Distributed Control Design for Systems Interconnected Over an Arbitrary Graph

Cédric Langbort, Ramu Sharat Chandra, and Raffaello D'Andrea, *Senior Member, IEEE*

***Abstract*—We consider the problem of synthesizing a distributed dynamic output feedback controller achieving $\mathcal{H}_\infty$ performance for a system composed of different interconnected sub-units, when the topology of the underlying graph is arbitrary. First, using tools inspired by dissipativity theory, we derive sufficient conditions in the form of finite-dimensional linear matrix inequalities when the interconnections are assumed to be ideal. These inequalities are coupled in a way that reflects the spatial structure of the problem and can be exploited to design distributed synthesis algorithms. We then investigate the case of lossy interconnection links and derive similar results for systems whose interconnection relations can be captured by a class of integral quadratic constraints that includes constant delays.**

***Index Terms*—Dissipativity, distributed control, large-scale systems, linear matrix inequalities (LMIs).**

I. Introduction

OVER THE PAST few years, there has been a renewal of interest in systems consisting in the interaction/co-operation of a large number of spatially interconnected units. Examples include unmanned aerial vehicles or satellites flying in formation to achieve a particular task [17], [43], automated highways [37], cross-directional control in the pulp/paper and chemical industry [23], [35], as well as problems posed by emerging fields such as control of communication networks or smart structures (large arrays of micromechanical and electrical actuators and sensors) [4], [27]. When controlling such systems, the classical approach that consists in lumping all subsystems together and controlling them as a single large-scale system is not always appropriate. For example, in the case of the automated highway or formation flight, it seems more reasonable to regulate the system by equipping each vehicle with its own controller (maybe sharing information with the neighboring ones) than to rely on a centralized control authority that observes and acts on all units at once. In general, there are two main reasons why one might want to avoid centralized controllers.

- First, using a centralized control scheme typically requires modifying the original interconnection topology of the system. This may be undesirable for applications such as communication networks, where introducing new channels from and to a remote subsystem can sometimes be less practical and more expensive than increasing transmission rates over the pre-existing ones. It would thus be more profitable to be able to respect the structure of the plant's physical interconnection for control design.
- Even for systems that are not physical networks but whose interconnection structure follows from mere coupling between the equations describing the different units, centralized control has a major drawback. By its very nature, a centralized controller is bound to have a large number of states, inputs and outputs, which is precisely the situation classical optimal control design algorithms cannot handle. Hence, for this reason also, one would like to use a control architecture that takes advantage of the problem's structure and helps reduce or distribute the computational effort of the design process.

In the recent literature, it has often been taken for granted that the natural way to address these two points is to adopt a fully *decentralized* architecture, where a controller is assigned to each plant's subsystem, providing it with corrective action based solely on local measurements. Successful synthesis methods have even been proposed that give convex conditions for the existence of a decentralized controller guaranteeing well-posedness, stability and performance of the closed-loop system [2], [3], [7], [16], [30], [33], [36]. However, these results are only valid when the interconnection structure has some special property (identical subsystems with nearest neighbors coupling [2], [3], hierarchical or nested coupling [40] or quadratic invariance [33]) and/or yield static state feedback controllers [7], [36].

The goal of the present paper is to show that both these limitations can be bypassed if one agrees to look for *distributed* rather than decentralized controllers. More precisely, we give sufficient conditions for the existence of a controller with the same structure as the plant, when the latter is made of possibly different linear time-invariant (LTI) subsystems connected over an arbitrary graph. Our results take the form of a set of coupled linear matrix inequalities (LMIs), the particular structure of which lends itself to distributed computation over the system's interconnection graph. With such an implementation, distributed control also appears to be a good alternative to centralized control, as the interconnection topology is respected and the design algorithm is scalable.

The approach taken here builds on recent papers [8], [9], [12], where the idea of distributed control was first introduced. We extend the results to arbitrary interconnections and in so doing,

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C. Langbort is with the Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853 USA (e-mail: cl263@cornell.edu).

R. S. Chandra and R. D'Andrea are with the Department of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY 14853 USA (e-mail: rsc28@cornell.edu; rd28@cornell.edu).

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clarify the link between these works and those applying dissipativity theory to large-scale systems [26], [36], [41]. This paper is closest in spirit to [36], in that we treat the interconnection as a set of constraints on the coupling signals and use the corresponding multipliers (the supply rates of dissipativity theory) as extra design variables. However, looking for distributed rather than decentralized controllers (i.e., allowing the controller to use structure information) enables us to formulate *convex* conditions for the existence of *output-feedback controllers*, in the exact same way as allowing for online parameter measurements in linear parameter-varying (LPV) synthesis yields a convex relaxation of the robust control design problem.

The point of view outlined above is also related to integral quadratic constraints (IQCs) analysis methods [25], [32], because our dissipativity conditions can be seen as a conservative full-block S-procedure when parameterizing the interconnection subset by a family of IQCs. Exploring this connection will allow us to treat systems with more general imperfect communication links.

The paper is organized as follows. In Section II, we introduce a first class of interconnected systems, along with a general framework for describing them. Analysis conditions for these so-called ideal systems are derived in Section III and used in Section IV for controller design. In Section V, we present numerical experiments in order to compare the performance achieved by our distributed controller with that obtained with a centralized one. In Section VI, we consider another type of systems, this time with nonideal interconnections, that can be described via IQCs. Analysis and synthesis results of the previous sections are extended to this class. We also quantify the gap between systems with ideal and directed interconnections.

Notation: The set of real numbers is denoted by $\mathbb{R}$, the nonnegative reals by $\mathbb{R}^+$ and the $n \times m$ real matrices by $\mathbb{R}^{n\times m}$. The $n \times n$ identity and $n \times m$ zero matrix are denoted by I_n and $0^{n\times m}$, respectively, or just I and 0 if dimension is clear from context. The set of real symmetric matrices is denoted by $\mathbb{R}^{n\times n}_{\mathbf{S}}$. If A, B belong to $\mathbb{R}^{n\times n}$, the matrix inequality $A < B$ (respectively, $A \leq B$) means that $B - A$ is symmetric positive definite (positive semidefinite). Also, given matrices $A_1, A_2 \dots A_L$, we define $\underset{k\leq i\leq l}{\mathbf{diag}} A_i$ by

$$\underset{k\leq i\leq l}{\mathbf{diag}} A_i := \begin{bmatrix} A_k & 0 & \dots & 0 \\ 0 & A_{k+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_l \end{bmatrix}.$$

Likewise, if $e_1, \dots, e_L$ are elements of sets $E_1, \dots, E_L$, $\underset{k\leq i\leq l}{\mathbf{cat}}\ e_i$ will designate the element $(e_k, \dots, e_l) \in E_k \times \dots \times E_l$ when $1 \leq k < l \leq L$. We will sometimes write $\mathbf{diag}_i$ and $\mathbf{cat}_i$ instead of $\underset{1\leq i\leq L}{\mathbf{diag}}$ and $\underset{1\leq i\leq L}{\mathbf{cat}}$.

The Euclidean norm of a vector is denoted $|\bullet|$. The closure of a set S is denoted $\mathrm{cl}(S)$, and the convex hull by $\mathrm{co}(S)$. The signals dealt with in this paper belong to the class $\mathcal{L}_2^n$ (vector-valued square integrable continuous time signals on $\mathbb{R}^+$ with n components), with the usual inner product and norm; this is abbreviated $\mathcal{L}_2$ when n is clear from context or not relevant. The space of finite-horizon signals $\mathcal{L}_2[T_1, T_2]$, where $0 \leq T_1 < T_2 < \infty$, is the set of square integrable functions defined on the subset $[T_1, T_2]$ of the positive real line. The truncation $P_{[T_1,T_2]} : \mathcal{L}_2 \to \mathcal{L}_2[T_1, T_2]$ is defined by

$$\left(P_{[T_1,T_2]}x\right)(t) = x(t), \qquad \text{if } T_1 \leq t \leq T_2 \tag{1}$$

and if $T_1 = 0$, $T_2 = T$, we abbreviate this by P_T. We will also make use of the class $\mathcal{L}_{2e}$ which consists of measurable functions v on $\mathbb{R}^+$ such that $\int_0^T |v(t)|^2 dt < \infty$, for all $T > 0$.

The norm of an $\mathcal{L}_2$ signal or that of an operator on $\mathcal{L}_2$ are both denoted by $\|\bullet\|$; the context will make it clear which meaning is intended. The inner product between two signals in $\mathcal{L}_2$ is denoted by $\langle\bullet,\bullet\rangle$. A bounded linear operator $\delta : \mathcal{L}_2 \to \mathcal{L}_2$ is said to be contractive if $\|\delta\| \leq 1$; it is said to be unitary if it is surjective and if $\langle\delta(p), \delta(q)\rangle = \langle p, q\rangle$ for all $p, q \in \mathcal{L}_2$. The adjoint of an operator δ is denoted δ^* and defined by the relation $\langle q, \delta(p)\rangle = \langle\delta^*(q), p\rangle$ for all $p, q \in \mathcal{L}_2$. Note that if operator δ is unitary, it has a bounded inverse δ^{-1} which is just its adjoint. When an operator F is self-adjoint (i.e. satisfies $F = F^*$), the operator inequality $F < 0$ means that there exists $\epsilon > 0$ such that $\langle Fv, v\rangle \leq -\epsilon\|v\|^2$ for all $v \in \mathcal{L}_2$. Finally, we extend the notation $\mathbf{diag}$ to operators. If $\delta_k, \dots, \delta_l$ are linear operators, the operator $\underset{k\leq i\leq l}{\mathbf{diag}}\ \delta_i$ is defined as mapping $v = \underset{k\leq i\leq l}{\mathbf{cat}}(v_i)$ to $w = \underset{k\leq i\leq l}{\mathbf{cat}}(w_i)$ if and only if $w_i = \delta_i v_i$, for all $k \leq i \leq l$.

II. Interconnected Systems

In this paper, we will concern ourselves with systems consisting of an assembly of L, possibly different LTI subsystems connected arbitrarily, as depicted for example in Fig. 1. In general, such interconnected systems can be described by an undirected graph, the state–space representation of the different subsystems and an interconnection condition, as we will now explain.

A. Underlying Graph Structure

We will use graph-theoretic terminology only for notational convenience. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a interconnected system is defined as follows. First, we identify the set $\mathcal{V}$ of vertices with the set $\{G_i, i = 1 \dots L\}$, where each G_i is an LTI finite-dimensional subsystem. Then, we let the set of nonoriented edges be defined as $\mathcal{E} := \{(G_i, G_j), i \leq j\}$. Note that we can have edges of the form (G_i, G_i), corresponding to cases where a subsystem feeds back into itself.

To every edge (G_i, G_j), we associate its size, an integer $n_{ij} \geq 0$. It will prove convenient to use the notation n_{ij} even when $i > j$ (recall that edge (G_i, G_j) is only defined for $i \leq j$). We do so by letting $n_{ij} = n_{ji}$ when $i > j$. By allowing $n_{ij} = 0$, we also capture the case where subsystem G_i and G_j are *not* interconnected.

For our purposes, all the relevant information regarding the topology of the interconnection can be summarized by a symmetric L by L matrix $\mathcal{N}$, with entries n_{ij}, the so-called (weighted) adjacency matrix of graph $\mathcal{G}$ [10]. We will denote by $\mathbb{R}^{\mathcal{N}}$ the vector space of partitioned vectors $\mathsf{v} = \mathbf{cat}_i \mathsf{v_i}$, where each $\mathsf{v_i}$ can itself be further partitioned as $\mathsf{v_i} = \mathbf{cat}_j \mathsf{v_{ij}}$, each $\mathsf{v_{ij}}$ being of size n_{ij}. When $\mathsf{v} \in \mathbb{R}^{\mathcal{N}}$, each $\mathsf{v_i}$ is of size

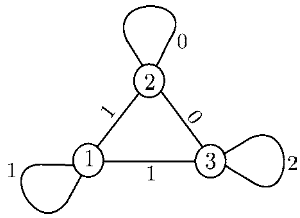

Fig. 1. Example of interconnected system with $L = 3$ subsystems. The integers indicated on the graph are the size n_{ij} of the edges. See text for details.

$n_i := \sum_{j=1}^{L} n_{ij}$ for all $i = 1 \dots L$. Likewise, we define $\mathcal{L}_2^{\mathcal{N}}$ for partitioned signals.

B. Interconnection Signals and Subsystems

Each subsystem G_i is captured by the following state-space equations:

$$\begin{bmatrix} \dot{x}_i(t) \\ w_i(t) \\ z_i(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{TT}}^i & A_{\mathrm{TS}}^i & B_{\mathrm{T}}^i \\ A_{\mathrm{ST}}^i & A_{\mathrm{SS}}^i & B_{\mathrm{S}}^i \\ C_{\mathrm{T}}^i & C_{\mathrm{S}}^i & D^i \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \\ d_i(t) \end{bmatrix}, \qquad \text{for all } t \geq 0$$
$$x_i(0) = x_i^0 \tag{2}$$

where $x_i(t) \in \mathbb{R}^{m_i}$, $d_i(t) \in \mathbb{R}^{p_i}$, $z_i(t) \in \mathbb{R}^{q_i}$, $v_i(t)$, and $w_i(t) \in \mathbb{R}^{n_i}$ for all $t \geq 0$. In (2), d_i is a disturbance acting on subsystem G_i, z_i is a performance output and v_i and w_i are the overall interconnection signals used by G_i. A comment on notation is in order. Our nomenclature for the various matrices follows that of [8], which was itself inspired by multidimensional systems theory. In this framework, subscript "T" refers to temporal variables (the states) while subscript "S" refers to spatial variables (the interconnection signals).

For each given i, we partition signals v_i and w_i further into v_{ij} and w_{ij}—the interconnection signals G_i shares with G_j. By adding zero-components to the signals of smaller size and modifying the state-space matrices of (2) if this is not the case, we can always assume that $v_{ij}(t)$, $v_{ji}(t)$, $w_{ij}(t)$ and $w_{ji}(t)$ all belong to $\mathbb{R}^{n_{ij}}$ for all $t \geq 0$. This, in turn, naturally defines two signals v and w such that $v(t)$, $w(t) \in \mathbb{R}^{\mathcal{N}}$ for all $t \geq 0$. Note that v_{ij} and v designate a $\mathbb{R}^{n_{ij}}$ and $\mathbb{R}^{\mathcal{N}}$-valued signal respectively, while v_{ij} is a component of the vector $\mathsf{v} \in \mathbb{R}^{\mathcal{N}}$. This construction allows us to think of A_{SS}^i as a square, n_i by n_i, matrix without loss of generality and will simplify the notations and proofs of Section III and IV.

C. Interconnection Relation

Once the relationships between v_i and w_i have been specified at each vertex, we can get a description of the interconnected system with input $d = (d_1, \dots, d_L)$ and output $z = (z_1, \dots, z_L)$ by closing all loops. The simplest way this can be done is by requiring that

$$\begin{bmatrix} v_{ij}(t) \\ w_{ij}(t) \end{bmatrix} = \begin{bmatrix} w_{ji}(t) \\ v_{ji}(t) \end{bmatrix}, \qquad \text{for all } i \geq j,\ t \geq 0 \tag{3}$$

or, equivalently, that

$$(w(t), v(t)) \in \mathcal{S}_{\mathcal{I}}, \qquad \text{for all } t \geq 0 \tag{4}$$

where the interconnection subspace $\mathcal{S}_{\mathcal{I}}$ is defined as

$$\mathcal{S}_{\mathcal{I}} := \left\{ (\mathsf{w}, \mathsf{v}) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}} : \mathsf{w}_{\mathsf{ji}} = \mathsf{v}_{\mathsf{ij}}, \qquad \text{for all } i, j = 1 \dots L \right\}. \tag{5}$$

An interconnection defined by (4) will be called *ideal* since in that case, every edge specifies an exact mathematical equality between two signals. Later in this paper, we will consider more general interconnected systems whose edges are, in some sense, "lossy," and whose interconnection relations can be described via IQCs. Our goal, for now, is to derive analysis and synthesis conditions for systems with ideal interconnections, using the underlying structure of the graph and the state-space representations of its subsystems.

III. Analysis

A. Well-Posedness, Stability, and Contractiveness

The construction of an interconnected system given in Section II is not always well defined because the signals satisfying the interconnection may not exist or be unique. Thus, we need the following.

Definition 1: An interconnected system defined by adjacency matrix $\mathcal{N} \in \mathbb{R}^{L \times L}$ and state-space matrices as per (2) is well-posed if the two subspaces $\mathcal{S}_{\mathcal{I}}$ and $\mathcal{S}_{\mathcal{B}}$ satisfy

$$\mathcal{S}_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{B}} = \{0\}$$

where

$$\mathcal{S}_{\mathcal{B}} := \left\{ (\mathsf{w}, \mathsf{v}) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}} : \begin{bmatrix} \mathsf{w}_{\mathsf{i}} \\ \mathsf{v}_{\mathsf{i}} \end{bmatrix} \in \mathcal{S}_{\mathcal{B}}^i, \text{ for all } i = 1 \dots L \right\} \tag{6}$$

and

$$\mathcal{S}_{\mathcal{B}}^i := \mathrm{Im} \begin{bmatrix} A_{\mathrm{SS}}^i \\ I \end{bmatrix}.$$

When an interconnected system is well-posed, initial conditions x_i^0 and disturbances d_i uniquely determine the signals x_i, v_i, w_i, and z_i. More precisely, if all disturbances belong to $\mathcal{L}_{2e}$, then (2)–(4) have a unique solution in $\mathcal{L}_{2e}$. We will say that a well-posed interconnected system is stable if, in the absence of disturbances ($d_i = 0$ for all $i = 1 \dots L$) and for any set $\{x_i^0, i = 1 \dots L\}$ of initial conditions, x_i is a smooth function of time and goes to zero as t approaches infinity, for all $i = 1 \dots L$. Finally, it is easy to see that if an interconnected system is well-posed and stable then x_i and z_i belong to $\mathcal{L}_2$ for all i when all d_i do. Such a system thus defines an input/output map (i.e. a linear operator) from $\mathcal{L}_2$ to $\mathcal{L}_2$ and its induced norm characterizes its performance. We will say that it is contractive if there exists $\epsilon > 0$ such that

$$\|z\|^2 = \sum_{i=1}^{L} \|z_i\|^2 \leq (1 - \epsilon) \sum_{i=1}^{L} \|d_i\|^2 = (1 - \epsilon)\|d\|^2.$$

B. Sufficient Conditions for Well-Posedness, Stability, and Contractiveness

For each $i = 1 \dots L$, we introduce the quadratic form on $\mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$

$$\mathcal{P}_i(\mathsf{w_i}, \mathsf{v_i}) := \sum_{j=1}^{L} \begin{bmatrix} \mathsf{w_{ij}} \\ \mathsf{v_{ij}} \end{bmatrix}^* X_{ij} \begin{bmatrix} \mathsf{w_{ij}} \\ \mathsf{v_{ij}} \end{bmatrix} \tag{7}$$

where, for all $i, j = 1 \dots L$, the symmetric $2n_{ij}$ by $2n_{ij}$ matrices X_{ij} satisfy

$$X_{ji} = -E_{ij} X_{ij} E_{ij} \quad \text{for } E_{ij} = \begin{bmatrix} 0 & I_{n_{ij}} \\ I_{n_{ij}} & 0 \end{bmatrix}. \tag{8}$$

Using (8), it is easy to show the following.

