Distributed Control of Heterogeneous Systems

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***Abstract*—This paper considers control design for distributed systems, where the controller is to adopt and preserve the distributed spatial structure of the nominal system. The specific feature of this work is that it does not require the underlying system dynamics to be homogeneous (shift invariant) with respect to spatial or temporal variables. Operator theoretic tools for working with these systems are developed, and lead to sufficient convex conditions for analysis and synthesis with respect to the ℓ_2-induced norm.**

***Index Terms*—Distributed control, interconnected systems, networks.**

I. Introduction

RECENTLY, there has been renewed research interest in distributed control of systems with an emphasis on synthesizing controllers that preserve the distributed structure of the nominal plant. The primary motivation for this work is 1) the increasing number of systems that are formed by the interconnection of interacting subsystems; and 2) the tremendous potential of emerging technologies that are making the deployment of large distributed sensor and actuator arrays possible. Much of this recent work has focused on systems that are shift invariant, or *homogeneous*, with respect to both spatial and temporal variables. Many systems occurring in both nature and engineering, due to boundary conditions, inhomogeneity in material, or the coupling of subsystems, fail to possess such an invariance property, particularly with respect to spatial variables. The major goal of this paper is to develop tools and results for distributed control, within the context of robust control, to deal with such *heterogeneity*.

We stress that the research literature is vast in the distinct area of control theory for *distributed parameter systems*, where aggregate infinite dimensional systems are considered; for instance, see [9], [30], and the references therein. This research field focuses on the many challenging and subtle technical issues associated with the generalization of optimal control from finite to infinite dimensional systems in this setting. In contrast, the concentration of this paper is the separate issue of the *structure* of controllers for systems that are spatially distributed.

Manuscript received April 17, 2003; revised January 19, 2004 and June 24, 2004. Recommended by Associate Editor F. M. Callier. The work of G. E. Dullerud was supported in part by the National Science Foundation under Grants ECS-9875244 and ITR-0085917, and by the Air Force Office of Scientific Research under Grants F49620-98-1-0416 and F49620-02-1-0325. The work of R. D'Andrea was supported by the National Science Foundation under Grant ECS-9983954, and by the Air Force Office of Scientific Research under Grants F49620-98-1-0416 and F49620-01-1-0119.

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Digital Object Identifier 10.1109/TAC.2004.838499

We consider distributed state–space systems using a generalization of the models proposed in [10] and [34], and develop stability and synthesis results using the ℓ_2-induced norm as the performance measure. In particular, we are concerned with developing controllers that have the same realization structure as the nominal system. Our approach makes significant use of operator theory, and the results obtained are all stated as convex feasibility problems. More specifically, in the paper we will extend robust control machinery to include heterogeneous systems. Using a relaxation we derive sufficient conditions for analyzing performance with respect to the ℓ_2-induced norm, and then provide sufficient conditions for the existence of controllers which stabilize the system and provide a guaranteed level of performance. The techniques in the paper are based on extending and combining those developed in [16] for time-varying systems, with recent advances in control of spatially invariant distributed systems [10], [15]. Also see the closely related earlier time-varying work in [2], [19], [23], and [22]. An operator inertia concept is central to obtaining the results in the paper, and is adapted for use with distributed systems from the important work in [5] on inertia and dichotomy for standard nonstationary systems. The basic methodology used in obtaining the synthesis conditions is closely related to the synthesis approach in [21], [32], and [24].

In [14], we have considered distributed control for homogeneous systems, obtaining sufficient synthesis results. We refer the reader to [14] for further motivation of the model class pursued here, and to [11], [20], and [27] for explicit practical application of our work on homogeneous systems. Reference [28] provides a computationally attractive method using this methodology for design of homogeneous systems with certain types of boundary conditions. Other researchers have considered the design of distributed controllers in the context of homogeneous dynamics [3], [4], [25], [26]. The former two recent papers use an approach based on Fourier transform theory on groups to obtain attractive necessary and sufficient conditions for controllers to exist; this approach may require multidimensional gridding and interpolation to explicitly construct a controller.

The current paper is organized as follows. After a basic notational section, in Section III the systems models to be considered are introduced, along with the important notion of a hyperdiagonal operator, which will be used extensively throughout the paper. Section IV is devoted to analysis of stability and performance. In Section V, we move on to deriving synthesis results. Section VI deals with the computational issues of finding finite dimensional conditions.

II. Preliminaries

We now introduce our notation and gather some elementary facts. The natural numbers (including zero), integers, real and complex numbers are denoted by $\mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$, respectively. Given an m-tuple $(k_1, \ldots, k_m) \in \mathbb{Z}^m$, we will frequently use the abbreviation $\bar{k}$. If we have two vectors of real numbers x and y, the notation $x \leq y$ will be used to mean the inequalities $x_j \leq y_j$ hold for all indices j; if x and y are infinite sequences we adopt a similar pointwise meaning.

The maximum singular value of a matrix M will be denoted by $\bar{\sigma}(M)$. Given a symmetric matrix H, its inertia $\mathrm{in}(H)$ is the triplet $(\mathrm{in}_+(H), \mathrm{in}_0(H), \mathrm{in}_-(H))$ giving the number of positive, zero, and negative eigenvalues of H, respectively. We state the following standard result.

Proposition 1: Suppose H is an $n \times n$ symmetric matrix, and that Q is an $n \times m$ matrix. Then, $\mathrm{in}_+(H) \geq \mathrm{in}_+(Q^*HQ)$ and $\mathrm{in}_-(H) \geq \mathrm{in}_-(Q^*HQ)$. Furthermore, if $m = n$ and Q is nonsingular, then $\mathrm{in}(H) = \mathrm{in}(Q^*HQ)$.

If V is a vector space, we will say that the linear mapping $M: V \to V$ has an *algebraic* inverse on V if there exists another linear mapping on V, denoted M^{-1} such that both MM^{-1} and $M^{-1}M$ are equal to the identity map. Given a Hilbert space H we denote its associated norm by $\|\cdot\|_H$ and its inner product by $\langle \cdot, \cdot \rangle_H$; for convenience we frequently suppress the subscript. The notation $H \oplus W$ will refer to the Hilbert space direct sum of the spaces H and W. Given two Hilbert spaces H and F we denote the space of bounded linear operators mapping H to F by $\mathcal{L}(H, F)$, and shorten this to $\mathcal{L}(H)$ when H equals F. If X is in $\mathcal{L}(H, F)$ we denote the H to F induced norm of X by $\|X\|_{H \to F}$; when $\|X\|_{H \to F} < 1$ we say that X is strictly *contractive*. The adjoint of X is written X^*. When X is in $\mathcal{L}(H)$ we denote its spectrum by $\mathrm{spec}(X)$.

An operator $X \in \mathcal{L}(H, F)$ is *coercive* if there exists an $\alpha > 0$ such that $\|Xu\|_F \geq \alpha \|u\|_H$ holds for all u in H. When an operator $X \in \mathcal{L}(H)$ is self-adjoint we use $X \prec 0$ to mean it is *negative definite*; that is there exists a number $\alpha > 0$, such that for all nonzero $x \in H$ the inequality $\langle x, Xx \rangle < -\alpha \|x\|^2$ holds.

We are interested in one main vector space in this paper, and in particular two of its subspaces, one of which is a Hilbert space. Let m be a fixed integer, $\mathbb{K}_1, \ldots, \mathbb{K}_m$ given subsets of $\mathbb{Z}$, and define $\mathbb{K} := \mathbb{K}_1 \times \cdots \times \mathbb{K}_m \subset \mathbb{Z}^m$. Also, suppose that $n(\bar{k})$ is an m-indexed sequence mapping $\mathbb{K}$ to the nonnegative integers $\mathbb{N}_0$. We define $\ell(\mathbb{K}, \{\mathbb{R}^{n(\bar{k})}\})$ to be the vector space of mappings w which satisfy $w: \bar{k} \in \mathbb{K} \mapsto w(\bar{k}) \in \mathbb{R}^{n(\bar{k})}$. When the sets $\mathbb{K}_i$ are all the same we write $\ell(\mathbb{K}_1^m; \{\mathbb{R}^{n(\bar{k})}\})$, and will further abbreviate to simply ℓ when contextually clear. We will use $\ell_2(\mathbb{K}, \{\mathbb{R}^{n(\bar{k})}\})$ to denote the subspace of $\ell(\mathbb{K}, \{\mathbb{R}^{n(\bar{k})}\})$ which is a Hilbert space under the norm

$$\|w\|_2 := \left(\sum_{\bar{k} \in \mathbb{K}} |w(k_1, \ldots, k_m)|_2^2 \right)^{\frac{1}{2}}$$

where $|\cdot|_2$ is the Euclidean norm. We will be regarding k_1 as a temporal variable, and with this in mind define $\ell_{2e}(\mathbb{K}, \{\mathbb{R}^{n(\bar{k})}\})$ to be the subset of ℓ satisfying for each fixed $k_1 \in \mathbb{K}_1$ the inequality

$$\sum_{k_2 \in \mathbb{K}_2} \cdots \sum_{k_m \in \mathbb{K}_m} |w(k_1, k_2, \ldots, k_m)|_2^2 < \infty.$$

That is, the elements of ℓ which, for each fixed value of the index k_1, are square summable over the remaining indexes $k_2, \ldots, k_m$. In other words, $w(k_1, \cdot)$ is in an ℓ_2 space for each k_1. Of the three spaces ℓ, ℓ_2 and ℓ_{2e} just defined, we will for most part be dealing with $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{n(\bar{k})}\})$ in the sequel.

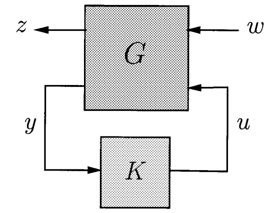

Fig. 1. Closed-loop system.

III. Distributed System Model and Formulation

In this paper, we will consider a controller *synthesis* problem, illustrated in Fig. 1, where the nominal system G is linear, causal and distributed in a certain sense, as is the controller K.

The system G is described by the difference equations

$$\begin{bmatrix} x_1(k_1+1, k_2, k_3, \ldots, k_m) \\ x_2(k_1, k_2+1, \ldots, k_m) \\ x_3(k_1, k_2-1, \ldots, k_m) \\ \vdots \\ x_{d-1}(k_1, k_2, \ldots, k_m+1) \\ x_d(k_1, k_2, \ldots, k_m-1) \end{bmatrix} = A(\bar{k})x(\bar{k}) + B(\bar{k}) \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}$$

$$\begin{bmatrix} z(\bar{k}) \\ y(\bar{k}) \end{bmatrix} = \begin{bmatrix} C_1(\bar{k}) \\ C_2(\bar{k}) \end{bmatrix} x(\bar{k}) + \begin{bmatrix} D_{11}(\bar{k}) & D_{12}(\bar{k}) \\ D_{21}(\bar{k}) & D_{22}(\bar{k}) \end{bmatrix} \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix} \tag{1}$$

where $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m, d = 2m - 1$, and theses matrices are compatibly partitioned as

$$A(\bar{k}) = \begin{bmatrix} A_{11}(\bar{k}) & \cdots & A_{1d}(\bar{k}) \\ \vdots & \ddots & \vdots \\ A_{d1}(\bar{k}) & \cdots & A_{dd}(\bar{k}) \end{bmatrix}$$

$$B(\bar{k}) = \begin{bmatrix} B_{11}(\bar{k}) & B_{12}(\bar{k}) \\ \vdots & \vdots \\ B_{d1}(\bar{k}) & B_{d2}(\bar{k}) \end{bmatrix}$$

$$\text{and } C(\bar{k}) = \begin{bmatrix} C_{11}(\bar{k}) & \cdots & C_{1d}(\bar{k}) \\ C_{21}(\bar{k}) & \cdots & C_{2d}(\bar{k}) \end{bmatrix}.$$

Throughout this paper, we assume that these matrix sequences are uniformly bounded. We will allow matrix dimensions that depend on $\bar{k}$, and thus define the sequences $c(\bar{k}), n_r(\bar{k}), n_c(\bar{k})$, and $b(\bar{k})$ so that for each $\bar{k} \in \mathbb{Z}^m$ we have that $C(\bar{k})$ is $c(\bar{k}) \times n_c(\bar{k}), A(\bar{k})$ is $n_r(\bar{k}) \times n_c(\bar{k})$, and $B(\bar{k})$ is $n_r(\bar{k}) \times b(\bar{k})$; note that n_c is equal to n_r by performing an appropriate shifting of the variables k_i. We will assume that this model admits a unique solution x in ℓ_{2e} given inputs $w, u \in \ell_{2e}$, and that the associated mapping $(w, u) \mapsto x$ is causal; in Section IV-A, this is made precise and conditions to ensure these properties are established.

Observe that in (1) each of the variables k_i has both forward and backward shifts associated with it, except for k_1 which has only a backward shift. Throughout the sequel, we will regard

k_1 as the *temporal* variable. Many systems that depend on both temporal and spatial variables can be modeled in this way, and this model is a generalization of the models introduced in [10], [34]. We will be seeking a controller synthesis K that also has this state–space structure. Its state space matrices will be denoted $(A_K(\bar{k}), B_K(\bar{k}), C_K(\bar{k}), D_K(\bar{k}))$, and are partitioned in a similar way to the corresponding plant matrices.

We have studied such models in the homogeneous distributed control context in [10], [14], where several explicit motivating examples were provided. This model can arise in a number of ways: see [11] for an example involving partial differential equation (PDE) discretization; [20] for an example where the subsystems are aerodynamically coupled wings; and [27] where the connected subsystems are telescope mirrors.

