L \geq 2\}$, which is a Schubert variety of codimension three in $LG(V)$. It is the intersection of $LG(V)$ with a Schubert variety $\Omega_{\boxplus}(L)$ of the Grassmannian $Gr(4, V)$. Set $X(L, M) := \begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}(L) \cap \begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}(M)$, a Richardson variety. If $H \in X(L, M)$, then $H \cap L \in Gr(2, L)$ and $H \cap M \in Gr(2, M)$. Let us write $h := H \cap L$ and $h' := H \cap M$, then $H = h \oplus h'$. As H is isotropic, $\langle h, h' \rangle \equiv 0$, which implies that h' is the annihilator $h^\perp$ of h in $L^* = M$.

Following work on the Pieri formula in isotropic Schubert calculus [2] (see also [1]), it is useful to define the union, $Z(L, M)$, of the linear spaces in $X(L, M)$,

$$Z(L, M) \;:=\; \bigcup \{H \;\mid\; H \in X(L, M)\}\,.$$

More formally and projectively, let $C(1, 4; V) = \{(\ell, H) \mid H \in LG(V) \text{ and } \ell \subset H\}$ be the symplectic flag variety of isotropic lines lying on Lagrangian subspaces in V. This is a fibre bundle over its projections to projective space, $pr\colon C(1, 4; V) \twoheadrightarrow \mathbb{P}(V)$, and to $LG(V)$, $\pi\colon C(1, 4; V) \twoheadrightarrow LG(V)$. Then $Z(L, M) := pr \circ \pi^{-1}(X(L, M))$. Define $Y(L, M) := \pi^{-1}(X(L, M))$.

Lemma 1.1. *In the coordinates $\{(u, v) \mid u \in L \text{ and } v \in M\}$ for V, the variety $Z(L, M)$ is the quadratic hypersurface with equation $\langle u, v \rangle = 0$. The map $Y(L, M) \to Z(L, M)$ is a $\mathbb{P}^1$-bundle with fibre over a point $(u, v) \in Z(L, M)$ identified with $\mathbb{P}(v^\perp/u)$. Given $(u, v) \in Z(L, M)$, and restricting the maps π, pr to $Y(L, M)$, the set*

$$\bigcup \{H \in X(L, M) \mid (u, v) \in H\} \;=\; pr \circ \pi^{-1} \circ \pi \circ pr^{-1}(u, v)\,,$$

is the quadric $Z(L, M, u, v) := Z(L, M) \cap (v^\perp + u^\perp)$ in $\mathbb{P}^5$, and the maps between $\pi \circ pr^{-1}(u, v) \subset LG(V)$ and $Z(L, M, u, v)$ is birational away from the exceptional divisor $v^\perp \cup u^\perp$.

Proof. Since Lagrangian planes $H \in X(L, M)$ have the form $h + h^\perp$ for $h \in Gr(2, L)$ and these are in direct sum, if $(u, v) \in H$, then $u \in h$ and $v \in h^\perp$, so that $\langle u, v \rangle = 0$. Given a point $(u, v) \in V$ with $\langle u, v \rangle = 0$, we see that $u \in v^\perp \subset l$. Let $h \in Gr(2, L)$ be any 2-plane in the Schubert variety $\Omega_{\boxplus}(u, v^\perp) = \mathbb{P}(v^\perp/u) \simeq \mathbb{P}^1$, then $v \in h^\perp$ so that $(u, v) \in h + h^\perp$, which is a Lagrangian plane in $X(L, M)$.

For the last statement, note that this set is contained in $Z(L, M, u, v)$. Given a point $(a, b) \in Z(L, M, u, v) \smallsetminus (v^\perp \cup u^\perp)$, we have that a and u are linearly independent, as are b and v. Furthermore, $\operatorname{span}\{a, u\}$ annihilates $\operatorname{span}\{b, v\}$, so that $\operatorname{span}\{a, b, u, v\}$ is the unique Lagrangian subspace containing these four points. $\square$

1.1. **The Galois group of $\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}^2 \cdot \square\square\square \cdot \square = 4$ is D_4.** Let L, M, and N be general Lagrangian subspaces in V as before, and let m be an isotropic 2-plane, also in general position. Observe that

$$\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}(L) \cap \begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}(M) \cap \square\square\square(m) \;=\; \pi\big(pr^{-1}(m \cap Z(L, M))\big)\,.$$

By Lemma 1.1, $Z(L, M)$ is a quadric. Thus it meets m in two points (u, v) and (u', v'), showing that this intersection has two components.

Let W be the component coming from (u, v). By Lemma 1.1 again, if we restrict π to $Y(L, M)$, then $pr(\pi^{-1}(W)) = Z(L, M) \cap (u^\perp + v^\perp) = Z(L, M, u, v)$ is a quadric hypersurface in the $\mathbb{P}^5 \simeq u^\perp + v^\perp$. Each point of intersection of N with $Z(L, M, u, v)$ gives a solution to the Schubert problem

$$\ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M) \cap \ydiagram{3}(m) \cap \ydiagram{1}(N)$$

With the other point (u', v') this gives four solutions to the Schubert problem. As its Galois group must preserve the partition coming from the two points (u, v) and (u', v'), it is a subgroup of D_4. Either by considering Frobenius elements or by fixing the two points x, y of intersection of N with $Z(L, M, u, v)$, while the other two points $N \cap Z(L, M, u', v')$ are allowed to vary, we see that the Galois group is D_4.[1]

1.2. **The Galois group of $\ydiagram{2,1+1}^2 \cdot \ydiagram{3,1+1} = 4$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.** Let L, M, N be as before, and consider a Lagrangian subspace $H \in \ydiagram{2,1+1}(L)\ydiagram{2,1+1}(M)\ydiagram{2,1+1}(N)$. As $H \in \ydiagram{2,1+1}(L)\ydiagram{2,1+1}(M)$, it has the form $h + h^\perp$, and it is not hard to see that $h^\perp = \varphi_N(h)$. These together imply that $(h, h)_N \equiv 0$, so that h is an isotropic 2-plane in the linear space $L \simeq \mathbb{C}^4$ equipped with the nondegenerate symmetric form $(\bullet, \bullet)_N$. Working projectively, h lies in one of the two families of lines that rule the quadric surface $Q\colon \{u \in \mathbb{P}(L) \mid (u, u)_N = 0\}$ in $\mathbb{P}(L)$. Now let $\ell \subset L$ be an isotropic 2-plane in L, which is a line in $\mathbb{P}(L)$. This will meet Q in two points, and through each point there will be two lines—one in each ruling. These four solutions h give the four solutions $h + h^\perp$ to the Schubert problem.

The partition of the four solutions by the corresponding points of intersection $\ell \cap Q$ show that the Galois group is a subgroup of D_4. However, there are two solution lines h in each ruling and the Galois group must preserve their intersections. Consequently, the Galois group is the Klein 4-group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

1.3. **The Galois group of $\ydiagram{2,1+1}^3 \cdot \ydiagram{1} = 8$ is not yet determined.**

References

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[1]need to make this more precise.