Proposition 1: Let the quadratic form $\mathcal{P}$ be defined on $\mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}}$ by

$$\mathcal{P}(\mathsf{w}, \mathsf{v}) := \sum_{i=1}^{L} \mathcal{P}_i(\mathsf{w_i}, \mathsf{v_i}) \tag{9}$$

then $\mathcal{P}$ vanishes on the interconnection subspace $\mathcal{S}_{\mathcal{I}}$.

Following the terminology of [41], we will say that the interconnection is neutral with respect to the supply rates $\mathcal{P}_i$, $i = 1 \dots L$. It will prove useful to partition each symmetric scale X_{ij} into four n_{ij} by n_{ij} blocks as

$$X_{ij} =: \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ \left(X_{ij}^{12}\right)^* & X_{ij}^{22} \end{bmatrix}, \qquad \text{for all } i, j = 1 \dots L. \tag{10}$$

Using (8) and symmetry of X_{ij}, we then get that

$$X_{ij}^{11} = \left(X_{ij}^{11}\right)^* = -X_{ji}^{22} \qquad \left(X_{ij}^{12}\right)^* = -X_{ji}^{12}$$

and, thus, the set $\{X_{ij} \in \mathbb{R}_{\mathbf{S}}^{2n_{ij} \times 2n_{ij}} : X_{ji} = -E_{ij} X_{ij} E_{ij}, \text{ for all } i, j = 1 \dots L\}$ is entirely parameterized by the two sets

$$\left\{X_{ij}^{11} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}, \qquad i, j = 1 \dots L\right\} \text{ and}$$
$$\left\{X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}} : X_{ii}^{12} \text{ skew-symmetric}, \ 1 \le j \le i \le L\right\}.$$

We are now in a position to state our first analysis conditions.

Theorem 1: An interconnected system is well-posed, stable and contractive if there exist symmetric matrices $X_{\mathrm{T}}^i \in \mathbb{R}_{\mathbf{S}}^{m_i \times m_i}$ and $X_{ij}^{11} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}$ for all $i, j = 1 \dots L$, and matrices $X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}$ for all $i \ge j$, with X_{ii}^{12} skew-symmetric, such that $X_{\mathrm{T}}^i > 0$ and LMI (11) is satisfied for all $i = 1 \dots L$, with

$$Z_i^{11} := -\operatorname*{\mathbf{diag}}_{1 \le j \le L} X_{ij}^{11}$$
$$Z_i^{22} := \operatorname*{\mathbf{diag}}_{1 \le j \le L} X_{ji}^{11}$$
$$Z_i^{12} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \le j \le i} X_{ij}^{12}, \operatorname*{\mathbf{diag}}_{i < j \le L} \left(X_{ji}^{12}\right)^*\right)$$

$$\begin{bmatrix} I & 0 & 0 \\ A_{\mathrm{TT}}^i & A_{\mathrm{TS}}^i & B_{\mathrm{T}}^i \\ \hline A_{\mathrm{ST}}^i & A_{\mathrm{SS}}^i & B_{\mathrm{S}}^i \\ 0 & I & 0 \\ \hline C_{\mathrm{T}}^i & C_{\mathrm{S}}^i & D^i \\ 0 & 0 & I \end{bmatrix}^* \left[\begin{array}{cc|cc|cc} 0 & X_{\mathrm{T}}^i & 0 & 0 & 0 & 0 \\ X_{\mathrm{T}}^i & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Z_i^{11} & Z_i^{12} & 0 & 0 \\ 0 & 0 & \left(Z_i^{12}\right)^* & Z_i^{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right] \times \begin{bmatrix} I & 0 & 0 \\ A_{\mathrm{TT}}^i & A_{\mathrm{TS}}^i & B_{\mathrm{T}}^i \\ \hline A_{\mathrm{ST}}^i & A_{\mathrm{SS}}^i & B_{\mathrm{S}}^i \\ 0 & I & 0 \\ \hline C_{\mathrm{T}}^i & C_{\mathrm{S}}^i & D^i \\ 0 & 0 & I \end{bmatrix} < 0. \tag{11}$$

Proof: We start with well-posedness. From (11), we see that, in particular

$$\begin{bmatrix} A_{\mathrm{SS}}^i \\ I \end{bmatrix}^* \begin{bmatrix} Z_i^{11} & Z_i^{12} \\ \left(Z_i^{12}\right)^* & Z_i^{22} \end{bmatrix} \begin{bmatrix} A_{\mathrm{SS}}^i \\ I \end{bmatrix} < -\left(C_{\mathrm{S}}^i\right)^* C_{\mathrm{S}}^i < 0.$$

By definition of the Z_i's and property (10), this means that each $\mathcal{P}_i$ is positive definite on $\mathcal{S}_{\mathcal{B}}^i$. In turn, $\mathcal{P}$ is also positive definite on $\mathcal{S}_{\mathcal{B}}$ while according to Proposition 1, $\mathcal{P}$ vanishes on $\mathcal{S}_{\mathcal{I}}$. Thus, $\mathcal{S}_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{B}} = \{0\}$.

Once we have established well-posedness, the state x_i and interconnection signals v_i and w_i of each subsystem are well-defined for all d_i and initial conditions x_i^0, and $(w(t), v(t))$ belongs to $\mathcal{S}_{\mathcal{I}}$ for all $t \ge 0$. Moreover, when d_i is chosen to be identically zero for all i, these signals are smooth functions of time and $z_i(t) = C_{\mathrm{T}}^i x_i(t) + C_{\mathrm{S}}^i v_i(t)$ for all t.

Pre- and postmultiplying (11) by the nonzero vector $[x_i^*(t) \ v_i^*(t) \ 0]$ and its transpose and summing over $i = 1 \dots L$, we get that

$$\frac{d}{dt}\left(\sum_{i=1}^{L} x_i^* X_{\mathrm{T}}^i x_i\right)(t) - \mathcal{P}\left(w(t), v(t)\right) + \sum_{i=1}^{L} z_i^*(t) z_i(t) < 0$$

for all $t \ge 0$. Hence, using again the fact that $\mathcal{P}$ vanishes on $\mathcal{S}_{\mathcal{I}}$, we see that $\sum_{i=1}^{L} x_i(t)^* X_{\mathrm{T}}^i x_i(t)$ is a Lyapunov function since $X_{\mathrm{T}}^i > 0$; this establishes stability of the system.

Finally, when a disturbance is applied in the form of a $\mathcal{L}_2$ signal d_i for each $i = 1 \dots L$, x_i, v_i, w_i, and z_i are all $\mathcal{L}_2$ signals, since the interconnected system is stable. Also, $(w(t), v(t))$ belongs to $\mathcal{S}_{\mathcal{I}}$ for almost all $t \ge 0$. Pre- and postmultiplying (11) by $[x_i^*(t) \ v_i^*(t) \ d_i^*(t)]$ and its transpose, summing over $i = 1 \dots L$ and using the strict inequality in (11), we see that there exists $\epsilon > 0$ such that $d/dt(\sum_{i=1}^{L} x_i^* X_{\mathrm{T}}^i x_i)(t) + \sum_{i=1}^{L} z_i^*(t) z_i(t) \le \sum_{i=1}^{L} d_i^*(t) d_i(t) + \mathcal{P}(w(t), v(t)) - \epsilon \sum_{i=1}^{L} d_i^*(t) d_i(t)$ for almost all $t \ge 0$. Then, using the fact that $\mathcal{P}$ vanishes on the interconnection subspace and integrating from 0 to ∞, we get

$$\|z\|^2 - (1-\epsilon)\|d\|^2 \le \sum_{i=1}^{L} x_i^*(0) X_{\mathrm{T}}^i x_i(0) - \sum_{i=1}^{L} x_i^*(\infty) X_{\mathrm{T}}^i x_i(\infty)$$

Now, use the fact that the system is stable to conclude that the right-hand side of the latter inequality is zero when $x_i^0 = 0$ for all i and, in turn, that the system is contractive. ■

Apart from the well-posedness statement, Theorem 1 can be seen as a dissipativity result [41]: It just states that the interconnected system is strictly dissipative with respect to supply rate $(d, z) \mapsto |d|^2 - |z|^2$, with storage function $(\mathsf{x}_1, \dots, \mathsf{x}_L) \mapsto \sum_{i=1}^{L} \mathsf{x}_i^* X_{\mathrm{T}}^i \mathsf{x}_i$, whenever each subsystem is strictly dissipative with respect to supply rate $\mathcal{P}_i + |d_i|^2 - |z_i|^2$, with storage function $\mathsf{x}_i \mapsto \mathsf{x}_i^* X_{\mathrm{T}}^i \mathsf{x}_i$. Hence, it is simply a particular case of results of [41] or [26] regarding neutral interconnections of dissipative systems. From this standpoint, Theorem 1 is even more conservative than these results because we have restricted ourselves to a particular type of quadratic supply rates, as per (7) and (8). However, the fact that we consider these supply rates as free parameters (apart from the constraint (8)) gives less conservative analysis conditions than the classical methods requiring each subsystem to have small-gain or be passive [39]. The latter conditions would yield decoupled LMIs via theKalman–Yakubovich–Popov (KYP) lemma [31], while LMIs (11) are coupled through the scales X_{ij}.

As we will see, constraint (8) plays an important role for synthesis. Imposing that the interconnection be neutral in this explicit way is indeed essential for obtaining a *convex* characterization of interconnected controllers, when we apply Theorem 1 to a closed-loop system in Section IV.

Before doing so, we would like to end this section by pointing out an interesting feature of analysis conditions (11): They can guarantee stability of the interconnected system even when the subsystems are unstable. For example, for three subsystems interconnected on a ring, with

$$A^1_{\rm TT} = A^3_{\rm TT} = 1 \quad A^1_{\rm TS} = A^3_{\rm TS} = [-3 \quad 3]$$
$$A^1_{\rm ST} = A^3_{\rm ST} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A^1_{\rm SS} = A^3_{\rm SS} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$
$$A^2_{\rm TT} = 1 \quad A^2_{\rm TS} = [3 \quad -3]$$

and

$$A^2_{\rm ST} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A^2_{\rm SS} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

the stability LMIs corresponding to (11) are feasible with

$$X^i_{\rm T} = 1, \qquad \text{for all } i$$
$$X_{12} = X_{23} = X_{31} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$
$$X_{13} = X_{21} = X_{32} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In fact, after reordering the interconnection signals of the second subsystem, one can rewrite the corresponding interconnected system as a periodic system in the sense of [8] and [19], i.e., as a system made of identical units, identically coupled to each other. One can then use the techniques of [8] to prove stability irrespective of the number of subsystems.

As a consequence of this remark, the distributed controllers that we design in Section IV will not ensure the stability of each subsystem in closed-loop, but only that of the entire closed-loop system. This should be compared with the results of Sections VI-B and C, where we consider systems with nonideal interconnection relations.

IV. Synthesis

A. Problem Formulation

Let G, the plant, be an interconnected system and assume that subsystem G_i now has a control input u_i and a measured output y_i in addition to the signals given in (2), and that

$$\begin{bmatrix} \dot{x}_i(t) \\ w_i(t) \\ z_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} A^i_{\rm TT} & A^i_{\rm TS} & B^i_{{\rm T}d} & B^i_{{\rm T}u} \\ A^i_{\rm ST} & A^i_{\rm SS} & B^i_{{\rm S}d} & B^i_{{\rm S}u} \\ C^i_{{\rm T}z} & C^i_{{\rm S}z} & D^i_{zd} & D^i_{zu} \\ C^i_{{\rm T}y} & C^i_{{\rm S}y} & D^i_{yd} & D^i_{yu} \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \\ d_i(t) \\ u_i(t) \end{bmatrix} \qquad (12)$$

for all $t \geq 0$ and $i = 1 \ldots L$. In the rest of this paper, we will assume that $D^i_{yu} = 0$ for all $i = 1 \ldots L$. This is without loss of generality since one can use loop-shifting (see, e.g., [8] and the references therein) to construct the appropriate controller when this situation is not at hand. We want to find another interconnected system (K, the controller) with adjacency matrix $\mathcal{N}_{\rm K}$ and subsystems K_i, $i = 1 \ldots L$, given by

$$\begin{bmatrix} \dot{x}^{\rm K}_i(t) \\ w^{\rm K}_i(t) \\ u_i(t) \end{bmatrix} = \begin{bmatrix} (A^i_{\rm TT})_{\rm K} & (A^i_{\rm TS})_{\rm K} & (B^i_{\rm T})_{\rm K} \\ (A^i_{\rm ST})_{\rm K} & (A^i_{\rm SS})_{\rm K} & (B^i_{\rm S})_{\rm K} \\ (C^i_{\rm T})_{\rm K} & (C^i_{\rm S})_{\rm K} & D^i_{\rm K} \end{bmatrix} \begin{bmatrix} x^{\rm K}_i(t) \\ v^{\rm K}_i(t) \\ y_i(t) \end{bmatrix} \qquad (13)$$

such that the closed-loop is well-posed, stable and contractive. In addition, we require that $n^{\rm K}_{ij} = 0$ whenever $n_{ij} = 0$. This means that if two subsystems cannot communicate in the plant, this is also the case in closed loop. We are thus aiming for a *distributed* control strategy, as pictured in Fig. 2.

B. Solution

Since the plant and controller are defined over the same graph $\mathcal{G}$, so is the closed-loop system. The entries of its adjacency matrix $\mathcal{N}_{\rm C}$ are given by $n^{\rm C}_{ij} = n_{ij} + n^{\rm K}_{ij}$ for all $i, j = 1 \ldots L$. It is also routine to establish that each subsystem of the closed-loop system admits the state-space matrices

$$\begin{bmatrix} (A^i_{\rm TT})_{\rm C} & (A^i_{\rm TS})_{\rm C} & (B^i_{{\rm T}d})_{\rm C} \\ (A^i_{\rm ST})_{\rm C} & (A^i_{\rm SS})_{\rm C} & (B^i_{{\rm S}d})_{\rm C} \\ (C^i_{{\rm T}z})_{\rm C} & (C^i_{{\rm S}z})_{\rm C} & (D^i_{zd})_{\rm C} \end{bmatrix} = \begin{bmatrix} \mathcal{A}^i & \mathcal{B}^i \\ \mathcal{C}^i & \mathcal{D}^i \end{bmatrix} + \begin{bmatrix} \mathcal{B}^i_{{\rm T}u} \\ \hline \mathcal{B}^i_{{\rm S}u} \\ \hline \mathcal{D}^i_{zu} \end{bmatrix} \Theta_i \left[\mathcal{C}^i_{{\rm T}y} \mid \mathcal{C}^i_{{\rm S}y} \mid \mathcal{D}^i_{yd} \right] \qquad (14)$$

where

$$\mathcal{A}^i := \left[\begin{array}{cc|cc} A^i_{\rm TT} & 0 & A^i_{\rm TS} & 0 \\ 0 & 0 & 0 & 0 \\ \hline A^i_{\rm ST} & 0 & A^i_{\rm SS} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (15)$$

$$\Theta_i := \begin{bmatrix} (A^i_{\rm TT})_{\rm K} & (A^i_{\rm TS})_{\rm K} & (B^i_{\rm T})_{\rm K} \\ (A^i_{\rm ST})_{\rm K} & (A^i_{\rm SS})_{\rm K} & (B^i_{\rm S})_{\rm K} \\ (C^i_{\rm T})_{\rm K} & (C^i_{\rm S})_{\rm K} & D^i_{\rm K} \end{bmatrix} \qquad (16)$$

$$\mathcal{B}^i := \left[\begin{array}{c} B^i_{{\rm T}d} \\ 0 \\ \hline B^i_{{\rm S}d} \\ 0 \end{array}\right]$$

$$\mathcal{C}^i := \left[C^i_{{\rm T}z} \quad 0 \mid C^i_{{\rm S}z} \quad 0 \right] \qquad (17)$$

$$\mathcal{B}^i_{{\rm T}u} := \begin{bmatrix} 0 & 0 & B^i_{{\rm T}u} \\ I & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}^i_{{\rm S}u} := \begin{bmatrix} 0 & 0 & B^i_{{\rm S}u} \\ 0 & I & 0 \end{bmatrix} \qquad (18)$$

$$\mathcal{C}^i_{{\rm T}y} := \begin{bmatrix} 0 & I \\ 0 & 0 \\ C^i_{{\rm T}y} & 0 \end{bmatrix} \quad \mathcal{C}^i_{{\rm S}y} := \begin{bmatrix} 0 & 0 \\ 0 & I \\ C^i_{{\rm S}y} & 0 \end{bmatrix} \qquad (19)$$

$$\mathcal{D}^i_{yd} := \begin{bmatrix} 0 \\ 0 \\ D^i_{yd} \end{bmatrix} \quad \mathcal{D}^i_{zu} := \begin{bmatrix} 0 & 0 & D^i_{zu} \end{bmatrix} \qquad (20)$$

$$\text{and} \quad \mathcal{D}^i := D^i_{zd} \qquad (21)$$

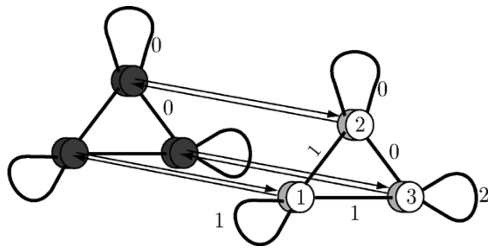

Fig. 2. Control architecture for the system pictured in Fig. 1. Inputs d_i and outputs z_i have been omitted for clarity. The controller's subsystems are represented by the dark elements. As illustrated, the size of each of the controller's edges should be zero whenever the corresponding plant's edge has size zero. We will show that the controller's edges' size is at most three times that of the plant's edges.