Example: String of Subsystems: To illustrate how such models arise we consider a string of interconnected systems as shown in Fig. 2. In the figure, we have a string of coupled subsystems $G^{(p)}$, each a linear state–space system, with inputs $d^{(p)}, v_+^{(p)}$, and $v_-^{(p)}$. As shown in the figure, the latter two inputs are outputs from the neighboring subsystems $G^{(p-1)}$ and $G^{(p+1)}$ respectively. The subsystem $G^{(p)}$ has outputs $e^{(p)}, v_+^{(p-1)}$, and $v_-^{(p+1)}$. Thus, the state–space equation for the subsystem can be written as

$$\begin{bmatrix} x(t+1,p) \\ v_-(t,p+1) \\ v_+(t,p-1) \\ e(t,p) \end{bmatrix} = \begin{bmatrix} A_{11}(t,p) & A_{12}(t,p) & A_{13}(t,p) & B_{1\bullet}(t,p) \\ A_{21}(t,p) & A_{22}(t,p) & A_{23}(t,p) & B_{2\bullet}(t,p) \\ A_{31}(t,p) & A_{32}(t,p) & A_{33}(t,p) & B_{3\bullet}(t,p) \\ C_{\bullet 1}(t,p) & C_{\bullet 2}(t,p) & C_{\bullet 3}(t,p) & D(t,p) \end{bmatrix} \times \begin{bmatrix} x(t,p) \\ v_-(t,p) \\ v_+(t,p) \\ d(t,p) \end{bmatrix} \tag{2}$$

where $x(t,p)$ is the state of the system. We note that the state dimension of each subsystem may be different as p changes, as could be the dimensions of the respective inputs and outputs. Thus the matrices A_{ij}, B_j, C_i, and D may vary in dimension for different values of p.

If we relabel the time and space variables (t,p) as (k_1,k_2) and define

$$\begin{bmatrix} x_1(k_1,k_2) \\ x_2(k_1,k_2) \\ x_3(k_1,k_2) \end{bmatrix} := \begin{bmatrix} x(k_1,k_2) \\ v_-(k_1,k_2) \\ v_+(k_1,k_2) \end{bmatrix}$$

we find that (2) is exactly of the form in (1).

We now, in the context of this example, informally discuss the properties we would like the model to have. First, let us assume that the system "starts" at $t=0$ with each subsystem in a given state $x(0,p)$. We restrict such initial states to be of finite energy in space; namely, $\sum_{p=-\infty}^{\infty} |x(0,p)|_2^2 < \infty$. With $d \in \ell_2$ we require that at each time $\tau \geq 0$ the spatial functions $x(\tau,\cdot), v_+(\tau,\cdot)$ and $v_-(\tau,\cdot)$ exist and have unique solutions in ℓ_2. That is, if the initial state is finite energy, and the input d is finite energy, then at each point in time the state variables and

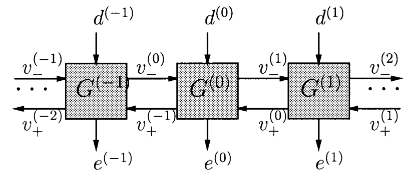

Fig. 2. Open-loop: one-dimensional spatial array.

interconnection variables are finite energy. In short we require that x, v_+, and v_- are well-defined elements of ℓ_{2e}, and will later refer to this requirement as a well-posedness assumption about our model.

Clearly considering systems in two or more spatial variables, with similarly interconnected lattice structures, will also yield models of the form in (1), and so from this example we see how the model of this paper arises naturally when systems are connected in a topological lattice. The results also apply to the case where any spatial variable k_i is defined on a cyclic set $\mathbb{Z}_{q_i}$, for some $q_i > 0$; see Remark 29. We will return to the current example later in the paper when discussing computation of the results obtained. $\square$

In Fig. 1, we are interested in the case where all the signals lie in the normed space ℓ_2, and are concerned with controllers that: a) stabilize the closed-loop state equations; and b) ensure $w \mapsto z$ is strictly contractive on ℓ_2 and is causal. When a controller meets these two objectives we refer to it as an *admissible* synthesis. We now introduce a useful technical concept.

A. Hyperdiagonal Operators

The machinery of this section will allow us to work more transparently with the heterogeneous equations in (1), for instance allowing us to express them in the compact form of (6) of the sequel, reminiscent of standard state–space equations.

Definition 2: Let v and n be sequences mapping $\mathbb{Z}^m$ to $\mathbb{N}_0$, and Q be a linear mapping from $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v(\bar{k})}\})$ to $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{n(\bar{k})}\})$. Then, Q is said to be a *hyperdiagonal* operator if there exists a uniformly bounded sequence of matrices $Q(\bar{k}) \in \mathbb{R}^{n(\bar{k}) \times v(\bar{k})}$ such that the equality $(Qw)(\bar{k}) = Q(\bar{k})w(\bar{k})$ holds for each $\bar{k} \in \mathbb{Z}^m$.

Hyperdiagonal operators are the direct generalization of block-diagonal operators to the m-indexed case, and when $m=1$, hyperdiagonal operators are the same as block-diagonal operators.

An example of a hyperdiagonal operator is the temporal-truncation operator P_τ. Given $\tau \in \mathbb{Z}$ it is defined by

$$P_\tau : w(\bar{k}) \mapsto y(\bar{k}) = \begin{cases} w(k_1,\ldots,k_m), & \text{for } k_1 < \tau \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Observe that the matrix sequence in Definition 2 which defines P_τ contains only identity and zero matrices. We will find the operator P_τ to be useful later in Section IV-A.

The idea of inertia is now generalized to hyperdiagonal operators; this follows the work in [5] where inertia is defined for block-diagonal operators. If a hyperdiagonal operator Q is self-adjoint, we define its inertia to be the mapping

$$\mathrm{In}(Q) : \mathbb{Z}^m \to \mathbb{N}_0^3 \text{ defined by } \mathrm{In}(Q)(\bar{k}) := \mathrm{in}(Q(\bar{k})).$$

In the same vein, define $\mathrm{In}_+(Q)(\bar{k}) := \mathrm{in}_+(Q(\bar{k}))$ and $\mathrm{In}_-(Q)(\bar{k}) := \mathrm{in}_-(Q(\bar{k}))$. We now state the following congruence result for hyperdiagonal operators.

Proposition 3: Suppose H and M are hyperdiagonal operators, with H self-adjoint. Then

$$\mathrm{In}_+(H) \geq \mathrm{In}_+(M^*HM) \quad \text{and} \quad \mathrm{In}_-(H) \geq \mathrm{In}_-(M^*HM).$$

Furthermore, if M is nonsingular, then $\mathrm{In}(H) = \mathrm{In}(M^*HM)$.

The proof follows easily by applying the definition of inertia and the matrix result in Proposition 1.

We now consider partitioned operators mapping the space $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v_1(\bar{k})}\}) \oplus \ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v_2(\bar{k})}\})$ to the space $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{q_1(\bar{k})}\}) \oplus \ell_2(\mathbb{Z}^m; \{\mathbb{R}^{q_2(\bar{k})}\})$. Let $W = \begin{bmatrix} H & P \\ G & J \end{bmatrix}$ be such an operator. We say that W is a *partitioned* hyperdiagonal operator if the constituent operators H, P, G, and J are hyperdiagonal. Given a partioned hyperdiagonal operator W, we define its *hyperdiagonal representation* $[\![W]\!] : \ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v_1(\bar{k})+v_2(\bar{k})}\}) \to \ell_2(\mathbb{Z}^m; \{\mathbb{R}^{q_1(\bar{k})+q_2(\bar{k})}\})$, as the hyperdiagonal operator given by

$$([\![W]\!]x)(\bar{k}) := \begin{bmatrix} H(\bar{k}) & P(\bar{k}) \\ G(\bar{k}) & J(\bar{k}) \end{bmatrix} x(\bar{k}).$$

Clearly, these concepts generalize to arbitrary partitions. Given vector-valued sequences $\bar{q} = (q_1, \ldots, q_r) : \mathbb{Z}^m \to \mathbb{N}^r$ and $\bar{v} = (v_1, \ldots, v_c) : \mathbb{Z}^m \to \mathbb{N}^c$ we denote by $\mathcal{P}(\bar{q}, \bar{v})$ the set of partitioned hyperdiagonal operators of the form

$$\begin{bmatrix} J_{11} & \cdots & J_{1c} \\ \vdots & \ddots & \vdots \\ J_{r1} & \cdots & J_{rc} \end{bmatrix} \tag{4}$$

where each J_{ij} is a hyperdiagonal operator mapping $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v_j(\bar{k})}\})$ to $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{q_i(\bar{k})}\})$. The following notation will be convenient: given a partitioned hyperdiagonal operator J in $\mathcal{P}(\bar{q}, \bar{v})$ we define

$$p(J) := (\bar{q}, \bar{v}). \tag{5}$$

Furthermore, if $\bar{q} = \bar{v}$ we will simply set $p(J) = \bar{q}$ and write $\mathcal{P}(\bar{q})$ in place of $\mathcal{P}(\bar{q}, \bar{v})$. Also when the partition dimensions $(\bar{q}, \bar{v})$ are not important we will use the abbreviation $\mathcal{P}$ to denote the set of partitioned hyperdiagonal operators.

The following basic properties are immediately verifiable and are worth noting here for use in the sequel. Given partitioned hyperdiagonal operators A and B in $\mathcal{P}(\bar{n}, \bar{v})$, and C in $\mathcal{P}(\bar{s}, \bar{n})$ it is easy to verify that $[\![A+B]\!] = [\![A]\!] + [\![B]\!]$, and $[\![AC]\!] = [\![A]\!][\![C]\!]$ hold. Also the induced norm satisfies $\|A\|_{\ell_2 \to \ell_2} = \sup_{\bar{k} \in \mathbb{Z}^m} \bar{\sigma}([\![A]\!](\bar{k}))$. Finally, suppose B is self-adjoint and that scalar $\beta > 0$. Then, the operator $B \succeq \beta I$ if and only if the matrices $[\![B]\!](\bar{k}) \succeq \beta I$ for each $\bar{k} \in \mathbb{Z}^m$. From this it is clear that $[\![\cdot]\!]$ is a homomorphism from $\mathcal{P}$ into the space of hyperdiagonal operators, which is isometric, and preserves products, addition, and ordering.

We now extend the definition of inertia to cover partitioned hyperdiagonal operators. If a partitioned hyperdiagonal operator $W \in \mathcal{P}(\bar{n})$ is self-adjoint, we define its inertia by

$$\mathrm{In}(W) := \mathrm{In}([\![W]\!]).$$

We now have the hyperdiagonal form of the Schur complement formula.

Proposition 4: Suppose that $T \in \mathcal{P}(\bar{n}), H \in \mathcal{P}(\bar{s})$, and $P \in \mathcal{P}(\bar{n}, \bar{s})$ are partitioned hyperdiagonal operators, with T invertible and both T and H self-adjoint. Then

$$\mathrm{In}\left(\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}\right) = \mathrm{In}(T) + \mathrm{In}(H - P^*T^{-1}P) \text{holds.}$$

Furthermore, the partitioned operator $\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}$ is invertible if and only if $H - P^*T^{-1}P$ is invertible.

Proof: The proof parallels the standard demonstration of the Schur complement, using the definition of inertia and the congruence result of Proposition 1. We have

$$\begin{aligned} \mathrm{In}\left(\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}\right) &= \mathrm{In}\left(\left[\!\!\left[\begin{matrix} T & P \\ P^* & H \end{matrix}\right]\!\!\right]\right) \\ &= \mathrm{In}\left(\left[\!\!\left[\begin{matrix} I & 0 \\ -P^*T^{-1} & I \end{matrix}\right]\!\!\right] \left[\!\!\left[\begin{matrix} T & P \\ P^* & H \end{matrix}\right]\!\!\right] \right. \\ &\quad \left. \times \left[\!\!\left[\begin{matrix} I & -T^{-1}P \\ 0 & I \end{matrix}\right]\!\!\right]\right) \\ &= \mathrm{In}\left(\left[\!\!\left[\begin{matrix} T & 0 \\ 0 & H - P^*T^{-1}P \end{matrix}\right]\!\!\right]\right) \\ &= \mathrm{In}(T) + \mathrm{In}(H - P^*T^{-1}P) \end{aligned}$$

Here we have defined hyperdiagonal operators on ℓ_2; however, a given hyperdiagonal operator on ℓ_2 has obvious extensions to ℓ and ℓ_{2e}, we will not distinguish between these objects. Also, we have made all the definitions in this section with respect to the index set $\mathbb{Z}^m$, but clearly the concepts can be defined on a more general index set $\mathbb{K}$; this generalization is needed in Section VI.

To complete this section, we show how the equations in (1) can be expressed in what will be an advantageous way using hyperdiagonal operators and shift operators. For a fixed integer $1 \leq i \leq m$, define the k_i-shift operator $S_i : \ell(\mathbb{Z}^m; \{\mathbb{R}^{n(\bar{k})}\}) \to \ell(\mathbb{Z}^m; \{\mathbb{R}^{q(\bar{k})}\})$, where the dimension sequences satisfy $q(k_1, k_2, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_m) = n(k_1, \ldots, k_m)$, by

$$(S_i v)(\bar{k}) = v(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_m).$$

It is straightforward to verify that S_i is invertible and $(S_i^{-1} v)(\bar{k}) = v(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_m)$.

Now observe that the matrix sequences $A_{ij}(\bar{k}), B_{jr}(\bar{k}), C_{li}(\bar{k})$, and $D_{lr}(\bar{k})$ in (1) define partitioned hyperdiagonal operators A, B, C, and D. From this, we readily see that the equations in (1) can be rewritten as

$$x = \Lambda A x + \Lambda B \begin{bmatrix} w \\ u \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = Cx + D \begin{bmatrix} w \\ u \end{bmatrix} \tag{7}$$

where $\Lambda := \mathrm{diag}(S_1, S_2, S_2^{-1}, \ldots, S_m, S_m^{-1})$, the *composite* shift operator. Note that when restricted to ℓ_2 this operator is unitary since each S_i is unitary.