for all $i = 1 \dots L$. The state of each subsystem thus has size $m_i^{\mathrm{C}} := m_i^{\mathrm{K}} + m_i$. Before applying our analysis results to the closed-loop, we make the convention that every symmetric matrix $T_{\mathrm{C}}^i \in \mathbb{R}_{\mathbf{S}}^{m_i^c \times m_i^c}$ is partitioned according to the plant's and controller's states as

$$T_{\mathrm{C}}^i = \begin{bmatrix} T_{\mathrm{G}}^i & T_{\mathrm{GK}}^i \\ T_{\mathrm{GK}}^{i\ *} & T_{\mathrm{K}}^i \end{bmatrix} \tag{22}$$

with $T_{\mathrm{G}}^i \in \mathbb{R}_{\mathbf{S}}^{m_i \times m_i}$ and $T_{\mathrm{K}}^i \in \mathbb{R}_{\mathbf{S}}^{m_i^{\mathrm{K}} \times m_i^{\mathrm{K}}}$. Likewise, if $(T_{ij})_{\mathrm{C}}$ belongs to $\mathbb{R}^{n_{ij}^c \times n_{ij}^c}$, we partition it according to the plant's and controller's interconnection signals as

$$(T_{ij})_{\mathrm{C}} = \begin{bmatrix} (T_{ij})_{\mathrm{G}} & (T_{ij})_{\mathrm{GK}} \\ (T_{ij})_{\mathrm{KG}} & (T_{ij})_{\mathrm{K}} \end{bmatrix} \tag{23}$$

with $(T_{ij})_{\mathrm{G}} \in \mathbb{R}^{n_{ij} \times n_{ij}}$ and $(T_{ij})_{\mathrm{K}} \in \mathbb{R}^{n_{ij}^{\mathrm{K}} \times n_{ij}^{\mathrm{K}}}$. Note that if $(T_{ij})_{\mathrm{C}}$ is symmetric, $(T_{ij})_{\mathrm{KG}} = (T_{ij})_{\mathrm{GK}}^*$. Applying Theorem 1 to the closed-loop system then yields the following.

Proposition 2: The closed-loop system is well-posed, stable, and contractive if there exist symmetric matrices $(X_{\mathrm{T}}^i)_{\mathrm{C}} \in \mathbb{R}_{\mathbf{S}}^{m_i^c \times m_i^c}$ and $(X_{ij}^{11})_{\mathrm{C}} \in \mathbb{R}_{\mathbf{S}}^{n_{ij}^c \times n_{ij}^c}$ for all $i, j = 1 \dots L$, and matrices $(X_{ij}^{12})_{\mathrm{C}} \in \mathbb{R}^{n_{ij}^c \times n_{ij}^c}$ for $i \geq j$, with $(X_{ii}^{12})_{\mathrm{C}}$ skew-symmetric such that $(X_{\mathrm{T}}^i)_{\mathrm{C}} > 0$ and LMI (24) is satisfied for all $i = 1 \dots L$

$$\begin{bmatrix} I & 0 & 0 \\ (A_{\mathrm{TT}}^i)_{\mathrm{C}} & (A_{\mathrm{TS}}^i)_{\mathrm{C}} & (B_{\mathrm{T}d}^i)_{\mathrm{C}} \\ \hline (A_{\mathrm{ST}}^i)_{\mathrm{C}} & (A_{\mathrm{SS}}^i)_{\mathrm{C}} & (B_{\mathrm{S}d}^i)_{\mathrm{C}} \\ 0 & I & 0 \\ \hline (C_{\mathrm{T}z}^i)_{\mathrm{C}} & (C_{\mathrm{S}z}^i)_{\mathrm{C}} & (D_{zd}^i)_{\mathrm{C}} \\ 0 & 0 & I \end{bmatrix}^*$$
$$\times \left[\begin{array}{cc|cc|cc} 0 & (X_{\mathrm{T}}^i)_{\mathrm{C}} & 0 & 0 & 0 & 0 \\ (X_{\mathrm{T}}^i)_{\mathrm{C}} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathrm{C}}^* & (Z_i^{22})_{\mathrm{C}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right]$$
$$\times \begin{bmatrix} I & 0 & 0 \\ (A_{\mathrm{TT}}^i)_{\mathrm{C}} & (A_{\mathrm{TS}}^i)_{\mathrm{C}} & (B_{\mathrm{T}d}^i)_{\mathrm{C}} \\ \hline (A_{\mathrm{ST}}^i)_{\mathrm{C}} & (A_{\mathrm{SS}}^i)_{\mathrm{C}} & (B_{\mathrm{S}d}^i)_{\mathrm{C}} \\ 0 & I & 0 \\ \hline (C_{\mathrm{T}z}^i)_{\mathrm{C}} & (C_{\mathrm{S}z}^i)_{\mathrm{C}} & (D_{zd}^i)_{\mathrm{C}} \\ 0 & 0 & I \end{bmatrix} < 0 \tag{24}$$

where

$$\left(Z_i^{11}\right)_{\mathrm{C}} := \begin{bmatrix} \left(Z_i^{11}\right)_{\mathrm{G}} & \left(Z_i^{11}\right)_{\mathrm{GK}} \\ \left(Z_i^{11}\right)_{\mathrm{GK}}^* & \left(Z_i^{11}\right)_{\mathrm{K}} \end{bmatrix}$$
$$\left(Z_i^{12}\right)_{\mathrm{C}} := \begin{bmatrix} \left(Z_i^{12}\right)_{\mathrm{G}} & \left(Z_i^{12}\right)_{\mathrm{GK}} \\ \left(Z_i^{12}\right)_{\mathrm{KG}} & \left(Z_i^{12}\right)_{\mathrm{K}} \end{bmatrix} \quad \text{and}$$
$$\left(Z_i^{22}\right)_{\mathrm{C}} := \begin{bmatrix} \left(Z_i^{22}\right)_{\mathrm{G}} & \left(Z_i^{22}\right)_{\mathrm{GK}} \\ \left(Z_i^{22}\right)_{\mathrm{GK}}^* & \left(Z_i^{22}\right)_{\mathrm{K}} \end{bmatrix}$$

with

$$\left(Z_i^{11}\right)_{\mathrm{G}} := -\operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ij}^{11}\right)_{\mathrm{G}} \tag{25}$$
$$\left(Z_i^{12}\right)_{\mathrm{G}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} \left(X_{ij}^{12}\right)_{\mathrm{G}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} \left(X_{ji}^{12}\right)_{\mathrm{G}}^*\right) \tag{26}$$
$$\left(Z_i^{22}\right)_{\mathrm{G}} := \operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ji}^{11}\right)_{\mathrm{G}} \tag{27}$$
$$\left(Z_i^{11}\right)_{\mathrm{K}} := -\operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ij}^{11}\right)_{\mathrm{K}} \tag{28}$$
$$\left(Z_i^{12}\right)_{\mathrm{K}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} \left(X_{ij}^{12}\right)_{\mathrm{K}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} \left(X_{ji}^{12}\right)_{\mathrm{K}}^*\right) \tag{29}$$
$$\left(Z_i^{22}\right)_{\mathrm{K}} := \operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ji}^{11}\right)_{\mathrm{K}} \tag{30}$$
$$\left(Z_i^{11}\right)_{\mathrm{GK}} := -\operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ij}^{11}\right)_{\mathrm{GK}} \tag{31}$$
$$\left(Z_i^{12}\right)_{\mathrm{GK}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} \left(X_{ij}^{12}\right)_{\mathrm{GK}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} \left(X_{ji}^{12}\right)_{\mathrm{KG}}^*\right) \tag{32}$$
$$\left(Z_i^{22}\right)_{\mathrm{GK}} := \operatorname*{\mathbf{diag}}_{1 \leq j \leq L} \left(X_{ji}^{11}\right)_{\mathrm{GK}} \tag{33}$$
$$\text{and} \quad \left(Z_i^{12}\right)_{\mathrm{KG}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} \left(X_{ij}^{12}\right)_{\mathrm{KG}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} \left(X_{ji}^{12}\right)_{\mathrm{GK}}^*\right). \tag{34}$$

Although the derivation of (24) is similar to that of (11), note that the structure of matrices $(Z_i)_{\mathrm{C}}$ and Z_i are different because the closed-loop system's interconnection signals are the concatenation of those of the plant and the controller.

It follows from (24) and (14) that the obtained conditions would be convex in the scalings $(X_{\mathrm{T}}^i)_{\mathrm{C}}$ and supply rate matrices $(X_{ij})_{\mathrm{C}}$ if the controller's data $\Theta_1, \dots, \Theta_L$ were known. Thus, we would like to find an equivalent formulation that does not involve the decision variables Θ_i, $i = 1 \dots L$. This elimination step requires two preliminary results that are related to the inertia of the synthesis supply rates.

Recall that the inertia of a symmetric matrix M is the triplet $in(M) = (in_-(M), in_0(M), in_+(M))$, where $in_-(M)$, $in_0(M)$ and $in_+(M)$ are the number of negative, zero, and positive eigenvalues of M. The following lemma is proved in [34].

Lemma 1 (Elimination Lemma): Let M be a symmetric matrix with inertia $in(M) = (m, 0, n)$ and $R \in \mathbb{R}^{n \times m}$. The matrix inequality

$$\begin{bmatrix} I_m \\ V^*\Theta U + R \end{bmatrix}^* M \begin{bmatrix} I_m \\ V^*\Theta U + R \end{bmatrix} < 0 \tag{35}$$

in the unstructured unknown Θ has a solution if and only if

$$U_\perp^* \begin{bmatrix} I \\ R \end{bmatrix}^* M \begin{bmatrix} I \\ R \end{bmatrix} U_\perp < 0 \tag{36a}$$

$$V_\perp^* \begin{bmatrix} -R^* \\ I \end{bmatrix}^* M^{-1} \begin{bmatrix} -R^* \\ I \end{bmatrix} V_\perp > 0 \tag{36b}$$

where $U_\perp$ and $V_\perp$ are arbitrary matrices whose columns span the null-space of U and V respectively.

When (36a) and (36b) are satisfied, a solution of (35) can be found as follows. First, introduce nonsingular matrices H and J such that $VH = [V_1 \ \ 0]$, $UJ = [U_1 \ \ 0]$ with V_1 and U_1 of full-column rank and let

$$N := \begin{bmatrix} J^* & 0 \\ 0 & H^{-1} \end{bmatrix} M \begin{bmatrix} J & 0 \\ 0 & (H^*)^{-1} \end{bmatrix}$$

$$Q := \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ T_{21} & 0 \end{bmatrix} \quad S := \begin{bmatrix} 0 \\ I \\ T_{12} \\ T_{22} \end{bmatrix}$$

where the matrix $T := H^* R J$ is partitioned as

$$T =: \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \ T_{11} \text{ being of the size of } V_1^* \Theta U_1.$$

Then, a satisfactory Θ is given by

$$Z = V_1^* \Theta U_1 + T_{11} \tag{37}$$

where $Z = Z_2 Z_1^{-1}$ and Z_1, Z_2 satisfy

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}^* \left(Q^* N Q - Q^* N S (S^* N S)^{-1} S^* N Q \right) \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} < 0.$$

The assumption regarding the inertia of M guarantees that such Z_1 and Z_2 exist. Also, note that solving this inequality only requires to diagonalize the symmetric matrix $(Q^*NQ - Q^*NS(S^*NS)^{-1}S^*NQ)$ in an orthonormal basis and take the columns of $\left[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right]$ to be eigenvectors associated with negative eigenvalues. If Z_1 thus obtained is singular, there always exists $\epsilon > 0$ such that $Z_1' := Z_1 + \epsilon I$ is invertible and Z_1', Z_2 satisfy the matrix inequality. In practice, one should use the largest such ϵ in order to avoid numerical conditioning problems.

Proposition 3: For a set of matrices such that the inequalities of Proposition 2 are satisfied, define matrices $(X_{ij}^{11})_{\mathrm{C}}$ and $(X_{ij}^{12})_{\mathrm{C}}$ as per (23). Then, let $(X_{ij})_{\mathrm{C}} \in \mathbb{R}_{\mathbf{S}}^{2n_{ij}^c \times 2n_{ij}^c}$ be given by

$$(X_{ij})_{\mathrm{C}} := \begin{cases} \begin{bmatrix} (X_{ij}^{11})_{\mathrm{C}} & (X_{ij}^{12})_{\mathrm{C}} \\ (X_{ij}^{12})_{\mathrm{C}}^* & -(X_{ji}^{11})_{\mathrm{C}} \end{bmatrix}, & \text{for } i \geq j \\ \begin{bmatrix} (X_{ij}^{11})_{\mathrm{C}} & -(X_{ji}^{12})_{\mathrm{C}}^* \\ -(X_{ji}^{12})_{\mathrm{C}} & -(X_{ji}^{11})_{\mathrm{C}} \end{bmatrix}, & \text{otherwise.} \end{cases}$$

Then, $\sum_{i,j=1}^L in_+((X_{ij})_{\mathrm{C}}) = \sum_{i,j=1}^L in_-((X_{ij})_{\mathrm{C}}) = \sum_{i=1}^L n_i^{\mathrm{C}}$ if and only if $(X_{ij})_{\mathrm{C}}$ is nonsingular for all $i, j = 1 \dots L$.

Proof: Clearly, $(X_{ij})_{\mathrm{C}} = -(E_{ij})_{\mathrm{C}} (X_{ji})_{\mathrm{C}} (E_{ij})_{\mathrm{C}}$ for all i, j, where

$$(E_{ij})_{\mathrm{C}} := \begin{bmatrix} 0 & I_{n_{ij}^c} \\ I_{n_{ij}^c} & 0 \end{bmatrix}$$

and, consequently, $in_\pm((X_{ji})_{\mathrm{C}}) = in_\mp((X_{ij})_{\mathrm{C}})$, since $(E_{ij})_{\mathrm{C}}$ is a permutation matrix. Hence

$$\begin{aligned} 2\sum_{i,j=1}^L in_+((X_{ij})_{\mathrm{C}}) + \sum_{i,j} in_0((X_{ij})_{\mathrm{C}}) &= \sum_{i,j=1}^L (2n_{ij}^c) \\ &= 2\sum_{i=1}^L n_i^c. \end{aligned}$$

■

With these results in hand, we can show our main synthesis theorem.

Theorem 2: There exist matrices such that (24) is satisfied for all $i = 1 \dots L$, with $n_{ij}^{\mathrm{K}} = 3n_{ij}$ if $i \neq j$ and $n_{ii}^{\mathrm{K}} = n_{ii}$, if and only if there exist symmetric matrices $(X_{\mathrm{T}}^i)_{\mathrm{G}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{G}} \in \mathbb{R}_{\mathbf{S}}^{m_i \times m_i}$ and $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}$ for all $i, j = 1 \dots L$, and matrices $(X_{ij}^{12})_{\mathrm{G}}$, $(Y_{ij}^{12})_{\mathrm{G}} \in \mathbb{R}^{n_{ij} \times n_{ij}}$ for $i \geq j$, with $(X_{ii}^{12})_{\mathrm{G}}$, $(Y_{ii}^{12})_{\mathrm{G}}$ skew-symmetric such that $(X_{\mathrm{T}}^i)_{\mathrm{G}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{G}} > 0$ and LMIs (47)–(49), as shown at the bottom of the next page, are satisfied for all $i = 1 \dots L$. In (47)–(49), N_X^i and N_Y^i are matrices whose columns span the null-space of

$$\begin{bmatrix} C_{\mathrm{T}y}^i & C_{\mathrm{S}y}^i & D_{yd}^i \end{bmatrix} \text{ and } \begin{bmatrix} (B_{\mathrm{T}u}^i)^* & (B_{\mathrm{S}u}^i)^* & (D_{zu}^i)^* \end{bmatrix}$$

respectively, and

$$(Z_i^{11})_{\mathrm{G}} := -\operatorname*{\mathbf{diag}}_{1 \leq j \leq L} (X_{ij}^{11})_{\mathrm{G}} \tag{38}$$

$$(Z_i^{22})_{\mathrm{G}} := \operatorname*{\mathbf{diag}}_{1 \leq j \leq L} (X_{ji}^{11})_{\mathrm{G}} \tag{39}$$

$$(Z_i^{12})_{\mathrm{G}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} (X_{ij}^{12})_{\mathrm{G}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} (X_{ji}^{12})_{\mathrm{G}}^* \right) \tag{40}$$

$$\left(\tilde{Z}_i^{11}\right)_{\mathrm{G}} := -\operatorname*{\mathbf{diag}}_{1 \leq j \leq L} (Y_{ij}^{11})_{\mathrm{G}} \tag{41}$$

$$\left(\tilde{Z}_i^{22}\right)_{\mathrm{G}} := \operatorname*{\mathbf{diag}}_{1 \leq j \leq L} (Y_{ji}^{11})_{\mathrm{G}} \tag{42}$$

$$\left(\tilde{Z}_i^{12}\right)_{\mathrm{G}} := \mathbf{diag}\left(-\operatorname*{\mathbf{diag}}_{1 \leq j \leq i} (Y_{ij}^{12})_{\mathrm{G}}, \operatorname*{\mathbf{diag}}_{i < j \leq L} (Y_{ji}^{12})_{\mathrm{G}}^* \right). \tag{43}$$

Before proving Theorem 2, a remark is in order regarding conditions (47)–(49). The first idea that may come to mind for solving this set of $3L$ matrix inequalities is to regroup them as a single LMI and use conventional semidefinite programming (SDP) packages to solve it. Note, however, that, even for moderate values of L and n_i, this LMI will typically be of large size. Another option is to realize that (47)–(49) are in fact structured, just like the underlying control problem they are derived from, and try to exploit this structure to reduce computational effort and design distributed algorithms that are directly implementable on the graph $\mathcal{G}$. Indeed, the way two different LMIs are coupled through the presence of a common scaling matrix $(X_{ij})_{\mathrm{G}}$ and $(Y_{ij})_{\mathrm{G}}$ completely parallels the spatial structure of the plant. We can think of two ways in which to take advantage of this structure.

— At a high and conceptually simple level, one can exploit the mere convexity of conditions (47)–(49) to decompose them into smaller problems that can be independently and repeatedly solved at each vertex of the graph, while some variables are transmitted along the edges. For the "method of alternating projections" proposed in [20], these variables are simply the coupling scales $(X_{ij})_{\mathrm{G}}$, $(Y_{ij})_{\mathrm{G}}$, while each subproblem consists in determining the projection of a point onto the feasible set of an LMI. We are currently working on another such decomposition method using subgradients [5], [21].

— On a lower level, one could start with a classical interior-point method for SDP and use the particular structure of the problem only to calculate Newton steps in a distributed fashion. This should result in more efficient algorithms than the higher level ones. This is also ongoing work.

Proof: • "Only if:" First, we note that (24) is satisfied for all i if and only if the following holds:

$$T^*\Lambda T < 0 \tag{44}$$

where

$$\Lambda = \operatorname*{\mathbf{diag}}_{1\leq i\leq L} \Lambda_i \quad T = \operatorname*{\mathbf{diag}}_{1\leq i\leq L} T_i$$

and, for all i

$$\Lambda_i = \left[\begin{array}{cc|cc|cc} 0 & (X_{\mathrm{T}}^i)_{\mathrm{C}} & 0 & 0 & 0 & 0 \\ (X_{\mathrm{T}}^i)_{\mathrm{C}} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathrm{C}}^* & (Z_i^{22})_{\mathrm{C}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right]$$

$$T_i = \left[\begin{array}{ccc} I & 0 & 0 \\ (A_{\mathrm{TT}}^i)_{\mathrm{C}} & (A_{\mathrm{TS}}^i)_{\mathrm{C}} & (B_{\mathrm{T}d}^i)_{\mathrm{C}} \\ \hline (A_{\mathrm{ST}}^i)_{\mathrm{C}} & (A_{\mathrm{SS}}^i)_{\mathrm{C}} & (B_{\mathrm{S}d}^i)_{\mathrm{C}} \\ 0 & I & 0 \\ \hline (C_{\mathrm{T}z}^i)_{\mathrm{C}} & (C_{\mathrm{S}z}^i)_{\mathrm{C}} & (D_{zd}^i)_{\mathrm{C}} \\ 0 & 0 & I \end{array}\right]. \tag{45}$$

It is easy but a little tedious to see that, for all $i = 1 \dots L$, there exists a permutation matrix Π_i such that

$$\begin{bmatrix} (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} \\ (Z_i^{12})_{\mathrm{C}}^* & (Z_i^{22})_{\mathrm{C}} \end{bmatrix} = -\Pi_i^{-1}\mathbf{diag}((X_{i1})_{\mathrm{C}}, \dots, (X_{iL})_{\mathrm{C}})\Pi_i. \tag{46}$$

$$\left(N_X^i\right)^* \left[\begin{array}{ccc} I & 0 & 0 \\ A_{\mathrm{TT}}^i & A_{\mathrm{TS}}^i & B_{\mathrm{T}d}^i \\ \hline A_{\mathrm{ST}}^i & A_{\mathrm{SS}}^i & B_{\mathrm{S}d}^i \\ 0 & I & 0 \\ \hline C_{\mathrm{T}z}^i & C_{\mathrm{S}z}^i & D_{zd}^i \\ 0 & 0 & I \end{array}\right]^* \left[\begin{array}{cc|cc|cc} 0 & (X_{\mathrm{T}}^i)_{\mathrm{G}} & 0 & 0 & 0 & 0 \\ (X_{\mathrm{T}}^i)_{\mathrm{G}} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (Z_i^{11})_{\mathrm{G}} & (Z_i^{12})_{\mathrm{G}} & 0 & 0 \\ 0 & 0 & (Z_i^{12})_{\mathrm{G}}^* & (Z_i^{22})_{\mathrm{G}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right]$$
$$\times \left[\begin{array}{ccc} I & 0 & 0 \\ A_{\mathrm{TT}}^i & A_{\mathrm{TS}}^i & B_{\mathrm{T}d}^i \\ \hline A_{\mathrm{ST}}^i & A_{\mathrm{SS}}^i & B_{\mathrm{S}d}^i \\ 0 & I & 0 \\ \hline C_{\mathrm{T}z}^i & C_{\mathrm{S}z}^i & D_{zd}^i \\ 0 & 0 & I \end{array}\right] N_X^i < 0 \tag{47}$$

$$\left(N_Y^i\right)^* \left[\begin{array}{ccc} (A_{\mathrm{TT}}^i)^* & (A_{\mathrm{ST}}^i)^* & (C_{\mathrm{T}z}^i)^* \\ -I & 0 & 0 \\ \hline 0 & -I & 0 \\ (A_{\mathrm{TS}}^i)^* & (A_{\mathrm{SS}}^i)^* & (C_{\mathrm{S}z}^i)^* \\ \hline 0 & 0 & -I \\ (B_{\mathrm{T}d}^i)^* & (B_{\mathrm{S}d}^i)^* & (D_{zd}^i)^* \end{array}\right]^* \left[\begin{array}{cc|cc|cc} 0 & (Y_{\mathrm{T}}^i)_{\mathrm{G}} & 0 & 0 & 0 & 0 \\ (Y_{\mathrm{T}}^i)_{\mathrm{G}} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \left(\tilde{Z}_i^{11}\right)_{\mathrm{G}} & \left(\tilde{Z}_i^{12}\right)_{\mathrm{G}} & 0 & 0 \\ 0 & 0 & \left(\tilde{Z}_i^{12}\right)_{\mathrm{G}}^* & \left(\tilde{Z}_i^{22}\right)_{\mathrm{G}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right]$$
$$\times \left[\begin{array}{ccc} (A_{\mathrm{TT}}^i)^* & (A_{\mathrm{ST}}^i)^* & (C_{\mathrm{T}z}^i)^* \\ -I & 0 & 0 \\ \hline 0 & -I & 0 \\ (A_{\mathrm{TS}}^i)^* & (A_{\mathrm{SS}}^i)^* & (C_{\mathrm{S}z}^i)^* \\ \hline 0 & 0 & -I \\ (B_{\mathrm{T}d}^i)^* & (B_{\mathrm{S}d}^i)^* & (D_{zd}^i)^* \end{array}\right] N_Y^i > 0 \tag{48}$$

$$\begin{bmatrix} (X_{\mathrm{T}}^i)_{\mathrm{G}} & I \\ I & (Y_{\mathrm{T}}^i)_{\mathrm{G}} \end{bmatrix} \geq 0 \tag{49}$$

Because Π_i is a permutation, it is an orthogonal matrix and, thus

$$in\left(\begin{bmatrix} (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} \\ (Z_i^{12})_{\mathrm{C}}^* & (Z_i^{22})_{\mathrm{C}} \end{bmatrix}\right) = -\sum_{j=1}^{L} in\left((X_{ij})_{\mathrm{C}}\right).$$

Hence

$$in(\Lambda) = \sum_{i=1}^{L} in\left(\begin{bmatrix} 0 & (X_{\mathrm{T}}^i)_{\mathrm{C}} \\ (X_{\mathrm{T}}^i)_{\mathrm{C}} & 0 \end{bmatrix}\right) - \sum_{i,j=1}^{L} in\left((X_{ij})_{\mathrm{C}}\right) + \sum_{i=1}^{L}(p_i, 0, q_i)$$

and, according to Proposition 3

$$in(\Lambda) = \left(\sum_{i=1}^{L}(m_i^c + n_i^c + p_i), 0, \sum_{i=1}^{L}(m_i^c + n_i^c + q_i)\right)$$

if all $(X_{ij})_{\mathrm{C}}$'s are invertible. If this situation is not at hand, we can perturb the $(X_{ij}^{11})_{\mathrm{C}}$ and/or $(X_{ij}^{12})_{\mathrm{C}}$ blocks of the singular $(X_{ij})_{\mathrm{C}}$'s by a small amount to obtain an invertible one and inequality (24) will remain satisfied. Permuting the rows of T and the rows and lines of Λ, we can put (44) in the form of (35) with M similar to Λ and

$$R = \operatorname*{\mathbf{diag}}_{1\le i\le L}\begin{bmatrix} \mathcal{A}^i & \mathcal{B}^i \\ \mathcal{C}^i & \mathcal{D}^i \end{bmatrix} \quad V^* = \operatorname*{\mathbf{diag}}_{1\le i\le L}\begin{bmatrix} \mathcal{B}_{\mathrm{T}u}^i \\ \hline \mathcal{B}_{\mathrm{S}u}^i \\ \hline \mathcal{D}_{zu}^i \end{bmatrix}$$
$$U = \operatorname*{\mathbf{diag}}_{1\le i\le L}\left[\mathcal{C}_{\mathrm{T}y}^i \mid \mathcal{C}_{\mathrm{S}y}^i \mid \mathcal{D}_{yd}^i\right] \quad \Theta = \operatorname*{\mathbf{diag}}_{1\le i\le L}\Theta_i.$$

M has the same inertia as Λ, namely $(\sum_{i=1}^{L}(m_i^{\mathrm{C}} + n_i^{\mathrm{C}} + p_i), 0, \sum_{i=1}^{L}(m_i^{\mathrm{C}} + n_i^{\mathrm{C}} + q_i))$ and we can thus apply Lemma 1 to eliminate the controller's data and obtain (36).

Permuting the lines and columns back gives inequalities involving block-diagonal matrices again, which can in turn be decoupled into $2L$ matrix inequalities. Note that, because of (46), Λ^{-1} is given by $\Lambda^{-1} = \operatorname*{\mathbf{diag}}_{1\le i\le L}\Lambda_i^{-1}$ where

$$\Lambda_i^{-1} = \left[\begin{array}{cc|cc|cc} 0 & (X_{\mathrm{T}}^i)_{\mathrm{C}}^{-1} & 0 & 0 & 0 & 0 \\ (X_{\mathrm{T}}^i)_{\mathrm{C}}^{-1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \left(\tilde{Z}_i^{11}\right)_{\mathrm{C}} & \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}} & 0 & 0 \\ 0 & 0 & \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}}^* & \left(\tilde{Z}_i^{22}\right)_{\mathrm{C}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \end{array}\right] \tag{50}$$

for each i and

$$\begin{bmatrix} \left(\tilde{Z}_i^{11}\right)_{\mathrm{C}} & \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}} \\ \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}}^* & \left(\tilde{Z}_i^{22}\right)_{\mathrm{C}} \end{bmatrix} := \begin{bmatrix} (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} \\ (Z_i^{12})_{\mathrm{C}}^* & (Z_i^{22})_{\mathrm{C}} \end{bmatrix}^{-1}.$$

One then obtains LMIs (47) and (48) by using the particular form of $V_\perp$ and $U_\perp$ and by taking $(X_{\mathrm{T}}^i)_{\mathrm{G}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{G}}$, $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}}$, $(X_{ij}^{12})_{\mathrm{G}}$, and $(Y_{ij}^{12})_{\mathrm{G}}$ to be the top-left blocks of $(X_{\mathrm{T}}^i)_{\mathrm{C}}$, $(X_{\mathrm{T}}^i)_{\mathrm{C}}^{-1}$, $(X_{ij}^{11})_{\mathrm{C}}$, $(X_{ij}^{11})_{\mathrm{C}}^{-1}$, $(X_{ij}^{12})_{\mathrm{C}}$, $(X_{ij}^{12})_{\mathrm{C}}^{-1}$, respectively. Inequality (49) is then satisfied and $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}}$ are symmetric for all i, j since $(X_{ij}^{11})_{\mathrm{C}}$ is. Likewise $(X_{ii}^{12})_{\mathrm{G}}$ and $(Y_{ii}^{12})_{\mathrm{G}}$ are skew-symmetric for all $i = 1 \ldots L$.

• "If:" Let LMIs (47)–(49) be feasible. We want to show that there exist *design scales* $(X_{ij}^{11})_{\mathrm{GK}}$, $(X_{ij}^{11})_{\mathrm{K}}$, $(Y_{ij}^{11})_{\mathrm{GK}}$, $(Y_{ij}^{11})_{\mathrm{K}}$, $(X_{ij}^{12})_{\mathrm{GK}}$, $(X_{ij}^{12})_{\mathrm{K}}$, $(Y_{ij}^{12})_{\mathrm{GK}}$, $(Y_{ij}^{12})_{\mathrm{K}}$, $(X_{ij}^{12})_{\mathrm{KG}}$, $(Y_{ij}^{12})_{\mathrm{KG}}$, $(X_{\mathrm{T}}^i)_{\mathrm{GK}}$, $(X_{\mathrm{T}}^i)_{\mathrm{K}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{GK}}$, and $(Y_{\mathrm{T}}^i)_{\mathrm{K}}$ such that (44) and (50) are satisfied for some $(X_{\mathrm{T}}^i)_{\mathrm{C}}$ and $(Z_i)_{\mathrm{C}}$ as given in Proposition 2. It will then follow that (24) holds.

We do this by first rewriting LMIs (47) and (48) in the form of (36) for M, R, V, and U as in the proof of necessity. At this point, one would want to use the necessity part of Lemma 1 to establish the existence of a controller. This, however, requires some care: even if matrix M satisfies the inertia assumption of the lemma, one can only conclude that there exists an *unstructured* matrix Θ such that the corresponding version of (36a) is satisfied. This is not enough, as we want a solution of the form $\Theta = \operatorname*{\mathbf{diag}}_{1\le i\le L}\Theta_i$.

This leads us to restrict our choice of design scales so that *each* Λ_i has the right inertia. In this case, the existence of each Θ_i can be established by applying Lemma 1 to each LMI

$$T_i^*\Lambda_i T_i < 0$$

separately. Recalling the expression of Λ_i, we see that it is enough to show that if LMIs (47)–(49) are feasible, the design scales can be chosen such that

i)

$$\begin{bmatrix} (X_{\mathrm{T}}^i)_{\mathrm{G}} & (X_{\mathrm{T}}^i)_{\mathrm{GK}} \\ (X_{\mathrm{T}}^i)_{\mathrm{GK}}^* & (X_{\mathrm{T}}^i)_{\mathrm{K}} \end{bmatrix}^{-1} = \begin{bmatrix} (Y_{\mathrm{T}}^i)_{\mathrm{G}} & (Y_{\mathrm{T}}^i)_{\mathrm{GK}} \\ (Y_{\mathrm{T}}^i)_{\mathrm{GK}}^* & (Y_{\mathrm{T}}^i)_{\mathrm{K}} \end{bmatrix} > 0$$

for all i;

ii)

$$(X_{ij})_{\mathrm{C}}^{-1} := \begin{bmatrix} \left(X_{ij}^{11}\right)_{\mathrm{C}} & \left(X_{ij}^{12}\right)_{\mathrm{C}} \\ \left(X_{ij}^{12}\right)_{\mathrm{C}}^* & -\left(X_{ji}^{11}\right)_{\mathrm{C}} \end{bmatrix}^{-1} = \begin{bmatrix} \left(Y_{ij}^{11}\right)_{\mathrm{C}} & \left(Y_{ij}^{12}\right)_{\mathrm{C}} \\ \left(Y_{ij}^{12}\right)_{\mathrm{C}}^* & -\left(Y_{ji}^{11}\right)_{\mathrm{C}} \end{bmatrix}$$

for all $i \ge j$;

iii) $in((X_{ij})_{\mathrm{C}}) = (n_{ij}^{\mathrm{C}}, 0, n_{ij}^{\mathrm{C}})$ for all $i \ge j$;

iv) $(X_{ii}^{12})_{\mathrm{C}}$ is skew-symmetric for all i.