Thus an input–output mapping G, defined by the state–space equations (1), can be formally expressed as

$$G = C(I - \Lambda A)^{-1} \Lambda B + D. \tag{8}$$

So, for instance, in Example 1 we would have $\Lambda = \mathrm{diag}(S_1, S_2, S_2^{-1})$. In the next section, we develop tools and results in the context of system analysis, before proceeding to synthesis in Section V.

Before beginning the rigorous analysis of the next section, we provide some insight into our model class defined in (1) by featuring it in a more standard state space setup. Let us make the following spatial–temporal partition, defining $\bar{A}_{22}, \bar{A}_{12}, \bar{A}_{21}, \bar{B}_2$, and $\bar{\Lambda}_2$ so that

$$A = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{1\bullet} \\ \bar{B}_2 \end{bmatrix} \quad \text{and}$$
$$\Lambda = \begin{bmatrix} S_1 & 0 \\ 0 & \bar{\Lambda}_2 \end{bmatrix}. \tag{9}$$

Now, referring to the update equation in (6) it is straightforward to arrive at the following formal expression for x_1:

$$x_1 = S_1(A_{11} + \bar{A}_{12}(I - \Lambda_2\bar{A}_{22})^{-1}\Lambda_2\bar{A}_{21})x_1 + S_1(\bar{A}_{12}(I - \Lambda_2\bar{A}_{22})^{-1}\Lambda_2\bar{B}_2 + B_{1\bullet}) \begin{bmatrix} w \\ u \end{bmatrix}.$$

If, for an element v in ℓ_{2e}, we now use the convention $v(k_1)$ to mean the ℓ_2-valued sequence $v(k_1, \cdot)$, then the previous equation can be written as

$$x_1(k_1+1) = \underline{A}(k_1)x_1(k_1) + \underline{B}(k_1) \begin{bmatrix} w(k_1) \\ u(k_1) \end{bmatrix} \tag{10}$$

for operator-valued sequences $\underline{A}(k_1)$ and $\underline{B}(k_1)$ defined in the obvious way.

Having established the basic setting for our work, and providing some connections based on formal manipulations, we now move on to a systematic analysis.

IV. Analysis Conditions

The purpose of this section is to develop some properties and results for systems described by state–space equations of the form in (1), which will be used when doing synthesis later in this paper. Our first goal is to make precise the assumptions about our open-loop system model.

A. Open-Loop: Well-Posedness and Causality

Here, we will establish conditions under which the open-loop model equations in (1), given inputs in ℓ_{2e}, are guaranteed to have a unique solution in ℓ_{2e} and define a causal linear mapping on ℓ_{2e}. From the conditions thus far, the following basic uniqueness condition can be easily proved.

Lemma 5: Suppose that u and w are elements of ℓ_{2e}. If $I - \Lambda A$ has an algebraic inverse on the space $\oplus_{j=1}^{d}\ell_{2e}$, then there exists a unique element $x \in \ell_{2e}$ that solves (1).

The invertibility condition in the above result ensures that G in (8) is a well-defined mapping. However, as we have already noted above, in the sequel we will give the variable k_1 the special role of *time*, and so are interested in systems that have a causality property. Recall the temporal truncation operator defined in (3). Then, we say that a linear mapping on ℓ_{2e} is *causal* if for all $\tau \in \mathbb{Z}$ we have $P_\tau R P_\tau = P_\tau R$. It is *memoryless* if $RP_\tau = P_\tau R$ for all integers τ. It is easy to verify that the set of causal linear mappings on ℓ_{2e} forms an algebra, as does the set of memoryless ones; also, every memoryless operator is causal. The set of memoryless mappings is clearly algebraically closed under the inverse operation; however, the set of causal operators is *not* inverse-closed, i.e., if R is a causal invertible mapping, R^{-1} may not be causal (an example is S_1).

We will say that the system described by (1) is *well-posed* on ℓ_{2e} when a) the mapping $I - \Lambda A$ has an algebraic inverse on $\oplus_{j=1}^{d}\ell_{2e}$; and b) the inverse $(I - \Lambda A)^{-1}$ is causal. When this is the case, and $w, u \in \ell_{2e}$, then the solution x to (1) exists in ℓ_{2e} and is unique by the previous lemma, but also satisfies (10). The remainder of this section is devoted to establishing conditions on the objects in (1) so that well-posedness is guaranteed.

Lemma 6: If R is a memoryless linear mapping on ℓ_{2e} satisfying $RP_0 = 0$, then the algebraic inverse $(I - S_1R)^{-1}$ is well-defined on ℓ_{2e} and is causal.

Proof: To begin, using the facts $P_{\tau+1}S_1 = S_1P_\tau$ for $\tau \in \mathbb{Z}$, and the hypothesis on R, it is straightforward to verify that

$$P_\tau(S_1R)^\tau = 0 \text{ holds for all } \tau \geq 1. \tag{11}$$

Thus, given two integers $\kappa \geq \tau \geq 1$, noticing $P_\tau P_\kappa = P_\tau$, we have that $P_\tau P_\kappa\{I + S_1R + \cdots + (S_1R)^{\kappa-1}\} = P_\tau\{I + S_1R + \cdots + (S_1R)^{\tau-1}\}$.

The aforementioned relationship allows us to unambiguously define the linear mapping Q on ℓ_{2e} via

$$P_\tau Q := \begin{cases} P_\tau\{I + S_1R + \cdots + (S_1R)^{\tau-1}\}, & \text{for } \tau > 1 \\ P_\tau, & \text{for } \tau \leq 1. \end{cases} \tag{12}$$

Observe that each term on the right-hand side above is causal and so it is immediate that $P_\tau Q = P_\tau Q P_\tau$; namely, Q is a causal mapping.

We will complete the proof by demonstrating that Q is the algebraic inverse of $I - S_1R$. It is sufficient to show that $P_\tau(I - S_1R)Q = P_\tau Q(I - S_1R) = P_\tau$ for each integer τ. To see the first equality holds simply notice that from its definition, Q commutes with S_1R. The second equality follows noting from (12) that: a) for $\tau \geq 1$ the equality $P_\tau Q(I - S_1R) = P_\tau(I - (S_1R)^\tau)$ holds, which by (11) equals P_τ; and b) for $\tau \leq 1$ we have $P_\tau Q(I - S_1R) = P_\tau(I - S_1R) = P_\tau$.

It will be useful to define the following condition; it uses the partition defined in (9).

Condition 7:

i) The equalities $AP_0 = 0, BP_0 = 0, CP_0 = 0$, and $DP_0 = 0$ all hold.
ii) The linear mapping $I - \bar{\Lambda}_2\bar{A}_{22}$ has an algebraic inverse on $\oplus_{j=2}^{d}\ell_{2e}$.

Part i) simply says that $A(k_1, \ldots, k_m)$, and the other matrices, are equal to zero when k_1 is negative. The condition in ii) says that the given memoryless operator is invertible. We will now show that these conditions are enough to guarantee that (1) has a unique solution in ℓ_{2e} when its input is also in this space. To end this section, we have the following important result.

Lemma 8: If Condition 7 is satisfied, then the system in (1) is well-posed on ℓ_{2e}.

Proof: Observe that $I - \Lambda A$ is a map on $\oplus_{j=1}^{d}\ell_{2e}$ and is equal to the product

$$\begin{bmatrix} I & -S_1\bar{A}_{12}(I - \bar{\Lambda}_2\bar{A}_{22})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - S_1R & 0 \\ -\bar{\Lambda}_2\bar{A}_{21} & I - \bar{\Lambda}_2\bar{A}_{22} \end{bmatrix}$$

where $R := A_{11} + \bar{A}_{12}(I - \bar{\Lambda}_2\bar{A}_{22})^{-1}\bar{\Lambda}_2\bar{A}_{21}$. Therefore, it suffices to demonstrate that $I - S_1R$ is invertible on ℓ_{2e} and is causal, because $S_1\bar{A}_{12}(I - \bar{\Lambda}_2\bar{A}_{22})^{-1}$ is causal, and both $\bar{\Lambda}_2\bar{A}_{21}$ and $I - \bar{\Lambda}_2\bar{A}_{22}$ are memoryless by virtue of the fact that they do not contain S_1. From its definition it is clear that R is memoryless, and satisfies $RP_0 = 0$ by Condition 7. Invoking Lemma 6 we obtain the result.

We now proceed to the next section where we will consider operators on ℓ_2.

B. Closed-Loop

The purpose of this section is to develop results which we will later deploy when considering the closed-loop equations for Fig. 1, which involve both G and K. However, to avoid unnecessary additional notation we will use the open-loop equations in (1) for developing the results in this section since they are of the same form as those of the closed-loop.

1) Stability: To begin, we state a standard result about Lyapunov inequalities for general operators.

Proposition 9: Suppose that H and X are operators on a Hilbert space, and that X is self-adjoint. If the inequality $H^*XH - X \prec 0$ is satisfied, then $I - H$ is coercive.

To prove the result simply note that $H^*XH - X = H^*X(H-I) + (H-I)^*XH - (H-I)^*X(H-I)$, and so $I - H$ is necessarily coercive when the left-hand side is negative definite.

Our goal is now to formulate Lyapunov-based tests for invertibility and performance of the realization operators shown previously. The following result is easily verified from the definitions so far, and states that under a similarity transformation by a shift operator a hyperdiagonal operator remains hyperdiagonal.

Lemma 10: Suppose that W is a hyperdiagonal operator on ℓ_2. Then both $S_i^*WS_i$ and $S_iWS_i^*$ are also hyperdiagonal, and given by the relationships $(S_i^*WS_i)(\bar{k}) = W(k_1,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots,k_m)$, and $(S_iWS_i^*)(\bar{k}) = W(k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_m)$.

The aforementioned result holds for hyperdiagonal operators and standard shifts, and we would like an analogous invariance property for operators in $\mathcal{P}$ when transformed by a similarity transformation involving Λ. Given a partitioned hyperdiagonal operator $X \in \mathcal{P}$, it is not in general true that $\Lambda^*X\Lambda$ will also be a partitioned hyperdiagonal operator. To guarantee that this transformed operator is in $\mathcal{P}$ the operator X must be block-diagonal with respect to its partition (also that of Λ). Accordingly define the set of *invertible* operators $\mathcal{X} \subset \mathcal{P}$ to be the self-adjoint, partitioned hyperdiagonal operators of the form

$$X = \mathrm{diag}(X_1, X_2, \ldots, X_d). \tag{13}$$

Having made this observation and the previous definition it is possible to prove a corollary.

Corollary 11: If $X \in \mathcal{X}$, then both $\Lambda^*X\Lambda$ and $\Lambda X\Lambda^*$ are in $\mathcal{P}$.

Proof: From the definition of $\mathcal{X}$ the operator X has the form $X = \mathrm{diag}(X_1,\ldots,X_d)$, therefore $\Lambda^*X\Lambda = \mathrm{diag}(S_1^*X_1S_1, S_2^*X_2S_2, \ldots, S_mX_dS_m^*)$. From Lemma 10, each block entry of this operator is hyperdiagonal and so it is in $\mathcal{P}$. A similar argument holds for $\Lambda X\Lambda^*$.

With this result established, we can now show a stability result.

Lemma 12: Suppose $X \in \mathcal{X}$ and that A is in $\mathcal{P}$. If $A^*\Lambda^*X\Lambda A - X \prec 0$ and $\mathrm{In}_-(\Lambda^*X\Lambda) = \mathrm{In}_-(X)$ both hold, then $1 \notin \mathrm{spec}(\Lambda A)$.

Proof: The Lyapunov inequality of the hypothesis guarantees that $I - \Lambda A$ is coercive, and therefore it is sufficient to prove that $I - A^*\Lambda^*$ is also coercive.

First note that by Corollary 11 we have that $\Lambda^*X\Lambda$ is a partitioned hyperdiagonal operator, and so $\mathrm{In}(\Lambda^*X\Lambda)$ is clearly well defined. Applying the Schur complement formula in Proposition 4 twice we have $\mathrm{In}(X - A^*\Lambda^*X\Lambda A) + \mathrm{In}(\Lambda^*X^{-1}\Lambda) = \mathrm{In}(\begin{bmatrix} \Lambda^*X^{-1}\Lambda & A \\ A^* & X \end{bmatrix}) = \mathrm{In}(\Lambda^*X^{-1}\Lambda - AX^{-1}A^*) + \mathrm{In}(X)$, and note that $\Lambda^*X^{-1}\Lambda - AX^{-1}A^*$ must be invertible. Now, $\mathrm{In}(\Lambda^*X^{-1}\Lambda) = \mathrm{In}(\Lambda^*X\Lambda)$, and therefore the above two equalities yield $\mathrm{In}(\Lambda^*X^{-1}\Lambda - AX^{-1}A^*) = \mathrm{In}(X - A^*\Lambda^*X\Lambda A) + \mathrm{In}(\Lambda^*X\Lambda) - \mathrm{In}(X)$. The first inertia term on the right-hand side has the form $(n,0,0)$ since its argument is positive definite, and by the supposition the next two terms sum to a triple of the form $(n_1 - n_2, 0, 0)$. Thus we have $\mathrm{In}(\Lambda^*X^{-1}\Lambda - AX^{-1}A^*) = (n + n_1 - n_2, 0, 0)$; namely, the operator has purely positive inertia.

As already noted, the operator is invertible and, therefore, we conclude $\Lambda^*X^{-1}\Lambda - AX^{-1}A^* \succ 0$, and so immediately $X^{-1} - \Lambda AX^{-1}A^*\Lambda^* \succ 0$ follows, implying $I - A^*\Lambda^*$ is coercive.