It is classical (see, e.g., [15]) that (49) guarantees that condition i) can be met. We now prove iv) and ii) when $i = j$. For any fixed i, let

$$S_1 := \begin{bmatrix} \left(X_{ii}^{11}\right)_{\mathrm{G}} & \left(X_{ii}^{12}\right)_{\mathrm{G}} \\ \left(X_{ii}^{12}\right)_{\mathrm{G}}^* & -\left(X_{ii}^{11}\right)_{\mathrm{G}} \end{bmatrix} \quad R_1 := \begin{bmatrix} \left(Y_{ii}^{11}\right)_{\mathrm{G}} & \left(Y_{ii}^{12}\right)_{\mathrm{G}} \\ \left(Y_{ii}^{12}\right)_{\mathrm{G}}^* & -\left(Y_{ii}^{11}\right)_{\mathrm{G}} \end{bmatrix}.$$

We can always assume that R_1 is invertible, after a small perturbation if necessary. Since $(X_{ii}^{12})_{\mathrm{G}}$ and $(Y_{ii}^{12})_{\mathrm{G}}$ are skew-symmetric by assumption, $E_{ii}S_1E_{ii} = -S_1$ and $E_{ii}R_1E_{ii} = -R_1$. Take

$$S_2 = \begin{bmatrix} I_{n_{ii}} & 0 \\ 0 & -I_{n_{ii}} \end{bmatrix} \quad S_3 = S_2\left(S_1 - R_1^{-1}\right)^{-1}S_2 \tag{51}$$

then easy algebra shows that

$$S^{-1} := \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}^{-1} = \begin{bmatrix} R_1 & -R_1\left(S_1 - R_1^{-1}\right)S_2 \\ -S_2\left(S_1 - R_1^{-1}\right)R_1 & S_2S_1R_1\left(S_1 - R_1^{-1}\right)S_2 \end{bmatrix}.$$

Furthermore, $E_{ii}S_2E_{ii} = -S_2$, $E_{ii}S_3E_{ii} = -S_3$ and S_3 is symmetric. One can thus partition these matrices as

$$S_2 =: \begin{bmatrix} (X_{ii}^{11})_{\mathrm{GK}} & (X_{ii}^{12})_{\mathrm{GK}} \\ (X_{ii}^{12})_{\mathrm{KG}}^{*} & -(X_{ii}^{11})_{\mathrm{GK}} \end{bmatrix}$$
$$S_3 =: \begin{bmatrix} (X_{ii}^{11})_{\mathrm{K}} & (X_{ii}^{12})_{\mathrm{K}} \\ (X_{ii}^{12})_{\mathrm{K}}^{*} & -(X_{ii}^{11})_{\mathrm{K}} \end{bmatrix}$$

with $(X_{ii}^{12})_{\mathrm{K}}$ skew-symmetric and $(X_{ii}^{12})_{\mathrm{KG}}^{*} = (X_{ii}^{12})_{\mathrm{GK}} = 0$ to satisfy conditions iv) and ii), since

$$(X_{ii}^{12})_{\mathrm{C}} =: \begin{bmatrix} (X_{ii}^{12})_{\mathrm{G}} & (X_{ii}^{12})_{\mathrm{GK}} \\ (X_{ii}^{12})_{\mathrm{KG}} & (X_{ii}^{12})_{\mathrm{K}} \end{bmatrix}$$

and S is similar to $(X_{ii})_{\mathrm{C}}$ via a permutation. Finally, note that S, as defined previously, satisfies

$$\begin{bmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{bmatrix} S = -S \begin{bmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{bmatrix}.$$

Hence, $in((X_{ii})_{\mathrm{C}}) = (2n_{ii}, 0, 2n_{ii})$, which is condition iii) when $i = j$, taking $n_{ii}^{\mathrm{K}} = n_{ii}$.

Let us now prove ii) and iii) for $i > j$. In this case, let S_1, $R_1 \in \mathbb{R}^{2n_{ij} \times 2n_{ij}}$ be defined as

$$S_1 := \begin{bmatrix} (X_{ij}^{11})_{\mathrm{G}} & (X_{ij}^{12})_{\mathrm{G}} \\ (X_{ij}^{12})_{\mathrm{G}}^{*} & -(X_{ji}^{11})_{\mathrm{G}} \end{bmatrix} \quad R_1 := \begin{bmatrix} (Y_{ij}^{11})_{\mathrm{G}} & (Y_{ij}^{12})_{\mathrm{G}} \\ (Y_{ij}^{12})_{\mathrm{G}}^{*} & -(Y_{ji}^{11})_{\mathrm{G}} \end{bmatrix}.$$

After perturbation if necessary, we can assume S_1 and R_1 nonsingular. It is proved in [12, Lemma 18] that one can then find symmetric S_3, $R_3 \in \mathbb{R}_{\mathbf{S}}^{m \times m}$ and S_2, $R_2 \in \mathbb{R}^{2n_{ij} \times m}$ such that

$$\begin{bmatrix} S_1 & S_2 \\ S_2^{*} & S_3 \end{bmatrix}^{-1} = \begin{bmatrix} R_1 & R_2 \\ R_2^{*} & R_3 \end{bmatrix}$$
$$\text{and } in\left(\begin{bmatrix} S_1 & S_2 \\ S_2^{*} & S_3 \end{bmatrix}\right) = (i^-, 0, i^+)$$

for some integers m, i^+, i^- satisfying $2n_{ij} + m = i^+ + i^-$ if and only if

$$in_+\left(\begin{bmatrix} S_1 & I \\ I & S_3 \end{bmatrix}\right) \le i^+; \quad in_-\left(\begin{bmatrix} S_1 & I \\ I & S_3 \end{bmatrix}\right) \le i^-. \quad (52)$$

If we choose $i^+ = i^- = 4n_{ij}$ and $m = 6n_{ij}$, (52) is always trivially satisfied and, thus, partitioning S_2, S_3 as

$$S_2 =: \begin{bmatrix} (X_{ij}^{11})_{\mathrm{GK}} & (X_{ij}^{12})_{\mathrm{GK}} \\ (X_{ij}^{12})_{\mathrm{KG}}^{*} & -(X_{ji}^{11})_{\mathrm{GK}} \end{bmatrix} \quad S_3 =: \begin{bmatrix} (X_{ij}^{11})_{\mathrm{K}} & (X_{ij}^{12})_{\mathrm{K}} \\ (X_{ij}^{12})_{\mathrm{K}}^{*} & -(X_{ji}^{11})_{\mathrm{K}} \end{bmatrix}$$

and similarly for R_2 and R_3, we see that we can satisfy (ii) and (iii), provided we choose $n_{ij}^{\mathrm{K}} = 3n_{ij}$ for $i \neq j$. ■

Note that what made it possible to obtain convex conditions when applying Lemma 1 and Proposition 3 to (24) was that we had enough free independent parameters to satisfy

$$\begin{bmatrix} \left(\tilde{Z}_i^{11}\right)_{\mathrm{C}} & \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}} \\ \left(\tilde{Z}_i^{12}\right)_{\mathrm{C}}^{*} & \left(\tilde{Z}_i^{22}\right)_{\mathrm{C}} \end{bmatrix} = \begin{bmatrix} (Z_i^{11})_{\mathrm{C}} & (Z_i^{12})_{\mathrm{C}} \\ (Z_i^{12})_{\mathrm{C}}^{*} & (Z_i^{22})_{\mathrm{C}} \end{bmatrix}^{-1}, \text{ for all } i.$$

This would not have been the case if there had been an additional constraint specifying neutrality of the closed-loop system's interconnection; a situation we avoided by restricting ourselves to scales satisfying property (8).

C. *Controller Reconstruction and Properties*

When LMIs (47)–(49) are feasible, one can construct the closed-loop scales $(X_{ij})_{\mathrm{C}}$ and $(X_{\mathrm{T}}^{i})_{\mathrm{C}}$ as indicated in the proof of Theorem 2 and then solve for the controller's data using the construction outlined in Lemma 1. Note that the controller so obtained is not necessarily well-posed, as is also the case in LPV synthesis [1]. However, if the plant's subsystems are such that

$$C_{\mathrm{S}y}^{i} = 0 \quad D_{yu}^{i} = 0, \qquad \text{for all } i = 1, \ldots, L. \qquad (53)$$

Equations (14) tell us that $(A_{\mathrm{SS}}^{i})_{\mathrm{C}}$ has the form

$$\left(A_{\mathrm{SS}}^{i}\right)_{\mathrm{C}} = \begin{bmatrix} A_{\mathrm{SS}}^{i} & B_{\mathrm{S}u}^{i}\left(C_{\mathrm{S}}^{i}\right)_{\mathrm{K}} \\ 0 & \left(A_{\mathrm{SS}}^{i}\right)_{\mathrm{K}} \end{bmatrix}.$$

Using this particular triangular structure and interpreting the definition (1) of well-posedness as requiring that some matrix be invertible, it is easy to show that the closed loop is well-posed if and only if both the plant and the controller are well-posed. A similar result is obtained with the assumption

$$B_{\mathrm{S}u}^{i} = 0 \quad D_{yu}^{i} = 0, \qquad \text{for all } i = 1, \ldots, L.$$

The physical meaning of conditions (53) is that neither the interconnection signals nor the actuator signal of a subsystem can directly affect the sensor signals. They must go through some temporal dynamics first. If this situation is not already at hand, one can enforce conditions (53) by inserting low-pass filters between every subsystem and its corresponding controller's subsystem. The presence of such filters is justified in practice.

V. Numerical Examples

In this section, we present numerical results that illustrate the tradeoff between performance and delocalization, when one uses a distributed controller. The files used can be downloaded from [22].

We generated four sets of 100 systems. Each of these interconnected systems was made of 15 subsystems with the following properties.

- The dimensions of $x_i(t)$, $d_i(t)$, $u_i(t)$, $z_i(t)$, and $y_i(t)$ was one.
- Subsystem G_i was interconnected to subsystem G_j with probability 0.1, 0.3, 0.5, or 1 (each set corresponds to a different probability of interconnection). When subsystem G_i was interconnected to subsystem G_j, $n_{ij} = n_{ji}$ was set to 1, otherwise it was set to 0.
- For each subsystem, the following were set to zero: A_{SS}^{i}, $B_{\mathrm{S}d}^{i}$, $B_{\mathrm{S}u}^{i}$, $C_{\mathrm{S}z}^{i}$, and $C_{\mathrm{S}y}^{i}$. In other words, systems were interconnected only through their states.
- The following were generated randomly from a normal distribution with zero mean and unit variance: A_{TT}^{i}, $B_{\mathrm{T}d}^{i}$, $C_{\mathrm{T}z}^{i}$, and D_{zd}^{i}.
- The following were generated randomly from a uniform distribution on the set $[-1.1, -0.1] \cup [0.1, 1.1]$: $B_{\mathrm{T}u}^{i}$, $C_{\mathrm{T}y}^{i}$, D_{zu}^{i}, D_{yd}^{i}. This was done to ensure that each subsystem was controllable and observable, and that the resulting $\mathcal{H}_\infty$ control problem was not singular [11].

For each system, we designed a distributed, interconnected control system using the algorithms presented in this paper;

we uniformly scaled each z_i and d_i until the synthesis LMIs in Section IV were within 1 percent of not being feasible. Then, we synthesized the corresponding distributed controller and calculated the $\mathcal{H}_\infty$-norm of the closed-loop system. Finally, we designed the optimal, centralized $\mathcal{H}_\infty$ controller using the standard formulas in [15].

The results may be found in Fig. 3. The plots have the following meaning.

In Fig. 3(a), a point (x,y) belongs to the graph if and only if the fraction of problems x for which the ratio of the centralized $\mathcal{H}_\infty$ performance to the distributed $\mathcal{H}_\infty$ performance is greater than or equal to y. For example, for a probability of interconnection of 10 percent, the ratio of the centralized $\mathcal{H}_\infty$ norm to the distributed $\mathcal{H}_\infty$ norm is greater than 0.7 in 60% of the cases.

Likewise, in Fig. 3(b), a point (x,y) belongs to the graph if and only if the fraction of problems x for which the ratio of the distributed $\mathcal{H}_\infty$ performance to the best upper bound guaranteed by the synthesis LMIs is greater than or equal to y. Note that, for both Figures, the point (0,1) must be on all graphs: For none of the problems is the distributed $\mathcal{H}_\infty$ norm better than the centralized $\mathcal{H}_\infty$ norm. Similarly, the distributed $\mathcal{H}_\infty$ norm can never be better than the best guaranteed upper bound.

Even though recent results on the gap between the complex structured singular value and its upper bound [38] make us expect that the upper bound guaranteed by LMIs (47)–(49) can also be arbitrarily conservative in the worst case, Fig. 3(b) suggests that, for a typical, randomly chosen interconnected system, this bound may not be so conservative. For example, for a rather densely interconnected graph with interconnection probability of 50%, the conservatism of the upper bound is less than 20% in more than 85% of the problems.

This is essentially saying that the performance degradation plotted in Fig. 3(a) really captures the inherent gap between distributed and centralized control, and is not a result of the conservatism of our techniques.

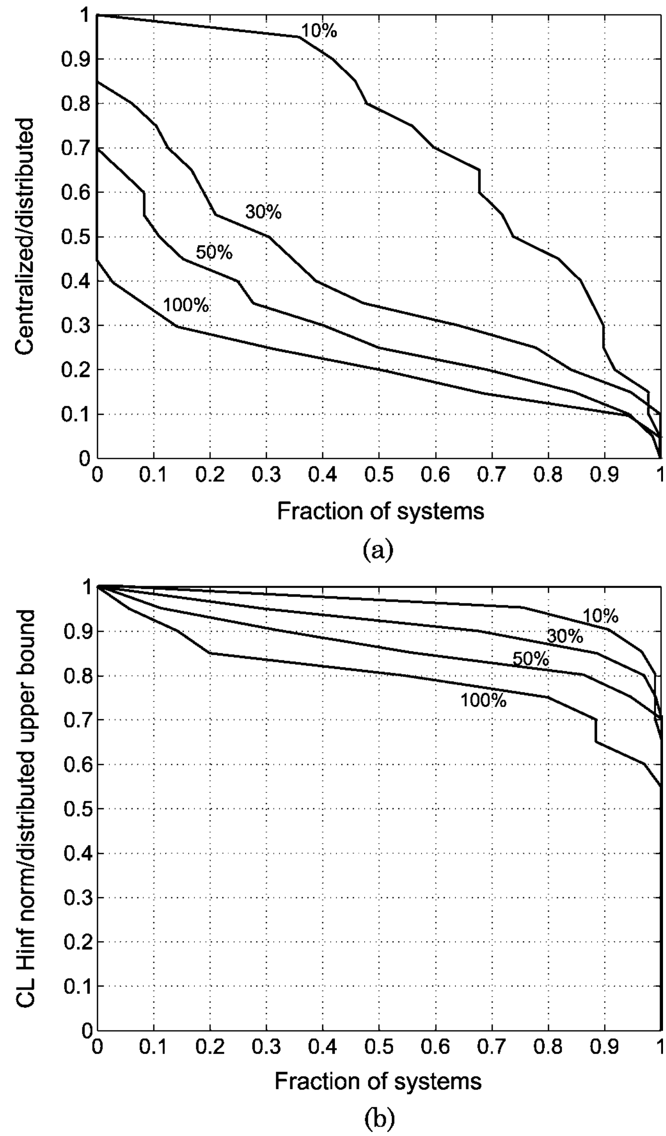

Fig. 3. Summary of synthesis results for 100 randomly generated systems. (a) Comparison between distributed and centralized performance, for the indicated probability of interconnection. (b) Gap between distributed $\mathcal{H}_\infty$ norm and best upper bound.

VI. Nonideal Interconnections

So far, we have only been interested in the design of distributed controllers for systems with ideal interconnection relations, as defined in Section II-C. However, in many cases of practical interest, interconnections are not ideal in the sense that information is degraded while passing from one subsystem to the other, so that relation (4) does not hold. For example, when each edge stands for an actual communication channel and signals between subsystem i and subsystem j can be thought of as being physically transmitted along this channel, one can reasonably expect that v_{ji} and w_{ij} are not rigorously equal but are, at the very least, related by a delay. A notable exception, however, is given by semidiscretized partial differential equations, where blocks represent finite elements and interactions between them correspond to pure mathematical coupling (see [8] for an example of how to put such discretized partial differential equations in the framework of this paper).

It is then natural to ask whether the results of the previous sections hold for or can be adapted to systems with nonideal interconnections. As one might expect, the answer depends crucially on the type of nonidealities allowed in the description of the system. The following example, in particular, shows that the analysis conditions of Theorem 1 (and, thus, Theorem 2) must be modified when dealing with interconnections subject to delays.

Consider the interconnected system of Fig. 4, where the block has the following transfer function between v_{11} and w_{11}:

$$F(s) = \frac{2s + 0.5}{s - 0.25}.$$

Taking $A_{\mathrm{TT}}^1 = 0.25$, $A_{\mathrm{TS}}^1 = 1$, $A_{\mathrm{ST}}^1 = 1$, and $A_{\mathrm{SS}}^1 = 2$ as a representation of the subsystem, it is easy to see that the conditions (11) for well-posedness and stability (there are no disturbances d or outputs z) are satisfied with $X_T^1 = 1$, $X_{11}^{11} = 1$, and $X_{11}^{12} = 0$. However, the Nyquist plot of $F(s)$ clearly demonstrates that the distributed system of Fig. 4 is unstable, when any small delay is introduced in the positive feedback loop between w_{11} and v_{11}.

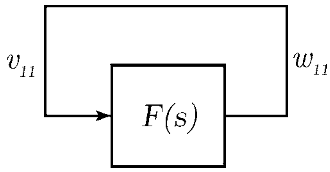

Fig. 4. Simple interconnected system. See text for details.

A. IQC Model of Directed Interconnection Relations

Motivated by the preceding discussion and example, we now turn our attention to a new class of interconnected systems with *directed* interconnections. For these systems, interconnection relations (4) are replaced by more general matrix inequality constraints on signals v_i and w_i. More precisely, the subspace $\mathcal{S}_{\mathcal{I}}$ of Section II-C is replaced by

$$\mathcal{S}_{\text{direct}} = \left\{ (w,v) \in \mathcal{L}_2^{\mathcal{N}} \times \mathcal{L}_2^{\mathcal{N}} : \int_0^\infty v_{ji}(t) v_{ji}^*(t) dt \right. \\ \left. \leq \int_0^\infty w_{ij}(t) w_{ij}^*(t) dt, \qquad \text{for all } i,j = 1 \dots L \right\}. \quad (54)$$

This type of interconnection subspace, defined as a subset of $\mathcal{L}_2^{\mathcal{N}} \times \mathcal{L}_2^{\mathcal{N}}$ instead of $\mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}}$, forces us to adopt an operator framework and redefine well-posedness, stability and contractiveness (see also [26]). We start with a lemma, proved in [29], that characterizes $\mathcal{S}_{\text{direct}}$ in terms of operators.

Lemma 2: Let ν and β be signals in $\mathcal{L}_2^k$. Then, there exists a contractive operator $\delta : \mathcal{L}_2^1 \to \mathcal{L}_2^1 (\|\delta\| \leq 1)$ such that $\nu = (\delta I_k)\beta$ if and only if $\int_0^\infty \nu(t)\nu^*(t)dt \leq \int_0^\infty \beta(t)\beta^*(t)dt$.

This shows that the IQC in (54) can be represented by

$$v_{ij} = \left(\delta_{ji} I_{n_{ji}}\right) w_{ji}; \quad \|\delta_{ji}\| \leq 1, \qquad \text{for all } i,j = 1 \dots L \quad (55)$$

for some choice of contractive operators $\delta_{ij} : \mathcal{L}_2^1 \to \mathcal{L}_2^1$. In turn, the interconnection relation specified by $\mathcal{S}_{\text{direct}}$ captures a wide range of nonideal interconnection channels, such as arbitrary delays, arbitrary attenuations, and low-pass filtering; all of which are present to some extent in physical communication channels. The adjective "directed" used for such interconnections stems from the fact that some of these nonidealities are causal with noncausal inverses, which imposes a direction for information transfer between subsystem [42].

Now, if we also look at the subsystems (2) of a directed interconnected system G in the input–output form

$$\begin{bmatrix} w_i \\ z_i \end{bmatrix} = \begin{bmatrix} G_i^{wv} & G_i^{wd} \\ G_i^{zv} & G_i^{zd} \end{bmatrix} \begin{bmatrix} v_i \\ d_i \end{bmatrix}, \qquad \text{for all } i = 1 \dots L \\ = G_i \begin{bmatrix} v_i \\ d_i \end{bmatrix} \quad (56)$$

it is possible to represent its input–output map as the linear fractional transformation (LFT) [28] pictured in the block diagram of Fig. 5(a). In this diagram, $\mathbf{G}$ denotes the bounded LTI operator mapping (v,d) to (w',z) where

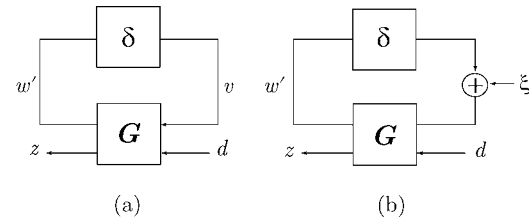

Fig. 5. (a) Linear fractional transformation representation of interconnected system. (b) System with external signal injected for definition of stability.

$w'(t) := \mathbf{cat}_i(\mathbf{cat}_j w_{ji}(t)) \in \mathbb{R}^{\mathcal{N}}$ for all $t \geq 0$, while, according to Lemma 2, δ is any operator of the form

$$\delta := \mathbf{diag}_i\left(\mathbf{diag}_j\left(\delta_{ji} I_{n_{ji}}\right)\right) \quad \|\delta_{ij}\| \leq 1, \text{ for all } i,j = 1 \dots L. \quad (57)$$

Following [13], we can then define stability as the property that all internal signals are bounded when an external signal is injected into the exposed interconnections between the subsystems, as shown in Fig. 5(b). More rigorously, we have the following.

Definition 2: A directed interconnected system consisting of subsystems (56) and interconnection constraint (54) is said to be well-posed and stable if the map $I - \delta \mathbf{G}_{11}$ has a bounded inverse on $\mathcal{L}_2$, for every choice of δ as per (57), where

$$\mathbf{G} =: \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \quad (58)$$

has been partitioned to conform with the vector (v,d).

Finally, we will say that such a system with directed interconnection is contractive if it is stable and if there exists $\epsilon > 0$ such that $\|z\| \leq (1-\epsilon)\|d\|$ for all $d \in \mathcal{L}_2$, and for all δ_{ij} of the form specified in (57).

B. Analysis of Systems With Directed Interconnections

The great advantage of representing a directed interconnected system as an LFT is that we can apply standard results in robust control theory to obtain *necessary and sufficient* LMI conditions for well-posedness, stability, and contractiveness. As was done for systems with ideal interconnections, these conditions can then be used to provide a convex characterization of all controllers that have the same distributed structure as the plant and guarantee that these properties are satisfied in closed-loop.

Theorem 3: A directed interconnected system G is well-posed, stable, and contractive if and only if there exist symmetric matrices $X_{\mathrm{T}}^i \in \mathbb{R}_{\mathbf{S}}^{m_i \times m_i}$ and $X_{ij}^{11} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}$ for all $i,j = 1 \dots L$ such that $X_{ij}^{11} < 0$, $X_{\mathrm{T}}^i > 0$ and LMI (11) are satisfied for all $i = 1 \dots L$, with $Z_i^{11} := -\mathbf{diag}_j(X_{ij}^{11})$, $Z_i^{22} := \mathbf{diag}_j(X_{ji}^{11})$ and $Z_i^{12} = 0$.

Note the differences between Theorem 3 and Theorem 1: Scales X_{ij}^{11} are assumed to be negative definite while X_{ij}^{12} is set to zero for all i, j. Also, the LMI conditions are now both necessary and sufficient.

Proof: We first prove sufficiency. The goal is to determine the LTI operator $\mathbf{G}$ such that system G is described by the LFT of Fig. 5 and apply classical results from robust control theory to it. Notice that we can assume $\mathbf{G}$ to be stable, since from the (1,1) block of the left-hand side of (11), we have

$$\left(A_{\mathrm{TT}}^i\right)^* X_{\mathrm{T}}^i + X_{\mathrm{T}}^i A_{\mathrm{TT}}^i + \left(A_{\mathrm{ST}}^i\right)^* Z_i^{11} A_{\mathrm{ST}}^i + \left(C_{\mathrm{T}}^i\right)^* C_{\mathrm{T}}^i < 0$$

where $X_{\rm T}^i$ and Z_i^{11} are positive definite by hypothesis. This implies that

$$\left(A_{\rm TT}^i\right)^* X_{\rm T}^i + X_{\rm T}^i A_{\rm TT}^i < 0$$

i.e. that each individual subsystem is stable and, therefore, so is $\mathbf{G}$. Let the structured operators δ_i be defined by

$$\delta_i := \mathbf{diag}_j\left(\delta_{ji} I_{n_{ji}}\right). \tag{59}$$

Define the operator $\mathbf{G}_{wv}$ as $\mathbf{diag}_i(G_i^{wv})$. Similarly, define $\mathbf{G}_{wd}$, $\mathbf{G}_{zv}$ and $\mathbf{G}_{zd}$. Note that G_i^{wv} maps v_i to w_i, G_i^{wd} maps d_i to w_i, G_i^{zv} maps v_i to z_i, and G_i^{zd} maps d_i to z_i. If P is the permutation matrix that maps w to w', then we see that $\mathbf{G}$ in Fig. 5 is given by

$$\begin{aligned}\mathbf{G} &= \begin{bmatrix} P\mathbf{G}_{wv} & P\mathbf{G}_{wd} \\ \mathbf{G}_{zv} & \mathbf{G}_{zd}\end{bmatrix} \\ &= \begin{bmatrix} P & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} \mathbf{G}_{wv} & \mathbf{G}_{wd} \\ \mathbf{G}_{zv} & \mathbf{G}_{zd}\end{bmatrix} =: \bar{P}\hat{\mathbf{G}}.\end{aligned} \tag{60}$$

Since we have assumed that $n_{ij} = n_{ji}$ for all i, j, it can be verified that P is always a symmetric matrix. Now, we can rewrite system $\hat{\mathbf{G}}$ as

$$\hat{\mathbf{G}} = Q\bar{\mathbf{G}}Q^* \tag{61}$$

where $\bar{\mathbf{G}} := \mathbf{diag}_i(G_i)$, and Q is a permutation matrix that reorders the rows and columns of $\bar{\mathbf{G}}$. Thus, the interconnected system can be finally written as the LFT shown in Fig. 6.

From the scaled small-gain theorem, a sufficient condition for well-posedness, stability, and contractiveness is therefore that there exists Z' in the positive commutant of

$$\mathcal{D} := \{\delta = \mathbf{diag}_i(\delta_i) : \|\delta_{ij}\| \le 1, \qquad \text{for all } i, j\} \tag{62}$$

the δ_i being defined in (59), such that the following operator inequality holds:

$$Q(\bar{\mathbf{G}}^*)(Q^*)(\bar{P}^*)\begin{bmatrix} Z' & 0 \\ 0 & I\end{bmatrix}\bar{P}Q\bar{\mathbf{G}}Q^* - \begin{bmatrix} Z' & 0 \\ 0 & I\end{bmatrix} < 0. \tag{63}$$

Using the structure of the commutant, we can write $Z' =: -\mathbf{diag}_i(Z_i^{22}) > 0$ where, for all i, Z_i^{22} has the form indicated in the statement of Theorem 3. Using the fact that Q is a permutation and, hence, an orthogonal matrix, we have

$$(\bar{\mathbf{G}}^*)(Q^*)(\bar{P}^*)\begin{bmatrix} Z' & 0 \\ 0 & I\end{bmatrix}\bar{P}Q\bar{\mathbf{G}} - (Q^*)\begin{bmatrix} Z' & 0 \\ 0 & I\end{bmatrix}Q < 0 \tag{64}$$

which means that

$$(\bar{\mathbf{G}}^*)(Q^*)\begin{bmatrix} (P^*)Z'P & 0 \\ 0 & I\end{bmatrix}Q\bar{\mathbf{G}} - (Q^*)\begin{bmatrix} Z' & 0 \\ 0 & I\end{bmatrix}Q < 0. \tag{65}$$

Using the definition of Q, we finally obtain

$$(\bar{\mathbf{G}}^*)\begin{bmatrix} Y_1 & 0 & \dots & 0 \\ 0 & Y_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Y_L\end{bmatrix}\bar{\mathbf{G}} - \begin{bmatrix} Y_1' & 0 & \dots & 0 \\ 0 & Y_2' & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Y_L'\end{bmatrix} < 0 \tag{66}$$

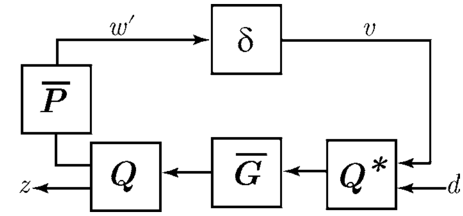

Fig. 6. Final LFT representation for analysis.

where

$$Y_i := \begin{bmatrix} Z_i^{11} & 0 \\ 0 & I\end{bmatrix} \quad Y_i' := \begin{bmatrix} -Z_i^{22} & 0 \\ 0 & I\end{bmatrix}, \qquad \text{for all } i = 1\dots L. \tag{67}$$

Recalling the way $\bar{\mathbf{G}}$ was defined, this means that the preceding conditions for robust performance reduce to a system of L coupled operator inequalities, namely

$$G_i^*\begin{bmatrix} Z_i^{11} & 0 \\ 0 & I\end{bmatrix}G_i - \begin{bmatrix} -Z_i^{22} & 0 \\ 0 & I\end{bmatrix} < 0. \tag{68}$$

An application of the KYP lemma now shows that the aforementioned inequality is satisfied if and only if the following LMI is satisfied:

$$\begin{aligned}&\begin{bmatrix} \left(A_{\rm TT}^i\right)^* X_{\rm T}^i + X_{\rm T}^i A_{\rm TT}^i & X_{\rm T}^i\left[\left(A_{\rm TS}^i\right)\left(B_{\rm T}^i\right)\right] \\ \begin{bmatrix}\left(A_{\rm TS}^i\right)^* \\ \left(B_{\rm T}^i\right)^*\end{bmatrix} X_{\rm T}^i & 0\end{bmatrix} \\ &+ \left[\begin{array}{c|cc} A_{\rm ST}^i & A_{\rm SS}^i & B_{\rm S}^i \\ C_{\rm T}^i & C_{\rm S}^i & D^i \\ \hline 0 & I & \end{array}\right]^* \begin{bmatrix} Y_i & 0 \\ 0 & -Y_i'\end{bmatrix} \\ &\times \left[\begin{array}{c|cc} A_{\rm ST}^i & A_{\rm SS}^i & B_{\rm S}^i \\ C_{\rm T}^i & C_{\rm S}^i & D^i \\ \hline 0 & I & \end{array}\right] < 0.\end{aligned} \tag{69}$$

Rearranging, we obtain (11) in the form given in the statement of the theorem. Also, the LMIs $X_{\rm T}^i > 0$ and $X_{ij}^{11} < 0$ are satisfied because of the way the scales were defined. This proves the sufficiency part of the theorem.

For necessity, we may invoke standard results [24], [29] which state that the scaled small-gain theorem is both necessary and sufficient for contractiveness when the interconnection operators are allowed to vary in the class of linear operators

$$\mathcal{D}_{\rm causal} = \{\delta \in \mathcal{D} : \ \delta_{ij} \text{ causal}, \qquad \text{for all } i, j\} \subset \mathcal{D}.$$

This proves Theorem 3. ■

C. Distributed Controller Synthesis for Directed Interconnections

We can now proceed as in Section IV and use Theorem 3 and the Elimination Lemma to derive convex conditions for the existence of distributed controllers for plants with directed interconnections. We are again looking for controllers "with the same structure as the plant" but this phrase has now gained a new meaning. The controller's interconnection relation is not only specified by the IQC of (54) but, more precisely, if each of the plant's communication channel is given by

$$v_{ij} = \left(\delta_{ji} I_{n_{ji}}\right) w_{ji}$$

for some contractive operator δ_{ji}, we are looking for controllers whose interconnection signals satisfy

$$v_{ij}^{\mathrm{K}} = \left(\delta_{ji} I_{n_{ji}^{\mathrm{K}}}\right) w_{ji}^{\mathrm{K}}$$

for some integer n_{ji}^{K} and *the same* operator δ_{ji}. This condition—which physically means that the plant and controller use identical communication channels—is of the same nature (and plays the same role in rewriting the closed-loop system as an LFT) as the basic assumption in LPV control design, where one is looking for controllers that are parameterized in the same way as the plant [1], [34].

The main synthesis result for systems with directed interconnection is as follows.

Theorem 4: Let G be a directed interconnected system consisting of subsystems (56) and interconnection relation (55). There exists a distributed controller with the same structure as G such that $n_{ij}^{\mathrm{K}} = n_{ij}$ for all $i, j = 1 \ldots L$ and the closed-loop system is well-posed, stable, and contractive if and only if there exist symmetric matrices $(X_{\mathrm{T}}^i)_{\mathrm{G}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{G}} \in \mathbb{R}_{\mathbf{S}}^{m_i \times m_i}$ and $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}$ for all $i, j = 1 \ldots L$ such that LMIs (47)–(49) hold for all i with $(Z_{12}^i)_{\mathrm{G}} = (\tilde{Z}_{12}^i)_{\mathrm{G}} = 0$ and, in addition $(X_{\mathrm{T}}^i)_{\mathrm{G}}$, $(Y_{\mathrm{T}}^i)_{\mathrm{G}} > 0$, $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}} < 0$, and

$$\begin{bmatrix} \left(X_{ij}^{11}\right)_{\mathrm{G}} & -I \\ -I & \left(Y_{ij}^{11}\right)_{\mathrm{G}} \end{bmatrix} \leq 0, \qquad \text{for all } i, j. \tag{70}$$

Proof: The proof is essentially identical to that of Theorem 2, starting with the analysis conditions of Theorem 3 applied to the closed-loop system. An important difference, however, is that definiteness of scales $(X_{ij}^{11})_{\mathrm{G}}$, $(Y_{ij}^{11})_{\mathrm{G}}$ and (70) make the necessity part simpler. Indeed, they guarantee that each Λ_i satisfies the inertia condition of Lemma 1 without having to take $n_{ij}^{\mathrm{K}} = 3n_{ij}$ for $i \neq j$. ■

D. Nonideal Interconnections and the Necessity of Analysis Conditions (11)

We now have analysis and synthesis tools for systems with ideal and directed interconnections. In the latter case, the conditions are even necessary and sufficient. An interesting question, however, remains.

We had already remarked in Section III-B that analysis conditions (11) were conservative for ideal interconnections. Yet, as the example at the beginning of this section showed, they are not sufficient to guarantee stability for directed interconnections, as arbitrarily small delays in the communication channels can destabilize an interconnected system satisfying (11). Thus, there seems to be a gap regarding how the properties of these two types of interconnected systems relate to Theorem 1.

For the sake of completeness, and because the construction is of independent interest, we shall now quantify this gap, characterizing the type of nonideal interconnections for which conditions (11) are necessary.

1) Another Class of Nonideal Interconnections: In this section, we consider yet another class of interconnection relations given by the IQCs

$$\Lambda\left(\begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix}\right) = \Lambda\left(\begin{bmatrix} w_{ji} \\ w_{ji} \end{bmatrix}\right), \qquad \text{for all } i \geq j \tag{71}$$

where for all signal $\beta \in \mathcal{L}_2^{2n_{ij}}$, $\Lambda(\beta)$ is the $2n_{ij} \times 2n_{ij}$ symmetric matrix defined by

$$\Lambda(\beta) := \int_0^\infty \beta(t)\beta(t)^* dt. \tag{72}$$

As was the case for directed interconnections, we need a characterization of IQC (71) via a class of linear operators in order to treat the corresponding nonideally interconnected systems as LFTs and define well-posedness, stability and contractiveness for them. The following was proved in [6].

Lemma 3: Given vector-valued signals β and ν with components in a separable Hilbert space, there exists a scalar unitary operator $\gamma : \mathcal{L}_2^1 \rightarrow \mathcal{L}_2^1$ such that $\nu = (\gamma I)\beta$ if and only if $\Lambda(\nu) = \Lambda(\beta)$, where $\Lambda(\bullet)$ is defined in (72).

Lemma 3 means that, for each communication channel, there exists a unitary linear operator $\delta_{ij} : \mathcal{L}_2^1 \rightarrow \mathcal{L}_2^1$, such that

$$\begin{aligned} \begin{bmatrix} v_{ij} \\ w_{ij} \end{bmatrix} &= \delta_{ji} I_{2n_{ij}} \left(\begin{bmatrix} w_{ji} \\ v_{ji} \end{bmatrix}\right), \text{ i.e.,} \\ v_{ij} &= (\delta_{ji} I) w_{ji} \\ v_{ji} &= \left(\delta_{ji}^{-1} I\right) w_{ij}, \qquad \text{for all } i \geq j. \end{aligned} \tag{73}$$

Note that if δ_{ij} is set to 1 for all $i, j = 1 \ldots L$, we recover the ideal interconnection (4). Using Lemma 3 and proceeding as for directed interconnected systems, we can represent a system G by the LFT of Fig. 5(a), where δ belongs to the set

$$\begin{aligned} \mathcal{D}_u := \big\{ \delta = \mathbf{diag}_i \left(\mathbf{diag}_j \left(\delta_{ji} I_{n_{ji}}\right)\right), \delta_{ij} : \mathcal{L}_2^1 \rightarrow \mathcal{L}_2^1 \\ \delta_{ij} \text{ unitary}, \ \delta_{ji} = \delta_{ij}^{-1}, \qquad \text{for } i \geq j \big\}. \end{aligned} \tag{74}$$

We can define well-posedness, stability, and contractiveness just like in Section VI-A.