The previous result provides a sufficient condition for invertibility of $I - \Lambda A$, which in general is not necessary and so some conservatism is introduced into our analysis. Whether a sharp quantitative assessment of this conservatism can be established is uncertain, and is certainly a formidable task; it is closely related to the simpler scenario of the gap between the matrix structured singular value (see, e.g., [31]) and its upper bound. A detailed study of stability of single variable systems (i.e., only k_1) is carried out in [5], where it is shown that if $\ker A = \{0\}$, then the condition is both necessary and sufficient; this paper develops a number of important stability results for the single variable bi-infinite systems. Also note above that the space of operators which satisfy $\mathrm{In}_-(\Lambda^*X\Lambda) = \mathrm{In}_-(X)$ forms a cone, but not a convex one. This is a significant obstacle we will overcome in the sequel.

2) Performance: The purpose of this short section is to assemble the results obtained so far into a version of the Kalman–Yacubovich–Popov (KYP) lemma, to be used throughout the sequel when pursuing convex synthesis. In Section IV-A, the issue of causality was considered for the open-loop model, here we provide a closed-loop test. To commence we have the following causality result in the context of Lemma 12.

Lemma 13: Suppose $X = \mathrm{diag}(X_1, X_2, \ldots, X_d) \in \mathcal{X}, \mathrm{In}_-(\Lambda^*X\Lambda) = \mathrm{In}_-(X)$, and that A is in $\mathcal{P}$. If $X_1 \succ 0$ and $\Lambda^*A^*X\Lambda A - X \prec 0$, then $1 \notin \mathrm{spec}(\Lambda A)$ and the operator $(I - \Lambda A)^{-1}$ is causal.

Proof: That $1 \notin \mathrm{spec}(\Lambda A)$ follows immediately from Lemma 12, so we only need to address causality. Using the same initial step as in the proof of Lemma 8 it is sufficient to show that $(I - S_1R)^{-1}$ is a well-defined element of $\mathcal{L}(\ell_2)$, and is causal, where $R := A_{11} + \bar{A}_{12}(I - \bar{\Lambda}_2\bar{A}_{22})^{-1}\bar{\Lambda}_2\bar{A}_{21}$ which is memoryless. Note that Lemma 12 ensures $(I - \bar{\Lambda}_2\bar{A}_{22})^{-1} \in \mathcal{L}(\oplus_{j=2}^{d}\ell_2)$.

Pre- and postmultiply the given inequality $\Lambda^*A^*X\Lambda A - X \prec 0$ by $[I \;\; \bar{A}_{21}^*\Lambda_2(I - \bar{A}_{22}^*\Lambda_2^*)^{-1}]$ and its adjoint, respectively, to get $S_1^*R^*X_1S_1R - X_1 \prec 0$. This inequality implies that we have $\|X_1^{(1/2)}S_1RX_1^{-(1/2)}\| < 1$, from which it follows that the spectral radius of S_1R is strictly less than one. Therefore, the standard power series $I+S_1R+(S_1R)^2+\cdots$ for $(I-S_1R)^{-1}$ converges (see, e.g., [6]). Each term in the series is causal and, therefore, their sum is causal.

Because of our desire to maintain closed-loop causality, the last result motivates a refinement of the set $\mathcal{X}$. Define

$$\mathcal{X}^c := \{X \in \mathcal{X} : \ X_1 \succ 0\}. \tag{14}$$

That is, each element of $\mathcal{X}^c$ is invertible and of the form $\mathrm{diag}(X_1, X_2, \ldots, X_d)$, where each X_i is a self-adjoint hyperdiagonal operator, and in particular $X_1 \succ 0$. We can now prove the desired version of the KYP lemma.

Lemma 14: Suppose X is in $\mathcal{X}^c$ and satisfies the inertia condition $\mathrm{In}_-(\Lambda^* X \Lambda) = \mathrm{In}_-(X)$. If the operator inequality

$$\begin{bmatrix} \Lambda A & \Lambda B \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda A & \Lambda B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{15}$$

holds, then $1 \notin \mathrm{spec}(\Lambda A)$ and the operator $M := C(I-\Lambda A)^{-1}\Lambda B + D$ is both causal and strictly contractive.

Proof: We note that the (1,1)-block of (15) is $A^*\Lambda^* X \Lambda A - X \prec 0$, and condition a) follows directly from Lemma 12.

To see that M is causal: first, invoke Lemma 13 to conclude that $(I-\Lambda A)^{-1}$ is causal. Also Λ is clearly causal, and B, C, and D are memoryless. Therefore, M must be causal since it is constructed from products and sums of causal operators.

To show that $\|M\| < 1$ holds, simply pre- and postmultiply (15) by the operator $[B^*\Lambda^*(I - A^*\Lambda^*)^{-1} \ \ I]$ and its adjoint, respectively, to get $M^*M - I \prec 0$.

We are now ready to proceed to considering synthesis of controllers, and will base our approach on the just proved analysis result.

V. Synthesis

We now turn to controller synthesis for the closed-loop system in Fig. 1. Our goal will be to derive conditions for the existence of a controller K which makes the closed-loop stable and causal, and the mapping $w \mapsto z$ strictly contractive. The conditions derived will be sufficient for such a controller to exist, but in general can be conservative. An appealing feature of the approach used is that the explicit controllers obtained are always state space systems of the form in (1). The approach we follow is closely related to the LMI solution to the standard H_∞-synthesis problem of [21], [32], particularly that of [32]. Throughout the sequel we assume that the objects in (1) satisfy Condition 7, so that by Lemma 8 the mapping G is well-posed on ℓ_{2e}; for convenience we assume that $D_{22} = 0$.

The controller K in Fig. 1 is realized by the hyperdiagonal operators (A_K, B_K, C_K, D_K), and we set $\bar{n}_K = p(A_K)$ which is the state dimensions of the control. To begin we write the closed-loop system equations for the configuration in Fig. 1, and obtain that the map

$$w \mapsto z = C_{cl}(I - \Lambda_{cl} A_{cl})^{-1} \Lambda_{cl} B_{cl} + D_{cl}$$

where $\Lambda_{cl} := \mathrm{diag}(\Lambda, \Lambda)$ and $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ is defined equal to

$$\begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{bmatrix}.$$

In the sequel, it will be important to have the closed-loop equations in a form compatible with the conditions of Lemma 13. To this end we define an associated set of scaling operators. Let X_{cl} be a self-adjoint hyperdiagonal operator of the form

$$X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix} \in \mathcal{P}(\bar{n}_{cl}) \tag{16}$$

where $X \in \mathcal{X}^c, X_{GK} = \mathrm{diag}(X_{GK1}, \ldots, X_{GKd}) \in \mathcal{P}(p(A); \bar{n}_K)$, $X_K = \mathrm{diag}(X_{K1}, \ldots, X_{Kd}) \in \mathcal{P}(\bar{n}_K)$, and $\bar{n}_{cl} := (p(A), \bar{n}_K)$. This partitioning ensures that $\Lambda_{cl}^* X_{cl} \Lambda_{cl}$ is hyperdiagonal, which is easily verified as in Lemma 10. Define $\mathcal{X}_{cl}^c$ to be the following set:

$$\mathcal{X}_{cl}^c := \left\{ X_{cl} \in \mathcal{P}(\bar{n}_{cl}) : X_{cl} = X_{cl}^* \text{ partitioned as in (16) with } \begin{bmatrix} X_1 & X_{GK1} \\ X_{GK1}^* & X_{K1} \end{bmatrix} \succ 0 \right\}.$$

With these new definitions, we can now state the following result.

Lemma 15: Suppose that K is a given controller for the configuration in Fig. 1. If there exists an operator X_{cl} in $\mathcal{X}_{cl}$, satisfying $\mathrm{In}_-(X_{cl}) = \mathrm{In}_-(\Lambda_{cl}^* X_{cl} \Lambda_{cl})$ and

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}^* \begin{bmatrix} \Lambda_{cl}^* X_{cl} \Lambda_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} - \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \prec 0$$

then K is an admissible controller.

The result provides a sufficient condition for the closed-loop system to have the desired performance and stability qualities. Its proof follows routinely as a notational variant of the proof of Lemma 14 and is, therefore, not included.

Looking at the closed-loop systems equations it follows for appropriately defined hyperdiagonal operators R, U, and V, which depend *only* on the plant, that

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = R + U^* Q V \quad \text{where } Q = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}. \tag{17}$$

To proceed, we require a number of intermediate results. These will allow us to determine when it is possible to find an operator Q, corresponding to an admissible controller. The first step is establishing a *matrix* lemma.

Lemma 16: Suppose

a) that $\tilde{R}, \tilde{U}$, and $\tilde{V}$ are matrices;
b) the matrices $\tilde{U}_\perp$ and $\tilde{V}_\perp$ are full rank, whose images are the null spaces of $\tilde{U}$ and $\tilde{V}$, respectively;
c) that $\tilde{H}$ and $\tilde{Z}$ are nonsingular, symmetric matrices satisfying $\mathrm{in}(\tilde{H}) \leq \mathrm{in}(\tilde{Z})$;

d) the matrix

$$\tilde{\Psi}(\tilde{Z},\tilde{H}) = \begin{bmatrix} \tilde{V}_\perp^* \tilde{Z} \tilde{V}_\perp & \tilde{V}_\perp^* \tilde{R} \tilde{U}_\perp \\ \tilde{U}_\perp^* \tilde{R} \tilde{V}_\perp & \tilde{U}_\perp^* \tilde{H}^{-1} \tilde{U}_\perp \end{bmatrix}$$

is nonsingular.

Then, there exists a matrix $\tilde{Q}$ such that $\tilde{\Omega} = (\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V})^*\tilde{H}(\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V}) - \tilde{Z} \prec 0$ if and only if the matrix inequalities $\tilde{U}_\perp^*(\tilde{R}\tilde{Z}^{-1}\tilde{R}^* - \tilde{H}^{-1})\tilde{U}_\perp \prec 0$ and $\tilde{V}_\perp^*(\tilde{R}^*\tilde{H}\tilde{R} - \tilde{Z})\tilde{V}_\perp \prec 0$ both hold.

Proof: This result is proved for the case $\tilde{H} = \tilde{Z}$ in [12, Prop. 1]; we will reduce to this case.

The matrix $\tilde{Z}$ has inertia no smaller than that of $\tilde{H}$, thus there exists a right-invertible matrix $\tilde{T}$ so that $\tilde{T}^*\tilde{Z}\tilde{T} = \tilde{H}$. Therefore, for any $\tilde{Q}$, we have

$$\tilde{\Omega} = (\tilde{T}\tilde{R} + \tilde{T}\tilde{U}^*\tilde{Q}\tilde{V})^*\tilde{Z}(\tilde{T}\tilde{R} + \tilde{T}\tilde{U}^*\tilde{Q}\tilde{V}) - \tilde{Z}$$

where $\tilde{\Omega}$ is defined in the lemma statement. Define $\bar{R} = \tilde{T}\tilde{R}, \bar{U} = \tilde{U}\tilde{T}^*$, and $\bar{V} = \tilde{V}$, and then the right-hand side of the previous equation becomes $(\bar{R}+\bar{U}^*\tilde{Q}\bar{V})^*\tilde{Z}(\bar{R}+\bar{U}^*\tilde{Q}\bar{V}) - \tilde{Z}$. In [12, Prop. 1], conditions are given for when this matrix can be made negative definite by choosing $\tilde{Q}$; these are easily seen to be exactly the conditions of the lemma.

The previous result is now strengthened, removing condition d).

Lemma 17: Suppose that conditions a)–c) in Lemma 16 hold. Then there exists a matrix $\tilde{Q}$ such that $\tilde{\Omega} = (\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V})^*\tilde{H}(\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V}) - \tilde{Z} \prec 0$ if and only if the both follow matrix inequalities hold:

$$\begin{aligned} &\tilde{U}_\perp^*(\tilde{R}\tilde{Z}^{-1}\tilde{R}^* - \tilde{H}^{-1})\tilde{U}_\perp \prec 0 \quad \text{and} \\ &\tilde{V}_\perp^*(\tilde{R}^*\tilde{H}\tilde{R} - \tilde{Z})\tilde{V}_\perp \prec 0. \end{aligned} \tag{18}$$

Proof: The "only if" direction is immediate, and so we must demonstrate "if." Without loss of generality, we make the assumption that $\tilde{U}^*\tilde{U} = I$ and $\tilde{V}^*\tilde{V} = I$, for convenience.

Clearly, there exist open neighborhoods $\mathcal{N}_{\tilde{Z}}$ and $\mathcal{N}_{\tilde{H}}$ around $\tilde{Z}$ and $\tilde{H}$, respectively, in which the conditions in b) and inequalities (18) all hold. Let $\beta > 0$ be such that all matrices $\bar{Z}$ and $\bar{H}$ satisfying $\bar{\sigma}(\bar{Z} - \tilde{Z}) \le \beta$ and $\bar{\sigma}(\bar{H} - \tilde{H}) \le \beta$ are, respectively, in these neighborhoods. Set

$$\tilde{H}_\beta := \tilde{H} + \frac{\beta}{2}I \in \mathcal{N}_{\tilde{H}}.$$

Now, choose $0 \le \epsilon \le (\beta/2)$ so that the matrix $\tilde{\Psi}(\tilde{Z},\tilde{H}_\beta) - \epsilon I$ is nonsingular, where $\tilde{\Psi}$ is defined as in d) of Lemma 16, *and* both $(I - \epsilon\tilde{H}_\beta)^{-1}$ exists and

$$-\frac{\beta}{2}I \preceq -\epsilon\tilde{H}_\beta^2(I - \epsilon\tilde{H}_\beta)^{-1} \preceq \frac{\beta}{2}I \text{ holds.} \tag{19}$$

Having done this, set $\tilde{Z}_\epsilon = \tilde{Z} - \epsilon I$ and $\tilde{H}_{\epsilon\beta} = \tilde{H}_\beta + \epsilon\tilde{H}_\beta^2(I - \epsilon\tilde{H}_\beta)^{-1}$, noting that these matrices are in the neighborhoods $\mathcal{N}_{\tilde{Z}}$ and $\mathcal{N}_{\tilde{H}}$, respectively.

It is routine to verify that $\tilde{H}_{\epsilon\beta}^{-1} = \tilde{H}_\beta^{-1} - \epsilon I$ and thus we have that $\tilde{\Psi}(\tilde{Z}_\epsilon,\tilde{H}_{\epsilon\beta}) = \tilde{\Psi}(\tilde{Z},\tilde{H}_\beta) - \epsilon I$ which by construction is nonsingular. As noted $\tilde{Z}_\epsilon \in \mathcal{N}_{\tilde{Z}}$ and $\tilde{H}_{\epsilon\beta} \in \mathcal{N}_{\tilde{H}}$, and so invoking Lemma 16 we have that there exists $\tilde{Q}$ such that

$$(\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V})^*\tilde{H}_{\epsilon\beta}(\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V}) - \tilde{Z}_\epsilon \prec 0.$$

Defining $\tilde{M} = \tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V}$ and substituting the definitions of $\tilde{Z}_\epsilon$ and $\tilde{H}_{\epsilon\beta} = \tilde{H}_\beta$, we get

$$\tilde{M}^*\tilde{H}\tilde{M} - \tilde{Z} \prec -\epsilon I - M^*\left(\frac{\beta}{2}I + \epsilon\tilde{H}_\beta^2(I - \epsilon\tilde{H}_\beta)^{-1}\right)M.$$

Recalling (19), we see that the right-hand side is negative definite.

We can use this result to prove a slightly stronger result which we will need.

Corollary 18: Suppose that conditions a) and b) in Lemma 16 hold, and that $\tilde{H}$ and $\tilde{Z}$ are nonsingular, symmetric matrices satisfying $\mathrm{in}_-(\tilde{H}) = \mathrm{in}_-(\tilde{Z})$. Then there exists a matrix $\tilde{Q}$ such that $\tilde{\Omega} = (\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V})^*\tilde{H}(\tilde{R} + \tilde{U}^*\tilde{Q}\tilde{V}) - \tilde{Z} \prec 0$ if and only if both the following matrix inequalities are satisfied:

$$\begin{aligned} &\tilde{U}_\perp^*(\tilde{R}\tilde{Z}^{-1}\tilde{R}^* - \tilde{H}^{-1})\tilde{U}_\perp \prec 0 \quad \text{and} \\ &\tilde{V}_\perp^*(\tilde{R}^*\tilde{H}\tilde{R} - \tilde{Z})\tilde{V}_\perp \prec 0. \end{aligned} \tag{20}$$

Proof: There are two cases to consider. The first is when $\dim(\tilde{Z}) \ge \dim(\tilde{H})$. In this case we conclude that $\mathrm{in}(\tilde{Z}) \ge \mathrm{in}(\tilde{H})$ since both matrices are nonsingular, and so we can invoke Lemma 17 to establish the desired equivalence.

We now consider the second case, where $\dim(\tilde{Z}) < \dim(\tilde{H})$. The "only if" direction follows by showing that if $\tilde{\Omega} \prec 0$, then $(\tilde{R}+\tilde{U}^*\tilde{Q}\tilde{V})\tilde{Z}^{-1}(\tilde{R}+\tilde{U}^*\tilde{Q}\tilde{V})^* - \tilde{H}^{-1} \prec 0$; this follows from a Schur complement argument similar to that used in Lemma 12.

Thus, it remains to show the "if" direction when $\dim(\tilde{Z}) < \dim(\tilde{H})$. Define the matrices

$$\bar{Z} := \begin{bmatrix} \tilde{Z} & 0 \\ 0 & I_s \end{bmatrix} \quad \bar{V} := \begin{bmatrix} \tilde{V} \\ 0 \end{bmatrix} \quad \bar{R} := [\tilde{R} \quad 0] \tag{21}$$

where $s := \mathrm{in}_+(\tilde{H}) - \mathrm{in}_+(\tilde{Z}) = \dim(\tilde{H}) - \dim(\tilde{Z}) > 0$. Note that with this definition we have $\mathrm{in}(\bar{Z}) = \mathrm{in}(\tilde{H})$. Using (20), it is routine to verify that $\bar{V}_\perp^*(\bar{R}^*\tilde{H}\bar{R} - \bar{Z})\bar{V}_\perp \prec 0$, and so by Lemma 17 there exists a matrix $\tilde{Q}$ satisfying $(\bar{R} + \tilde{U}^*\tilde{Q}\bar{V})^*\tilde{H}(\bar{R} + \tilde{U}^*\tilde{Q}\bar{V}) - \bar{Z} \prec 0$. Substituting the definitions in (21) verifies that the left-hand side above is equal to $\mathrm{diag}(\tilde{\Omega}, -I)$.

We can now extend these results to operators in $\mathcal{P}$.

Corollary 19: For given sequences $\bar{n}, \bar{m}, \bar{s}$, and $\bar{q}$, suppose

a) that the operators $R \in \mathcal{P}(\bar{n},\bar{m}), U \in \mathcal{P}(\bar{s},\bar{n})$, and $V \in \mathcal{P}(\bar{q},\bar{m})$;
b) the partitioned hyperdiagonal operators $U_\perp$ and $V_\perp$ are coercive, and satisfy $\mathrm{Im}U_\perp = \mathrm{Ker}U$ and $\mathrm{Im}V_\perp = \mathrm{Ker}V$;
c) that $H \in \mathcal{P}(\bar{n})$ and $Z \in \mathcal{P}(\bar{m})$ are invertible operators, which are self-adjoint and satisfy $\mathrm{In}_-(H) = \mathrm{In}_-(Z)$.

Then, there exists a partitioned hyperdiagonal operator $Q \in \mathcal{P}(\bar{s},\bar{q})$ such that $(R + U^*QV)^*H(R + U^*QV) - Z \prec 0$ if and only if both the operator inequalities hold

$$U_\perp^*(RZ^{-1}R^* - H^{-1})U_\perp \prec 0 \quad \text{and} \quad V_\perp^*(R^*HR - Z)V_\perp \prec 0.$$

Proof: (Only if): This is immediate from the hypothesis, noting that $V_\perp$ and $U_\perp$ are coercive.

(If): First, choose $\epsilon > 0$, such that $Z_\epsilon := Z - \epsilon I$ is invertible, $\mathrm{In}(Z) = \mathrm{In}(Z_\epsilon)$, and satisfies both $M := U_\perp^*(RZ_\epsilon{}^{-1}R^* - H^{-1})U_\perp \prec 0$ and $P := V_\perp^*(R^*HR - Z_\epsilon)V_\perp \prec 0$. In particular, this means that $[\![M]\!](\bar{k}) \prec 0$ and $[\![P]\!](\bar{k}) \prec 0$, hold for all $\bar{k} \in \mathbb{Z}^m$. Invoking Corollary 18, we therefore can conclude that there exists a sequence of matrices $\tilde{Q}(\bar{k})$ such that $([\![R]\!](\bar{k}) + [\![U]\!]^*(\bar{k})\tilde{Q}(\bar{k})[\![V]\!](\bar{k}))^*[\![H]\!](\bar{k})([\![R]\!](\bar{k}) + [\![U]\!]^*(\bar{k})\tilde{Q}(\bar{k})[\![V]\!](\bar{k})) - [\![Z_\epsilon]\!](\bar{k}) \prec 0$, for all $\bar{k} \in \mathbb{K}$. Now, define the partitioned hyperdiagonal operator Q via the relationship $[\![Q]\!](\bar{k}) = \tilde{Q}(\bar{k})$, and then the sequence of matrix inequalities implies the following operator inequality:

$$(R + U^*QV)^*H(R + U^*QV) - Z_\epsilon \preceq 0.$$

Substituting the definition of Z_ϵ yields the strict inequality required.

Given symbols $E_1, \ldots, E_4, Q_1, Q_2$, and N, we define the notation $\mathbf{L}(\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}, Q_1, Q_2, N) :=$

$$N^* \left\{ \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}^* \begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} - \begin{bmatrix} Q_2 & 0 \\ 0 & I \end{bmatrix} \right\} N. \tag{22}$$

The technical results shown previously are now applied to yield the following intermediate synthesis result.

Proposition 20: Suppose the following.

a) The self-adjoint operator $X_{cl} \in \mathcal{X}_{cl}^c$ satisfies $\mathrm{In}_-(X_{cl}) = \mathrm{In}_-(\Lambda_{cl}^* X_{cl} \Lambda_{cl})$.

b) The partitioned hyperdiagonal operators X and Y are in $\mathcal{X}^c$ and satisfy

$$X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix} \quad \text{and} \quad X_{cl}^{-1} = \begin{bmatrix} Y & Y_{GK} \\ Y_{GK}^* & Y_K \end{bmatrix}.$$

c) The operators N_c and N_o are coercive with $\mathrm{Im} N_c = \mathrm{Ker}[B_2^* \ D_{12}^*]$ and $\mathrm{Im} N_o = \mathrm{Ker}[C_2 \ D_{21}]$.

If the inequalities $\mathbf{L}(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \Lambda^* X \Lambda, X, N_c) \prec 0$ and

$$\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^*, Y, \Lambda^* Y \Lambda, N_o\right) \prec 0 \text{ both hold} \tag{23}$$

then there exists an admissible controller K, with state dimension $p(A_K) = \bar{n}_K$.

Proof: We recall the definitions of R, U and V in (17), where also $Q \in \mathcal{P}$ signifies the realization for the controller K. Using this closed-loop parametrization, from Lemma 15, we know that if

$$(R + U^*QV)^* \underbrace{\begin{bmatrix} \Lambda_{cl}^* X_{cl} \Lambda_{cl} & 0 \\ 0 & I \end{bmatrix}}_{H} (R + U^*QV) - \underbrace{\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix}}_{Z} \prec 0 \tag{24}$$

then Q provides a realization for an admissible controller.

Using the supposition that $\mathrm{In}_-(\Lambda_{cl}^* X_{cl} \Lambda_{cl}) = \mathrm{In}_-(X_{cl})$, its clear that $\mathrm{In}_-(H) = \mathrm{In}_-(Z)$. We can now invoke Corollary 19 to conclude that a Q satisfying (24) exists if and only if

$$V_\perp^*(R^*HR - Z)V_\perp \prec 0 \quad \text{and} \quad U_\perp^*(RZ^{-1}R^* - H^{-1})U_\perp \prec 0.$$

Using the operator definitions of R, U, and V it is routine to show that the previous two inequalities are equivalent, respectively, to those in (23).

To obtain convex conditions we must further examine the conditions under which operators X_{cl} exist, which satisfy the conditions in the proposition. We begin with the following matrix result, which is motivated by [32, Lemma 6.2].

Lemma 21: Suppose the matrices $\tilde{X}$ and $\tilde{Y}$ are symmetric, $n \times n$, and nonsingular, and that the nonnegative integers i_+, i_-, and h, satisfy $n + h = i_+ + i_-$. Then there exist matrices $\tilde{X}_2, \tilde{Y}_2$ in $\mathbb{R}^{n \times h}$, and symmetric $h \times h$ matrices $\tilde{X}_3, \tilde{Y}_3$ satisfying $\begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{X}_3 \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{Y} & \tilde{Y}_2 \\ \tilde{Y}_2^* & \tilde{Y}_3 \end{bmatrix}$ and $\mathrm{in}(\begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{X}_3 \end{bmatrix}) = (i_+, 0, i_-)$ if and only if $\mathrm{in}_+(\begin{bmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{bmatrix}) \leq i_+$ and $\mathrm{in}_-(\begin{bmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{bmatrix}) \leq i_-$. Furthermore, when the latter inequalities are satisfied, the matrices $\tilde{X}_2$ and $\tilde{X}_3$ can be chosen so that $\bar{\sigma}(\begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{X}_3 \end{bmatrix}) \leq \bar{\sigma}(\tilde{X}) + \bar{\sigma}^{(1/2)}(\tilde{X} - \tilde{Y}^{-1}) + 1$ and $\bar{\sigma}(\begin{bmatrix} \tilde{Y} & \tilde{Y}_2 \\ \tilde{Y}_2^* & \tilde{Y}_3 \end{bmatrix}) \leq \bar{\sigma}(\tilde{Y})(1 + \bar{\sigma}^{(1/2)}(\tilde{X} - \tilde{Y}^{-1}))^2 + 1$.

Proof: (Only if): From the relationship relating $\tilde{X}$ and $\tilde{Y}$, we get

$$\begin{bmatrix} I & 0 \\ \tilde{Y} & \tilde{Y}_2 \end{bmatrix}^* \begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{X}_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{Y} & \tilde{Y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{bmatrix}.$$

Using this and invoking Proposition 1 gives the desired conclusion.

(If): First note that

$$\begin{bmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{Y}^{-1} & I \end{bmatrix}^* \begin{bmatrix} \tilde{X} - \tilde{Y}^{-1} & 0 \\ 0 & \tilde{Y} \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{Y}^{-1} & I \end{bmatrix}$$

and so by Proposition 4 we have $\mathrm{in}(\begin{bmatrix} \tilde{X} & I \\ I & \tilde{Y} \end{bmatrix}) = \mathrm{in}(\tilde{X} - \tilde{Y}^{-1}) + \mathrm{in}(\tilde{Y})$. By assumption we therefore have that $\mathrm{in}_+(\tilde{X} - \tilde{Y}^{-1}) + \mathrm{in}_+(\tilde{Y}) \leq i_+$ and $\mathrm{in}_-(\tilde{X} - \tilde{Y}^{-1}) + \mathrm{in}_-(\tilde{Y}) \leq i_-$. From this, it follows that the rank of $\tilde{X} - \tilde{Y}^{-1}$ is at most m.