In the remainder of this paper, we assume that each subsystem G_i of a nonideally interconnected system is a stable LTI system and, to simplify notations, that signals d_i and z_i have the same dimension, i.e., $p_i = q_i$, for all i. We also let $q_{zd} := \sum_{i=1}^L q_i$.

2) Necessity Results via an S-Procedure: In this subsection, we use ideas similar to those of [24] to prove that the LMI conditions (11) are necessary for the well-posedness, stability, and contractiveness of an interconnected system with subsystems given by (56) and nonideal interconnection (71). In particular, we prove the following.

Theorem 5: A nonideally interconnected system G is well-posed, stable and contractive only if there exist symmetric matrices $X_{ij}^{11} \in \mathbb{R}_{\mathbf{S}}^{n_{ij} \times n_{ij}}$ for all $i, j = 1 \ldots L$ and $X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}$ for all $i \geq j$, with X_{ii}^{12} skew-symmetric, such that the input–output map G_i of each subsystem satisfies

$$\begin{bmatrix} G_i \\ I \end{bmatrix}^* \begin{bmatrix} Z_i^{11} & Z_i^{12} \\ \left(Z_i^{12}\right)^* & Z_i^{22} \end{bmatrix} \begin{bmatrix} G_i \\ I \end{bmatrix} < 0 \tag{75}$$

for all $i = 1 \ldots L$, where

$$Z_i^{11} := \mathbf{diag}\left(-\mathop{\mathbf{diag}}_{1 \leq j \leq L} X_{ij}^{11}, I_{p_i}\right) \tag{76}$$

$$Z_i^{12} := \mathbf{diag}\left(-\mathop{\mathbf{diag}}_{1 \leq j \leq i} X_{ij}^{12}, \mathop{\mathbf{diag}}_{i < j \leq L} \left(X_{ji}^{12}\right)^*, 0^{p_i \times p_i}\right) \tag{77}$$

$$Z_i^{22} := \mathbf{diag}\left(\mathop{\mathbf{diag}}_{1 \leq j \leq L} X_{ji}^{11}, -I_{p_i}\right). \tag{78}$$

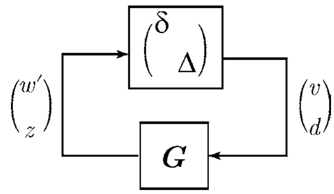

Fig. 7. LFT representation, Theorem 5.

Note that since all G_i are assumed to be stable, the adjoint used in the above theorem does exist and we can apply the KYP lemma to yield LMI (11). The aforementioned necessity result can also be stated as follows: If the operator inequality (75), and hence LMI (11) are *not* satisfied, then there exists an operator $\delta \in \mathcal{D}_u$, and hence signals satisfying (56) and (71) which renders the LFT of Fig. 5 ill-posed.

The proof of this result extends standard S-procedure results [13] to the case where the operator δ in Fig. 5(a). is unitary (instead of the usual contractive—see [24] and [29]). Another extension is the fact that we allow both an operator δ_{ij} and its inverse to appear in the structured operator δ, as per (74).

Proof: Consider the system shown in Fig. 5(a), with the operator

$$\delta := \mathbf{diag}_i(\delta_i) \in \mathcal{D}_u.$$

The well-posedness, stability and contractiveness of this system are well-known to be equivalent, via the main loop theorem [28], to the robust well-posedness of the LFT shown in Fig. 7, where $\Delta : \mathbb{R}^{q_{zd}} \to \mathbb{R}^{q_{zd}}$ is an arbitrary contractive operator. If $\mathbf{\Delta} := \mathbf{diag}(\delta, \Delta)$, this means that the map $I - \mathbf{\Delta G}$ must have a (uniformly) bounded inverse. Suppose that this is the case. Let $\bar{\mathcal{D}}$ be the class of all operators $\mathbf{\Delta}$. Then there does not exist a nonzero signal $\eta := (v, d)$ such that $(I - \mathbf{\Delta G})\eta = 0$ for any $\mathbf{\Delta} \in \bar{\mathcal{D}}$. In order the simplify the proof, we find it convenient to write $\eta_i := (v_i, d_i)$ and introduce the maps $H_{ij} : \mathcal{L}_2^{n_i+p_i} \to \mathcal{L}_2^{n_{ij}}$, such that $H_{ij}\eta_i = v_{ij}$ and the map $H_i^{zd} : \mathcal{L}_2^{n_i+p_i} \to \mathcal{L}_2^{p_i}$, such that $H_i^{zd}\eta_i = d_i$, for any $\eta_i \in \mathcal{L}_2^{n_i+p_i}$ which has been appropriately partitioned to conform with the components of (v_i, d_i).

By the IQC characterization of unitary δ_{ij} of Lemma 3, the fact that $I - \mathbf{\Delta G}$ has a bounded inverse implies that there is no η such that $\|\eta\| = 1$ and equations

$$\Lambda\left(\begin{bmatrix} w_{ij} \\ v_{ij} \end{bmatrix}\right) = \Lambda\left(\begin{bmatrix} v_{ji} \\ w_{ji} \end{bmatrix}\right), \qquad i \le j \tag{79}$$

$$\|z\|^2 - \|d\|^2 \ge 0 \tag{80}$$

are satisfied, where $w_{ij} = H_{ij}G_i\eta_i$, $z = \mathbf{cat}_i(z_i)$, and $z_i = H_i^{zd}G_i\eta_i$. Thus, there are no solutions $\eta(\|\eta\| = 1)$ to

$$\Lambda\left(\begin{bmatrix} H_{ij}G_i\eta_i \\ H_{ij}\eta_i \end{bmatrix}\right) - \Lambda\left(\begin{bmatrix} H_{ji}\eta_j \\ H_{ji}G_j\eta_j \end{bmatrix}\right) = 0, \qquad i \le j \tag{81}$$

$$\sum_{i=1}^{L}\left(\left\|H_i^{zd}G_i\eta_i\right\|^2 - \left\|H_i^{zd}\eta_i\right\|^2\right) \ge 0. \tag{82}$$

Define the vector space $\mathbb{V}$ by

$$\mathbb{V} := \left(\prod_{i \le j} \mathbb{R}_{\mathbf{S}}^{2n_{ij} \times 2n_{ij}}\right) \times \mathbb{R} \tag{83}$$

where $\prod$ denotes Cartesian product, and endow it with the usual inner product that, to a pair (X, Y) of elements of $\mathbb{V}$ such that $X = (X_1, \ldots, X_{L(L+1)/2}, r)$, $Y = (Y_1, \ldots, Y_{L(L+1)/2}, q)$, associates $\sum_{i=1}^{L(L+1)/2} \mathbf{trace}(X_i Y_i) + rq$.

Let $\mathcal{S}_{ij}(i \le j)$ and $\mathcal{S}_{zd}$ be the sets of quadratic forms on $\mathcal{L}_2$ defined by

$$\mathcal{S}_{ij} := \left\{\Lambda\left(\begin{bmatrix} H_{ij}G_i\eta_i \\ H_{ij}\eta_i \end{bmatrix}\right) - \Lambda\left(\begin{bmatrix} H_{ji}\eta_j \\ H_{ji}G_j\eta_j \end{bmatrix}\right) : \|\eta\| = 1\right\} \tag{84}$$

$$\mathcal{S}_{zd} := \left\{\sum_{i=1}^{L}\left(\left\|H_i^{zd}G_i\eta_i\right\|^2 - \left\|H_i^{zd}\eta_i\right\|^2\right) : \|\eta\| = 1\right\} \tag{85}$$

and let $\mathcal{S}_{ij}(\eta)$, $\mathcal{S}_{zd}(\eta)$ be the elements of $\mathcal{S}_{ij}$ and $\mathcal{S}_{zd}$, respectively, generated by a specific signal $\eta \in \mathcal{L}_2$ of unit norm. Embed the above sets of quadratic forms in $\mathbb{V}$ in the natural way:

$$\mathbb{V} \supset \mathcal{S} := \left(\prod_{i \le j} \mathcal{S}_{ij}\right) \times \mathcal{S}_{zd}. \tag{86}$$

Now, it is known that the set $\mathcal{S}$ of time-invariant (matrix-valued) quadratic forms on $\mathcal{L}_2$ is convex [29], and by the hypothesis on nonexistence of unit-norm solutions η to (81) and (82), $\mathcal{S}$ does not intersect the (trivially closed and convex) set $\mathcal{C}$ in $\mathbb{V}$ given by $\mathcal{C} := (\prod_{i \ge j}\{0^{2n_{ij} \times 2n_{ij}}\}) \times \mathbb{R}_+$. However, since $\mathcal{S}$ is not necessarily closed, it may happen that $\mathrm{cl}(\mathcal{S})$ intersects $\mathcal{C}$. On the other hand, the following proposition, proved in the Appendix, shows that this is not possible if the LFT in Fig. 7 is robustly well-posed.

Proposition 4: If the closure $\mathrm{cl}(\mathcal{S})$ of $\mathcal{S}$ intersects $\mathcal{C}$, then the operator $I - \mathbf{\Delta G}$ does not have a bounded inverse.

Invoking the Hahn–Banach theorem on the separation of strictly separated convex sets in finite dimensional space, we see that there exist matrices $P_{ij} \in \mathbb{R}_{\mathbf{S}}^{2n_{ij} \times 2n_{ij}}$, $i \le j$, such that

$$\sum_{i \le j} \mathbf{trace}\left((\mathcal{S}_{ij}(\eta))(P_{ij})\right) + \mathcal{S}_{zd}(\eta) \le -\epsilon \tag{87}$$

for some $\epsilon > 0$ and all η such that $\|\eta\| = 1$. Therefore

$$\begin{aligned} -\epsilon \ge \sum_{i=1}^{L} &\left[\sum_{i \le j} \mathbf{trace}\left(\Lambda\left(\begin{bmatrix} H_{ij}G_i\eta_i \\ H_{ij}\eta_i \end{bmatrix}\right) P_{ij}\right)\right. \\ &\left. - \sum_{i \ge j} \mathbf{trace}\left(\Lambda\left(\begin{bmatrix} H_{ij}\eta_i \\ H_{ij}G_i\eta_i \end{bmatrix}\right) P_{ji}\right)\right] \\ &+ \sum_{i=1}^{L}\left[\left\|H_i^{zd}G_i\eta_i\right\|^2 - \left\|H_i^{zd}\eta_i\right\|^2\right]. \end{aligned} \tag{88}$$

Using the definition of $\Lambda(\bullet)$, and noting that the above inequality decouples over i, we have, for each $i = 1 \ldots L$

$$\left\langle \eta_i, \begin{bmatrix} G_i \\ I \end{bmatrix}^* \left(\sum_{i<j} \begin{bmatrix} H_{ij}^* P_{ij}^{11} H_{ij} & H_{ij}^* P_{ij}^{12} H_{ij} \\ H_{ij}^* \left(P_{ij}^{12}\right)^* H_{ij} & H_{ij}^* P_{ij}^{22} H_{ij} \end{bmatrix}\right.\right.$$
$$+ \begin{bmatrix} H_{ii}^* \left(P_{ii}^{11} - P_{ii}^{22}\right) H_{ii} & H_{ii}^* \left(P_{ii}^{12} - \left(P_{ii}^{12}\right)^*\right) H_{ii} \\ H_{ii}^* \left(\left(P_{ii}^{12}\right)^* - P_{ii}^{12}\right) H_{ii} & H_{ii}^* \left(P_{ii}^{22} - P_{ii}^{11}\right) H_{ii} \end{bmatrix}$$

$$
-\sum_{i>j}\begin{bmatrix} H_{ij}^{*}P_{ji}^{22}H_{ij} & H_{ij}^{*}\left(P_{ji}^{12}\right)^{*}H_{ij}\\ H_{ij}^{*}P_{ji}^{12}H_{ij} & H_{ij}^{*}P_{ji}^{22}H_{ij}\end{bmatrix}
+\begin{bmatrix}\left(H_i^{zd}\right)^{*}H_i^{zd} & 0\\ 0 & -\left(H_i^{zd}\right)^{*}H_i^{zd}\end{bmatrix}\Bigg)\begin{bmatrix}G_i\\ I\end{bmatrix}\eta_i\Bigg\rangle \leq -\epsilon \quad (89)
$$

for all $\eta_i \in \mathcal{L}_2$ such that $\|\eta_i\| = 1$, where $P_{ij} = \begin{bmatrix} P_{ij}^{11} & P_{ij}^{12}\\ P_{ij}^{21} & P_{ij}^{22}\end{bmatrix}$ has been partitioned into four n_{ij} by n_{ij} blocks for each $i \leq j$. Now, set

$$
X_{ij}^{11} := \begin{cases} -P_{ij}^{11}, & \text{if } i<j\\ -P_{ij}^{11}+P_{ij}^{22}, & \text{if } i=j\\ P_{ji}^{22}, & \text{otherwise}\end{cases} \quad (90)
$$

and

$$
X_{ij}^{12} := \begin{cases} -P_{ij}^{12}+(P_{ij}^{12})^{*}, & \text{if } i=j\\ \left(P_{ji}^{12}\right)^{*}, & \text{if } i>j\end{cases} \quad (91)
$$

and notice that $X_{ij}^{11} \in \mathbb{R}_{\mathbf{S}}^{n_{ij}\times n_{ij}}$ for all $i,j = 1\ldots L$, $X_{ij}^{12} \in \mathbb{R}^{n_{ij}\times n_{ij}}$ for $i \geq j$ and that X_{ii}^{12} is an arbitrary skew-symmetric matrix. From the definitions of H_{ij} and H_i^{zd}, it is immediately apparent that (89) implies (75). ■

VII. Conclusion

We have shown how the tools of [8], originally developed for spatially invariant interconnected systems, can be extended to derive sufficient convex synthesis conditions for heterogeneous systems interconnected over arbitrary graphs.

While dissipativity theory [26], [41] was used to motivate and derive the first set of analysis conditions (11), IQCs theory allowed us to show that the LMIs become nonconservative for a particular model (Section VI-D) of the communication channels between subsystems. The problem of synthesis of controllers subject to having the same structure as the plant was cast as a convex semidefinite program with an interesting coupled structure in Theorem 2, the main result of this paper. It was also observed that this may lead to efficient ways of solving the semidefinite programs (see [21] for more details).

Since the class of IQCs in Section VI-D does not capture delays (even arbitrarily small) that may be present between communicating subsystems, we introduced another choice of IQCs (Section VI-A) to model the communication channels and provide robustness to nonidealities such as delays and attenuations. This had the additional attractive property (at the expense of conservatism) that the number of communication channels between controller subsystems is smaller in general than that required by Theorem 2.

It is an open question as to whether other, less conservative IQC models of time delays (see, e.g., [14]) can be used to design less conservative structured controllers. So, it is the development of SDP algorithms that takes full advantage of the structure imposed to LMIs by the distributed nature of the underlying control problem.

Appendix

Proof of Proposition 4

In what follows, the notation $|A| < B$ for symmetric matrices A and B with $B > 0$ will mean that $-B < A < B$. Where required, we use the notation established in Section VI for the various signals relevant to the interconnected system $\mathbf{G}$ in Fig. 7. The following lemma is, for example, proved in [13, App. B].

Lemma 4: Given two signals β and ν in $\mathcal{L}_2^n$ such that $\|\beta\|^2 - \|\nu\|^2 > -\epsilon$ and $\|\nu\| \leq 1$, there exists a contractive full-block operator $\Delta : \mathcal{L}_2^n \to \mathcal{L}_2^n$ such that $\|\Delta(\beta) - \nu\| < o(\epsilon)$, where $o(\epsilon)$ is some positive function that tends to zero as $\epsilon \to 0$.

We also require the following proposition.

Proposition 5: Given two signals β and ν in $\mathcal{L}_2^n$ such that $|\Lambda(\beta) - \Lambda(\nu)| < \epsilon I_n$, $\|\nu\| = 1$, there exists a (scalar) unitary operator $\delta : \mathcal{L}_2^1 \to \mathcal{L}_2^1$ such that $\|\delta I_n(\beta) - \nu\| < o(\epsilon)$.

Proof: Notice that $\Lambda(\nu)$ is the matrix whose (i,j) entry is the number $\langle \nu_i, \nu_j\rangle$, where the ν_i are the scalar components of ν. Consider the subspace S of $\mathcal{L}_2^1$ spanned by the ν_i, $i = 1\ldots n$. Let $\mathcal{B} := \{v_1, v_2, \ldots, v_n\}$ be any orthonormal basis for S (if the dimension of S is less than n, $\mathcal{B}$ may be augmented with elements not in S so that its dimension can be assumed to be n without loss of generality). Associate to each ν the vectors $\nu^i \in \mathbb{R}^n$ such that ν^i is the vector of the n components of ν_i with respect to the ordered basis $\mathcal{B}$. Thus, the (i,j) element of $\Lambda(\nu)$ is nothing but the number $\langle \nu^i, \nu^j\rangle$, where $\langle \bullet, \bullet\rangle$ now denotes the Euclidean inner product on $\mathbb{R}^n$ (we use the same notation for the inner products on $\mathbb{R}^n$ and $\mathcal{L}_2$, since this will be clear from context).