Now, because of this rank condition we conclude that there exist $\tilde{X}_2 \in \mathbb{R}^{n \times m}$ and a matrix $\tilde{J} = \mathrm{diag}(I, -I) \in \mathbb{R}^{m \times m}$ such that $\tilde{X} - \tilde{Y}^{-1} = \tilde{X}_2 \tilde{J} \tilde{X}_2^*$ and $\mathrm{in}(\tilde{J}) = (i_+, 0, i_-) - \mathrm{in}(\tilde{Y})$. The first equation says, equivalently, that $\tilde{X} - \tilde{X}_2 \tilde{J} \tilde{X}_2^* = \tilde{Y}^{-1}$ and so an application of the Schur complement formula yields $\mathrm{in}(\begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{J}^{-1} \end{bmatrix}) = \mathrm{in}(\tilde{Y}) + \mathrm{in}(\tilde{J}) = (i_+, 0, i_-)$. Via algebraic verification we get

$$\begin{bmatrix} \tilde{X} & \tilde{X}_2 \\ \tilde{X}_2^* & \tilde{J}^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{Y} & -\tilde{Y}\tilde{X}_2\tilde{J} \\ -\tilde{J}\tilde{X}_2^*\tilde{Y} & \tilde{J} + \tilde{J}\tilde{X}_2^*\tilde{Y}\tilde{X}_2\tilde{J} \end{bmatrix}$$

and so set $\tilde{X}_3 = \tilde{J}^{-1}$.

Finally, the inequalities given follow by noting, from the spectral theorem for symmetric matrices, that $\tilde{X}_2$ can be chosen to satisfy $\bar{\sigma}(\tilde{X}_2) \leq \bar{\sigma}^{-(1/2)}(\tilde{X} - \tilde{Y}^{-1})$, and then exploiting the triangle and submultipicative inequalities of the maximum singular value; we omit these routine details.

This lemma gives exact conditions under which a matrix can be completed with specified inertia, while satisfying an inversion equation. We will now use this result to determine when it is possible to complete a hyperdiagonal operator so that it has shift invariant inertia with respect to a specific index k_j. To this end, it is convenient to define the following notation. Given a partitioned hyperdiagonal operator $H \in \mathcal{P}$, an index $1 \leq j \leq m$, and $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z}^{m-j}$, define

$$\overline{\mathrm{In}}_-^j(H)(\bar{k}_1, \bar{k}_2) := \max_{k_j \in \mathbb{Z}}(\mathrm{in}_-\{[\![H]\!](\bar{k}_1, k_j, \bar{k}_2)\}).$$

That is, the left-hand side is the maximum number of negative eigenvalues of $[\![H]\!](k_1, \ldots, k_n)$, when the index k_j is varied over all integers, when the remaining indices are specified. The next important lemma can now be stated.

Lemma 22: Suppose j is an integer in $\{1, \ldots m\}$, that $n(\bar{k})$ and $h(\bar{k})$ are sequences, and that the hyperdiagonal operators X and Y are in $\mathcal{P}(n)$. Then there exist hyperdiagonal operators X_2, Y_2 in $\mathcal{P}(n,h)$, and self-adjoint operators $X_3, Y_3 \in \mathcal{P}(h)$ satisfying $\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$ and

$$\mathrm{In}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right]\right) = \mathrm{In}_-\left(S_j^* \left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right] S_j\right) \tag{25}$$

if and only if for each $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z}^{m-j}$ the inequality

$$\mathrm{In}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)(\bar{k}_1, k_j, \bar{k}_2) + \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right)(\bar{k}_1, \bar{k}_2) \leq (n+h)(\bar{k}_1, k_j, \bar{k}_2) \text{ holds for all } k_j \in \mathbb{Z}. \tag{26}$$

Proof: To begin, we observe that by Lemma 10, the condition in (25) is equivalent to

$$\begin{aligned} &\mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right](k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_m)\right) \\ &\quad = \mathrm{in}_-\left(\left[\!\!\left[\begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix}\right]\!\!\right](k_1, \ldots, k_{j-1}, k_j, k_{j+1}, \ldots, k_m)\right) \end{aligned} \tag{27}$$

for all $\bar{k} \in \mathbb{Z}^m$. Namely, the negative inertia is *constant* with respect to the jth index.

(Only if): From (27) and (27), we conclude for each $\bar{k} = (\bar{k}_1, k_j, \bar{k}_2) \in \mathbb{Z}^m$, that $(n+h)(\bar{k}) =$

$$\mathrm{In}_+\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(\bar{k}) + \overline{\mathrm{In}}_-^j\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right)(\bar{k}_1, \bar{k}_2). \tag{28}$$

We have the relationship

$$\begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix}^* \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}.$$

From this and an application of Proposition 3, we obtain for every $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z}^{m-j}$ that $\overline{\mathrm{In}}_-^j(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix})(\bar{k}_1, \bar{k}_2) \geq \overline{\mathrm{In}}_-^j(\begin{bmatrix} X & I \\ I & Y \end{bmatrix})(\bar{k}_1, \bar{k}_2)$ and $\mathrm{In}_+(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}) \geq \mathrm{In}_+(\begin{bmatrix} X & I \\ I & Y \end{bmatrix})$, so the conclusion follows from (28).

(If): Fix $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z}^{m-j}$. Then, by hypothesis there exist a nonnegative q_- and a sequence $q_+(k_j)$, satisfying $(n+h)(\bar{k}) = q_+(k_j) + q_-$, such that $\mathrm{in}_-(\left[\!\!\left[\begin{matrix} X & I \\ I & Y \end{matrix}\right]\!\!\right](\bar{k}_1, k_j, \bar{k}_2)) \leq q_-$ and $\mathrm{in}_+(\left[\!\!\left[\begin{matrix} X & I \\ I & Y \end{matrix}\right]\!\!\right](\bar{k}_1, k_j, \bar{k}_2)) \leq q_+(k_j)$, for all $k_j \in \mathbb{Z}$. Invoking Lemma 21, we see that there exist matrix sequences with $\tilde{X}_2(\bar{k}_1, k_j, \bar{k}_2), \tilde{Y}_2(\bar{k}_1, k_j, \bar{k}_2)$ being $n(\bar{k}) \times h(\bar{k})$, and symmetric $h(\bar{k}) \times h(\bar{k})$ matrices $\tilde{X}_3(\bar{k}_1, k_j, \bar{k}_2), \tilde{Y}_3(\bar{k}_1, k_j, \bar{k}_2)$ such that

$$\begin{aligned} &\begin{bmatrix} X(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_2(\bar{k}_1, k_j, \bar{k}_2) \\ \tilde{X}_2^*(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_3(\bar{k}_1, k_j, \bar{k}_2) \end{bmatrix}^{-1} \\ &\quad = \begin{bmatrix} Y(\bar{k}_1, k_j, \bar{k}_2) & \tilde{Y}_2(\bar{k}_1, k_j, \bar{k}_2) \\ \tilde{Y}_2^*(\bar{k}_1, k_j, \bar{k}_2) & \tilde{Y}_3(\bar{k}_1, k_j, \bar{k}_2) \end{bmatrix} \end{aligned} \tag{29}$$

and

$$\mathrm{in}_-\left(\begin{bmatrix} X(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_2(\bar{k}_1, k_j, \bar{k}_2) \\ \tilde{X}_2^*(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_3(\bar{k}_1, k_j, \bar{k}_2) \end{bmatrix}\right) = q_- \tag{30}$$

holds for all $k_j \in \mathbb{Z}$. Furthermore, from Lemma 21, these sequences can be chosen so that $\|X\| + \|X - Y^{-1}\|^{(1/2)} + 1 \geq$

$$\bar{\sigma}\left(\begin{bmatrix} X(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_2(\bar{k}_1, k_j, \bar{k}_2) \\ \tilde{X}_2^*(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_3(\bar{k}_1, k_j, \bar{k}_2) \end{bmatrix}\right) \tag{31}$$

and $\|Y\| \cdot (1 + \|X - Y^{-1}\|^{(1/2)})^2 + 1 \geq$

$$\bar{\sigma}\left(\begin{bmatrix} X(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_2(\bar{k}_1, k_j, \bar{k}_2) \\ \tilde{X}_2^*(\bar{k}_1, k_j, \bar{k}_2) & \tilde{X}_3(\bar{k}_1, k_j, \bar{k}_2) \end{bmatrix}^{-1}\right) \text{ both hold.}$$

This procedure can now be repeated for all other values $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z}^{m-j}$, thus completely specifying the required hyperdiagonal operators.

From (29) and (31), we see that the required invertibility and boundedness conditions are satisfied, and from (30) that the inertia condition of (27) is met.

We now have the following corollary, which states that for h sufficiently large, the condition in (26) is always satisfied. Define the quantity

$$n_j^{\max}(\bar{k}) := \max_{\kappa \in \mathbb{Z}} n(k_1, \ldots, k_{i-1}, \kappa, k_{i+1}, \ldots, k_m) \tag{32}$$

where $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ and i is the smallest integer satisfying $i \geq (j + 1/2)$.

Proposition 23: Assume that the hypothesis of Lemma 22 holds, and further that $h \geq n + 2n_j^{\max}$. Then, the condition in (26) is always satisfied.

Proof: The result follows by simply noting that the left-hand of (26) is at most equal to $2n + 2n_j^{\max}$.

We next have the following specialized result for blocks constrained to be positive definite.

Lemma 24: Suppose n and h are sequences of nonnegative integers, and that the operators X and Y are in $\mathcal{P}(n)$. Then there exist operators X_2, Y_2 in $\mathcal{P}(n,h)$, and self-adjoint operators $X_3, Y_3 \in \mathcal{P}(h)$ satisfying

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix} \succ 0 \tag{33}$$

if and only if

$$\mathrm{In}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right) \leq n + h \quad \text{and} \quad \mathrm{In}_-\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right) = 0. \tag{34}$$

Notice, in particular, that if $h \geq n$ then (34) reduces to the simple condition $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$. The proof of this result is a simplified version of that given for Lemma 22, and is formally equivalent to that of [32, Lemma 6.2]; accordingly, we do not include it here. The main synthesis result of the paper is now stated.

Theorem 25: Given nominal system as in (1), a vector-valued sequence $(n_{K1}, \ldots, n_{Kd})$, and $(n_1, \ldots, n_d) := p(A)$, suppose the following.

i) The partitioned hyperdiagonal operators $X =: \mathrm{diag}(X_1, \ldots, X_d)$ and $Y =: \mathrm{diag}(Y_1, \ldots Y_d)$ are both elements of the set $\mathcal{X}^c$ defined in (14).

ii) The coercive operators N_c and N_o satisfy $\mathrm{Im} N_c = \mathrm{Ker}[B_2^* \; D_{12}^*]$ and $\mathrm{Im} N_o = \mathrm{Ker}[C_2 \; D_{21}]$.

iii) For each $2 \leq j \leq d$, setting i equal to the smallest integer satisfying $i \geq (j + 1/2)$, the inequality $(n_j + n_{Kj})(\bar{k}) \geq$

$$\mathrm{In}_+ \left(\begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} \right) (\bar{k}) + \overline{\mathrm{In}}_-^i \left(\begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} \right) (\bar{k}_1, \bar{k}_2)$$

holds for all $\bar{k} = (\bar{k}_1, k_j, \bar{k}_2) \in \mathbb{Z}^{j-1} \times \mathbb{Z} \times \mathbb{Z}^{m-j}$.

If the following four conditions are satisfied:

$$\begin{aligned} &\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \Lambda^* X \Lambda, X, N_c \right) \prec 0 \\ &\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^*, Y, \Lambda^* Y \Lambda, N_o \right) \prec 0 \\ &\mathrm{In}_+ \left(\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \right) \leq n_1 + n_{K1} \quad \text{and} \\ &\mathrm{In}_- \left(\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \right) = 0 \end{aligned} \tag{35}$$

then an admissible controller K exists with realization dimensions satisfying $p(A_K) \leq (n_{K1}, \ldots, n_{Kd})$.

Proof: Our goal is to show that all the conditions in Proposition 20 are met.

From the inertia criterion in (35) and Lemma 24, we know that there exist $X_{GK1} \in \mathcal{P}(n_1, n_{K1})$ and $X_{K1} \in \mathcal{P}(n_{K1})$ such that

$$\begin{bmatrix} X_1 & X_{GK1} \\ X_{GK1}^* & X_{K1} \end{bmatrix}^{-1} = \begin{bmatrix} Y_1 & - \\ - & - \end{bmatrix} > 0.$$

For each $2 \leq j \leq d$, the maximum inertia inequality in iii), together with the result of Lemma 22, guarantee the existence of hyperdiagonal operators $X_{GKj} \in \mathcal{P}(n_j, n_{Kj})$ and $X_{Kj} \in \mathcal{P}(n_{Kj})$ such that

$$\begin{aligned} &\begin{bmatrix} X_j & X_{GKj} \\ X_{GKj}^* & X_{Kj} \end{bmatrix}^{-1} = \begin{bmatrix} Y_j & - \\ - & - \end{bmatrix} \quad \text{and} \\ &\mathrm{In}_- \left(\left[\!\left[\begin{matrix} X_j & X_{GKj} \\ X_{GKj}^* & X_{Kj} \end{matrix} \right]\!\right] \right) \\ &\quad = \mathrm{In}_- \left(S_i^* \left[\!\left[\begin{matrix} X_j & X_{GKj} \\ X_{GKj}^* & X_{Kj} \end{matrix} \right]\!\right] S_i \right) \end{aligned}$$

where i is the largest integer satisfying $i \leq (j + 1/2)$.

With X_{GKj} and X_{Kj} thus chosen, set the partitioned operators $X_K := \mathrm{diag}(X_{K1}, \ldots X_{Kd}) \in \mathcal{P}(\bar{n}_K)$ and $X_{GK} := \mathrm{diag}(X_{GK1}, \ldots, X_{GKd}) \in \mathcal{P}(\bar{n}, \bar{n}_K)$, which directly from their construction satisfy a) and b) in Proposition 20. Referring to conditions ii) and (35), we see that conditions c) and (23) of Proposition 20 are met. Since we have now showed that all the conditions of Proposition 20 are implied by the supposition here, we may invoke Proposition 20 to conclude that an admissible controller exists satisfying $p(A_K) \leq (n_{K1}, \ldots, n_{Kd})$.

This theorem gives sufficient conditions for the existence of an admissible controller; if there is only one independent variable, time, then we recover the exact synthesis conditions of the H_∞ control problem. Condition iii) of the theorem provides information on the order possible for the controller. We note the following. Suppose that X and Y satisfying the theorem have been found. If $n_{Kj}(\bar{k}) < n_j(\bar{k})$, then the matrices $\begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix}(\bar{k})$ must be singular at each $\bar{k}$ (dropping rank by an appropriate value) in order to satisfy iii); thus there is no neighborhood around X and Y for which property iii) holds uniformly. In contrast, if $n_{Kj} \geq n_j$, then it is readily verified that there must exist neighborhoods around X and Y where iii) holds uniformly. Thus, we see a substantial difference in the condition when n_{Kj} crosses this threshold. However, by way of Proposition 23, we have the following result which eliminates condition iii) of the theorem.

Corollary 26: Given nominal system G as in (1), and definitions i) and ii) in Theorem 25. If the following inequalities are satisfied $\mathbf{L}(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}, \Lambda^* X \Lambda, X, N_c) \prec 0$

$$\begin{aligned} &\mathbf{L}\left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^*, Y, \Lambda^* Y \Lambda, N_o \right) \prec 0 \quad \text{and} \\ &\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \succeq 0 \end{aligned} \tag{36}$$

then there exists an admissible controller K, with realization dimensions of satisfying $p(A_K) \leq (n_1, n_2 + 2n_2^{\max}, n_3 + 2n_3^{\max} \ldots, n_d + 2n_d^{\max})$ where $(n_1, \ldots, n_d) := p(A)$ and $n_j^{\max}$ is defined in (32).

Proof: Start by setting $(n_{K1}, n_{K2}, \ldots n_{Kd}) = (n_1, n_2 + 2n_2^{\max}, n_3 + 2n_3^{\max} \ldots, n_d + 2n_d^{\max})$. Note therefore that the operator inequality $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \succeq 0$, together with the fact that $n_1 = n_{K1}$, guarantees that the inertia condition in (35) is satisfied.

Also, by Proposition 23 we see that $n_j + 2n_j^{\max} = n_{Kj}$ ensures that the conditions in iii) of Theorem 25 are trivially met. Thus, by the theorem there exists an admissible controller with state dimension $p(A_K)$ as required.

This corollary states that an admissible controller exists if partitioned hyperdiagonal operators X and Y in $\mathcal{X}^c$ can be found solving the three linear operator inequalities in (36). This corresponds in general to a convex feasibility problem over an infinite dimensional vector space; in the subsequent section we will discuss some important cases where these conditions can be reduced to finite dimensional semidefinite programs. The corollary also asserts that if solutions X and Y exist, then a feasible controller exists with state dimension at most n_1 in the temporal variable and $n_j + 2n_j^{\max}$ in the spatial variables. The state dimension of the temporal variable is standard in the robust synthesis literature (see, e.g., [21] and [32]), but the bound of $n_j + 2n_j^{\max}$ for the spatial variables is somewhat surprising. Whether it can be in general reduced to n_j using the inequality approach of this paper is still unknown; however, it can be reduced to n_j in some special cases. For example, when the nominal system G is homogeneous, or finite, in the corresponding variable it is straightforward to reduce these bounds to the usual case of n_j; see Remark 30.

VI. Computational Issues

This section is devoted to addressing the application of Theorem 25 and Corollary 26.

A. Controller

We start with the issue of explicitly constructing a controller, which once solutions X and Y have been found, amounts to a sequence of *pointwise* computations. Namely, in contrast to finding solutions to X and Y where the matrices $[\![X]\!](\bar{k})$ and $[\![Y]\!](\bar{k})$ may be coupled a different points in $\bar{k} \in \mathbb{Z}^d$, finding the state space matrices for a controller amounts to solving a sequence of uncoupled matrix problems. We provide the following summarizing outline.

1) Algorithm: Controller Construction: Given solutions X and Y to (36), the following hold.

1) Set n_{Kj} and n_{K1} to satisfy the inertia conditions of Theorem 25 in (35) and iii), respectively.
2) Use the pointwise-in-$\bar{k}$ procedure of Lemma 22 to construct X_{GK} and X_K such that $X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix}$ satisfies a) and b) of Proposition 20.
3) Using the definitions in (24), for each $\bar{k}$, solve in the variable $Q(\bar{k})$ the matrix inequality

$$T(\bar{k})^* H(\bar{k}) T(\bar{k}) - Z(\bar{k}) \prec 0 \tag{37}$$

where $T(\bar{k}) := R(\bar{k}) + U^*(\bar{k})Q(\bar{k})V(\bar{k})$. This can be achieved by converting, via a bilinear transformation, the above to an linear matrix inequality. Alternatively, it can be explicitly obtained using the decompositions in the proofs of Lemma 16 and [12, Prop. 1].

We now turn to discussing when the conditions of Corollary 26 can be reduced to finite dimensional computations.

B. Finite Interval Variables

As aforementioned, in general the conditions in Theorem 25 and Corollary 26 are infinite dimensional in both the dimension of the variable spaces and constraint spaces. However, if the dynamics of interest, with respect to a *fixed* variable k_i, are restricted to a finite horizon, then the conditions can be reduced to being finite dimensional in that variable. Specifically, we say that the dynamics in k_i are restricted to a finite (discrete) interval $\{a_i, a_i+1, \ldots, b_i-1, b_i\} \subset \mathbb{Z}$ if the state space operators for the nominal system G satisfy

$$\left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix}\right]\!\!\right](k_1, \ldots, k_m) = 0 \tag{38}$$

when $k_i \notin \{a_i, a_i+1, \ldots, b_i-1, b_i\}$. In this case, a routine examination of the operator inequalities in (36) yields a reduction in their dimensions, which we will now characterize.

Given that the above condition holds with respect to the variable k_i, define the interval

$$\mathbb{I}_i = \{a_i - 1, a_i, a_i+1, \ldots, b_i-1, b_i, b_i+1\}$$

and then define the hyperdiagonal operators $\hat{A}_{ts}$ on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-i})$, from the operators A_{ts}, via

$$\hat{A}_{ts}(k_1, \ldots, k_m) = \begin{cases} A_{rs}(k_1, \ldots, k_m), \text{for } a_i \le k_i \le b_i \\ 0, \text{for either } k_i = a_i - 1 \text{ or } k_i = b_i \end{cases}.$$

Together these define the partitioned operator $\hat{A}$. The operators $\hat{B}, \hat{C}$, and $\hat{D}$ are defined in a similar way, as are the variables $\hat{X}$ and $\hat{Y}$. Further, define the shift operators $\hat{S}_j$ by

$$\hat{S}_j : \ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-i}) \to \ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-i})$$

$$(\hat{S}_j x)(\bar{k}) := \begin{cases} x(k_1, \ldots, k_j - 1, \ldots, k_m), & j \ne i \\ x(k_1, \ldots, k_i - 1, \ldots, k_m), & j = i, k_i \ne a_i \\ 0, & j = i, k_i = a_i. \end{cases}$$

The reduction is now summarized.

Proposition 27: Suppose $i > 1$. When the nominal system G has dynamics with respect to k_i restricted to a finite interval, as in (38), then the inequalities in (36) are solvable if and only if the inequalities

$$\mathbf{L}\left(\begin{bmatrix} \hat{A} & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_{11} \end{bmatrix}, \hat{\Lambda}^* \hat{X} \hat{\Lambda}, \hat{X}, \hat{N}_c\right) \prec 0$$

$$\mathbf{L}\left(\begin{bmatrix} \hat{A} & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_{11} \end{bmatrix}^*, \hat{Y}, \hat{\Lambda}^* \hat{Y} \hat{\Lambda}, \hat{N}_o\right) \prec 0$$

$$\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \succeq 0, \begin{bmatrix} \hat{X}_j & I \\ I & \hat{Y}_j \end{bmatrix} \succeq 0 \quad \text{and}$$

$$\begin{bmatrix} \hat{X}_{j+1} & I \\ I & \hat{Y}_{j+1} \end{bmatrix} \succeq 0$$

are solvable, where j is the largest integer satisfying $2i - 2 \ge j$. In the case of $i = 1$, the result holds but the latter 2 inequalities are not required.

Note that these reduced inequalities are simply those in (36) with A, B, C, D, X, and Y, replaced by $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{X}$, and $\hat{Y}$, and the additional constraint. The proof is routine, and amounts to comparing the inequalities and observing that they characterize the same pointwise-in-$\bar{k}$ constraints.

If there are several variables, say $k_{i_1}, \ldots, k_{i_f}$, in which the system dynamics are finitely restricted, then the above result can be applied successively to each, to obtain a reduction in each variable. Note that in particular, if these are *all* the variables then the result of such a successive reduction will be an inequality on a finite dimensional space, a linear matrix inequality (LMI).

If however, some of the variables do not satisfy a finite extent condition, then the reduction will instead yield an infinite dimensional set of inequalities. We now look at another property that can be exploited to reduce the remaining variables.

C. Periodic Variable Reduction

A reduction is also possible when the system dynamics are periodic with respect to one of the variables k_i. In particular we require a periodic realization, and say that it is q_i-periodic in k_i if

$$\left[\!\!\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\!\!\right] S_i^{q_i} = S_i^{q_i} \left[\!\!\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\!\!\right].$$

If this condition is satisfied, we can adopt the averaging approach used in [16], and earlier work in [8], to show that if solutions X and Y exist to the inequalities (36), then they can always be chosen to be q_i-periodic in the variable k_i.

The fact that such periodic solutions exist can be used to rewrite the conditions of (36) in terms of equivalent conditions on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{Z}_{q_i} \times \mathbb{Z}^{m-i})$, where $\mathbb{Z}_{q_i}$ denotes the finite set of the integers modulo q_i. Simply define the following operators on $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{Z}_{q_i} \times \mathbb{Z}^{m-i})$:

$$\tilde{A}_{ts} : x \mapsto y, \text{where } y(\bar{k}) = A_{ts}(\bar{k})x(\bar{k}).$$
$$\tilde{S}_j : x \mapsto y, \text{with } y(\bar{k}) = x(k_1, \ldots, k_j - 1, \ldots, k_m).$$

Similarly define $\tilde{B}, \tilde{C}, \tilde{D}$, and the variables $\tilde{X}$ and $\tilde{Y}$. We have the following result.

Proposition 28: When the nominal system G has a q_i-periodic realization with respect to the variable k_i, then the inequalities in (36) are solvable if and only if a solution exists to

$$\begin{gathered}\begin{bmatrix} \tilde{X}_1 & I \\ I & \tilde{Y}_1 \end{bmatrix} \succeq 0 \\ \mathbf{L}\left(\begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix}, \tilde{\Lambda}^* \tilde{X} \tilde{\Lambda}, \tilde{X}, \tilde{N}_c\right) \prec 0 \quad \text{and} \\ \mathbf{L}\left(\begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix}^*, \tilde{Y}, \tilde{\Lambda}^* \tilde{Y} \tilde{\Lambda}, \tilde{N}_o\right) \prec 0.\end{gathered}$$

This result reduces the conditions in Corollary 26, which are stated in terms of operator inequalities with respect to the space $\ell_2(\mathbb{Z}^m)$, to conditions with respect to $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{Z}_{q_i} \times \mathbb{Z}^{m-i})$. Again, if there are several variables in which the state space realization is periodic, then this result can be used iteratively to reduce each of them.

Remark 29: We point out that the conditions of Proposition 28 apply directly to the case where the variable k_i in (1) is originally chosen to be in $\mathbb{Z}_{q_i}$. This corresponds to the scenario where the system is finite but cyclic in that spatial direction. Referring to the one-dimensional spatial example of Section III, the cyclic version of this example would be that there are q_i subsystems $G^{(p)}$, and that they are connected to form a topological circle.

Finally, if the system dynamics are either finite extent, periodic, or cyclic, in each of the variables $k_1, \ldots k_m$, then Propositions 27 and 28 can be used to reduce the conditions of Corollary 26 to a finite dimensional condition. In summary, in this case, the operator inequalities in Corollary 26 can be converted to equivalent LMIs. We now have an observation regarding the controller order.

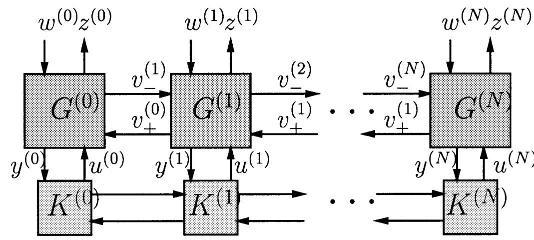

Fig. 3. Closed-loop: finite one-dimensional array.

Remark 30: In particular, if the system dynamics are shift invariant (i.e., one-periodic) with respect to a variable k_i, then the corresponding controller dimension n_{Ki} need not be chosen greater than n_i. This is seen from Theorem 25 by simply noting that iii) is satisfied if one-periodic solutions X_{per} and Y_{per} are obtained.

D. Case Study

Here, we return to the example of Section III shown in Fig. 2, with the aim of providing some perspectives of the results in the paper. The system to be considered has been chosen so that standard centralized synthesis results can alternatively be used, and this will allow us in Section VI-D3 to compare aspects of the two approaches.