Suppose the proposition were not true, that is, that there exist sequences $\{\nu(i)\}_{i=1}^{\infty}$ and $\{\beta(i)\}_{i=1}^{\infty}$ of signals in $\mathcal{L}_2^n$ such that $\|\nu(i)\| = 1$ and

$$
|\Lambda\left(\nu(i)\right) - \Lambda\left(\beta(i)\right)| < \epsilon_i I \quad (92)
$$

but such that

$$
\|\delta I\left(\beta(i)\right) - \nu(i)\| > c > 0 \quad (93)
$$

for some constant c, for some sequence of $\epsilon_i \to 0$, and for all unitary δ. Let $\tilde{\nu}_i$ and $\tilde{\beta}_i$ be the vectors in $\mathbb{R}^{n^2}$ given by

$$
\tilde{\nu}_i = \underset{1\leq j\leq n^2}{\mathbf{cat}}\left(\nu(i)^j\right)
$$
$$
\tilde{\beta}_i = \underset{1\leq j\leq n^2}{\mathbf{cat}}\left(\beta(i)^j\right).
$$

Note that the hypothesis (92) and the fact that $\|\nu(i)\| = 1$ imply that $|\tilde{\nu}_i| = 1$ and that $\sqrt{(1-\epsilon_i)} < |\tilde{\beta}_i| < \sqrt{(1+\epsilon_i)}$, so that $\tilde{\beta}_i$ and $\tilde{\nu}_i$ vary in compact sets. Choose subsequences $\tilde{\beta}_{n_i}$, $\tilde{\nu}_{n_i}$ that converge to points $\tilde{\beta}$ and $\tilde{\nu}$ in $\mathbb{R}^{n^2}$. By the continuity of inner products on $\mathbb{R}^n$, these satisfy $\|\tilde{\nu}\| = 1$ and

$$
\langle \beta^k, \beta^l\rangle = \langle \nu^k, \nu^l\rangle \quad (94)
$$

for all $k,l = 1\ldots n$. Here, β^k, $\nu^k \in \mathbb{R}^n$ are the vectors formed by components $(k-1)n+1$ through kn of $\tilde{\beta}$ and $\tilde{\nu}$, respectively.

Equation (94) just means that there exists a unitary linear transformation $\hat{\delta}$ on $\mathbb{R}^n$ which maps β^i to ν^i for each $i = 1 \dots n$. By the continuity of $\hat{\delta}$, this means that for a large enough index N, we have that $|\hat{\delta}(\beta(N)^i) - \nu(N)^i| < (c/\sqrt{n})$, where $|\bullet|$ denotes vector norm on $\mathbb{R}^n$. This relation can be used to construct a unitary operator δ on $\mathcal{L}_2^1$ which violates (93). Let T_N and S_N be the subspaces of $\mathcal{L}_2^1$ spanned by $\beta(N)_i$ and $\nu(N)_i$, respectively, and let $\mathcal{B}_\beta = \{v_1 \dots v_n\}$ and $\mathcal{B}_\nu := \{w_1, \dots, w_n\}$ be the orthonormal bases of T_N and S_N, respectively which were used to construct the vectors $\beta(N)^i$ and $\nu(N)^i$, possibly augmented to ensure that their dimension is n. Extend these to (countable) orthonormal bases for the separable Hilbert space $\mathcal{L}_2^1$ by adjoining to them the sets of orthonormal vectors $\mathcal{B}'_\beta = \{v_{n+1}, v_{n+2}, \dots\}$ and $\mathcal{B}'_\nu = \{w_{n+1}, w_{n+2}, \dots\}$, respectively. The map δ is defined on the basis $\mathcal{B}_\beta \cup \mathcal{B}'_\beta$ of $\mathcal{L}_2^1$ as follows.

- Suppose e_i is the i^{th} standard basis vector in $\mathbb{R}^n$, for $i = 1, 2 \dots n$; let $(\alpha_{i1}, \alpha_{i2} \dots, \alpha_{in}) := \hat{\delta}(e_i)$. Then, $\delta(v_i)$ is defined as $\sum_{j=1}^n \alpha_{ij} w_j$ for $i = 1, 2 \dots n$.
- $\delta(v_i) := w_i$ for $i > n$.

The definition of δ on the subspace T_N is seen to imply that $\|\delta(\beta(N)_i) - \nu(N)_i\| < (c/\sqrt{n})$ for each $i = 1, 2 \dots n$. Therefore $\|\delta I(\beta(N)) - \nu(N)\| < c$, contradicting (93). Also since the finite-dimensional linear transformation $\hat{\delta}$ is unitary, we have by the previous construction that δ is unitary. ■

The previous lemma and proposition now imply the following

Lemma 5: Suppose $|\mathcal{S}_{ij}(\eta)| < \epsilon I_{2n_{ij}}$ and that $\mathcal{S}_{zd}(\eta) > -\epsilon$ for some $\epsilon > 0$ and $\eta \in \mathcal{L}_2^{n+q_{zd}}$ $(i \leq j)$. Also, suppose that $\|\eta\| = 1$ and that η is supported on the time interval $[0, \hat{T}]$ for some $\hat{T} > 0$. Then, there exists a number $T \geq \hat{T}$ and unitary operators $\delta_{ij} : \mathcal{L}_2^1[0,T] \to \mathcal{L}_2^1[0,T]$ $(i \leq j)$ such that

$$\left\| \begin{bmatrix} v_{ji} \\ w_{ji} \end{bmatrix} - \delta_{ij} I_{2n_{ij}} \left(\begin{bmatrix} P_T w_{ij} \\ P_T v_{ij} \end{bmatrix} \right) \right\| < o(\epsilon) \tag{95}$$

and a contractive operator $\Delta : \mathcal{L}_2^{q_{zd}}[0,T] \to \mathcal{L}_2^{q_{zd}}[0,T]$ such that

$$\|\Delta(P_T z) - d\| < o(\epsilon) \tag{96}$$

on the interval $[0, T]$.

Here, as before, $w_{ij} = H_{ij} G_i \eta_i$, and $z = \mathbf{cat}_i(H_i^{zd} G_i \eta_i)$. Moreover, by choosing T large enough, the following inequalities can also be satisfied:

$$\|P_T z - z\| < \epsilon \tag{97}$$
$$\|P_T w - w\| < \epsilon. \tag{98}$$

We remark that although $P_T z$ and z live in different spaces, the notation $P_T z - z$ makes sense by embedding the space $\mathcal{L}_2[0,T]$ in $\mathcal{L}_2$ in the natural way. We find it convenient to abuse notation in this way; we do so without further comment. For the proof of Lemma 5, note that though the w_{ij} and z do not have finite support in general, they can be truncated (with an arbitrarily long but finite support T) so that they satisfy the conditions of the lemma. Also, note that Lemma 4 and Proposition 5 hold when the signal space considered is $\mathcal{L}_2[0,T]$ rather than $\mathcal{L}_2$.

Now, suppose that $\mathcal{S}$ and $\mathcal{C}$ are not strictly separated, i.e., there exists a sequence of $\epsilon_k > 0$ $(k = 1, 2, \dots)$, and vectors $\eta^k \in \mathcal{L}_2^{n+q_{zd}}$, $\|\eta^k\| = 1$ such that $\epsilon_k \to 0$, that

$$\left|\mathcal{S}_{ij}(\eta^k)\right| < \epsilon_k I_{2n_{ij}} \tag{99}$$

for $i \leq j$, and that

$$\mathcal{S}_{zd}(\eta^k) > -\epsilon_k. \tag{100}$$

Obviously, the η^k can be assumed to have finite support since one can truncate and rescale η^k to have unit norm; using a sufficiently large time horizon for the truncation will ensure that the hypotheses (99) and (100) are still satisfied. From Lemma 5, there exist positive numbers T_k $(k = 0, 1 \dots \infty)$ satisfying $T_{k-1} < T_k$, $T_0 = 0$, $T_\infty = \infty$, unitary operators $\delta_{ij}^k : \mathcal{L}_2^1[T_{k-1}, T_k] \to \mathcal{L}_2^1[T_{k-1}, T_k]$ and contractive operators $\Delta_k : \mathcal{L}_2^{q_{zd}}[T_{k-1}, T_k] \to \mathcal{L}_2^{q_{zd}}[T_{k-1}, T_k]$ such that the η^k are supported in the time intervals $[T_{k-1}, T_k]$ and such that

$$\left\| \begin{bmatrix} v_{ji}^k \\ w_{ji}^k \end{bmatrix} - \delta_{ij}^k I_{2n_{ij}} \left(\begin{bmatrix} P_{[T_{k-1},T_k]} w_{ij}^k \\ P_{[T_{k-1},T_k]} v_{ij}^k \end{bmatrix} \right) \right\| < o(\epsilon_k) \tag{101}$$
$$\left\|\Delta_k \left(P_{[T_{k-1},T_k]} z^k\right) - d^k\right\| < o(\epsilon_k) \tag{102}$$
$$\left\|P_{[T_{k-1},T_k]} z^k - z^k\right\| < \epsilon_k \tag{103}$$
$$\left\|P_{[T_{k-1},T_k]} w^k - w^k\right\| < \epsilon_k. \tag{104}$$

By assembling together the operators δ_{ij}^k and Δ_k as shown in (105) and (106), we can construct the following operators δ_{ij} $(i \leq j)$ and Δ, where δ is unitary and Δ is contractive:

$$\delta_{ij} := \sum_{k=1}^{\infty} P_{[T_{k-1},T_k]}^* \delta_{ij}^k P_{[T_{k-1},T_k]} \tag{105}$$
$$\Delta := \sum_{k=1}^{\infty} P_{[T_{k-1},T_k]}^* \Delta_k P_{[T_{k-1},T_k]}. \tag{106}$$

Define $\mathbf{\Delta} := \mathbf{diag}(\delta, \Delta)$. Since the sequences of operators $\{\delta_{ij}^k\}_{k=1}^\infty$ and $\{\Delta_k\}_{k=1}^\infty$ are uniformly bounded (by the number 1), it is easily seen that (101)–(104) imply that $\|(I - \mathbf{\Delta G})\eta^k\| \to 0$ as $k \to \infty$ with $\|\eta^k\| = 1$, where we emphasize that η_k are supported in the intervals $[T_{k-1}, T_k]$. Therefore, $(I - \mathbf{\Delta G})^{-1}$ cannot be bounded. (In fact, by Banach's theorem [18], $(I - \mathbf{\Delta G})^{-1}$ does not exist, since the inverse of a bounded operator on a Banach space, if it exists, is bounded). This completes the proof of Proposition 4.

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References

[1] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 853–864, May 1995.

[2] G. A. de Castro and F. Paganini, "Control of distributed arrays with recursive information flow: Some case studies," in *Proc. IEEE Conf. Decision and Control*, 1999, pp. 191–196.

[3] B. Bamieh, F. Paganini, and M. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 1091–1118, July 2002.

[4] H. T. Banks, R. C. Smith, and Y. Wang, *Smart Material Structures Modeling, Estimation and Control*. New York: Wiley, 1996.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[6] R. S. Chandra and R. D'Andrea, "Necessity of the small gain theorem for multidimensional systems," in *Proc. IEEE Conf. Decision and Control*, 2003, pp. 2859–2864.

[7] B. D. Chen and S. Lall, "Dissipation inequalities for distributed systems on graphs," in *Proc. IEEE Conf. Decision and Control*, 2003, pp. 3084–3090.

[8] R. D'Andrea and G. E. Dullerud, "Distributed control of spatially interconnected systems," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 1478–1495, Sept. 2003.

[9] R. D'Andrea, C. Langbort, and R. S. Chandra, "A state space approach to control of interconnected systems," in *Mathematical Systems Theory in Biology, Communications, Computation and Finance*, J. Rosenthal and D. S. Gilliam, Eds. New York: Springer-Verlag, 2003.

[10] R. Diestel, *Graph Theory*, 2nd ed. New York: Springer-Verlag, 2000.

[11] J. C. Doyle, K. Glover, P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, Aug. 1989.

[12] G. E. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *IEEE Trans. Automat. Contr.*, 2004, to be published.

[13] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory*. New York: Springer-Verlag, 2000.

[14] M. Fu, H. Li, and S.-I. Niculescu, "Robust stability and stabilization of time-delay system via integral quadratic constraint approach," in *Stability and Control of Time-Delay Systems*, L. Dugard and E. I. Verriest, Eds. London, U.K.: Springer-Verlag, vol. 228, pp. 101–116.

[15] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to $\mathcal{H}_\infty$ control," *Int. J. Robust Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[16] Z. Hu, "Decentralized stabilization of large-scale interconnected systems with delays," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 180–182, Jan. 1994.

[17] V. Kapila, A. G. Sparks, J. Buffington, and Q. Yan, "Spacecraft formation flying: Dynamics and control," in *Proc. Amer. Control Conf.*, 1999, pp. 4137–4141.

[18] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis* New York, 1970.

[19] C. Langbort and R. D'Andrea, "Imposing boundary conditions for a class of spatially interconnected systems," in *Proc. Amer. Control Conf.*, 2003, pp. 107–112.

[20] ——, "Distributed control of heterogeneous systems interconnected over an arbitrary graph," in *Proc. IEEE Conf. Decision and Control*, 2003, pp. 2835–2840.

[21] C. Langbort, L. Xiao, R. D'Andrea, and S. Boyd, "A decomposition approach to distributed analysis of networked systems," in *Proc. IEEE Conf. Decision and Control*, 2004, to be published.

[22] C. Langbort. Graph_Control. Cornell Univ., Ithaca, NY. [Online]http://www.tam.cornell.edu/~cl263/dist_cont.html

[23] D. Laughlin, M. Morari, and R. D. Braatz, "Robust performance of cross-directional basis-weight control in paper machines," *Automatica*, vol. 29, no. 6, pp. 1395–1410, 1993.

[24] A. Megretski and S. Treil, "Power distribution inequalities in optimization and robustness of uncertain systems," *J. Math. Syst., Estimat. Control*, vol. 3, no. 3, pp. 301–319, Mar. 1993.

[25] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 819–830, June 1997.

[26] P. J. Moylan and D. J. Hill, "Stability criteria for large scale systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 143–149, Feb. 1978.

[27] M. Napoli and B. Bamieh, "Modeling and observed design for an array of electrostatically actuated microcantilevers," in *Proc. IEEE Conf. Decision and Control*, 2001, pp. 4274–4279.

[28] A. Packard and J. C. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.

[29] F. Paganini and J. C. Doyle, "Analysis of implicitly defined systems," in *Proc. IEEE Conf. Decision and Control*, 1994, pp. 2782–2786.

[30] J. H. Park, "Design of robust decentralized dynamic controller for uncertain large-scale interconnected systems with time-delays," *IEICE Trans. Fundamentals*, vol. E84-A, no. 7, July 2001.

[31] A. Rantzer, "On the Kalman-Yakubovich-Popov Lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, 1996.

[32] A. Rantzer and A. Megretski, "System analysis via integral quadratic constraints," in *Proc. IEEE Conf. Decision and Control*, 1994, pp. 3062–3067.

[33] M. Rotkowitz and S. Lall, "Decentralized control information structures preserved under feedback," in *Proc. IEEE Conf. Decision and Control*, 2002, pp. 569–575.

[34] C. Scherer, "LPV control and full block multipliers," *Automatica*, vol. 37, pp. 361–375, 2001.

[35] G. E. Stewart, D. M. Gorinevsky, and G. A. Dumont, "Spatial loop-shaping: a case study on cross-directional profile control," in *Proc. Amer. Control Conf.*, 1999, pp. 3098–3103.

[36] G. Scorletti and G. Duc, "An LMI approach to decentralized control," *Int. J. Control*, vol. 74, no. 3, pp. 211–224, 2001.

[37] D. Swaroop and J. K. Hedrick, "Constant spacing strategies for platooning in automated highway systems," *J. Dyna. Syst., Measure., Control*, vol. 121, pp. 462–470, 1999.

[38] S. Treil. (1999) The gap between the complex structured singular value and its upper bound is infinite. Brown University, Providence, RI. [Online] www.math.brown.edu/faculty/treil.html

[39] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd ed. London, U.K.: Springer-Verlag, 2000.

[40] P. Voulgaris, "A convex characterization of classes of problems in control with specific interaction and communication structures," in *Proc. Amer. Control Conf.*, 2001, pp. 3128–3133.

[41] J. C. Willems, "Dissipative dynamical systems I & II," *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–393, 1972.

[42] ——, "Paradigms and puzzles in the theory of dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 259–294, Mar. 1991.

[43] J. D. Wolfe, D. F. Chichka, and J. L. Speyer, "Decentralized controllers for unmanned aerial vehicle formation flight," presented at the AIAA Guidance, Navigation, and Control Conf., San Diego, CA, 1996, AIAA-1996–3833.

Cédric Langbort received the Aerospace Engineering Degree, with highest honors, from the École Nationale Supérieure de l'Aéronautique et de l'Espace-Supaero, Toulouse, France, in 1999, and the Diplôme d'Etudes Approfondies in dynamical systems from the Institut Non-Linéaire de Nice, Nice, France, in 2000. He is currently working toward the Ph.D. degree in the Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY.

His current research interests concern all aspects of the dynamics and control of complex systems.

Mr. Langbort was a finalist in the Best Student Paper Award competition at the 2003 American Control Conference, Denver, CO.

Ramu Sharat Chandra was born in India in 1977. He received the B.Tech. degree in mechanical engineering from the Indian Institute of Technology, Madras, India, in 1999. He is currently working toward the Ph.D. in the Department of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY.

His interests are robust control and applied mathematics.

Raffaello D'Andrea (S'93–A'96–SM'01) was born in Pordenone, Italy, in 1967. He received the BASc. degree in engineering science from the University of Toronto, Toronto, ON, Canada, in 1991 and the M.S. and Ph.D. degrees in electrical engineering from the California Institute of Technology, Pasadena, in 1992 and 1997, respectively.

Since 1997, he has been with the Department of Mechanical and Aerospace Engineering at Cornell University, Ithaca, NY, where he is an Associate Professor.

Dr. D'Andrea has been a recipient of the University of Toronto W.S. Wilson Medal in Engineering Science, an IEEE Conference on Decision and Control Best Student Paper Award, an American Control Council O. Hugo Schuck Best Paper Award, a National Science Foundation CAREER Award, and a DOD sponsored Presidential Early Career Award for Scientists and Engineers. He was the System Architect and Faculty Advisor of the RoboCup World Champion Cornell Autonomous Robot Soccer team in 1999 (Stockholm, Sweden), 2000 (Melbourne, Australia), 2002 (Fukuoka, Japan), and 2003 (Padova, Italy), and third place winner in 2001 (Seattle, WA). His recent collaboration with Canadian artist Max Dean, "The Table," an interactive installation, appeared in the Biennale di Venezia in 2001. It was on display at the National Gallery of Canada in Ottawa, ON, in 2002 and 2003, and is now part of its permanent collection.