Shown in Fig. 3 is the closed-loop system with controllers $K^{(p)}$ connected. We focus on the arrangement as shown, of a finite string of N subsystems $G^{(p)}$ together with controllers, where each is a standard linear *time-invariant* state space system. These subsystems need not be the same; however, for ease of exposition we will assume that they each have the same state dimension n. Similarly, it is also assumed that the dimensions of the interconnection vectors $v_-^{(p)}$ and $v_+^{(p)}$, the input vectors $w^{(p)}$ and $u^{(p)}$, and the output vectors $z^{(p)}$ and $y^{(p)}$ are *all* n.

Thus, for the system in (3), referring to (2), we have that

$$\begin{bmatrix} x_1(k_1,k_2) \\ x_2(k_1,k_2) \\ x_3(k_1,k_2) \end{bmatrix} := \begin{bmatrix} x(k_1,k_2) \\ v_-(k_1,k_2) \\ v_+(k_1,k_2) \end{bmatrix} \in \mathbb{R}^{3n}. \tag{39}$$

Our first goal will be to consider the synthesis feasibility conditions for this setup.

1) Feasibility: Associated with the feasibility result of Corollary 26 we have the elements X and Y. These operators belong to the set $\mathcal{X}_c$, defined in (14), and are both of the form and dimension $[\![X(k_1,k_2)]\!] = \text{diag}(X_1(k_1,k_2), X_2(k_1,k_2), X_3(k_1,k_2)) \in \mathbb{R}^{3n \times 3n}$, where $(k_1,k_2) \in \mathbb{Z}^2$. Now referring to Section VI-B, the dynamics of the system are restricted to the finite interval $\{1,\ldots,N\}$ in the variable k_2. And so we can obtain an equivalent set of inequalities to those in Corollary 26, which are finite with respect to k_2, by invoking Proposition 27. These are in terms of the variables $\hat{X}$ and $\hat{Y}$ where $[\![\hat{X}]\!](k_1,k_2) = \text{diag}(\hat{X}_1(k_1,k_2), \hat{X}_2(k_1,k_2), \hat{X}_3(k_1,k_2)) \in \mathbb{R}^{3n \times 3n}$, with $(k_1,k_2) \in \mathbb{Z} \times \mathbb{I}_2$, and $\hat{Y}$ has a similar form. Here, $\mathbb{I}_2$ is the index set $\{0,1,\ldots,N+1\}$.

As already stated each $G^{(p)}$ is assumed to be time-invariant, namely 1-periodic in k_1, and thus we can use the reduction result of Section VI-C to obtain a condition that is finite in k_1. Proposition 28 states that k_1 can be restricted to $\mathbb{Z}_1$ which is simply $\{1\}$, so instead of the infinite dimensional inequalities in $\hat{X}$ and $\hat{Y}$ we can reduce to equivalent finite-dimensional inequalities in the variable $\tilde{X}(k_2) = \mathrm{diag}(\tilde{X}_1(k_2), \tilde{X}_2(k_2), \tilde{X}_3(k_2)) \in \mathbb{R}^{3n\times 3n}$, where $k_2 \in \mathbb{I}_2$, and $\hat{Y}(k_2) \in \mathbb{R}^{3n\times 3n}$ is defined in the obvious way. That is, in this 1-periodic case the variable k_1 can be completely eliminated. Thus, we are left with five pointwise coupled LMIs which must hold for all $k_2 \in \mathbb{I}_2$

$$\mathbf{L}\left(\begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix}(k_2), (\tilde{\Lambda}^* \tilde{X}\tilde{\Lambda})(k_2), \tilde{X}(k_2), \tilde{N}_c(k_2)\right) \prec 0$$
$$\mathbf{L}\left(\begin{bmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{C}_1 & \tilde{D}_{11} \end{bmatrix}^*(k_2), \tilde{Y}(k_2), (\tilde{\Lambda}^* \tilde{Y}\tilde{\Lambda})(k_2), \tilde{N}_o(k_2)\right) \prec 0$$
$$\begin{bmatrix} \tilde{X}_1(k_2) & I \\ I & \tilde{Y}_1(k_2) \end{bmatrix} \succeq 0, \begin{bmatrix} \tilde{X}_2(k_2) & I \\ I & \tilde{Y}_2(k_2) \end{bmatrix} \succeq 0 \qquad (40)$$

and $\begin{bmatrix} \tilde{X}_3(k_2) & I \\ I & \tilde{Y}_3(k_2) \end{bmatrix} \succeq 0$, where $(\tilde{\Lambda}^* \tilde{X}\tilde{\Lambda})(k_2) = \mathrm{diag}(\hat{X}_1(k_2), \hat{X}_2(k_2+1), \hat{X}_3(k_2-1))$ and similarly for $(\tilde{\Lambda}^* \tilde{Y}\tilde{\Lambda})(k_2)$. Note that the two additional inequalities come from Proposition 27.

We will now discuss some computational aspects of these LMIs. First, for fixed k_2, the size of the variable space associated with $\tilde{X}(k_2)$, noting that it has three $n \times n$ symmetric blocks, is $(3/2)n(n+1)$. This is the same for the variable $\tilde{Y}(k_2)$. Because k_2 ranges over $\mathbb{I}_2$ we arrive at the total number of variables as $3n(n+1)(N+2)$ the three LMIs in (40).

Next, let us consider the dimension of the constraint space associated with (40). Recalling that the vector dimensions of the measurement variable $y^{(p)}$ and the control input $u^{(p)}$ have both been assumed to be n, we see that the domains of N_o and N_c can each be at most be of dimension $4n$, and that the latter three LMIs in (40) are each $2n \times 2n$. This yields $28n^2 + 10n$ for the dimension of the constraint space at each k_2. Again since we have a list of $N+2$ such inequalities we arrive at the total dimension of the constraint space as $(28n^2 + 10n)(N+2)$. An important salient feature of both this dimension and that of the variable space is that they grow *affinely* in the variable N.

2) Controller: We now move on to looking at controller construction. To obtain the controller, the following sequence of matrices needs to be constructed:

$$\tilde{Q}(k_2) = \begin{bmatrix} \tilde{A}_K(k_2) & \tilde{B}_K(k_2) \\ \tilde{C}_K(k_2) & \tilde{D}_K(k_2) \end{bmatrix}, \text{ for each } k_2 \in \{0, 1, \ldots, N\}.$$

Note that as in (39) the matrix A_K is a 3×3 partitioned matrix in which only the (1, 1)-block will form the A-matrix of the standard state space realization for $K^{(p)}$. To find the matrices $\tilde{Q}(k_2)$ the algorithm given in Section VI-A is followed. In Steps 1) and 2), the scaling matrix sequence $\tilde{X}_{cl}(k_2)$ is constructed. We use the constructive technique from the proof of Lemma 21 to complete the $n \times n$ matrix sequences $\tilde{X}_1, \tilde{X}_2$ and $\tilde{X}_3$ so that for each $k_2 \in \mathbb{I}_2$ the following inequalities are satisfied:

$$\begin{bmatrix} \tilde{X}_j(k_2) & \tilde{X}_{j2}(k_2) \\ \tilde{X}_{j2}^*(k_2) & \tilde{X}_{j3}(k_2) \end{bmatrix} \succ 0, \text{ where } j = \{1, 2, 3\}.$$

Performing this amounts to a Cholesky (or Schur form) factorization of each $\tilde{X}_j(k_2)$. Following this, we set the equation shown at the bottom of the next page, for each $k_2 \in \mathbb{I}_2$, noting that by Corollary 26 each matrix in the sequence is at most $6n \times 6n$.

In Step 3) of the algorithm, we have a list of N LMIs of the form in (37) to solve. Recalling our assumption that the vector dimensions of $w^{(p)}$ and $z^{(p)}$ are both n, we have that each LMI in the list will be $7n \times 7n$, and thus is in a constraint space of dimension $(1/2)7n(7n+1)$. Further, the variable $\tilde{Q}(k_2)$ in each LMI is of dimension $4n \times 4n$, and so has $16n^2$ variables.

3) Centralized Comparison: We will outline the calculations associated with posing the given synthesis problem using a centralized approach, and base this on the application of the by now standard H_∞ synthesis procedure using LMIs as described in [21], [24], [32]. We remark that the use of LMIs for approaching synthesis is attractive because it allows for the consideration of more sophisticated performance criteria and design problems; for instance linear parameter-varying control [1], [32], mixed objective control [33], or control design with integral quadratic constraint (IQC) constraints [13]. Also see the recent survey [18].

Again referring to Fig. 3, we have N subsystems $G^{(p)}$ each with state dimension n, and so the total state dimension for this system is nN. If this interconnection is written as a centralized state space system say G, the corresponding A-matrix will be of dimension $nN \times nN$. If we apply the conditions from [21], [24], [32] we will obtain three LMIs similar to those in Corollary 26, with symmetric-matrix variables X^{cent} and Y^{cent} each of dimension $nN \times nN$. Thus the variable space dimension is $nN(nN+1)$ for this centralized setting. Each of the three LMIs is of dimension $2nN \times 2nN$ and so the associated constraint space is $3nN(2nN+1)$. Finally, the associated scaling matrix $X_{\mathrm{cl}}^{\mathrm{cent}}$ would be of dimension $3nN \times 3nN$ and the controller construction variable Q^c would be $2nN \times 2nN$ and thus of dimension $4n^2N^2$, and the LMI constraint $2nN \times 2nN$. We summarize in Table I.

We now try to illustrate this further. In order to avoid going into the specific complexity measures of any particular LMI solving algorithm, let us suppose that we have an LMI solver which has polynomial-time complexity with computational cost of the form

$$\text{computation time} = O(c^\alpha v^\beta)$$

where c is the dimension of the constraint space, v is the number of scalar variables in the problem, and α and β are two con-

$$\tilde{X}_{cl}(k_2) = \begin{bmatrix} \tilde{X}(k_2) & \mathrm{diag}(\tilde{X}_{12}, \tilde{X}_{22}, \tilde{X}_{32})(k_2) \\ \mathrm{diag}(\tilde{X}_{12}, \tilde{X}_{22}, \tilde{X}_{32})(k_2) & \mathrm{diag}(\tilde{X}_{13}, \tilde{X}_{23}, \tilde{X}_{33})(k_2) \end{bmatrix}$$

TABLE I
COMPUTATIONAL COST COMPARISON

	distributed	centralized
Existence	1 LMI problem	1 LMI problem
# variables	$3n(n+1)(N+2)$	$nN(nN+1)$
constraint dimension	$(28n^2+10n)(N+2)$	$3nN(2nN+1)$
X_{cl}	$3(N+2)$ Cholesky	1 Cholesky
matrix size	$n \times n$	$nN \times nN$
Controller	N LMI problems	1 LMI problem
# variables	$16n^2$	$4n^2N^2$
constraint dimension	$\frac{7n(7n+1)}{2}$	$3nN(2nN+1)$

stants that are algorithm dependent. Using the dimensions from Table I, we see that for feasibility

$$\frac{\text{c.t. centralized}}{\text{c.t. distributed}} = \frac{O\left(N^{2(\alpha+\beta)}n^{2(\alpha+\beta)}\right)}{O\left(N^{(\alpha+\beta)}n^{2(\alpha+\beta)}\right)} = O(N^{\alpha+\beta}).$$

That is the computation time for checking feasibility of centralized synthesis is $O(N^{\alpha+\beta})$ greater than using the distributed approach. For controller construction, we obtain

$$\frac{\text{c.t. centralized}}{\text{c.t. distributed}} = \frac{O\left(N^{2(\alpha+\beta)}n^{2(\alpha+\beta)}\right)}{O\left(Nn^{2(\alpha+\beta)}\right)} = O(N^{2(\alpha+\beta)-1}).$$

That is, for reconstruction the proportional cost of centralized synthesis is even greater (nearly power of two) than that obtained for verifying feasibility. In summary we see that even if the sum $\alpha+\beta$ is of fairly modest size there is a very significant computational advantage to using the distributed approach as the number of subsystems N increases. Typical numbers for $\alpha+\beta$ range between 3.5 and 5 depending on the implementation; see, for instance, [29]. We also remark that this analysis is done assuming a generic solver; however, there is considerable structure in the distributed synthesis conditions obtained which it may be possible to further exploit by developing appropriate specialized algorithms; see [7] for more on algorithm development. Note for instance that the controller construction computations can be processed in parallel.

Finally, we comment on scenarios different from that of this case study as depicted in Fig. 3. If for instance we consider, instead of a finite spatial extent problem, one that is spatially N-periodic so that $G^{(p+N)} = G^{(p)}$, the computational cost of distributed synthesis will have the same order in N as for the current example. For m variables $(k_1, \dots k_m)$ the order of the growth rates in each size $N_1, \dots, N_m$ will be as given in this example. Thus, this case study provides a representative overview of the computational aspects of the synthesis results.

VII. CONCLUSION

In this paper, we have studied distributed control of systems whose representations are explicit in temporal and spatial variables. Sufficient conditions were obtained for the existence of controllers that stabilize the closed-loop system and make it strictly contractive. The conditions obtained are convex; however, in general they are infinite dimensional. When the initial state space system is either finite extent or periodic in every variable the infinite dimensional operator inequalities obtained become linear matrix inequalities. Thus, the approach given is attractive from a computational perspective.

We have conducted our investigation focusing on purely discrete time systems. However, by using the Fourier transform the results can be readily extended to treat nominal systems which have a continuous time variable and discrete spatial variables, where the system is *time-invariant* but not necessarily spatially invariant. Further, it should be possible to extend this work so that a similar synthesis can be performed when the matrices in this hybrid problem vary with time, but some additional techniques are required.

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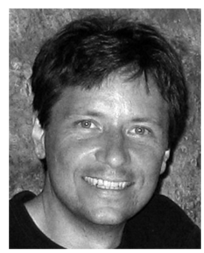

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